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SINGULAR ELLIPTIC EQUATIONS WITH NONLINEARITIES OF SUBCRITICAL AND CRITICAL GROWTH

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Abstract In this paper, we consider the problem $-\Delta u = -u^{-\beta}\chi_{\{u>0\}} + f(u)$ in Ω with $u = 0$ on $\partial\Omega$, where $0 < \beta < 1$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. We are able to solve this problem provided f has subcritical growth and satisfy certain hypothesis. We also consider this problem with $f(s) = \lambda s + s^{\frac{N+2}{N-2}}$ and $N \geq 3$. In this case, we are able to obtain a solution for large values of λ. We replace the singular function $u^{-\beta}$ by a function $g_\epsilon(u)$ which pointwisely converges to $u^{-\beta}$ as $\epsilon \to 0$. The corresponding energy functional to the perturbed equation $-\Delta u + g_\epsilon(u) = f(u)$ has a critical point u_ϵ in $H_0^1(\Omega)$, which converges to a non-trivial non-negative solution of the original problem as $\epsilon \to 0$.

Keywords: singular problem; variational methods; a priori estimates; critical growth

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1. Introduction

In this paper, we show that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + f(u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

has a non-negative solution when f has subcritical growth. The expression $\chi_{\{u>0\}}$ denotes the characteristic function corresponding to the set $\{x \in \Omega : u(x) > 0\}$ and by convention $u^{-\beta}\chi_{\{u>0\}} = 0$ if $u = 0$. Hereafter, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, $0 < \beta < 1$ and $2^* = \frac{2N}{N-2}$ for $N \geq 3$.

By a solution of problem (1), we mean a function $u \in H_0^1(\Omega)$ such that

$$u^{-\beta}\chi_{\{u>0\}} \in L^1_{loc}(\Omega)$$

and

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega\cap\{u>0\}} \left(\left(-u^{-\beta} + f(u)\right)\varphi\right) \text{ for every } \varphi \in C_c^1(\Omega).$$

Here, $C_c^1(\Omega)$ stands for the functions belonging to $C^1(\Omega)$ with compact support.

We consider the perturbed problem

$$\begin{cases} -\Delta u + g_\epsilon(u) = f(u) \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where the perturbation g_ϵ is given by

$$g_\epsilon(s) = \begin{cases} \dfrac{s^q}{(s+\epsilon)^{q+\beta}} \text{ for } s \geq 0 \\ 0 \text{ for } s < 0, \end{cases} \tag{3}$$

and $0 < q < \frac{1}{2}$. We say that $u_\epsilon \in H_0^1(\Omega)$ is a weak solution of problem (2) if

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon) v = \int_\Omega f(u_\epsilon) v \text{ for all } v \in H_0^1(\Omega). \tag{4}$$

We define the functional $I_\epsilon : H_0^1(\Omega) \to \mathbb{R}$ associated to problem (2) by

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_\epsilon(u) - \int_\Omega F(u), \tag{5}$$

where $G_\epsilon(s) = \int_0^s g_\epsilon(t)\,\mathrm{d}t$ and $F(s) = \int_0^s f(t)\,\mathrm{d}t$. It turns out that certain solutions of problem (2) converge to a solution of problem (1). Initially, we make the following assumptions on f.

$$f \text{ is of class } C^{1,\nu}(0,\infty) \cap C[0,\infty) \text{ and } \sup_{s\in[0,1]} s^{1-q_1}|f'(s)| < \infty, \tag{6}$$

for some $0 < \nu < 1$ and $0 < q_1 < 1$, and

$$f(s) = 0 \text{ for } s \leq 0. \tag{7}$$

We also assume that there exist constants $0 < \epsilon_0, \delta < 1$ such that

$$g_{\epsilon_0}(s) \geq f(s) \text{ for all } s \leq \delta, \tag{8}$$

and that there exists a constant $C > 0$ such that

$$|f(s)| \leq C(1 + s^p) \text{ for all } s \geq 0, \tag{9}$$

where $0 < p < 2^* - 1$ $(0 < p < \infty$ when $N = 2)$. We also assume that there exists constants $0 < \theta < 1/2$, $R > 0$ and $c > 0$ such that

$$(1-\theta)f(s) \leq \theta s f'(s) - c \text{ for } s \geq R, \tag{10}$$

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and that there exists $\phi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$I_\epsilon(\phi_0) < 0 \text{ for all } 0 < \epsilon < \epsilon_0. \tag{11}$$

Condition (11) holds provided

$$\lim_{s \to \infty} \frac{F(s)}{s^2} = \infty.$$

Assumptions (6) and (9) imply that I_ϵ is of class C^1 and

$$I'_\epsilon(u)(v) = \int_\Omega \nabla u \nabla v + \int_\Omega g_\epsilon(u)v - \int_\Omega f(u)v, \text{ for all } u, v \in H_0^1(\Omega). \tag{12}$$

Our first result is as the following

Theorem 1.1. *Assume that* (6)–(11) *hold. Then, problem* (1) *has a non-trivial non-negative solution.*

Examples: Let $\lambda > 0$ and $\mu \geq 0$ be constants. Conditions (6)–(11) hold for the following examples of f.

- $f(s) = \lambda s^p \pm \mu s^q$ with $0 < q < p < 2^* - 1$ and $p > 1$;
- $f(s) = \lambda s^p \pm \mu s^q$ with $N = 3, 0 < q < p$ and $1 < p < 5$;
- $f(s) = \lambda s^p \pm \mu s^q$ with $N = 2, 0 < q < p < \infty$ and $p > 1$;
- $f(s) = \lambda s^p \pm \mu s^q \log s$ with $1 < p < 2^* - 1$ and $0 < q < p$;
- $f(s) = \lambda s^p \log s \pm \mu s^q$ with $1 < p < 2^* - 1$ and $0 < q < p$.

Indeed, condition (10) will hold for $\frac{1}{1+p} < \theta < 1/2$ and condition (8) will hold because for each $\lambda > 0$ and $0 < \tau < 1$ there exists $0 < \delta_{\tau,\lambda} < 1$ and $0 < \epsilon_{\tau,\lambda} < 1$ such that

$$g_\epsilon(s) \geq \lambda s^\tau \text{ for } 0 \leq s \leq \delta_{\tau,\lambda} \text{ and } 0 < \epsilon < \epsilon_{\tau,\lambda},$$

provided $0 < q < \tau$ in (3).

Next, we study problem (1) with $N \geq 3$ and $f(s) = \lambda s + s^{2^*-1}$, where $\lambda > 0$. We prove

Theorem 1.2. *Assume that $N \geq 3$ and $f(s) = \lambda s + s^{2^*-1}$. Then, there exists $\lambda_0 > 0$ such that problem* (1) *has a non-trivial non-negative solution for all $\lambda > \lambda_0$.*

We recall the works of [1, 5], where the authors studied the problem

$$\begin{cases} -\Delta u = \lambda u^p + u^{2^*-1} \text{ in } \Omega \\ u > 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{13}$$

In [5], it was assumed that $1 < p < 2^* - 1$ in (13). They proved that problem (13) has a positive solution for every $\lambda > 0$ provided $N \geq 4$. The same result holds if $N = 3$ and $3 < p < 5$. In the case $N = 3$ and $1 < p \leq 3$, the authors proved in [5] that (13) possesses a positive solution provided $\lambda > 0$ is sufficiently large. In [1], the authors studied problem (13) when $0 < p < 1$. They showed that there exists a constant $\Lambda_1 > 0$ such that problem (13) has at least two solutions if $0 < \lambda < \Lambda_1$ and has no solution for $\lambda > \Lambda_1$. In [12], the authors studied the problem $-\Delta u = -u^{-\beta} + f(x)$ in Ω, $u = 0$ in $\partial\Omega$, the sub-supersolution method was used and positive solutions were obtained. Equation $-\Delta u + K(x)u^{-\beta} = \lambda u^p$ with $0 < p < 1$ and zero boundary condition was studied in [27], where K was assumed to be of class $C^{2,\alpha}(\overline{\Omega})$. For more on singular problems with sublinear nonlinearities, see [23, 30]. Theorem 1.1 asserts that the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p \pm \mu u^q \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{14}$$

is solvable for each $\lambda > 0$ and $\mu \geq 0$, provided $0 < q < p < 2^* - 1$ and $p > 1$. Problems (13) and (14) are similar in essence, the latter being a singular version of the former. Theorem 1.1 should also be compared with the results of [10, 22], where the authors studied the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p \text{ in } \Omega \\ u \geq 0, u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{15}$$

with $\lambda > 0$. In [10], the authors assumed that $p > 1$ and they obtained one solution of (15) for each $\lambda > 0$. The case $p = 1$ was also studied in [10], and they obtained one solution for $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$. In [22], the authors assumed that $0 < p < 1$ and they obtained two distinct solutions of (15) for large values of λ. See also [9], where the authors obtained sharper regularity results for solutions u_λ of problem (15) with $0 < p < 1$. In this work, we consider general nonlinearities f with subcritical growth, and we do not make use of parameters. Observe also that in Theorem 1.1, we make no assumptions on the sign of f.

Theorem 1.2 should be compared with the results of [14], where the authors studied the problem

$$\begin{cases} -\Delta u = -u^{-\beta}\chi_{\{u>0\}} + \lambda u^p + u^{\frac{N+2}{N-2}} \text{ in } \Omega \\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{16}$$

When $0 < p < 1$ in (16), the authors obtained a constant $\Lambda_0 > 0$ such that problem (16) has two distinct non-trivial and non-negative solutions for $0 < \lambda < \Lambda_0$. If $1 < p < \frac{N+2}{N-2}$, the authors obtained a constant $\Lambda_0^* > 0$ such that problem (16) admits a solution provided $\lambda > \Lambda_0^*$. Theorem 1.2 addresses the case $p = 1$ in (16).

Problems similar to (1) and (2) arise in the context of heterogeneous catalysis. Consider a reaction R which converts a given gas to useful products, and suppose that R occurs only in the presence of a catalyst that comes in the form of a porous pellet Ω. For the pellet to be useful, the gas must diffuse inside it. In this context, two entities arise: the rate of reaction k_R and the rate of diffusion k_D of the gas in regions of Ω. If k_D is large compared to k_R, then the reaction occurs throughout Ω and no free boundary arises. However, when k_D is small compared to k_R, then there are zones within Ω in which no reaction takes place, these are known as dead cores. The rates of adsorption k_a and desorption k_d of gas

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in the surface of the pellet must also be considered, for the equilibrium is reached when k_a equals k_d. Let A_1, A_2, $\ldots A_S$ be chemical species involved in the reaction

$$\sum_{j=1}^{S} \alpha_j A_j = 0,$$

where α_j denote the number of molecules of A_j being formed ($\alpha_j > 0$) or consumed ($\alpha_j < 0$) in Ω. Then, under certain assumptions about the mechanism of the reaction, the concentration $c_j = c_j(x)$ of A_j at $x \in \Omega$ satisfies the following elliptic equation

$$\begin{cases} D_j \Delta c_j + \alpha_j \pi_S k_R = 0 \text{ in } \Omega \\ c_j = c_{js} \text{ on } \partial\Omega, \end{cases}$$

for each $j \in \{1, 2, \ldots, S\}$. Here, D_j denotes the diffusion coefficient of A_j and π_S the catalytic area per unit volume. At equilibrium, the reaction rate k_R can be calculated as a function of the concentrations c_j. Using a suitable change of variables (see [3], p.168), we get an equation of the form

$$\begin{cases} \Delta u = \lambda^2 R(u) \text{ in } \Omega \\ u = 1 \text{ on } \partial\Omega, \end{cases} \tag{17}$$

where $\lambda > 0$ is a constant called Thiele Modulus, $R : \mathbb{R} \to \mathbb{R}$ is a rational function and $0 \leq u \leq 1$ represents a 'normalized dimensionless concentration'. We see that equations (2) and (17) are similar in essence.

For more applications in catalysis and in other fields of research, such as biochemistry, see [3, 11]. See [15, 26] for studies of the free boundary of solutions of some elliptic equations.

Singular equations are related to phase field models, see [6, 8, 13, 16]. For more results on singular elliptic equations, see [2, 4, 17, 19, 21, 24, 25, 28].

Our paper is organized as follows. In § 2, we give some preliminary results. Next, we study problem (2) by considering two different scenarios; in § 3, we consider the subcritical case and in § 4, we study problem (2) with $f(s) = \lambda s + s^{\frac{N+2}{N-2}}$. In both cases, we show that the associated functional satisfy the assumptions of the Mountain Pass Theorem. We thus obtain solutions of problem (2). These solutions will be shown to be bounded in $H_0^1(\Omega)$ by a constant that does not depend on ϵ. Such an estimate will be crucial in § 5, where we will establish regularity results for the solutions of problem (2) obtained in § 3 and § 4. In § 6, we prove Theorems 1.1 and 1.2.

2. Preliminary results

First, we show that critical points of the functional I_ϵ defined in (5) must be non-negative.

Lemma 2.1. *Assume that* (6), (7) *and* (9) *hold. Let u_ϵ be a critical point of the functional I_ϵ defined by* (5). *Then $u_\epsilon \geq 0$ and u_ϵ is a weak solution of problem* (2).

