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Shuffles and concatenations in the construction of graphs†

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This paper reports on an investigation into the role of shuffling and concatenation in the theory of graph drawing. A simple syntactic description of these and related operations is proved to be complete in the context of finite partial orders, and as general as possible. An explanation based on this result is given for a previously investigated collapse of the permutohedron into the associahedron, and for collapses into other less familiar polyhedra, including the cyclohedron. Such polyhedra have been considered recently in connection with the notion of tubing, which is closely related to tree-like finite partial orders, which are defined simply here and investigated in detail. Like the associahedron, some of these other polyhedra are involved in categorial coherence questions, which will be treated elsewhere.

1. Introduction

Shuffles and concatenations, which are usually considered only for finite linear orders, are here defined for arbitrary binary relations (see Section 4). Shuffles serve to define an associative and commutative partial operation on sets of relations, which we call the shuffle sum; analogously, concatenations serve to define an associative partial operation on sets of relations, which we call the concatenation product. We are interested in the shuffle sum and concatenation product because the one-to-one map L, which assigns to a partial order all its linear extensions, maps the disjoint union and concatenation of partial orders into the shuffle sum and concatenation product, respectively (see Section 4). We shall now explain why we are interested in the disjoint union and concatenation of partial orders.

We associate with a given graph Γ a set of terms representing tree-like finite partial orders $T(\Gamma)$, each of which may be understood as a possible history of the construction, or, conversely, the destruction, of Γ. The set $T(\Gamma)$ determines the graph Γ uniquely, that is, the map T is one-to-one. The tree-like partial orders of $T(\Gamma)$ are closely related to the *tubings* of Devadoss and Forcey (2008), but they are defined more simply (the connection with tubings is explained in detail in Došen and Petrić (2011) – see, in particular, Section 3 and Appendix A).

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The members of $T(\Gamma)$ are built inductively in a simple manner with the help of two operations corresponding to disjoint union and concatenation. These operations correspond through the map L mentioned above to shuffle sum and concatenation product.

The members of $T(\Gamma)$ label vertices of polyhedra that are obtained from permutohedra by collapsing connected families of vertices into a single vertex. (Historical references concerning permutohedra, associahedra and cyclohedra may be found in Došen and Petrić (2011); for three-dimensional versions of these polyhedra, see Section 5 of the current paper.) We use the map L to assign to a member of $T(\Gamma)$ the permutations in a connected family of vertices of the permutohedron, which are collapsed into a single vertex. The collapsing in question that produces associahedra were studied in Tonks (1997), and later generalised in Forcey and Springfield (2010). For a suitable choice of Γ, we obtain a collapsing that produces cyclohedra, and other choices yield less familiar polyhedra, which are described in Section 5. Some of these polyhedra (which are called graph associahedra in Carr and Devadoss (2006)) are interesting because of their connections with operad theory and category theory, which we will mention in the concluding section (Section 6).

Our examples of collapsing depend on specific graphs Γ, but we show that we have a general phenomenon, which is not just restricted to our examples. The maps T and L for a given graph Γ with n vertices induce an equivalence relation on the set of vertices of the $(n-1)$-dimensional permutohedron (see Section 5). While the collapsing studied in Tonks (1997) and Forcey and Springfield (2010) involves all the faces, our collapsing is a particular case of collapsing involving only the vertices. The concluding results of Section 5 (Propositions 5.6 and 5.7, and their consequence mentioned in the last paragraph) may be inferred from issues considered in Forcey and Springfield (2010, Section 3.3)[†].

Our approach through $T(\Gamma)$ is restricted to vertices. For other faces, we must switch to other notions (tubings or their equivalents considered in Došen and Petrić (2011)). However, restricting consideration just to the vertices does not make our results poorer, because, as explained in Došen and Petrić (2011), the abstract polytopes in question are completely determined by their vertices. These abstract polytopes and their realisations are treated in detail in Došen and Petrić (2011).

Our tree-like partial orders are easily described syntactically using two partial binary operations: one, corresponding to disjoint union, is associative and commutative; the other, corresponding to concatenation, is just associative. The novelty that this approach brings for the remaining part of the paper is a simplification of notation, for which we will give a theoretical grounding[‡].

[†] We are grateful to an anonymous referee for telling us about Forcey and Springfield (2010).

[‡] *Added on 30 August 2011.* After completing this paper, we learned that shuffles and concatenations have previously been studied in a manner related to ours, as reported in a number of papers cited in Aceto and Fokkink (2006, Section 3). In particular, the results of our Sections 2 and 3 were anticipated in Grabowski (1981) and Valdes *et al.* (1982, Theorem 1) – see also Tschantz (1994, Lemma 1) and Gischer (1988, Theorem 3.1).