Proof of Lemma 2.1. Let $u_\epsilon^- = \max\{-u_\epsilon, 0\}$. Taking $v = u_\epsilon^-$ in (12) and using (7), we obtain

$$0 = I'_\epsilon(u_\epsilon)(u_\epsilon^-) = -\|u_\epsilon^-\|^2_{H_0^1(\Omega)}.$$

Hence, $u_\epsilon \geq 0$ and

$$0 = I'_\epsilon(u_\epsilon)(v) = \int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v - \int_\Omega f(u_\epsilon)v, \text{ for all } v \in H_0^1(\Omega).$$

Hence, (4) holds. This proves Lemma 2.1. □

We will need estimates of the perturbation g_ϵ defined in (3). Note that

$$g_\epsilon(s) = \frac{s^q}{(s+\epsilon)^{q+\beta}} \geq \frac{s^q}{(s+1)^{q+\beta}} = \frac{s^{q-\frac{1}{2}}}{(s+1)^{q+\beta}} s^{\frac{1}{2}} \text{ for } s \geq 0.$$

Hence,

$$g_\epsilon(s) \geq \frac{1}{2^{q+\beta}} s^{q-\frac{1}{2}} s^{\frac{1}{2}} \text{ for } 0 \leq s < 1.$$

Therefore, from the fact that $0 < q < \frac{1}{2}$, it follows that, for each $M > 0$, there exists $\overline{\delta} = \overline{\delta}(M) < 1$ such that

$$g_\epsilon(s) \geq Ms \text{ for } 0 \leq s < \overline{\delta} < 1.$$

We thus obtain

$$G_\epsilon(s) = \int_0^s g_\epsilon(t)\,\mathrm{d}t \geq \int_0^s Mt\,\mathrm{d}t = \frac{M}{2}s^2 \text{ for } 0 \leq s < \overline{\delta} < 1. \tag{18}$$

Observe that

$$g'_\epsilon(s) = \frac{qs^{q-1}(s+\epsilon)^{q+\beta} - (q+\beta)s^q(s+\epsilon)^{q+\beta-1}}{(s+\epsilon)^{2(q+\beta)}}.$$

Hence,

$$sg'_\epsilon(s) = \frac{qs^q}{(s+\epsilon)^{q+\beta}} - \frac{(q+\beta)s^{q+1}}{(s+\epsilon)^{q+\beta+1}}. \tag{19}$$

The following lemma will play a crucial role in § 4.

Lemma 2.2.

$$G_\epsilon(s) \geq \frac{1}{2}g_\epsilon(s)s, \textit{ for every } s \geq 0.$$

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Proof of Lemma 2.2. Indeed, let $\widetilde{B}_\epsilon(s) = G_\epsilon(s) - \frac{1}{2}g_\epsilon(s)s$. We have that $B_\epsilon(0) = 0$ and

$$\widetilde{B}'_\epsilon(s) = g_\epsilon(s) - \frac{1}{2}g_\epsilon(s) - \frac{s}{2}g'_\epsilon(s) = \frac{1}{2}\left(g_\epsilon(s) - sg'_\epsilon(s)\right).$$

Therefore, $\widetilde{B}'_\epsilon(s) \geq 0$ if and only if

$$g_\epsilon(s) \geq sg'_\epsilon(s).$$

From (19), this inequality will be true if

$$\frac{s^q}{(s+\epsilon)^{q+\beta}} \geq \frac{qs^q}{(s+\epsilon)^{q+\beta}}. \tag{20}$$

Since $q < 1/2$, (20) holds for each $s \geq 0$. This proves Lemma 2.2. □

Now, we show that a version of the Ambrosetti–Rabinowitz condition holds. We define

$$j_\epsilon(s) = f(s) - g_\epsilon(s) \text{ for } s \in \mathbb{R}.$$

Consequently,

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega J_\epsilon(u),$$

where $J_\epsilon(s) = \int_0^s j_\epsilon(t)\,\mathrm{d}t$. For simplicity of notation, we denote J_ϵ and j_ϵ merely by J and j respectively.

Lemma 2.3. *Suppose that* (6) *and* (10) *hold. Let* $0 < \theta < 1/2$ *be given by* (10). *There exists a constant* $\overline{R} > 0$ *such that*

$$J(s) \leq \theta s j(s) \textit{ for } s \geq \overline{R}.$$

Proof of Lemma 2.3. Let $B_\epsilon(s) = J(s) - \theta s j(s)$. We have

$$B'_\epsilon(s) = (1-\theta)j(s) - \theta s j'(s).$$

Hence,

$$B'_\epsilon(s) = -(1-\theta)g_\epsilon(s) + \theta s g'_\epsilon(s) + (1-\theta)f(s) - \theta s f'(s).$$

From (19) we obtain

$$|sg'_\epsilon(s)| \leq q|s|^{-\beta} + (q+\beta)|s|^{-\beta} \to 0 \text{ as } s \to \infty.$$

It is also clear that

$$(1-\theta)g_\epsilon(s) \to 0 \text{ as } s \to \infty \text{ uniformly for } \epsilon.$$

Hence, for each $0 < \tau < 1$ there exists $R_\tau > 0$ that does not depend on ϵ such that

$$|(1-\theta)g_\epsilon(s)| + |sg'_\epsilon(s)| < \tau \text{ for } s \geq R_\tau.$$

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Therefore,

$$B'_\epsilon(s) < \tau + (1-\theta)f(s) - \theta s f'(s). \text{ for } s \geq R_\tau.$$

Consequently, from (10), we get

$$B'_\epsilon(s) < \tau - c \text{ for } s \geq \max\{R, R_\tau\},$$

where $c > 0$ and $R > 0$ are given by (10). Choosing $\tau = c/2$, we get

$$B'_\epsilon(s) < -\frac{c}{2} \text{ for } s \geq R_2, \tag{21}$$

where

$$R_2 = \max\{R, R_{c/2}\}.$$

Note that

$$B_\epsilon(R_2) \leq C_1$$

where

$$C_1 = |F(R_2)| + |\theta R_2 f(R_2)| + \theta R_2^{1-\beta}.$$

Therefore, (21) implies that there exists a constant $T > 0$ such that

$$B_\epsilon(s) \leq -\frac{cs}{2} + T \text{ for } s \geq R_2.$$

Hence, $B_\epsilon(s) \leq 0$ for $s \geq \max\{R_2, 2T/c\}$. This proves Lemma 2.3. □

Let $\phi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be given by (11). We have

Lemma 2.4. *Assume that* (6), (9) *and* (11) *hold. There exist a constant* $a_2 > 0$ *that does not depend on* ϵ *such that*

$$\sup_{0 \leq s \leq 1} I_\epsilon(s\phi_0) < a_2 \text{ for every } 0 < \epsilon < \epsilon_0. \tag{22}$$

Proof of Lemma 2.4. We have

$$I_\epsilon(s\phi_0) \leq \frac{s^2}{2}\|\phi_0\|_{H_0^1(\Omega)} + \int_\Omega G_\epsilon(s\phi_0) - \int_\Omega F(s\phi_0)\,\mathrm{d}x, \text{ for every } s \geq 0.$$

Consequently, we get

$$I_\epsilon(s\phi_0) \leq \frac{s^2}{2}\|\phi_0\|_{H_0^1(\Omega)} + \frac{1}{1-\beta}\int_\Omega |s\phi_0|^{1-\beta} - \int_\Omega F(s\phi_0)\,\mathrm{d}x, \text{ for every } s \geq 0.$$

We conclude that

$$\sup_{0 \leq s \leq 1} I_{\epsilon,\lambda}(s\phi_0) < a_2,$$

where

$$a_2 = \frac{1}{2}\|\phi_0\|_{H_0^1(\Omega)} + |\Omega|\left(\frac{\sup_\Omega |\phi_0|^{1-\beta}}{1-\beta} + \sup_{0 \leq s \leq \sup \phi_0} |F(s)|\right).$$

This proves (22). We have proved Lemma 2.4. □

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3. Existence of solution of the perturbed subcritical problem

Throughout this section, we will assume that f satisfies (9) with $0 < p < 2^* - 1$. Our aim is to show that problem (2) has a non-negative non-trivial solution. We recall that given a Banach space E and a functional $\Psi \in C^1(E;\mathbb{R})$, we say that a sequence (u_n) in E is a Palais–Smale sequence of Ψ if there exists $c \in \mathbb{R}$ such that $\Psi(u_n) \to c$ and $\|\Psi'(u_n)\| \to 0$ as $n \to \infty$. We say that Ψ satisfies the Palais–Smale condition if every Palais–Smale sequence of Ψ has a convergent subsequence.

Lemma 3.1. *Assume that* (6)–(10) *hold. Fix* $0 < \epsilon < 1$ *and let* (u_n^ϵ) *be a Palais–Smale sequence for* I_ϵ *in* $H_0^1(\Omega)$. *Assume that there exists a constant* $C > 0$ *that does not depend on* ϵ *such that*

$$|I_\epsilon(u_n^\epsilon)| < C \text{ for all } n \in \mathbb{N}. \tag{23}$$

Then, there exists $D > 0$ *that does not depend on* ϵ *such that*

$$\|u_n^\epsilon\|_{H_0^1(\Omega)} < D \text{ for all } n \in \mathbb{N}. \tag{24}$$

Furthermore, there exists $u_\epsilon \in H_0^1(\Omega)$ *such that, up to a subsequence,* $u_n^\epsilon \to u_\epsilon$ *strongly in* $H_0^1(\Omega)$. *Consequently,* u_ϵ *is a critical point of* I_ϵ.

Proof of Lemma 3.1. Throughout this proof, we denote $\|\cdot\|_{H_0^1(\Omega)}$ by $\|\cdot\|$. Let $(u_n^\epsilon)_{n\in\mathbb{N}}$ be a Palais–Smale sequence for I_ϵ satisfying (23). Consequently,

$$\frac{1}{2}\|u_n^\epsilon\|^2 \le C + \int_\Omega J(u_n^\epsilon)\,\mathrm{d}x \text{ for all } n \in \mathbb{N}, \tag{25}$$

and there is a sequence $\tau_n \to 0$ such that

$$\left|\int_\Omega \nabla u_n^\epsilon \nabla w\,\mathrm{d}x - \int_\Omega j(u_n^\epsilon) w\,\mathrm{d}x\right| \le \tau_n \|w\| \text{ for each } w \in H_0^1(\Omega). \tag{26}$$

Let $0 < \theta < 1/2$ be given by (10). From Lemma 2.3, there is a constant $\overline{R} > 0$ that does not depend on ϵ such that

$$J(t) \le \theta t j(t) \text{ for } t \ge \overline{R}.$$

Since there exists $D_1 > 0$ that does not depend on ϵ such that

$$\sup_{0\le s\le \overline{R}} \max\{|J(s)|, |sj(s)|\} < D_1,$$

we may find a constant $D_2 > 0$ such that

$$J(u_n^\epsilon) < D_2 + \theta u_n^\epsilon j(u_n^\epsilon).$$

We know from (25) that there is a constant $D_3 > 0$ such that

$$\frac{1}{2}\|u_n^\epsilon\|^2 \le D_3 + \theta \int_\Omega u_n^\epsilon j(u_n^\epsilon)\,\mathrm{d}x.$$

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Taking $w = u_n^\epsilon$ in (26), we also conclude that

$$\int_\Omega j(u_n^\epsilon) u_n^\epsilon \, \mathrm{d}x < \|u_n^\epsilon\|^2 + \tau_n \|u_n^\epsilon\|.$$

Hence,

$$\frac{1}{2}\|u_n^\epsilon\|^2 < D_3 + \theta \|u_n^\epsilon\|^2 + \tau_n \theta \|u_n^\epsilon\|.$$

Since $0 < \theta < \frac{1}{2}$, (24) follows. Consequently, there exists $u_\epsilon \in H_0^1(\Omega)$ such that $u_n^\epsilon \rightharpoonup u_\epsilon$ weakly in $H_0^1(\Omega)$. Since J_ϵ has subcritical growth at infinity (see [7], Theorem 3.4 and Remark 2.2.1), we conclude that, up to a subsequence, $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. Since $I_\epsilon'(u_n^\epsilon) \to 0$ as $n \to \infty$ and I_ϵ is of class C^1, we conclude that $I_\epsilon'(u_\epsilon) = 0$. This proves the result. □

Now, we obtain one solution for problem (2).

Proposition 3.2. *Assume that* (6)–(11) *hold and let* $a_2 > 0$ *be given by Lemma 2.4. Then, there is a non-negative solution* u_ϵ *of problem* (2) *and there exist constants* $a_1 > 0$ *and* $D > 0$ *that do not depend on* ϵ *such that*

$$0 < a_1 \le I_\epsilon(u_\epsilon) \le a_2,$$

and

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Proof of Proposition 3.2. Let $\delta > 0$ and ϵ_0 be given by (8). Note that

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\{u \le \delta\}} (G_\epsilon(u) - F(u)) \, \mathrm{d}x + \int_{\{u > \delta\}} (G_\epsilon(u) - F(u)) \, \mathrm{d}x \text{ for every } u \in H_0^1(\Omega).$$

The fact that g_ϵ is monotone in ϵ implies that

$$g_\epsilon(s) \ge f(s) \text{ for all } 0 \le s < \delta \text{ and } 0 < \epsilon < \epsilon_0.$$

Using the fact that $G_\epsilon \ge 0$, we get

$$I_\epsilon(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\{u > \delta\}} F(u) \text{ for every } u \in H_0^1(\Omega).$$

From (9), we have

$$|F(s)| \le \int_0^s |f(t)| \, \mathrm{d}t \le C \int_0^s (1 + t^p) \, \mathrm{d}t = Cs + \frac{Cs^{1+p}}{1+p}.$$

Consequently,

$$I_\epsilon(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 - C\int_{\{u > \delta\}} u - \frac{C}{1+p}\int_{\{u > \delta\}} u^{p+1} \text{ for every } u \in H_0^1(\Omega).$$

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We conclude that there exists $\widetilde{C} > 0$ such that

$$I_\epsilon(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \widetilde{C}\int_\Omega |u|^\sigma \text{ for every } u \in H_0^1(\Omega),$$

where $\sigma > 2$ is chosen such that $1 + p < \sigma < 2^*$. The Sobolev embedding implies that there is a constant $C_3 > 0$ such that

$$I_\epsilon(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_3\|u\|^\sigma_{H_0^1(\Omega)}.$$

Therefore,

$$I_\epsilon(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} \text{ for } \|u\|_{H_0^1(\Omega)} \leq \rho,$$

where

$$\rho = \left(\frac{1}{4C_3}\right)^{\frac{1}{\sigma-2}}.$$

Also,

$$I_\epsilon(u) \geq a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho,$$

where

$$a_1 = \frac{\rho^2}{4}.$$

Let ϕ_0 be given by (11) and $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \phi_0\}$. We know from (23) that $I_\epsilon(\phi_0) < 0$. Consequently, we may apply the Mountain Pass Theorem [29], page 12) to conclude that there is a Palais–Smale sequence $(u_n^\epsilon$ for I_ϵ in $H_0^1(\Omega)$ and a number

$$c_\epsilon = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I_\epsilon(\gamma(s)),$$

such that

$$\lim_{n\to\infty} I_\epsilon(u_n^\epsilon) = c_\epsilon \text{ and } \lim_{n\to\infty} I'_\epsilon(u_n^\epsilon) = 0.$$

From Lemma 2.4, we know that $a_1 \leq c_\epsilon \leq a_2$. From Lemma 3.1, we conclude that there exist $D > 0$ (that does not depend on ϵ) and $u_\epsilon \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$ and

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D.$$

Consequently, $I'_\epsilon(u_\epsilon) = 0$ and

$$a_1 \leq I_\epsilon(u_\epsilon) < a_2.$$

This proves the result. □

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4. Existence of solution of the perturbed problem when $p = 2^* - 1$

In this section, we study problem (2) with $f(s) = \lambda s + s^{2^*-1}$ for $s \geq 0$. We assume that $f(s) = 0$ for $s \leq 0$. This function satisfies (6), (7), (8) and (10). The difficulty here is that f no longer satisfies (9), so that Lemma 3.1 does not hold. The associated functional then becomes

$$I_{\epsilon,\lambda}(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega G_\epsilon(u) - \frac{\lambda}{2}\int_\Omega (u^+)^2 \, \mathrm{d}x - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} \text{ defined for } u \in H_0^1(\Omega), \tag{27}$$

where $u^+ = \max\{u, 0\}$. The functional $I_{\epsilon,\lambda}$ is of class C^1 and

$$\begin{aligned} I'_{\epsilon,\lambda}(u)(v) = &\int_\Omega \nabla u \nabla v \, \mathrm{d}x + \int_\Omega g_\epsilon(u) v \, \mathrm{d}x \\ &- \lambda \int_\Omega (u^+) v \, \mathrm{d}x - \int_\Omega (u^+)^{2^*-1} v \, \mathrm{d}x \text{ for all } u, v \in H_0^1(\Omega). \end{aligned} \tag{28}$$

The same argument given by Lemma 2.1 implies that critical points of $I_{\epsilon,\lambda}$ are non-negative solutions of problem (2). Observe also that zero is a local minimum of the functional $I_{\epsilon,\lambda}$. Indeed, let $0 < \overline{\delta} < 1$ be given by (18). Note that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_{\{u<\overline{\delta}\}} G_\epsilon(u) - \frac{\lambda}{2}\int_\Omega (u^+)^2 - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} \text{ for every } u \in H_0^1(\Omega)$$

Choosing $M = \lambda$ in (18), we obtain

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\{u>\overline{\delta}\}} u^2 - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} \text{for every } u \in H_0^1(\Omega).$$

Observe that there exists a constant $C_1 > 0$ such that

$$s^2 \leq C_1 s^{2^*} \text{ for } s \geq \overline{\delta}.$$

Hence, there exists a constant $C_2 > 0$ such that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_2 \int_\Omega |u|^{2^*} \text{ for every } u \in H_0^1(\Omega).$$

Consequently, the Sobolev embedding implies that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_3\|u\|^{2^*}_{H_0^1(\Omega)} \text{ for all } u \in H_0^1(\Omega)$$

We conclude that

$$I_{\epsilon,\lambda}(u) \geq \frac{1}{4}\|u\|^2_{H_0^1(\Omega)} \text{ for } \|u\|_{H_0^1(\Omega)} \leq \rho,$$

where

$$\rho = \left(\frac{1}{4C_3}\right)^{\frac{1}{2^*-2}}.$$

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Also,

$$I_{\epsilon,\lambda}(u) \geq a_1 \text{ for } \|u\|_{H_0^1(\Omega)} = \rho, \tag{29}$$

where

$$a_1 = \frac{\rho^2}{4}.$$

We now show that there exists $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$I_{\epsilon,\lambda}(\phi) < 0.$$

Indeed, let $\phi_1 \in H_0^1(\Omega)$ be the first eigenfunction of the operator $-\Delta$ with $\|\phi_1\|_{H_0^1(\Omega)} = 1$. We have

Lemma 4.1. *There exist constants $N_0 > 0$, $a_2 > 0$ and $b_1 > 0$ such that*

$$I_{\epsilon,\lambda}(N_0\phi_1) < -b_1 < 0, \text{ for every } 0 < \epsilon < 1, \tag{30}$$

and

$$\sup_{0\leq s\leq 1} I_{\epsilon,\lambda}(sN_0\phi_1) < a_2 \text{ for every } \lambda > 0,\ 0 < \epsilon < 1. \tag{31}$$

Moreover, these constants do not depend on λ.

Proof of Lemma 2.4. For each $t > 0$, we have

$$I_{\epsilon,\lambda}(t\phi_1) = \frac{t^2}{2} + \int_\Omega G_\epsilon(t\phi_1) - \frac{\lambda t^2}{2}\int_\Omega \phi_1^2 \,\mathrm{d}x - \frac{t^{2^*}}{2^*}\int \phi_1^{2^*} \,\mathrm{d}x.$$

From the fact that $G_\epsilon(s) \leq \frac{s^{1-\beta}}{1-\beta}$ for all $s \geq 0$, we get

$$I_{\epsilon,\lambda}(t\phi_1) \leq \frac{t^2}{2} + \frac{t^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta} - \frac{t^{2^*}}{2^*}\int \phi_1^{2^*} \,\mathrm{d}x. \tag{32}$$

Since $2^* > 2 > 1-\beta$, inequality (30) then follows by taking t large enough in (32). We also have

$$I_{\epsilon,\lambda}(sN_0\phi_1) \leq \frac{s^2N_0^2}{2} + \int_\Omega G_\epsilon(sN_0\phi_1) - \frac{s^{2^*}N_0^{2^*}}{2^*}\int \phi_1^{2^*} \,\mathrm{d}x, \text{ for every } s \geq 0.$$

Consequently, we get

$$I_{\epsilon,\lambda}(sN_0\phi_1) \leq \frac{s^2N_0^2}{2} + \frac{s^{1-\beta}N_0^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}, \text{ for every } s \geq 0.$$

We conclude that

$$\sup_{0\leq s\leq 1} I_{\epsilon,\lambda}(sN_0\phi_1) < a_2,$$

where

$$a_2 = \frac{N_0^2}{2} + \frac{N_0^{1-\beta}}{1-\beta}\int_\Omega \phi_1^{1-\beta}.$$

This proves (31). We have proved Lemma 4.1. □

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Lemma 4.1, (29) and the Mountain Pass Theorem imply that there is a sequence (u_n^ϵ) in $H_0^1(\Omega)$ and a number

$$c_{\epsilon,\lambda} = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I_{\epsilon,\lambda}(\gamma(s)), \tag{33}$$

such that

$$\lim_{n\to\infty} I_{\epsilon,\lambda}(u_n^\epsilon) = c_{\epsilon,\lambda} \text{ and } \lim_{n\to\infty} I'_{\epsilon,\lambda}(u_n^\epsilon) = 0, \tag{34}$$

where $\Gamma = \{\gamma \in C([0,\,1],\, H_0^1(\Omega)) : \gamma(0) = 0,\, \gamma(1) = N_0\phi_1\}$. It is clear that the function $f(s) = \lambda s + s^{2^*-1}$ satisfies (10). Consequently, the same computations developed in the proof of Lemma 2.3 imply that there exists $0 < \theta < 1/2$ such that

$$J(s) \leq \theta s j(s),$$

where $j(s) = \lambda s + s^{2^*-1} - g_\epsilon(s)$ and $J(s) = \int_0^s j(t)\,\mathrm{d}t$. Consequently, by a similar argument given in the proof of Lemma 3.1, we obtain a constant $D > 0$ such that

$$\|u_n^\epsilon\|_{H_0^1(\Omega)} < D \text{ for all } n \in \mathbb{N}, 0 < \epsilon < 1. \tag{35}$$

Furthermore, we have

Lemma 4.2. *Let $c_{\lambda,\epsilon}$ be given by* (33). *Then*

$$\lim_{\lambda\to\infty} c_{\lambda,\epsilon} = 0 \textit{ uniformly on } 0 < \epsilon < 1. \tag{36}$$

Proof of Lemma 4.2. Fix $0 < \epsilon < 1$ and let $t_{\lambda,\epsilon} \geq 0$ be such that

$$I_{\epsilon,\lambda}(t_{\lambda,\epsilon}\phi_1) = \max_{0\leq t\leq 1} I_{\epsilon,\lambda}(tN_0\phi_1).$$

From Lemmas 2.4 and (29), we get

$$I_{\epsilon,\lambda}(t\phi_1) > 0 \text{ for } 0 < t < \rho \quad \text{ and } I_{\epsilon,\lambda}(N_0\phi_1) < 0.$$

Hence, $0 < t_{\lambda,\epsilon} < 1$. Consequently,

$$\frac{d}{\mathrm{d}t} I_{\epsilon,\lambda}(tN_0\phi_1)\bigg|_{t=t_{\lambda,\epsilon}} = 0.$$

Equivalently, from (28),

$$\begin{aligned} 0 = I'_{\epsilon,\lambda}(t_{\lambda,\epsilon}N_0\phi_1)(N_0\phi_1) = N_0^2 t_{\lambda,\epsilon} \int_\Omega |\nabla\phi_1|^2 + N_0^{1+q} t_{\lambda,\epsilon}^q \int_\Omega \frac{\phi_1^{1+q}}{(N_0 t_{\lambda,\epsilon}\phi_1 + \epsilon)^{q+\beta}} \\ - N_0^2 \lambda t_{\lambda,\epsilon} \int_\Omega \phi_1^2 - N_0^{2^*} t_{\lambda,\epsilon}^{2^*-1} \int_\Omega \phi_1^{2^*}. \end{aligned} \tag{37}$$

Fix a sequence (λ_n) in $\mathbb{R}$ such that $\lambda_n \to \infty$. Since $0 < t_{\lambda_n,\,\epsilon} < 1$, we know that for each $0 < \epsilon < 1$ there exists an element $0 \leq t_{0,\epsilon} \leq 1$ such that

$$t_{\lambda_n,\epsilon} \to t_{0,\epsilon} \text{ as } n \to \infty.$$

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We will show that $t_{0,\epsilon} = 0$. Indeed, from (37) there exists a constant $M_0 > 0$ that does not depend on λ nor on ϵ such that

$$\lambda_n t_{\lambda_n,\epsilon}^2 \int_\Omega \phi_1^2 \le t_{\lambda_n,\epsilon}^2 \int_\Omega |\nabla \phi_1|^2 + t_{\lambda_n,\epsilon}^{1-\beta} \int_\Omega \phi_1^{1-\beta} \le M_0. \tag{38}$$

Letting $n \to \infty$ in (38), it follows that $t_{\lambda_n,\epsilon} \to 0$ as $n \to \infty$ uniformly on ϵ. Hence, $t_{0,\epsilon} = 0$. Consequently,

$$0 < c_{\lambda_n,\epsilon} \le \max_{0\le t\le 1} I_{\epsilon,\lambda_n}(tN_0\phi_1) = I_{\epsilon,\lambda_n}(t_{\lambda_n,\epsilon}\phi_1) \le t_{\lambda_n,\epsilon}^2 \int_\Omega |\nabla \phi_1|^2 \,\mathrm{d}x + t_{\lambda_n,\epsilon}^{1-\beta} \int_\Omega \phi_1^{1-\beta} \,\mathrm{d}x.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} c_{\lambda_n,\epsilon} = 0 \text{ uniformly on } 0 < \epsilon < 1.$$

Since the sequence (λ_n) was arbitrarily chosen, (36) follows. □

Consequently, there exist $\lambda_0 > 0$ and $0 < \epsilon_0 < 1$ such that

$$c_{\lambda,\epsilon} < \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{N}{2}} \text{ for all } \lambda > \lambda_0, 0 < \epsilon < \epsilon_0, \tag{39}$$

where

$$S = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\left(\displaystyle\int_\Omega |u|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}}}. \tag{40}$$

We now obtain the main result of this section.