Further references to previous related work are given in the concluding section. We also give there an indication of the work reported in our papers Došen and Petrić (2010; 2011), which develops the research we started in the current paper, and gives motivation for it.

2. Disjoint union and concatenation of relations

In this section we present some preliminary issues concerning the partial operations of disjoint union and concatenation of binary relations. These operations are partial because we require disjointness of domains. We will apply these operations in particular to partial orders that satisfy a property we call trifunctionality, which generalises difunctionality (see references below). The results of this section prepare the ground for the isomorphism result of the next section.

A *relation* on a set X is, as usual, an ordered pair $\langle R, X\rangle$ such that $R \subseteq X^2$. (We only deal with binary relations in this paper.) The set X is the *domain* of $\langle R, X\rangle$.

For the relations $\langle R, X\rangle$ and $\langle S, Y\rangle$ such that $X \cap Y = \emptyset$, we have:

$$\langle R, X\rangle + \langle S, Y\rangle =_{df} \langle R \cup S, X \cup Y\rangle$$
$$\langle R, X\rangle \cdot \langle S, Y\rangle =_{df} \langle R \cup S \cup (X \times Y), X \cup Y\rangle.$$

The operation $+$ is disjoint union, while $\cdot$ could be called *concatenation*, because this is what it is when $\langle R, X\rangle$ and $\langle S, Y\rangle$ are linear orders on finite domains, that is, finite sequences. It is clear that $+$ is associative and commutative, while $\cdot$ is associative without being commutative for X and Y non-empty (for X or Y empty, $+$ and $\cdot$ coincide). It is easy to verify the following results.

Remark 2.1 (+). If $\langle R, X\rangle$ is such that

$$X = X_1 \cup X_2$$
$$X_1 \cap X_2 = \emptyset$$

and for every x_1 in X_1 and every x_2 in X_2 we have $(x_1, x_2) \notin R$ and $(x_2, x_1) \notin R$, then there are relations $\langle R_1, X_1\rangle$ and $\langle R_2, X_2\rangle$ such that

$$\langle R, X\rangle = \langle R_1, X_1\rangle + \langle R_2, X_2\rangle.$$

Remark 2.2 (·). If $\langle R, X\rangle$ is such that

$$X = X_1 \cup X_2$$
$$X_1 \cap X_2 = \emptyset$$

and for every x_1 in X_1 and every x_2 in X_2 we have $(x_1, x_2) \in R$ and $(x_2, x_1) \notin R$, then there are relations $\langle R_1, X_1\rangle$ and $\langle R_2, X_2\rangle$ such that

$$\langle R, X\rangle = \langle R_1, X_1\rangle \cdot \langle R_2, X_2\rangle.$$

Partial orders in this paper will be *strict* partial orders – that is, relations that are irreflexive and transitive. Note that if $\langle R, X\rangle$ is a partial order, we may omit the conjunct

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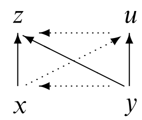
$(x_2, x_1) \notin R$ in Remark 2.2 (on $\cdot$) since it follows from $(x_1, x_2) \in R$. We can trivially prove the following proposition.

Proposition 2.3. If $X \cap Y = \emptyset$, then $\langle R, X\rangle$ and $\langle S, Y\rangle$ are partial orders if and only if $\langle R, X\rangle + \langle S, Y\rangle$ is a partial order, and similarly with $\cdot$ instead of $+$.

We say a relation $\langle R, X\rangle$ is *trifunctional* when for every x, y, z and u in X we have that if (x, z) and (y, z) and (y, u) are in R, then either (x, u) or (y, x) or (u, z) is in R. The following picture illustrates this implication:



If in this implication we omit the disjuncts $(y, x) \in R$ and $(u, z) \in R$ from the consequent, we obtain the implication that defines difunctional relations (see Riguet (1948) and Schmidt and Ströhlein (1993, Section 4.4); our term 'trifunctional' is motivated by the term 'difunctional' and the fact that we have three conjuncts in the antecedent and three disjuncts in the consequent). We can prove the following proposition.

Proposition 2.4. If $X \cap Y = \emptyset$, then $\langle R, X\rangle$ and $\langle S, Y\rangle$ are trifunctional relations if and only if $\langle R, X\rangle + \langle S, Y\rangle$ is trifunctional, and similarly with $\cdot$ instead of $+$.

Proof. For $+$ the proof is trivial, and for $\cdot$ the direction from right to left is trivial, which means we just need to prove that if $\langle R, X\rangle$ and $\langle S, Y\rangle$ are trifunctional, then $\langle R, X\rangle \cdot \langle S, Y\rangle$ is trifunctional. So we suppose $\langle R, X\rangle$ and $\langle S, Y\rangle$ are trifunctional, and that for x, y, z and u in $X \cup Y$, we have (x, z), (y, z) and (y, u) in $R \cup S \cup (X \times Y)$. We have the following cases:

(1) $z \in X$:
So $x, y \in X$, and we have the subcases:

(1.1) $u \in X$:
We appeal to the trifunctionality of $\langle R, X\rangle$.

(1.2) $u \in Y$:
We have $(x, u) \in X \times Y$.

(2) $z \in Y$:
We have the subcases:

(2.1) $u \in X$:
So $(u, z) \in X \times Y$.

(2.2) $u \in Y$:
We have the subcases:

(2.2.1) $x \in X$:
So $(x, u) \in X \times Y$.

(2.2.2) $x \in Y$:
We have the subcases:

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(2.2.2.1) $y \in X$:
So $(y, x) \in X \times Y$.

(2.2.2.2) $y \in Y$:
We appeal to the trifunctionality of $\langle S, Y \rangle$. □

For a relation $\langle R, X \rangle$ and x_1 and x_n, where $n \geqslant 2$, distinct elements of X, we write $x_1 \sim_R x_n$ when there is a sequence $x_1 \ldots x_n$ such that for every $i \in \{1, \ldots, n-1\}$ we have $(x_i, x_{i+1}) \in R$ or $(x_{i+1}, x_i) \in R$. We say that $\langle R, X \rangle$ is *connected* when for every two distinct x and y in X we have $x \sim_R y$. The proof of the following proposition is trivial.

Proposition 2.5. If for the relations $\langle R, X \rangle$ and $\langle S, Y \rangle$ we have $X \cap Y = \emptyset$, $X \neq \emptyset$ and $Y \neq \emptyset$, then $\langle R, X \rangle + \langle S, Y \rangle$ is not connected and $\langle R, X \rangle \cdot \langle S, Y \rangle$ is connected.

The following proposition follows easily from Proposition 2.5, and will be applied in the proof of the completeness proposition in the next section (Proposition 3.1).

Proposition 2.6. If

$$\langle R_1, X_1 \rangle + \ldots + \langle R_n, X_n \rangle = \langle S_1, Y_1 \rangle + \ldots + \langle S_m, Y_m \rangle,$$

and for every $i \in \{1, \ldots, n\}$ and every $j \in \{1, \ldots, m\}$ we have that both R_i and S_j are connected, and X_i and Y_j are not empty, then $n = m$ and there is a bijection

$$\pi : \{1, \ldots, n\} \to \{1, \ldots, m\}$$

such that for every $i \in \{1, \ldots, n\}$, we have

$$\langle R_i, X_i \rangle = \langle S_{\pi(i)}, Y_{\pi(i)} \rangle.$$

We also have the following proposition.

Proposition 2.7. If for the relations $\langle R_1, X \rangle$, $\langle R_2, X \rangle$ and $\langle S, Y \rangle$ we have $X \cap Y = \emptyset$, then either the equation

$$\langle R_1, X \rangle \cdot \langle S, Y \rangle = \langle R_2, X \rangle \cdot \langle S, Y \rangle$$

or the equation

$$\langle S, Y \rangle \cdot \langle R_1, X \rangle = \langle S, Y \rangle \cdot \langle R_2, X \rangle$$

implies $R_1 = R_2$.

Proof. Suppose

$$\langle R_1, X \rangle \cdot \langle S, Y \rangle = \langle R_2, X \rangle \cdot \langle S, Y \rangle.$$

If $(x, y) \in R_1$, then

$$(x, y) \in R_1 \cup S \cup (X \times Y),$$

so

$$(x, y) \in R_2 \cup S \cup (X \times Y).$$

But then $(x, y) \in R_2$ because $x, y \in X$, so $R_1 \subseteq R_2$, and we can demonstrate in the same manner that $R_2 \subseteq R_1$. The fact that

$$\langle S, Y \rangle \cdot \langle R_1, X \rangle = \langle S, Y \rangle \cdot \langle R_2, X \rangle$$

implies $R_1 = R_2$ is demonstrated analogously. □

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We can use this proposition to establish the following proposition, which will be applied in the proof of the completeness proposition in the next section (Proposition 3.1).