Proposition 4.3. *Let $a_1 > 0$, $a_2 > 0$ and $\lambda_0 > 0$ be given by (29), Lemmas 4.1 and (39), respectively. If $\lambda > \lambda_0$, then problem (2) has a non-negative solution u_ϵ such that*

$$0 < a_1 \le c_{\lambda,\epsilon} = I_{\epsilon,\lambda}(u_\epsilon) \le a_2,$$

where $c_{\lambda,\epsilon}$ is given by (33). Furthermore, there exists a constant $D > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D \text{ for all } 0 < \epsilon < 1.$$

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Proof of Proposition 4.3. Inequality (35) implies that there is $u_\epsilon \in H_0^1(\Omega)$ with $\|u_\epsilon\|_{H_0^1(\Omega)} < D$ such that, up to a subsequence,

$$u_n^\epsilon \rightharpoonup u_\epsilon \text{ weakly in } H_0^1(\Omega), \quad u_n^\epsilon \to u_\epsilon \text{ in } L^r(\Omega) \text{ for all } 1 \le r < 2^*, \quad u_n^\epsilon \to u_\epsilon \text{ a.e in } \Omega. \tag{41}$$

We claim that

$$\int_\Omega ((u_n^\epsilon)^+)^{2^*} \to \int_\Omega ((u_\epsilon)^+)^{2^*} \text{ as } n \to \infty. \tag{42}$$

To do this, we use the ideas given in [14]. Note that there exist positive measures μ, ν in Ω such that

$$|\nabla (u_n^\epsilon)^+|^2 \rightharpoonup |\nabla u_\epsilon^+|^2 + \mu \text{ and } ((u_n^\epsilon)^+)^{2^*} \rightharpoonup (u_\epsilon^+)^{2^*} + \nu$$

Using the concentration-compactness principle due to Lions (cf. [18], Lemma 1.1), we obtain at most a countable set of indexes denoted by Λ, sequences $x_i \in \overline{\Omega}$, μ_i, $\nu_i \in (0, \infty)$ such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \mu \ge \sum_{i\in\Lambda} \mu_i \delta_{x_i} \quad \text{and } S\nu_i^{\frac{2}{2^*}} \le \mu_i,$$

for every $i \in \Lambda$, where S is given by (40). Now, for every $\sigma > 0$ and $i \in \Lambda$, we define

$$\psi_{\sigma,i}(x) = \psi\left(\frac{x - x_i}{\sigma}\right),$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ is a function satisfying

$$0 \le \psi \le 1, \quad \psi \equiv 1 \text{ in } B_1(0), \quad \psi \equiv 0 \text{ in } \mathbb{R}^n \setminus B_2(0) \quad \text{and } \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)} \le 2.$$

Since the function $\psi_{\sigma,i}(u_n^\epsilon)^+$ is bounded in $H_0^1(\Omega)$, we know that $I'_{\lambda,\epsilon}(u_n)(\psi_{\sigma,i}(u_n^\epsilon)^+) \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}\int_\Omega \nabla u_n^\epsilon \nabla(\psi_{\sigma,i}(u_n^\epsilon)^+) &+ \int_\Omega g_\epsilon(u_n^\epsilon)\psi_{\sigma,i}((u_n^\epsilon)^+) \\ &= \lambda \int_\Omega ((u_n^\epsilon)^+)^2 \psi_{\sigma,i} + \int_\Omega ((u_n^\epsilon)^+)^{2^*} \psi_{\sigma,i} + o_n(1).\end{aligned}$$

Consequently,

$$\int_\Omega |\nabla(u_n^\epsilon)^+|^2 \psi_{\sigma,i} + \int_\Omega (u_n^\epsilon)^+ \nabla u_n^\epsilon \nabla\psi_{\sigma,i} \le \lambda \int_\Omega ((u_n^\epsilon)^+)^2 \psi_{\sigma,i} + \int_\Omega ((u_n^\epsilon)^+)^{2^*} \psi_{\sigma,i} + o_n(1). \tag{43}$$

Note that

$$\lim_{n\to\infty} \int_\Omega ((u_n^\epsilon)^+)^{2^*} \psi_{\sigma,i} = \int_\Omega (u_\epsilon^+)^{2^*} \psi_{\sigma,i} + \int_\Omega \psi_{\sigma,i}\, d\nu.$$

Hence,

$$\lim_{\sigma\to 0} \lim_{n\to\infty} \int_\Omega ((u_n^\epsilon)^+)^{2^*} \psi_{\sigma,i} = \nu_i.$$

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It is also clear that

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\int_\Omega ((u_n^\epsilon)^+)^2\psi_{\sigma,i}=\lim_{\sigma\to 0}\int_\Omega (u_\epsilon^+)^2\psi_{\sigma,i}=0$$

and

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\int_\Omega |\nabla(u_n^\epsilon)^+|^2\psi_{\sigma,i}\geq \mu_i.$$

We claim that

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\int_\Omega (u_n^\epsilon)^+\nabla u_n^\epsilon\nabla\psi_{\sigma,i}=0. \tag{44}$$

Indeed,

$$|(u_n^\epsilon)^+\nabla u_n^\epsilon\nabla\psi_{\sigma,i}|\leq \frac{((u_n^\epsilon)^+)^2|\nabla\psi_{\sigma,i}|^2}{2}+\frac{|\nabla(u_n^\epsilon)^+|^2}{2}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega (u_n^\epsilon)^+\nabla u_n^\epsilon\nabla\psi_{\sigma,i}\leq J_{1,\sigma}+J_{2,\sigma}$$

where

$$J_{1,\sigma}=\frac{1}{2}\int_\Omega (u_\epsilon^+)^2|\nabla\psi_{\sigma,i}|^2 \quad\text{and}\quad J_{2,\sigma}=\frac{1}{2}\int_{\{\sigma<|x-x_i|<2\sigma\}}|\nabla u_\epsilon^+|^2+\int_{\{\sigma<|x-x_i|<2\sigma\}}d\mu.$$

Using the Lebesgue Differentiation Theorem and the bound on $\nabla\psi$, we obtain

$$2\lim_{\sigma\to 0}J_{1,\sigma}\leq \lim_{\sigma\to 0}\frac{4}{\sigma^2}\int_{\{\sigma<|x-x_i|<2\sigma\}}(u_\epsilon^+)^2\leq C_N\lim_{\sigma\to 0}\sigma^{N-2}\frac{1}{V(B_{2\sigma}(x_i))}\int_{B_{2\sigma}(x_i)}(u_\epsilon^+)^2=0,$$

where $V(B_{2\sigma}(x_i))$ denotes the volume of the ball $B_{2\sigma}(x_i)$. Hence, $\lim_{\sigma\to 0}J_{1,\sigma}=0$. It is also clear that $\lim_{\sigma\to 0}J_{2,\sigma}=0$. This proves (44). Letting $n\to\infty$ and $\sigma\to 0$ in (43), we get

$$\mu_i\leq\nu_i \text{ for every } i\in\Lambda.$$

Hence,

$$\nu_i^{\frac{2}{N}}\geq S \text{ for each } i\in\Lambda.$$

Since

$$I_{\epsilon,\lambda}(u_n^\epsilon)=\frac{1}{2}\int_\Omega|\nabla u_n^\epsilon|^2+\int_\Omega G_\epsilon(u_n^\epsilon)-\frac{\lambda}{2}\int_\Omega((u_n^\epsilon)^+)^2-\frac{1}{2^*}\int_\Omega((u_n^\epsilon)^+)^{2^*}.$$

and

$$I'_{\epsilon,\lambda}(u_n^\epsilon)(u_n^\epsilon)=\int_\Omega\nabla u_n^\epsilon\nabla u_n^\epsilon+\int_\Omega g_\epsilon(u_n^\epsilon)u_n^\epsilon-\lambda\int_\Omega(u_n^\epsilon)^+u_n^\epsilon-\int_\Omega((u_n^\epsilon)^+)^{2^*-1}u_n^\epsilon,$$

it follows that

$$I_{\epsilon,\lambda}(u_n^\epsilon)-\frac{1}{2}I'_{\epsilon,\lambda}(u_n^\epsilon)(u_n^\epsilon)=\int_\Omega\left(G_\epsilon(u_n^\epsilon)-\frac{1}{2}g_\epsilon(u_n^\epsilon)u_n^\epsilon\right)+\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_\Omega((u_n^\epsilon)^+)^{2^*}.$$

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From Lemmas 2.2, (34) and from the definition of $\psi_{\sigma,i}$, we obtain

$$c_{\lambda,\epsilon} + o_n(1) \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega ((u_n^\epsilon)^+)^{2^*} \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{B_\sigma(x_i)} \psi_{\sigma,i}((u_n^\epsilon)^+)^{2^*} \text{ for each } i \in \Lambda. \tag{45}$$

Note that

$$\lim_{n\to\infty} \int_{B_\sigma(x_i)} \psi_{\sigma,i}((u_n^\epsilon)^+)^{2^*} = \int_{B_\sigma(x_i)} \psi_{\sigma,i}(u_\epsilon^+)^{2^*} + \int_{B_\sigma(x_i)} \psi_{\sigma,i}\, d\nu \geq \nu_i \geq S^{\frac{N}{2}}.$$

Hence, letting $n \to \infty$ in (45), we obtain

$$c_{\lambda,\epsilon} \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{N}{2}}.$$

This contradicts (39). This proves that $\Lambda = \emptyset$ and therefore (42) holds. We will now show that $u_n^\epsilon \to u_\epsilon$ in $H_0^1(\Omega)$. Indeed

$$\begin{aligned} I'_{\epsilon,\lambda}(u_n^\epsilon)(u_n^\epsilon) - I'_{\epsilon,\lambda}(u_n^\epsilon)(u^\epsilon) = &\int_\Omega |\nabla u_n^\epsilon|^2 - 2\int_\Omega \nabla u_n^\epsilon \nabla u_\epsilon + \int_\Omega |\nabla u_\epsilon|^2 \\ &- L_{1,n}^\epsilon + L_{2,n}^\epsilon - L_{3,n}^\epsilon - L_{4,n}^\epsilon, \end{aligned}$$

where

$$\begin{aligned} L_{1,n}^\epsilon &= \int_\Omega \left(|\nabla u_\epsilon|^2 - \nabla u_n^\epsilon \nabla u_\epsilon\right), \\ L_{2,n}^\epsilon &= \int_\Omega \left(g_\epsilon(u_n^\epsilon)(u_n^\epsilon - u_\epsilon)\right), \\ L_{3,n}^\epsilon &= \lambda \int_\Omega \left((u_n^\epsilon)^+(u_n^\epsilon - u_\epsilon)\right) \end{aligned}$$

and

$$L_{4,n}^\epsilon = \int_\Omega \left(((u_n^\epsilon)^+)^{2^*-1}(u_n^\epsilon - u_\epsilon)\right) = \int_\Omega ((u_n^\epsilon)^+)^{2^*} - \int_\Omega ((u_n^\epsilon)^+)^{2^*-1} u_\epsilon$$

Using the Dominated Convergence Theorem, (41) and (42), we obtain that

$$\lim_{n\to\infty} \max\{L_{1,n}^\epsilon, L_{2,n}^\epsilon, L_{3,n}^\epsilon, L_{4,n}^\epsilon\} = 0.$$

Therefore, it follows from (34) that

$$\|u_n^\epsilon - u_\epsilon\|^2_{H_0^1(\Omega)} = o_n(1).$$

Therefore, $u_n^\epsilon \to u_\epsilon$ strongly in $H_0^1(\Omega)$. From (34), it follows that u_ϵ is a critical point of $I_{\epsilon,\lambda}$ with

$$0 < a_1 < I_{\epsilon,\lambda}(u_\epsilon) < a_2.$$

In particular, we know that $u_\epsilon \geq 0$. This proves Proposition 4.3. □

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5. Regularity results and gradient estimates

We will need the following a priori bound in $L^\infty(\Omega)$.

Lemma 5.1. *Let $u_{\epsilon,\lambda} \in H_0^1(\Omega)$ be a non-negative solution of problem (2) with $f(s) = \lambda s + s^p$ and assume that there exists a constant $D > 0$ independent of ϵ such that*

$$\|u_{\epsilon,\lambda}\|_{H_0^1(\Omega)} \leq D \text{ for each } 0 < \epsilon < 1. \tag{46}$$

Then the following assertions hold

(i) *If $1 < p < 2^* - 1$ then $u_{\epsilon,\lambda} \in L^\infty(\Omega)$ and there exists a constant $K_1 > 0$ such that*

$$\|u_{\epsilon,\lambda}\|_{L^\infty(\Omega)} \leq K_1 \text{ for each } 0 < \epsilon < 1. \tag{47}$$

(ii) *If $p = 2^* - 1$ and*

$$\lim_{\lambda \to \infty} I_{\epsilon,\lambda}(u_{\epsilon,\lambda}) = 0 \text{ uniformly on } \epsilon, \tag{48}$$

then there exists $\widehat{\lambda_0} > 0$ such that $u_{\epsilon,\lambda} \in L^\infty(\Omega)$ for each $\lambda > \widehat{\lambda_0}$ and (47) holds.

Proof of Lemma 5.1. For simplicity, we denote $u_{\epsilon,\lambda}$ by u_ϵ. For $s \geq 0$, define $h(s) = \lambda s + s^p$. From (4), we get

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon) v = \int_\Omega h(u_\epsilon) v \text{ for all } v \in H_0^1(\Omega). \tag{49}$$

Note that

$$\frac{h(s)}{g_\epsilon(s)} = \frac{(s+\epsilon)^{q+\beta}}{s^q}(\lambda s + s^p) \leq (s+1)^{q+\beta}(\lambda s^{1-q} + s^{p-q}) \to 0 \text{ as } s \to 0.$$

Hence, the exists $0 < \delta_\lambda < 1$ that does not depend on ϵ such that

$$\frac{h(s)}{g_\epsilon(s)} < \frac{1}{2} \text{ for } s \leq \delta_\lambda. \tag{50}$$

Also,

$$\frac{h(s)}{s^p} = \lambda s^{1-p} + 1 \text{ for } s \geq \delta_\lambda.$$

Therefore, we conclude that

$$\frac{h(s)}{s^p} \leq 2 \text{ for } s \geq A_\lambda, \tag{51}$$

where

$$A_\lambda = \lambda^{\frac{1}{p-1}}.$$

It is also clear that

$$h(s) = \lambda s + s^p \leq \lambda A_\lambda + A_\lambda^p = C_\lambda \text{ for } \delta_\lambda \leq s \leq A_\lambda. \tag{52}$$

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Using (50), (51) and (52), we obtain

$$h(s) < \left(\frac{g_\epsilon(s)}{2}\right)\chi_{\{0\le s\le \delta_\lambda\}} + C_\lambda \chi_{\{\delta_\lambda \le s\le A_\lambda\}} + 2s^p\chi_{\{s\ge A_\lambda\}} \text{ for } s\ge 0$$

Hence, from (70), we get

$$\int_\Omega \nabla u_\epsilon \nabla v < C_\lambda \int_{\{\delta_\lambda \le u_\epsilon \le A_\lambda\}} v + 2\int_\Omega u_\epsilon^p v \text{ for all } v\in H_0^1(\Omega), v\ge 0. \tag{53}$$

We will now prove assertions (i) and (ii) separately. The proof of (ii) is more intricate, because we need to study the dependence of certain constants on λ, so that we can let $\lambda\to\infty$.