Proposition 2.8. If

$$\langle R_1, X_1\rangle \cdot \langle S_1, Y_1\rangle = \langle R_2, X_2\rangle \cdot \langle S_2, Y_2\rangle,$$

where $\langle S_1, Y_1\rangle$ and $\langle S_2, Y_2\rangle$ are either not connected or their domains are singletons, then

$$\langle R_1, X_1\rangle = \langle R_2, X_2\rangle$$
$$\langle S_1, Y_1\rangle = \langle S_2, Y_2\rangle.$$

Proof. Let

$$\langle T, Z\rangle = \langle R_1, X_1\rangle \cdot \langle S_1, Y_1\rangle = \langle R_2, X_2\rangle \cdot \langle S_2, Y_2\rangle.$$

It is clear from the definition of $\cdot$ that

$$(x \in Y_1 \;\&\; (x, y) \in T) \Rightarrow y \in Y_1 \tag{1}$$
$$(x \in X_2 \;\&\; y \in Y_2) \Rightarrow (x, y) \in T. \tag{2}$$

Note also that from the assumption that $\langle S_1, Y_1\rangle$ and $\langle S_2, Y_2\rangle$ are either not connected or their domains are singletons, it follows that Y_1 and Y_2 are not empty.

We show by *reductio ad absurdum* that $Y_1 \subseteq Y_2$. Suppose $Y_1 \subseteq Y_2$ is not true. So there is an x in Y_1 such that $x \notin Y_2$, which implies that $x \in X_2$. Then we show that $Y_2 \subseteq Y_1$:

$$\begin{aligned} y \in Y_2 &\Rightarrow (x, y) \in T && \text{(by (2))} \\ &\Rightarrow y \in Y_1 && \text{(by (1))}. \end{aligned}$$

The set Y_1 cannot be a singleton, because if it were, Y_2, which is not empty, would be the same singleton, and we supposed that we do not have $Y_1 \subseteq Y_2$. So Y_1 is not a singleton, and thus $\langle S_1, Y_1\rangle$ is not connected.

Let y_1 and y_2 be two distinct elements of Y_1 such that we do not have $y_1 \sim_{S_1} y_2$. The following three cases exhaust all the possibilities for y_1 and y_2 as elements of $X_2 \cup Y_2$:

(a) One of y_1 and y_2 is in X_2 and the other is in Y_2:
Let y_1 be in X_2 and y_2 in Y_2. Then by (2) we obtain $(y_1, y_2) \in T$, and since y_1 and y_2 are in Y_1, we have $(y_1, y_2) \in S_1$, which gives a contradiction.

(b) y_1 and y_2 are both in X_2:
Since Y_2 is not empty, for some y in Y_2 we have that (y_1, y) and (y_2, y) are in T. Since $Y_2 \subseteq Y_1$, we have $y \in Y_1$, from which we infer that (y_1, y) and (y_2, y) are in S_1, which gives a contradiction.

(c) y_1 and y_2 are both in Y_2:
Since $x \in X_2$, we have that (x, y_1) and (x, y_2) are in T, and since $x \in Y_1$, we have that (x, y_1) and (x, y_2) are in S_1, which gives a contradiction.

So we have established that $Y_1 \subseteq Y_2$, and we can establish in an analogous manner that $Y_2 \subseteq Y_1$. Hence we have $Y_1 = Y_2$.

We now show that $S_1 = S_2$. We have first that

$$\begin{aligned}(x,y) \in S_1 &\Rightarrow (x,y) \in T \;\;\&\;\; x,y \in Y_1 \\ &\Rightarrow (x,y) \in T \;\;\&\;\; x,y \in Y_2 \\ &\Rightarrow (x,y) \in S_2.\end{aligned}$$

So $S_1 \subseteq S_2$, and we can show analogously that $S_2 \subseteq S_1$.

Let $Y = Y_1 = Y_2$. Since X_1 and Y are disjoint, and X_2 and Y are also disjoint, we can infer from

$$X_1 \cup Y = X_2 \cup Y$$

that $X_1 = X_2$, and can then, by Proposition 2.7, conclude that

$$\langle R_1, X_1 \rangle = \langle R_2, X_2 \rangle,$$

which completes the proof. □

3. Diversified S-terms and relations in FTP

In this section we give very simple syntactic characterisations of trifunctional partial orders on finite sets. This is a freely generated structure, that is, it is an algebra in the sense of universal algebra, with two partial operations, one associative and commutative, corresponding to disjoint union, and the other associative, corresponding to concatenation. The operations are partial because we require that every free generator occurs just once in an element of our structure. We prove that this syntactically defined structure is isomorphic to the structure of trifunctional partial orders on finite sets with the operations of disjoint union and concatenation.