Proof of (i): From (53), we obtain a constant $C_{\delta,\lambda}>0$ such that

$$\int_\Omega \nabla u_\epsilon \nabla v < C_{\delta,\lambda}\int_\Omega u_\epsilon^p v \text{ for all } v\in H_0^1(\Omega), v\ge 0. \tag{54}$$

For $L>1$, we define,

$$u_{L,\epsilon}(x) = \begin{cases} u_\epsilon(x), \text{if } u_\epsilon(x)\le L\\ L, \text{if } u_\epsilon(x)\ge L,\end{cases}$$

$$z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)}u_\epsilon \quad \text{and} \quad w_{L,\epsilon} = u_\epsilon u_{L,\epsilon}^{\sigma-1},$$

with $\sigma>1$ to be determined later. Note that $z_{L,\epsilon}\in H_0^1(\Omega)$, $z_{L,\epsilon}\ge 0$ and

$$\nabla z_{L,\epsilon} = u_{L,\epsilon}^{2(\sigma-1)}\nabla u_\epsilon + 2(\sigma-1)u_\epsilon u_{L,\epsilon}^{2\sigma-3}\nabla u_{L,\epsilon}.$$

Taking $v=z_{L,\epsilon}$ in (54) we obtain

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)}|\nabla u_\epsilon|^2 + 2(\sigma-1)\int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3}\nabla u_\epsilon \nabla u_{L,\epsilon} < C_{\lambda,\delta}\int_\Omega u_\epsilon^{p+1}u_{L,\epsilon}^{2(\sigma-1)}.$$

Since $\sigma>1$ and

$$\int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3}\nabla u_\epsilon\nabla u_{L,\epsilon} = \int_{\{u_\epsilon<L\}} u_\epsilon^{2(\sigma-1)}|\nabla u_\epsilon|^2\ge 0,$$

we conclude that

$$\int_\Omega u_{L,\epsilon}^{2(\sigma-1)}|\nabla u_\epsilon|^2 < C_{\lambda,\delta}\int_\Omega u_\epsilon^{p+1}u_{L,\epsilon}^{2(\sigma-1)} < C_{\lambda,\delta}\int_\Omega u_\epsilon^{p-1}u_\epsilon^{2\sigma}. \tag{55}$$

On the other hand, from the Sobolev embedding, we know that there is a constant $C_1>0$ such that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le C_1\int_\Omega |\nabla w_{L,\epsilon}|^2\,\mathrm{d}x.$$

Since

$$\nabla w_{L,\epsilon} = u_{L,\epsilon}^{\sigma-1}\nabla u_\epsilon + (\sigma-1)u_\epsilon u_{L,\epsilon}^{\sigma-2}\nabla u_{L,\epsilon},$$

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it follows that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le C_1\int_\Omega u_{L,\epsilon}^{2(\sigma-1)}|\nabla u_\epsilon|^2\,\mathrm{d}x + C_1(\sigma-1)^2\int_\Omega u_\epsilon^2 u_{L,\epsilon}^{2(\sigma-2)}|\nabla u_{L,\epsilon}|^2$$
$$+\,2C_1(\sigma-1)\int_\Omega u_\epsilon u_{L,\epsilon}^{2\sigma-3}\nabla u_\epsilon \nabla u_{L,\epsilon}.$$

From the definition of $u_{L,\epsilon}$, we conclude that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le C_1\sigma^2\int_\Omega u_{L,\epsilon}^{2(\sigma-1)}|\nabla u_\epsilon|^2\,\mathrm{d}x.$$

Using (55), we obtain

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le C_1\sigma^2 C_{\delta,\lambda}\int_\Omega u_\epsilon^{p-1}u_\epsilon^{2\sigma}. \tag{56}$$

Now, observe that

$$\left(\int_\Omega w_{L,\epsilon}^{p+1}\,\mathrm{d}x\right)^{\frac{2}{p+1}} = \left(\int_\Omega u_\epsilon^{p+1}u_{L,\epsilon}^{(p+1)(\sigma-1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \ge \left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}}.$$

Hence, there is a constant $\widetilde{C_{\delta,\lambda}} > 0$ such that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le \sigma^2\widetilde{C_{\delta,\lambda}}\int_\Omega u_\epsilon^{p-1}u_\epsilon^{2\sigma}. \tag{57}$$

Let $\alpha_1, \alpha_2 > 1$ be constants such that $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=1$ and $p+1<\alpha_1(p-1)<2^*$. From (57) and Hölder's inequality it follows that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le \sigma^2\widetilde{C_{\delta,\lambda}}\left(\int_\Omega u_\epsilon^{\alpha_1(p-1)}\,\mathrm{d}x\right)^{\frac{1}{\alpha_1}}\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,\mathrm{d}x\right)^{\frac{1}{\alpha_2}}.$$

Using (46) and the Sobolev Embedding, we obtain a constant $\widetilde{C} > 0$ such that

$$\int_\Omega u_\epsilon^{\alpha_1(p-1)}\,\mathrm{d}x \le \widetilde{C}.$$

Hence, there exists a constant $\widehat{C} > 0$ that does not depend on σ nor on ϵ such that

$$\left(\int_\Omega u_{L,\epsilon}^{\sigma(p+1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le \widehat{C}\sigma^2\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,\mathrm{d}x\right)^{\frac{1}{\alpha_2}}. \tag{58}$$

Letting $L\to\infty$ in (58) and using Fatou's Lemma, we conclude that

$$\left(\int_\Omega u_\epsilon^{\sigma(p+1)}\,\mathrm{d}x\right)^{\frac{2}{p+1}} \le \widehat{C}\sigma^2\left(\int_\Omega u_\epsilon^{2\sigma\alpha_2}\,\mathrm{d}x\right)^{\frac{1}{\alpha_2}} \quad \text{for each } \sigma > 1,$$

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provided $u_\epsilon \in L^{2\sigma\alpha_2}(\Omega)$. Equivalently,

$$\|u_\epsilon\|_{L^{\sigma(p+1)}} \leq C^{\frac{1}{\sigma}}\sigma^{\frac{1}{\sigma}}\|u_\epsilon\|_{L^{2\sigma\alpha_2}} \text{ for each } \sigma > 1, \tag{59}$$

where $C = \sqrt{\widehat{C}}$. Observe that the choices of α_1 and α_2 imply that $\sigma(p+1) > 2\sigma\alpha_2$. The result now follows from an iterative argument. Indeed, take

$$\sigma_1 = \frac{p+1}{2\alpha_2}.$$

Using the Sobolev embedding and (46), we obtain a constant $\widetilde{D} > 0$ such that

$$\|u_\epsilon\|_{L^{\sigma_1(p+1)}(\Omega)} \leq C^{\frac{1}{\sigma_1}}\sigma_1^{\frac{1}{\sigma_1}}\|u_\epsilon\|_{L^{p+1}(\Omega)} \leq \widetilde{D}C^{\frac{1}{\sigma_1}}\sigma_1^{\frac{1}{\sigma_1}}$$

Now, take $\sigma_2 = \sigma_1^2$ in (59). We get

$$\|u_\epsilon\|_{L^{\sigma_1^2(p+1)}(\Omega)} \leq C^{\frac{1}{\sigma_1^2}}\sigma_2^{\frac{1}{\sigma_2}}\|u_\epsilon\|_{L^{\sigma_1(p+1)}} \leq \widetilde{D}C^{\frac{1}{\sigma_1}+\frac{1}{\sigma_1^2}}\left(\sigma_1^{\frac{1}{\sigma_1}}\sigma_2^{\frac{1}{\sigma_2}}\right)$$

Taking $\sigma_k = \sigma_1^k$ in (59), we get

$$\|u_\epsilon\|_{L^{\sigma_1^k(p+1)}(\Omega)} \leq \widetilde{D}C^{\sum_{i=1}^{k}\frac{1}{\sigma_1^i}}(\Pi_{i=1}^k\sigma_i^{\frac{1}{\sigma_i}}). \tag{60}$$

It is clear that

$$\lim_{k\to\infty}\left(\Pi_{i=1}^k\sigma_i^{\frac{1}{\sigma_i}}\right) = \lim_{k\to\infty}\left(\Pi_{i=1}^k\sigma_1^{\frac{i}{\sigma_1^i}}\right) < \infty \text{ and } \lim_{k\to\infty} C^{\sum_{i=1}^{k}\frac{1}{\sigma_1^i}} < \infty.$$

Letting $k \to \infty$ in (60), it follows that $u_\epsilon \in L^\infty(\Omega)$ and we obtain a constant $K_1 > 0$ that does not depend on ϵ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_1.$$

This proves (47).

Proof of (ii). Suppose that $p = 2^* - 1$. This case is much more complicated. We have

$$\int_\Omega \nabla u_\epsilon \nabla v \,\mathrm{d}x + \int_\Omega g_\epsilon(u_\epsilon)v \,\mathrm{d}x = \int_\Omega (\lambda u_\epsilon + u_\epsilon^{2^*-1})v \,\mathrm{d}x \text{ for all } v \in H_0^1(\Omega).$$

Consequently, since $g_\epsilon \geq 0$, we know that

$$\int_\Omega \nabla u_\epsilon \nabla v \leq \int_\Omega u_\epsilon(\lambda + u_\epsilon^{2^*-2})v \text{ for all } v \in H_0^1(\Omega), v \geq 0, 0 < \epsilon < 1. \tag{61}$$

For each $0 < \epsilon < 1$, we define

$$a_\epsilon(x) = \lambda + u_\epsilon(x)^{2^*-2}.$$

Observe that

$$\int_\Omega a_\epsilon(x)^{N/2} \leq C(N)\left(\lambda^{N/2}|\Omega| + \int_\Omega \left(u_\epsilon^{\frac{4}{N-2}}\right)^{N/2}\right) = C(N)(\lambda^{N/2}|\Omega| + \|u_\epsilon\|_{L^{2^*}(\Omega)}^{2^*}),$$

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where $C(N)$ is a constant that depends only on N. From (46) and the Sobolev embedding, we get a constant $C > 0$ such that

$$\|a_\epsilon\|_{L^{N/2}(\Omega)} \leq C \text{ for all } 0 < \epsilon < 1.$$

Let $\sigma \geq 0$ be a constant to be fixed later and consider the function $z_{L,\epsilon} = u_\epsilon \min\{u_\epsilon^{2\sigma}, L^2\} \in H_0^1(\Omega)$, with $L > 0$. Observe that

$$\nabla u_\epsilon \nabla z_{L,\epsilon} = |\nabla u_\epsilon|^2 \min\{u_\epsilon^{2\sigma}, L^2\} + 2\sigma u_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 \chi_{\{u_\epsilon^\sigma \leq L\}}.$$

Taking $v = z_{L,\epsilon}$ in (61), we get

$$\int_\Omega |\nabla u_\epsilon|^2 \min\{u_\epsilon^{2\sigma}, L^2\} + 2\sigma \int_{\{u_\epsilon^s \leq L\}} u_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 \leq \int_\Omega a_\epsilon(x) u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\}. \tag{62}$$

Define $w_{L,\epsilon}$ by $w_{L,\epsilon} = u_\epsilon \min\{u_\epsilon^\sigma, L\} \in H_0^1(\Omega)$. We have

$$\nabla w_{L,\epsilon} = \min\{u_\epsilon^\sigma, L\} \nabla u_\epsilon + \sigma u_\epsilon^\sigma \nabla u_\epsilon \chi_{\{u_\epsilon^\sigma \leq L\}}.$$

We thus get

$$|\nabla w_{L,\epsilon}|^2 = \sigma^2 u_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 \chi_{\{u_\epsilon^\sigma \leq L\}} + \min\{u_\epsilon^{2\sigma}, L^2\} |\nabla u_\epsilon|^2 + \sigma u_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 \chi_{\{u_\epsilon^\sigma \leq L\}}.$$

Hence,

$$|\nabla w_{L,\epsilon}|^2 = (1 + \sigma(\sigma + 1)) \min\{u_\epsilon^{2\sigma}, L^2\} |\nabla u_\epsilon|^2 \text{ in } \{u_\epsilon^\sigma \leq L\}$$

and

$$|\nabla w_{L,\epsilon}|^2 = \min\{u_\epsilon^{2\sigma}, L^2\} |\nabla u_\epsilon|^2 \text{ in } \{u_\epsilon^\sigma > L\}.$$

We conclude that

$$|\nabla w_{L,\epsilon}|^2 \leq c(\sigma) \min\{u_\epsilon^{2\sigma}, L^2\} |\nabla u_\epsilon|^2 \text{ in } \Omega,$$

where $c(\sigma) = 1 + \sigma(\sigma + 1)$. From (62), we get

$$\int_\Omega |\nabla w_{L,\epsilon}|^2 \leq c(\sigma) \int_\Omega \min\{u_\epsilon^{2\sigma}, L^2\} |\nabla u_\epsilon|^2 \leq c(\sigma) \int_\Omega a_\epsilon(x) u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\}.$$

Now, fix $K > 0$. We have

$$\int_\Omega |\nabla w_{L,\epsilon}|^2 \leq c(\sigma) K \int_\Omega u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\} + c(\sigma) \int_{\{a_\epsilon \geq K\}} a_\epsilon(x) u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\}.$$

Hence,

$$\int_\Omega |\nabla w_{L,\epsilon}|^2 \leq c(\sigma) K \int_\Omega u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\} \\ + c(\sigma) \left(\int_{\{a_\epsilon \geq K\}} a_\epsilon(x)^{\frac{N}{2}} \right)^{\frac{2}{N}} \left(\int_\Omega (u_\epsilon \min\{u_\epsilon^\sigma, L\})^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}.$$

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Consequently,

$$\int_\Omega |\nabla w_{L,\epsilon}|^2 \le c(\sigma)K\int_\Omega u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\} + c(\sigma)\left(\int_{\{a_\epsilon \ge K\}} a_\epsilon(x)^{\frac{N}{2}}\right)^{\frac{2}{N}}\left(\int_\Omega w_{L,\epsilon}^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}.$$

From the Sobolev embedding Theorem, we get

$$\int_\Omega |\nabla w_{L,\epsilon}|^2 \le c(\sigma)K\int_\Omega u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\} + Cc(\sigma)\left(\int_{\{a_\epsilon \ge K\}} a_\epsilon(x)^{\frac{N}{2}}\right)^{\frac{2}{N}}\int_\Omega |\nabla w_{L,\epsilon}|^2. \tag{63}$$

Choose $K > 0$ such that

$$\left(\int_{\{a_\epsilon \ge K\}} a_\epsilon(x)^{\frac{N}{2}}\right)^{\frac{2}{N}} \le \frac{1}{2Cc(\sigma)}.$$

Claim 1: K can be chosen independently of ϵ, provided λ is sufficiently large.

Assuming the claim to be true, we obtain

$$\int_\Omega |\nabla w_\epsilon|^2 \,\mathrm{d}x \le Cc(\sigma)K\int_\Omega u_\epsilon^2 \min\{u_\epsilon^{2\sigma}, L^2\}.$$

Consequently,

$$\int_{\{u_\epsilon^\sigma \le L\}} |\nabla(u_\epsilon^{\sigma+1})|^2 \,\mathrm{d}x \le Cc(\sigma)K\int_\Omega (u_\epsilon \min\{u_\epsilon^\sigma, L\})^2 \,\mathrm{d}x.$$

Suppose that $u_\epsilon \in L^{2\sigma+2}(\Omega)$ and let $u_{L,\epsilon} = (u_\epsilon \min\{u_\epsilon^\sigma,\, L\})^2$. Observe that

$$\lim_{L\to\infty} u_{L,\epsilon} = u_\epsilon^{2+2\sigma} \text{ almost everywhere in } \Omega$$

Furthermore,

$$L_1 < L_2 \text{ implies that } u_{L_1,\epsilon} < u_{L_2,\epsilon} \text{ in } \Omega.$$

The Monotone Convergence Theorem implies that

$$\int_{\{u_\epsilon^\sigma \le L\}} |\nabla(u_\epsilon^{\sigma+1})|^2 \,\mathrm{d}x \le Cc(\sigma)K\int_\Omega u_\epsilon^{2+2\sigma} \,\mathrm{d}x.$$

From Fatou's Lemma, we get

$$\int_\Omega |\nabla(u_\epsilon^{\sigma+1})|^2 \,\mathrm{d}x \le Cc(\sigma)K\int_\Omega u_\epsilon^{2+2\sigma} \,\mathrm{d}x \text{ for all } \sigma \ge 0. \tag{64}$$

Consequently, $u_\epsilon^{\sigma+1} \in H_0^1(\Omega)$ and $u_\epsilon \in L^{\frac{(2+2\sigma)N}{N-2}}(\Omega)$. Now let $q > 1$. We will show that

$$u_\epsilon \in L^q(\Omega) \text{ and } \|u_\epsilon\|_{L^q(\Omega)} < C_q \text{ for all } 0 < \epsilon < 1. \tag{65}$$

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Indeed, this follows by choosing adequate values for σ in (64). Let $\sigma_0 = 0$. From (64), we get

$$\int_\Omega |\nabla u_\epsilon|^2 \, \mathrm{d}x \le CK \int_\Omega u_\epsilon^2 \, \mathrm{d}x \le C_0.$$

Let $\sigma_1 = \frac{N}{N-2} - 1$. From (64), we get

$$\int_\Omega |\nabla (u_\epsilon^{\frac{N}{N-2}})|^2 \, \mathrm{d}x \le CK \int_\Omega u_\epsilon^{\frac{2N}{N-2}} \, \mathrm{d}x \le C_1.$$

Consequently, $u_\epsilon^{\frac{N}{N-2}} \in H_0^1(\Omega)$ and

$$\int_\Omega u_\epsilon^{(\frac{N}{N-2})\frac{2N}{N-2}} \le C_1.$$

Let $\sigma_2 = \frac{4N-4}{(N-2)^2}$. From (64), we get

$$\int_\Omega |\nabla (u_\epsilon^{\frac{N^2}{(N-2)^2}})|^2 \, \mathrm{d}x \le CK \int_\Omega u_\epsilon^{\frac{2N^2}{(N-2)^2}} \, \mathrm{d}x \le C_2.$$

Consequently, $u_\epsilon^{\frac{N^2}{(N-2)^2}} \in H_0^1(\Omega)$ and

$$\int_\Omega u_\epsilon^{(\frac{N^2}{(N-2)^2})\frac{2N}{N-2}} \le C_2.$$

Assertion (65) then follows by choosing

$$\sigma_i + 1 = (\sigma_{i-1} + 1)\frac{N}{N-2}.$$

and iterating up until $s_M > q$ for some $M \in \mathbb{N}$. Now let w_ϵ be the solution of the non-singular problem

$$\begin{cases} -\Delta w = \lambda u_\epsilon + u_\epsilon^{2^*-1} \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega, \end{cases} \tag{66}$$

Assertion (65) and elliptic regularity theory implies that $w_\epsilon \in W^{2,q}(\Omega)$ and

$$\|w_\epsilon\|_{W^{2,q}(\Omega)} \le C|\lambda u_\epsilon + u_\epsilon^{2^*-1}|_{L^q(\Omega)} = C_q,$$

where C_q does not depend on ϵ. Consequently, the Sobolev embedding assures that $w_\epsilon \in C^1(\overline{\Omega})$ and

$$\|w_\epsilon\|_{C^1(\Omega)} \le C \text{ for all } 0 < \epsilon < 1.$$

Observe that

$$\int_\Omega \nabla w_\epsilon \nabla v = \int_\Omega (\lambda u_\epsilon + u_\epsilon^{2^*-1})v \text{ for all } v \in H_0^1(\Omega), 0 < \epsilon < 1.$$

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Consequently,

$$\int_\Omega \nabla(u_\epsilon - w_\epsilon)\nabla v \leq 0 \text{ for all } v \in H_0^1(\Omega), v \geq 0, 0 < \epsilon < 1.$$

The weak maximum principle implies

$$\sup_\Omega (u_\epsilon - w_\epsilon) = 0.$$

Hence,

$$u_\epsilon \leq w_\epsilon.$$

Consequently,

$$\|u_\epsilon\|_{L^\infty(\Omega)} < C \text{ for all } 0 < \epsilon < 1.$$

This proves the result. We need to only show that the claim holds. Indeed, from (46), there exists an element $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_\epsilon \to u \text{ in } L^r(\Omega) \text{ for } 1 < r < 2^*, \\ u_\epsilon \to u \text{ a.e in } \Omega. \end{cases} \tag{67}$$

We first show that

$$\int_\Omega u_\epsilon^{2^*} \to \int_\Omega u^{2^*} \text{ as } \epsilon \to 0. \tag{68}$$

Again using the Concentration-compactness principle of Lions, we get positive measures μ, ν in Ω such that

$$|\nabla u_\epsilon|^2 \rightharpoonup |\nabla u|^2 + \mu \text{ and } u_\epsilon^{2^*} \rightharpoonup u^{2^*} + \nu.$$

Furthermore, there is at most a countable set of indexes denoted by Λ, sequences $x_i \in \overline{\Omega}$, μ_i, $\nu_i \in (0, \infty)$ such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \quad \mu \geq \sum_{i \in \Lambda} \mu_i \delta_{x_i} \quad \text{ and } S\nu_i^{\frac{2}{2^*}} \leq \mu_i,$$

for every $i \in \Lambda$, where S is given by (40). Now, for every $\sigma > 0$ and $i \in \Lambda$, we define

$$\psi_{\sigma,i}(x) = \psi\left(\frac{x - x_i}{\sigma}\right),$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$ is a function satisfying

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_1(0), \quad \psi \equiv 0 \text{ in } \mathbb{R}^n \setminus B_2(0) \quad \text{ and } \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)} \leq 2.$$

Proceeding as in the proof of Proposition 4.3 and using the hypothesis $\lim_{\lambda\to\infty} I_\epsilon(u_{\epsilon,\lambda}) = 0$, we conclude that $\Lambda = \emptyset$, provided $\lambda > 0$ is sufficiently large, thus proving (68).

Consequently, $a_\epsilon^{\frac{N}{2}}$ converges in $L^1(\Omega)$ to a, where

$$a(x) = (\lambda + u(x)^{2^*-2})^{\frac{N}{2}}.$$

We now show that for each $\delta^* > 0$ there exists $\eta > 0$ and $\epsilon_0 > 0$ such that

$$\int_B a_\epsilon(x)^{\frac{N}{2}} < \delta^* \text{ for all sets } B \subset \Omega \text{ with } |B| < \eta \text{ and } 0 < \epsilon < \epsilon_0. \tag{69}$$

Since $a \in L^1(\Omega)$, there exists $\eta > 0$ such that

$$\int_B a(x) < \delta^*/2 \text{ for all sets } B \subset \Omega \text{ with } |B| < \eta.$$

We write

$$\int_B a_\epsilon(x)^{\frac{N}{2}} = \int_B (a_\epsilon(x)^{\frac{N}{2}} - a(x)) + \int_B a(x).$$

and we choose $\epsilon_0 > 0$ such that

$$\int_\Omega |a_\epsilon(x)^{\frac{N}{2}} - a(x)| \leq \delta^*/2 \text{ for all } 0 < \epsilon < \epsilon_0.$$

Consequently,

$$\int_B a_\epsilon(x)^{\frac{N}{2}} \leq \delta^* \text{ for all sets } B \subset \Omega \text{ with } |B| < \eta \text{ and } 0 < \epsilon < \epsilon_0.$$

This proves (69). We now finally prove Claim 1. Indeed, we choose

$$\delta^* = \left(\frac{1}{2Cc(\sigma)}\right)^{\frac{N}{2}}$$

and we choose $K > 0$ such that

$$\left|\left\{u > \frac{(K-\lambda)^{\frac{1}{2^*-2}}}{2}\right\}\right| < \eta.$$

Observe that

$$\{a_\epsilon \geq K\} = \{u_\epsilon(x) \geq (K-\lambda)^{\frac{1}{2^*-2}}\}.$$

The choice of K implies that

$$|\{a_\epsilon \geq K\}| \leq \left|\left\{u > \frac{(K-\lambda)^{\frac{1}{2^*-2}}}{2}\right\}\right| < \eta \text{ for sufficiently small } \epsilon.$$

Consequently, from (69) and the choice of δ^*, we get

$$\left(\int_{\{a_\epsilon \geq K\}} a_\epsilon(x)^{\frac{N}{2}}\right)^{\frac{2}{N}} \leq \frac{1}{2Cc(\sigma)} \text{ for sufficiently small } \epsilon.$$

This proves Claim 1 and the result. □

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We also have

Lemma 5.2. *Let $u_\epsilon \in H_0^1(\Omega)$ be a non-negative solution of problem* (2) *and assume that there exists a constant $D > 0$ independent of ϵ such that*

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq D \text{ for each } 0 < \epsilon < 1.$$

If f satisfies (8) *and* (9) *for $0 < p < \frac{N+2}{N-2}$, then $u_\epsilon \in L^\infty(\Omega)$ and there exists a constant $K_2 > 0$ such that*

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq K_2 \text{ for each } 0 < \epsilon < 1.$$

Proof of Lemma 5.2. From (4), we get

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega g_\epsilon(u_\epsilon)v - \int_\Omega f(u_\epsilon)v = 0 \text{ for all } v \in H_0^1(\Omega), v \geq 0 \tag{70}$$

From (8), we get

$$\int_\Omega \nabla u_\epsilon \nabla v + \int_{\Omega\cap\{u_\epsilon \geq \delta\}} g_\epsilon(u_\epsilon)v - \int_{\Omega\cap\{u_\epsilon \geq \delta\}} f(u_\epsilon)v \leq 0 \text{ for all } v \in H_0^1(\Omega), v \geq 0.$$

Consequently,

$$\int_\Omega \nabla u_\epsilon \nabla v \leq \int_{\Omega\cap\{u_\epsilon \geq \delta\}} f(u_\epsilon)v \text{ for all } v \in H_0^1(\Omega), v \geq 0.$$

From (9), we get

$$\int_\Omega \nabla u_\epsilon \nabla v \leq \int_{\Omega\cap\{u_\epsilon \geq \delta\}} C(1 + u_\epsilon^p)v \text{ for all } v \in H_0^1(\Omega), v \geq 0.$$

Consequently,

$$\int_\Omega \nabla u_\epsilon \nabla v \leq C \int_{\Omega\cap\{u_\epsilon \geq \delta\}} \left(\frac{1}{\delta^p} + 1\right) u_\epsilon^p v \text{ for all } v \in H_0^1(\Omega), v \geq 0.$$

Consequently, there exists $\widetilde{C} > 0$ and $1 < \widetilde{p} < 2^* - 1$ such that

$$\int_\Omega \nabla u_\epsilon \nabla v \leq \widetilde{C} \int_\Omega u_\epsilon^{\widetilde{p}} v \text{ for all } v \in H_0^1(\Omega), v \geq 0.$$

The proof then follows as in item (i) of Lemma 5.1. □

Now, we obtain gradient estimates for solutions u_ϵ of problem (2).