Consider terms built out of an infinite set of variables, which we denote by $x, y, z, \ldots, x_1, \ldots$ with the binary operations $+$ and $\cdot$, which we call *sum* and *product*. Consider structures, that is, algebras, with two binary operations $+$ and $\cdot$ such that $+$ is associative and commutative, while $\cdot$ is associative. Let **S** be the structure of this kind freely generated by infinitely many generators. We may take it that the elements of **S** are equivalence classes of the terms introduced above, which we thus call **S**-*terms*, while the variables $x, y, z, \ldots$ are called **S**-*variables*. The equivalence relation by which we obtain these equivalence classes is the one that exists between two terms when they may differ only with respect to the associativity and commutativity of $+$ and the associativity of $\cdot$. We define the operations $+$ and $\cdot$ on these equivalence classes by $[t] + [s] = [t + s]$ and $[t] \cdot [s] = [t \cdot s]$.

An **S**-term is said to be *diversified* when no **S**-variable occurs in it more than once. Since associativity and commutativity preserve diversification, it is clear that if the equivalence class $[t]$ is an element of **S** for t a diversified **S**-term, then every element of $[t]$ is diversified. We say that the element $[t]$ of **S** is *diversified* when t is a diversified **S**-term.

Let FTP be the set of trifunctional partial orders on non-empty finite sets of **S**-variables. We define by induction on complexity a map κ from the set of diversified **S**-terms to the set FTP:

$$\kappa(x) = \langle \emptyset, \{x\} \rangle$$
$$\kappa(t+s) = \kappa(t) + \kappa(s)$$
$$\kappa(t \cdot s) = \kappa(t) \cdot \kappa(s).$$

That $\kappa(t)$ is indeed a member of FTP for every diversified **S**-term t follows from the fact that the relation $\langle \emptyset, \{x\} \rangle$ is in FTP, together with Propositions 2.3 and 2.4.

Since the operation $+$ on relations is associative and commutative, while $\cdot$ is associative, the map κ induces a map K from the set of diversified elements of **S** to FTP, which is defined by:

$$K[t] = \kappa(t).$$

We use $K[t]$ as an abbreviation for $K([t])$. We can prove the following completeness proposition.

Proposition 3.1. The map K is one-to-one.

Proof. Suppose $\kappa(t) = \kappa(s)$. We proceed by induction on the number k of **S**-variables in t. Since the domains of the relations $\kappa(t)$ and $\kappa(s)$ are the same, the same **S**-variables occur in t and s, so k is also the number of **S**-variables in s.

If $k = 1$, then t and s are the same **S**-variable. If $k > 1$, let t be of the form $t_1 + \ldots + t_n$ and s be of the form $s_1 + \ldots + s_m$, for $n \geqslant 1$ and $m \geqslant 1$, with t_i, for $i \in \{1, \ldots, n\}$, and s_j, for $j \in \{1, \ldots, m\}$, **S**-variables or products. (Since $k > 1$, it is impossible that $n = 1$ and t_1 is an **S**-variable.) Since we have

$$\kappa(t) = \kappa(t_1) + \ldots + \kappa(t_n) = \kappa(s_1) + \ldots + \kappa(s_m) = \kappa(s),$$

by Proposition 2.5, we conclude that $n = m$, and that there is a bijection

$$\pi : \{1, \ldots, n\} \to \{1, \ldots, m\}$$

such that for every $i \in \{1, \ldots, n\}$, we have $\kappa(t_i) = \kappa(s_{\pi(i)})$.

If $n = m > 1$, then, by the induction hypothesis, we have $[t_i] = [s_{\pi(i)}]$ for every $i \in \{1, \ldots, n\}$, and hence $[t] = [s]$, by the associativity and commutativity of $+$ in **S**.

If $n = m = 1$, then t and s are products, and by the associativity of $\cdot$ in **S**, we have $[t] = [t_1 \cdot t_2]$ and $[s] = [s_1 \cdot s_2]$ for t_2 and s_2 either sums or **S**-variables. Since we have

$$\kappa(t) = \kappa(t_1) \cdot \kappa(t_2) = \kappa(s_1) \cdot \kappa(s_2) = \kappa(s),$$

by Proposition 2.8 and the induction hypothesis we then get $[t] = [s]$. □

For the proof of Proposition 3.3 below, which will help us to establish that K, besides being one-to-one, is also onto, we need the notion of an *inner element* of X for a relation $\langle R, X \rangle$; this is an element y of X such that for some x and z in X we have $(x, y) \in R$ and $(y, z) \in R$.