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Lemma 5.3. *Assume that f satisfies* (6). *For each $0<\epsilon<1$, let $u_\epsilon \in H_0^1(\Omega)\cap L^\infty(\Omega)$ be a non-negative solution of problem* (2) *and assume that there exists a constant $T>0$ such that*

$$\sup_{0<\epsilon<1} \|u_\epsilon\|_{L^\infty(\Omega)} < T < \infty. \tag{71}$$

Let ψ be such that

$$\psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \text{ and } \frac{|\nabla\psi|^2}{\psi} \text{ is bounded in } \Omega.$$

Then, there exist constants $M>0$ and $\epsilon_0>0$ such that

$$\psi(x)|\nabla u_\epsilon(x)|^2 \le M(u_\epsilon(x)^{1-\beta} + u_\epsilon(x)) \text{ for every } x\in\Omega, \quad 0<\epsilon<\epsilon_0.$$

Proof of Lemma 5.3. From (6), we obtain constants $C_1>0$ and $0<t_0<1$ such that

$$|\widetilde{f}'(s)| \le C_1 s^{q_1-1} \text{ for } 0\le s\le t_0. \tag{72}$$

From (71) we obtain that Δu_ϵ is bounded in $L^\infty(\Omega)$. Thus, by standard elliptic regularity, u_ϵ belongs to $C^{1,\nu}(\overline{\Omega})$. We define

$$\overline{h}_\epsilon(u) = g_\epsilon(u) - f(u).$$

We shall denote u_ϵ simply by u. Define the functions

$$Z(u) = u^{1-\beta} + u + a, \qquad w = \frac{|\nabla u|^2}{Z(u)}, \qquad v = w\psi,$$

where $a>0$ is small. Note that v is C^2 at all points $x\in\Omega$ such that $u(x)>0$. Indeed, let $x\in\Omega$ be one such point. By continuity, there must exist an open ball $B\subset\Omega$ centred at x such that $u>0$ in $\overline{B}$. Consequently, we know that $g_\epsilon(u)\in C^{1,\nu}(B)$ and $f(u)\in C^{1,\nu}(B)$. Hence, $\overline{h}_\epsilon(u)\in C^{1,\nu}(B)$. Since u satisfies the equation $-\Delta u + \overline{h}_\epsilon(u) = 0$ in B, we conclude that $u\in C^3(B)$, implying that $Z(u)$ and w are C^2 in B.

The function v is continuous in $\overline{\Omega}$, hence it attains its maximum at some point $x_0\in\overline{\Omega}$. Thus, we obtain

$$v(x_0) > 0.$$

Note that $x_0\in\Omega$, because $v=0$ on $\partial\Omega$. Furthermore, $u(x_0)>0$, since otherwise x_0 would be a critical point of u and $w(x_0)=0$. Hence,

$$\nabla v(x_0) = 0$$

and

$$\Delta v(x_0) \le 0. \tag{73}$$

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The computations already carried out in [20, 22] lead to the following expression evaluated at x_0

$$\begin{aligned}\Delta v \geq \frac{1}{Z(u)}\Bigg[&\psi w^2\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right)\\ &+ w(2\psi Z(u)\overline{h}_\epsilon'(u) - \psi\overline{h}_\epsilon(u)Z'(u) - K_0Z(u))\\ &-K_0Z'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2}\Bigg],\end{aligned} \tag{74}$$

where

$$K_0 = \max\left(\sup_\Omega\left(\frac{|\nabla\psi|}{\psi^{1/2}}\right), \sup_\Omega\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right)\right) > 0.$$

We will show that if $v(x_0)$ is large enough then the right-hand side of (74) must be positive, which would contradict (73).

We will establish the following estimates uniformly for every ϵ sufficiently small.

$$Z'(u)Z(u)^{1/2} \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right), \tag{75}$$

$$Z(u)|\overline{h}_\epsilon'(u)| \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right), \tag{76}$$

$$Z'(u)|\overline{h}_\epsilon(u)| \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right), \tag{77}$$

$$Z(u) \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right), \tag{78}$$

for every $0 \leq u \leq T$. The constant C depends only on T, but not on ϵ nor on a.

Assuming for a moment that (75)–(78) are true. Inequality (74) implies that

$$\begin{aligned}\Delta v &\geq \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)}(\psi w^2 - C(w + \psi^{1/2}w^{3/2}))\\ &= \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)\psi}(v^2 - C(v + v^{3/2})).\end{aligned}$$

Since $\Delta v(x_0) \leq 0$ and

$$\frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)\psi} > 0 \text{ in } \Omega,$$

we conclude that

$$v(x_0)^2 - C(v(x_0) + v(x_0)^{3/2}) \leq 0.$$

Consequently, there exists $M > 0$ that does not depend on a such that

$$\sup_\Omega v = v(x_0) < M.$$

Consequently,

$$|\nabla u(x)|^2\psi(x) \leq M(u(x)^{1-\beta} + u(x) + a) \text{ for all } x \in \Omega.$$

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The result then follows by letting $a \to 0$.

We prove now the relations (75)–(78). In the course of this proof, C, $\widetilde{C}$, C_i, $i \in \{1, 2, 3, \ldots\}$ denote various positive constants independent of ϵ and a, we obtain gradient

$$Z(u) = u^{1-\beta} + u + a,$$

$$Z'(u) = (1-\beta)u^{-\beta} + 1, \qquad Z''(u) = -\beta(1-\beta)u^{-\beta-1}.$$

Hence,

$$\frac{1}{2}Z'(u)^2 - Z''(u)Z(u) \geq \frac{(1-\beta)^2}{2}(u^{-2\beta}+1) + a\beta(1-\beta)u^{-1-\beta}. \text{ for } u > 0. \tag{79}$$

We first prove (78). Indeed, there is a constant $C > 0$ such that

$$Z(u) = u^{1-\beta} + u + a \leq C \text{ for } 0 \leq u \leq T.$$

Hence, (78) follows from (79).

We now prove (77). Note that there exists a constant $\widetilde{C} > 0$ such that

$$\begin{aligned} Z'(u)|\overline{h}_\epsilon(u)| &\leq ((1-\beta)u^{-\beta}+1)(g_\epsilon(u) + |f(u)|) \\ &\leq (1-\beta)u^{-2\beta} + (1-\beta)u^{-\beta}\sup_{0\leq s\leq T}|f(s)| + u^{-\beta} + \sup_{0\leq s\leq T}|f(s)| \\ &\leq \widetilde{C}(1+u^{-2\beta}). \end{aligned}$$

Inequality (77) then follows from (79).

Now, we prove (76). Note that

$$\overline{h}_\epsilon'(u) = \frac{u^{q-1}}{(u+\epsilon)^{q+\beta+1}}\,(q\epsilon - \beta u) - f'(u).$$

We split the proof of (76) in three cases.

Case I. Suppose that $0 < u < \min\{\frac{q\epsilon}{2\beta}, t_0\}$, where $0 < t_0 < 1$ is given by (72). We define

$$\omega_\epsilon(u) = \frac{u^{q-1}}{(u+\epsilon)^{q+\beta+1}}\,(q\epsilon - \beta u) - C_1 u^{q_1-1},$$

where $C_1 > 0$ is given by (72). We claim that there exists $\epsilon_0 > 0$ such that $\omega_\epsilon(u) > 0$ for each $0 < \epsilon < \epsilon_0$. Indeed, assume by contradiction that $\omega_\epsilon(u) < 0$ for some $0 < u < \frac{q\epsilon}{2\beta}$.

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We then have

$$q\epsilon u^{q-1} < \beta u^q + C_1 u^{q_1-1}(u+\epsilon)^{q+\beta+1} < \beta u^q + C_1 u^{q_1-1}\epsilon^{q+\beta+1}\left(1+\frac{q}{2\beta}\right)^{q+\beta+1}.$$

Now take $\epsilon_0 > 0$ such that

$$C_1\epsilon^{q+\beta+1}\left(1+\frac{q}{2\beta}\right)^{q+\beta+1} < \frac{\epsilon q}{2} \text{ for } 0<\epsilon<\epsilon_0.$$

We may assume that $0 < q < q_1$. Consequently,

$$q\epsilon u^{q-1} < \beta u^q + \frac{\epsilon q u^{q_1-1}}{2} < \beta u^q + \frac{\epsilon q u^{q-1}}{2}.$$

Hence,

$$\frac{q\epsilon u^{q-1}}{2} < \beta u^q,$$

which implies that

$$u > \frac{q\epsilon}{2\beta}.$$

This contradicts our initial assumption. The claim is proven. Since

$$\frac{q\epsilon u^{q-1}}{(u+\epsilon)^{q+\beta+1}} \le q\frac{u^q}{u(u+\epsilon)^q}\frac{\epsilon}{(u+\epsilon)^{\beta+1}} \le \frac{q}{u^{\beta+1}},$$

we obtain

$$|\overline{h}_\epsilon'(u)| = \overline{h}_\epsilon'(u) \le \frac{q+C_1u^{q_1+\beta}}{u^{\beta+1}} \text{ for } 0<u<\min\left\{\frac{q\epsilon}{2\beta}, t_0\right\}.$$

Hence,

$$|\overline{h}_\epsilon'(u)| \le \frac{2q}{u^{\beta+1}} \text{ for } 0<u<\min\left\{\frac{q\epsilon}{2\beta}, t_0, t_1\right\},$$

where $t_1 > 0$ is chosen such that

$$C_1u^{q_1+\beta} < q \text{ for } 0 \le u \le t_1.$$

Therefore,

$$Z(u)|\overline{h}_\epsilon'(u)| \le (u^{1-\beta}+u+a)\left(\frac{2q}{u^{\beta+1}}\right) \le \frac{2q}{u^{2\beta}} + \frac{2qa}{u^{\beta+1}} \text{ for } 0 \le u \le \min\left\{\frac{q\epsilon}{2\beta}, t_0, t_1\right\}.$$

Comparing with (79), it follows that there exists a constant $C > 0$ that does not depend on a such that

$$Z(u)|\overline{h}_\epsilon'(u)| \le C(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)) \text{ for } 0<u<\min\left\{\frac{q\epsilon}{2\beta}, t_0, t_1\right\}, \quad 0<\epsilon<\epsilon_0. \tag{80}$$

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Case II. Suppose that $\frac{\epsilon q}{2\beta} \leq u \leq \min\{t_0,\, t_1\}$. We have

$$|\overline{h}_\epsilon'(u)| \leq \frac{u^{q-1}|q\epsilon - \beta u|}{(u+\epsilon)^{q+\beta+1}} + C_1 u^{q_1-1} \text{ for } \frac{q\epsilon}{2\beta} \leq u \leq t_0.$$

Note that $|q\epsilon - \beta u| \leq \beta u$ if $2\beta u \geq q\epsilon$. We then obtain

$$|\overline{h}_\epsilon'(u)| \leq \frac{\beta u^q + C_1 u^{q+q_1+\beta}\left(1+\frac{2\beta}{q}\right)^{q+\beta+1}}{(u+\epsilon)^{q+\beta+1}} \text{ for } \frac{q\epsilon}{2\beta} \leq u \leq t_0.$$

Now, observe that there exists $0 < t_2 < \min\{t_0,\, t_1\}$ that does not depend on ϵ such that

$$C_1 u^{q_1+\beta}\left(1+\frac{2\beta}{q}\right)^{q+\beta+1} < \beta \text{ for } 0 \leq u \leq t_2.$$

Therefore,

$$|\overline{h}_\epsilon'(u)| \leq \frac{2\beta}{u^{\beta+1}} \text{ for } \frac{q\epsilon}{2\beta} \leq u < t_2.$$

Comparing with (79), we obtain

$$Z(u)|\overline{h}_\epsilon'(u)| \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right) \text{ for } \frac{q\epsilon}{2\beta} \leq u < t_2. \tag{81}$$

Case III. Assume that $t_2 \leq u \leq T$. Since there exists a constant $C > 0$ such that $|\overline{h}_\epsilon'(u)| \leq C$ for $t_2 \leq u \leq T$, it follows from (78) that

$$Z(u)|\overline{h}_\epsilon'(u)| \leq C\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right) \text{ for } t_2 \leq u \leq T. \tag{82}$$

Hence, (76) follows from (80), (81) and (82).

We now prove (75). Observe that

$$Z'(u)Z(u)^{1/2} = ((1-\beta)u^{-\beta}+1)\sqrt{u^{1-\beta}+u+a}.$$

Hence,

$$Z'(u)Z(u)^{1/2} \leq \sqrt{3T}((1-\beta)u^{-\beta}+1).$$

When $0 \leq u \leq 1$ we know that $u^2 \leq u$. Hence $u^{-\beta} \leq u^{-2\beta}$. Therefore, from (79), there exist constants $C_3 > 0$ and $C_4 > C_3$ such that

$$Z'(u)Z(u)^{1/2} \leq C_3(u^{-2\beta}+1) \leq C_4\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right) \text{ for } 0 \leq u \leq 1. \tag{83}$$

If $1 \leq u \leq T$, we know that there exists a constant $C_5 > 0$ such that $Z'(u)Z(u)^{1/2} \leq C_5$. Hence, from (79), there exists a constant $C_6 > 0$ such that

$$Z'(u)Z(u)^{1/2} \leq C_6\left(\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\right) \text{ for } 1 \leq u \leq T. \tag{84}$$

Inequality (75) then follows from (83) and (84). We have proved Lemma 5.3. □

Consequently, we obtain

Corollary 5.4. *For each $0 < \epsilon < 1$, let u_ϵ be the solution of problem* (2) *obtained in Propositions* 3.2 *and* 4.3. *Let ψ be as in the hypothesis of Lemma* 5.3. *Then there exist constants $M > 0$ and $\epsilon_0 > 0$ such that*

$$\psi(x)|\nabla u_\epsilon(x)|^2 \leq M(u_\epsilon(x)^{1-\beta} + u_\epsilon(x)) \text{ for every } x \in \Omega, \quad 0 < \epsilon < \epsilon_0.$$

Proof of Corollary 5.4. From Propositions 3.2 and 4.3, we know that there is a constant $D > 0$ such that

$$\|u_\epsilon\|_{H_0^1(\Omega)} < D \text{ for each } 0 < \epsilon < 1.$$

From Lemmas 4.2, 5.1 and 5.2 we conclude that the solutions u_ϵ of (2) are bounded in $L^\infty(\Omega)$ by constant $K_1 > 0$ and $K_2 > 0$ independent of ϵ. Corollary 5.4 then follows by Lemma 5.3. □

6. The limit of approximate solutions

Now we will study the convergence as $\epsilon \to 0$ of the solutions u_ϵ of problem (2) obtained in Propositions 3.2 and 4.3. First, we obtain the existence of a non-trivial limit u. Next, we prove that u is a solution of problem (1).