For a relation $\langle R, X \rangle$ and y an element of X, let the relation $\langle R{-}y, X{-}\{y\} \rangle$ be defined by

$$R{-}y = \{(u, v) \in R \mid u \neq y \ \ \& \ \ v \neq y\}.$$

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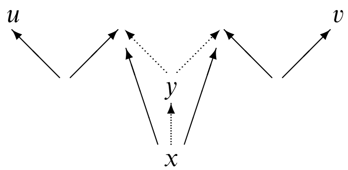
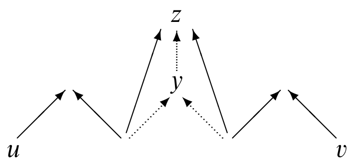
Then we can formulate the following, which is easy to establish.

Remark 3.2 (inner elements). If y is an inner element of X for $\langle R, X\rangle$ in FTP and connected, then $\langle R-y, X-\{y\}\rangle$ is in FTP and connected.

The fact that $\langle R-y, X-\{y\}\rangle$ is connected is clear from the following pictures of chains that ensure connectedness:





For every such chain connecting u and v in $\langle R, X\rangle$ that involves the inner element y, there is a substitute chain connecting u and v in $\langle R-y, X-\{y\}\rangle$, which does not involve y. Hence, we have the following proposition.

Proposition 3.3. If $\langle R, X\rangle$ is in FTP and connected, and there are at least two elements in X, then for some relations $\langle R_1, X_1\rangle$ and $\langle R_2, X_2\rangle$ with X_1 and X_2 non-empty $\langle R, X\rangle = \langle R_1, X_1\rangle \cdot \langle R_2, X_2\rangle$.

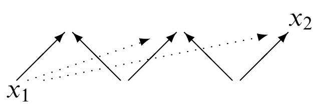
Proof. We proceed by induction on the number k of inner elements of X for $\langle R, X\rangle$. If $k = 0$, let

$$X_1 = \{x \in X \mid (\exists y \in X)\,(x, y) \in R\}$$
$$X_2 = \{x \in X \mid (\exists y \in X)\,(y, x) \in R\}.$$

Then $X = X_1 \cup X_2$, since $\langle R, X\rangle$ is transitive and connected, and there are at least two elements in X. We have $X_1 \cap X_2 = \emptyset$ since there are no inner elements in X. We also have $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, since $\langle R, X\rangle$ is connected and there are at least two elements in X. We can conclude that

$$(\forall x_1 \in X_1)(\forall x_2 \in X_2)\,(x_1, x_2) \in R, \qquad (\in)$$

because trifunctionality here implies difunctionality[†], and by difunctionality, we may, roughly speaking, shorten chains that ensure connectedness. This is clear from the following picture:



We can conclude also that

$$(\forall x_1 \in X_1)(\forall x_2 \in X_2)\,(x_2, x_1) \notin R. \qquad (\notin)$$

[†] As a matter of fact, a relation $\langle R, X\rangle$ in FTP is difunctional if and only if there are no inner elements in X for $\langle R, X\rangle$.

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Otherwise, $\langle R, X\rangle$ would not be irreflexive. We now only need to apply Remark 2.2 (on $\cdot$) to establish the basis of our induction.

Suppose the number k of inner elements of X for $\langle R, X\rangle$ is greater than 0. If x is such an element, then, by Remark 3.2 (on inner elements), we also have that $\langle R{-}x, X{-}\{x\}\rangle$ is in FTP and connected, and $X-\{x\}$ has $k{-}1$ inner elements for $\langle R{-}x, X{-}\{x\}\rangle$. So, by the induction hypothesis, there are relations $\langle R_1', X_1'\rangle$ and $\langle R_2', X_2'\rangle$ with X_1' and X_2' non-empty such that $\langle R{-}x, X{-}\{x\}\rangle = \langle R_1', X_1'\rangle \cdot \langle R_2', X_2'\rangle$. We may assume that $\langle R_2', X_2'\rangle$ is prime with respect to $\cdot$, in the sense that there are no relations $\langle S, Y\rangle$ and $\langle T, Z\rangle$ with Y and Z non-empty such that $\langle R_2', X_2'\rangle = \langle S, Y\rangle \cdot \langle T, Z\rangle$. (If there were such relations, we would pass to $\langle T, Z\rangle$ instead of $\langle R_2', X_2'\rangle$, and rely on the associativity of $\cdot$; and we may iterate that.)

Since x is an inner element of X for $\langle R, X\rangle$, there is a w in X such that $(x, w) \in R$.

(1) If $w \in X_1'$, we take

$$\begin{aligned} X_1 &= X_1' \cup \{x\} \\ X_2 &= X_2', \end{aligned} \tag{x1}$$

and we can conclude that $(\in)$ and $(\notin)$ hold. We now only need to apply Remark 2.2.

(2) If $w \in X_2'$, we have the following subcases:

(2.1) For every y in X_1' we have $(y, x) \in R$. Then we take

$$\begin{aligned} X_1 &= X_1' \\ X_2 &= X_2' \cup \{x\}, \end{aligned} \tag{x2}$$

and we can conclude that $(\in)$ and $(\notin)$ hold. We now only need to apply Remark 2.2.

(2.2) For some element y in X_1', we have $(y, x) \notin R$. Then we take $(x1)$, and to conclude that $(\in)$ and $(\notin)$ hold, it is enough to establish that

$$(\forall x_2 \in X_2')\ (x, x_2) \in R. \tag{$*$}$$

Suppose $(*)$ is not true; that is, for some v in X_2' we have $(x, v) \notin R$. Let

$$\begin{aligned} Y &= \{x_2 \in X_2' \mid (x, x_2) \notin R\} \\ Z &= \{x_2 \in X_2' \mid (x, x_2) \in R\}. \end{aligned}$$

The sets Y and Z are non-empty, since $v \in Y$ and $w \in Z$. Take an arbitrary u from Y and an arbitrary z from Z. We have that $(x, z) \in R$ by the definition of Z, and (y, z) and (y, u) are in R because $y \in X_1'$ and $z, u \in X_2'$. We have $(y, x) \notin R$ by assumption, and $(x, u) \notin R$ by the definition of Y. So, by trifunctionality, we may conclude that $(u, z) \in R$, which implies $(u, z) \in R{-}x$. This implies that for every u in Y and every z in Z, we have $(u, z) \in R_2'$, which, by Remark 2.2, contradicts the assumption that $\langle R_2', X_2'\rangle$ is prime with respect to $\cdot$.

So we have $(*)$, and hence $(\in)$ and $(\notin)$ hold. We now only need to apply Remark 2.2 to complete the proof. □

We can now prove the following proposition.

Proposition 3.4. The map K is onto.

Proof. We want to show that for $\langle R, X\rangle$ in FTP there is a diversified **S**-term t such that $\kappa(t) = \langle R, X\rangle$. We proceed by induction on the number of **S**-variables in X. For the basis, if $X = \{x\}$, then $R = \emptyset$, and t is x. Now we suppose for the induction step that there are at least two **S**-variables in X.

If $\langle R, X\rangle$ is not connected, then, by Remark 2.1 on $+$, for some relations $\langle R_1, X_1\rangle$ and $\langle R_2, X_2\rangle$ with X_1 and X_2 non-empty $\langle R, X\rangle = \langle R_1, X_1\rangle + \langle R_2, X_2\rangle$. So the cardinality of X_1 and X_2 is strictly smaller than the cardinality of X. By Propositions 2.3 and 2.4, we can conclude that $\langle R_1, X_1\rangle$ and $\langle R_2, X_2\rangle$ are in FTP, and then we apply the induction hypothesis.

If $\langle R, X\rangle$ is connected, we apply Proposition 3.3, and reason as in the preceding paragraph. □

So, by the definition of K and Propositions 3.1 and 3.4, we can conclude that K is an isomorphism between a substructure of **S** made of diversified elements and a structure on FTP. This is an isomorphism of two algebras with partial operations $+$ and $\cdot$.

4. Shuffle sums and concatenation products on relationships

In this section and the next we develop the main results of the paper, which are summarised in the Introduction (see Section 1). In this section we consider shuffles of arbitrary binary relations, and with their help we define two partial operations on sets of relations with the same domain. These operations, which we call shuffle sum and concatenation product, are partial because we again require disjointness of domains. The one-to-one map L, which assigns to a partial order all its linear extensions, maps disjoint union and concatenation of partial orders into shuffle sum and concatenation product. With L, and with two other related one-to-one maps, which are more general, we obtain other isomorphic representations of the partial algebras of Section 3.