Lemma 6.1. *Let (ϵ_n) be a sequence in $(0, 1)$ such that $\epsilon_n \to 0$ as $n \to \infty$. Let $(u^1_{\epsilon_n})$ and $(u^2_{\epsilon_n})$ be the sequences of solutions obtained in Propositions* 3.2 *and* 4.3 *respectively. Then there exist non-trivial functions $u_1 \in H_0^1(\Omega)$ and $u_2 \in H_0^1(\Omega)$ such that, up to a subsequence, $u^i_{\epsilon_n} \rightharpoonup u_i$ weakly in $H_0^1(\Omega)$, where $i \in \{1, 2\}$.*

Proof of Lemma 6.1. From Propositions 3.2 and 4.3, we know that there exist constants $D_i > 0$ such that

$$\|u^i_{\epsilon_n}\|_{H_0^1(\Omega)} < D_i \text{ for each } n \in \mathbb{N}, i \in \{1, 2\}$$

Hence, there exist functions $u_i \in H_0^1(\Omega)$ such that

$$\begin{cases} u^i_{\epsilon_n} \rightharpoonup u_i \text{ weakly in } H_0^1(\Omega); \\ u^i_{\epsilon_n} \to u_i \text{ in } L^r(\Omega) \text{ for every } 1 \leq r < 2^*; \\ u^i_{\epsilon_n} \to u_i \text{ a.e in } \Omega; \end{cases} \tag{85}$$

Lemmas 5.1 and 5.2 imply that $u^i_{\epsilon_n} \in L^\infty(\Omega)$ with $\|u^i_{\epsilon_n}\|_{L^\infty(\Omega)} < K_i$ for all $n \in \mathbb{N}$. Consequently, the Dominated Convergence Theorem implies that

$$u^i_{\epsilon_n} \to u_i \text{ in } L^r(\Omega) \text{ for every } r \geq 1.$$

We prove the result for $i = 1$ and denote $(u^1_{\epsilon_n})$ and u_1 merely by (u_{ϵ_n}) and u respectively. From Proposition 3.2, we have

$$0 < a_1 \leq I_{\epsilon_n}(u_{\epsilon_n}) = \frac{1}{2}\int_\Omega |\nabla u_{\epsilon_n}|^2 + \int_\Omega G_{\epsilon_n}(u_{\epsilon_n}) - \int_\Omega F(u_{\epsilon_n}).$$

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Since u_{ϵ_n} is a non-negative critical point of I_{ϵ_n}, we have

$$\|u_{\epsilon_n}\|^2_{H^1_0(\Omega)} + \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n} = \int_\Omega u_{\epsilon_n} f(u_{\epsilon_n}).$$

Hence,

$$I_{\epsilon_n}(u_{\epsilon_n}) = \int_\Omega \left(G_{\epsilon_n}(u_{\epsilon_n}) - \frac{1}{2}g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\right)\,\mathrm{d}x - \int_\Omega \left(F(u_{\epsilon_n}) - \frac{1}{2}u_{\epsilon_n} f(u_{\epsilon_n})\right)\,\mathrm{d}x > a_1. \tag{86}$$

The Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_\Omega g_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n}\,\mathrm{d}x = \int_\Omega u^{1-\beta}\,\mathrm{d}x,$$
$$\lim_{n\to\infty} \int_\Omega G_{\epsilon_n}(u_{\epsilon_n})\,\mathrm{d}x = \frac{1}{1-\beta}\int_\Omega u^{1-\beta}\,\mathrm{d}x,$$

and

$$\int_\Omega \left(F(u_{\epsilon_n}) - \frac{1}{2}u_{\epsilon_n} f(u_{\epsilon_n})\right)\,\mathrm{d}x \to \int_\Omega \left(F(u) - \frac{1}{2}uf(u)\right)\,\mathrm{d}x.$$

Taking the above claims into account and letting $n \to \infty$ in (86), we obtain

$$\int_\Omega \left(\frac{u^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{2}\right)\,\mathrm{d}x - \int_\Omega \left(F(u) - \frac{1}{2}uf(u)\right)\,\mathrm{d}x \geq a_1.$$

We proved that u is non-trivial. The proof for $i = 2$ is analogous. □

We now show that the functions u_1 and u_2 defined in Lemma 6.1 satisfy the following property.

Lemma 6.2. *Let u_1 and u_2 be the functions given by Lemma 6.1. The function $u_i^{-\beta}\chi_{\{u_i>0\}}$ belongs to $L^1_{loc}(\Omega)$ for $i \in \{1, 2\}$.*

Proof of Lemma 6.2. We again prove the result for $i = 1$. The proof for $i = 2$ is analogous. Let (u_{ϵ_n}) and u be given by (85) with $i = 1$. Let $V \subset \Omega$ be a open set such that $\overline{V} \subset \Omega$. Take $\zeta \in C^1_c(\Omega)$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ in V. Since u_{ϵ_n} is a critical point of I_{ϵ_n}, we obtain

$$\int_{\{u_{\epsilon_n}<1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta = \int_\Omega f(u_{\epsilon_n})\zeta - \int_\Omega \nabla u_{\epsilon_n}\nabla\zeta - \int_{\{u_{\epsilon_n}\geq 1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta.$$

Corollary 5.4 implies that $u_{\epsilon_n} \to u$ uniformly in compact subsets of Ω. Since $u_{\epsilon_n} \rightharpoonup u$ weakly in $H^1_0(\Omega)$, we get

$$\int_{\{u_{\epsilon_n}<1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta \to \int_\Omega f(u)\zeta - \int_\Omega \nabla u\nabla\zeta - \int_{\{u\geq 1\}} u^{-\beta}\zeta \text{ as } \epsilon \to 0 \tag{87}$$

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Define the set $\Omega_\rho = \{x \in \Omega : u(x) \geq \rho\}$ for $\rho > 0$. It follows from (87) that there exists a constant $C > 0$ that does not depend on n nor on ρ such that

$$\int_{V\cap\Omega_\rho} \frac{u_{\epsilon_n}^q}{(u_{\epsilon_n}+\epsilon_n)^{q+\beta}} \chi_{\{u_{\epsilon_n}<1-\epsilon_n\}} \zeta \leq \int_{\{u_{\epsilon_n}<1-\epsilon_n\}} g_{\epsilon_n}(u_{\epsilon_n})\zeta < C \text{ for all } n \in \mathbb{N}, \quad \rho > 0,$$

Letting $n \to \infty$ and using Fatou's Lemma, we then get

$$\int_V u^{-\beta} \chi_{\Omega_\rho} < C.$$

Letting $\rho \to 0$ and applying Fatou's Lemma again, we conclude that

$$\int_V u^{-\beta} \chi_{\{u>0\}} < \infty.$$

Since V was arbitrarily chosen, Lemma 6.2 is proved. □

Proof of Theorem 1.1. The proof of this result is very similar to the one given in [14], but for the sake of completeness, we give the proof with details. We will show that the sequences $(u^1_{\epsilon_n})$ given by Lemma 6.1 converge to a solution u_1 of (1) as $n \to \infty$. In doing so, we obtain a solution u_1 of (1) which is non-trivial. The non-triviality of u_1 is guaranteed by Lemma 6.1. From now on, we denote u_{ϵ_n} and u_1 merely by u_ϵ and u respectively. Let $\varphi \in C^1_c(\Omega)$. From Proposition 3.2, we have

$$\int_\Omega \nabla u_\epsilon \nabla \varphi = \int_\Omega (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi. \tag{88}$$

Let $\eta \in C^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\eta(s) = 0$ for $s \leq 1/2$, $\eta(s) = 1$ for $s \geq 1$. For $m > 0$ the function $\varrho := \varphi\eta(u_\epsilon/m)$ belongs to $C^1_c(\Omega)$.

From Corollary 5.4, we know that $|\nabla u_\epsilon|$ is locally bounded independent on $0 < \epsilon < \epsilon_0$. It then follows from (47) and the Arzelà-Ascoli Theorem that $u_\epsilon \to u$ in $C^0_{loc}(\Omega)$, and the set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ is open. Let $\tilde{\Omega}$ be an open set such that $\overline{support(\varphi)} \subset \tilde{\Omega}$ and $\overline{\tilde{\Omega}} \subset \Omega$. Let $\Omega_0 = \Omega_+ \cap \tilde{\Omega}$. For every $m > 0$, there is an $\epsilon_1 > 0$ such that

$$u_\epsilon(x) \leq m/2 \text{ for every } x \in \tilde{\Omega} \setminus \Omega_0 \text{ and } 0 < \epsilon \leq \epsilon_1. \tag{89}$$

Replacing φ by ϱ in (88), we obtain

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m). \tag{90}$$

We break the previous integral as

$$X_\epsilon := \int_{\Omega_0} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m)$$

and

$$Y_\epsilon := \int_{\tilde{\Omega}\setminus\Omega_0} (-g_\epsilon(u_\epsilon) + f(u_\epsilon))\varphi\eta(u_\epsilon/m).$$

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Clearly, $Y_\epsilon = 0$, whenever $0 < \epsilon \le \epsilon_1$ by (89) and the definition of η. From (47), the Dominated Convergence Theorem and from the fact that $u_\epsilon \to u$ uniformly in Ω_0, we get

$$X_\epsilon \to \int_{\Omega_0} (-u^{-\beta} + f(u))\varphi\eta(u/m) \text{ as } \epsilon \to 0.$$

We take the limit in m to conclude that

$$\int_{\Omega_0} (-u^{-\beta} + f(u))\varphi\eta(u/m) \to \int_{\Omega_0} (-u^{-\beta} + f(u))\varphi \text{ as } m \to 0, \tag{91}$$

since $\eta(u/m) \le 1$ and $u^{-\beta}\chi_{\Omega^+} + f(u) \in L^1(\tilde{\Omega})$, according to Lemma 6.2.

What follows next is identical to [20]. We proceed with the integral on the left side of (90). We have

$$\int_\Omega \nabla u_\epsilon \nabla(\varphi\eta(u_\epsilon/m)) = \int_{\tilde{\Omega}} (\nabla u_\epsilon \nabla\varphi)\eta(u_\epsilon/m) + W_\epsilon, \tag{92}$$

where

$$W_\epsilon = \int_{\tilde{\Omega}} \frac{|\nabla u_\epsilon|^2}{m}\eta'(u_\epsilon/m)\varphi.$$

Consequently,

$$\int_{\tilde{\Omega}} (\nabla u_\epsilon \nabla\varphi)\eta(u_\epsilon/m) \to \int_{\tilde{\Omega}} (\nabla u \nabla\varphi)\eta(u/m) \text{ as } \epsilon \to 0,$$

since $u_\epsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_\epsilon \to u$ uniformly in $\tilde{\Omega}$. Hence, by the Dominated Convergence Theorem,

$$\int_{\tilde{\Omega}} (\nabla u \nabla\varphi)\eta(u/m) \to \int_{\tilde{\Omega}} \nabla u \nabla\varphi \text{ as } m \to 0. \tag{93}$$

Now we only need to show that

$$W_\epsilon \to 0 \text{ as } \epsilon \to 0 \quad (\text{ and then as } m \to 0). \tag{94}$$

Let $Z_0(u_\epsilon) = u_\epsilon^{1-\beta} + u_\epsilon$. The estimate $|\nabla u_\epsilon|^2 \le M Z_0(u_\epsilon)$ in $\tilde{\Omega}$ provided by Corollary 5.4 yields

$$\begin{aligned}\limsup_{\epsilon\to 0} |W_\epsilon| &\le \frac{M}{m} \lim_{\epsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u_\epsilon \le m\}} Z_0(u_\epsilon)|\eta'(u_\epsilon/m)\varphi| \\ &\le M \lim_{\epsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u_\epsilon \le m\}} \frac{Z_0(u_\epsilon)|\eta'(u_\epsilon/m)\varphi|}{u_\epsilon}.\end{aligned}$$

Consequently

$$\begin{aligned}\limsup_{\epsilon\to 0} |W_\epsilon| &\le M \sup|\eta'| \sup|\varphi| \lim_{\epsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u_\epsilon \le m\}} \frac{Z_0(u_\epsilon)}{u_\epsilon} \\ &\le M \sup|\eta'| \sup|\varphi| \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u \le m\}} (1 + u^{-\beta})\chi_{\{u>0\}},\end{aligned}$$

for every $m > 0$.

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The claim follows by letting $m \to 0$ and by using Lemma 6.2.

As a immediate consequence of (90), (91),(92), (93) and (94), we have

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} (-u^{-\beta} + f(u))\varphi$$

for every $\varphi \in C_c^1(\Omega)$. This concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof of this result is entirely analogous to the proof of Theorem 1.1. We only need to use Proposition 4.3 instead of Proposition 3.2 and consider $f(s) = \lambda s + s^{2^*-1}$. □

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