While a relation on X is an ordered pair $\langle R, X\rangle$ such that $R \subseteq X^2$, that is, $R \in \mathscr{P}(X^2)$, let a *relationship* on X be an ordered pair $[U, X]$ such that $U \subseteq \mathscr{P}(X^2)$, that is, $U \in \mathscr{P}(\mathscr{P}(X^2))$. In a relationship $[U, X]$ the set U is a family of the form $\{R_i \mid i \in I \;\&\; R_i \subseteq X^2\}$. The set X is the *domain* of $[U, X]$.

For the relationships $[U, X]$ and $[V, Y]$ such that $X \cap Y = \emptyset$, we have

$$[U, X] + [V, Y] =_{df} [\{Q \subseteq (X \cup Y)^2 \mid (\exists R \in U)(\exists S \in V) \\ (Q \cap X^2 = R \;\;\&\;\; Q \cap Y^2 = S)\}, X \cup Y]$$

$$[U, X] \cdot [V, Y] =_{df} [\{Q \subseteq (X \cup Y)^2 \mid (\exists R \in U)(\exists S \in V) \\ \langle Q, X \cup Y\rangle = \langle R, X\rangle \cdot \langle S, Y\rangle\}, X \cup Y],$$

where the $\cdot$ in $\langle R, X\rangle \cdot \langle S, Y\rangle$ is the concatenation introduced in Section 2. We call the $\cdot$ in $[U, X] \cdot [V, Y]$, which we have just defined, the *concatenation product*.

When for $\langle R, X\rangle$, $\langle S, Y\rangle$ and $\langle Q, X \cup Y\rangle$ such that $X \cap Y = \emptyset$, we have $Q \cap X^2 = R$ and $Q \cap Y^2 = S$, we say that $\langle Q, X \cup Y\rangle$ is a *shuffle* of $\langle R, X\rangle$ and $\langle S, Y\rangle$, because this is what is usually called a shuffle when $\langle R, X\rangle$, $\langle S, Y\rangle$ and $\langle Q, X \cup Y\rangle$ are finite linear orders. We call the $+$ in $[U, X] + [V, Y]$, defined above, the *shuffle sum*.

The disjoint union $\langle R, X\rangle + \langle S, Y\rangle$ and the concatenation $\langle R, X\rangle \cdot \langle S, Y\rangle$ of $\langle R, X\rangle$ and $\langle S, Y\rangle$ are shuffles of $\langle R, X\rangle$ and $\langle S, Y\rangle$; they are limit cases of shuffles. The disjoint union is a shuffle $\langle Q, X \cup Y\rangle$ such that for every x in X and every y in Y we have $(x, y) \notin Q$ and $(y, x) \notin Q$, while the concatenation is a shuffle $\langle Q, X \cup Y\rangle$ such that for every x in X and every y in Y we have $(x, y) \in Q$ and $(y, x) \notin Q$ (see Remarks 2.1 and 2.2).

Consider the map E from the set of relations on X to the set of relationships on X defined by:

$$E\langle R, X\rangle =_{df} [\{R' \subseteq X^2 \mid R \subseteq R'\}, X].$$

We use $E\langle R, X\rangle$ as an abbreviation for $E(\langle R, X\rangle)$, and will omit the parentheses in the same way in analogous situations below.

It is trivial to show that E is one-to-one because $\bigcap\{R' \subseteq X^2 \mid R \subseteq R'\} = R$. We can also show that the image by E of disjoint union is shuffle sum; namely, we have the following proposition.

Proposition 4.1. $E(\langle R, X\rangle + \langle S, Y\rangle) = E\langle R, X\rangle + E\langle S, Y\rangle$.

Proof. We have to prove that $R \cup S \subseteq Q \subseteq (X \cup Y)^2$ if and only if

$$\exists R' \exists S' (R \subseteq R' \subseteq X^2 \;\;\&\;\; S \subseteq S' \subseteq Y^2 \;\;\&\;\; Q \cap X^2 = R' \;\;\&\;\; Q \cap Y^2 = S').$$

For the left to right implication, it is enough to note that from the left-hand side we can infer that $R \subseteq Q \cap X^2 \subseteq X^2$ and $S \subseteq Q \cap Y^2 \subseteq Y^2$. The right to left implication is trivial. □

On the other hand, we cannot show that $E(\langle R, X\rangle \cdot \langle S, Y\rangle)$ is the concatenation product $E\langle R, X\rangle \cdot E\langle S, Y\rangle$. This is because

$$R \cup S \cup (X \times Y) \subseteq Q \subseteq (X \cup Y)^2 \tag{Q1}$$

need not imply

$$\exists R' \exists S' (R \subseteq R' \subseteq X^2 \;\;\&\;\; S \subseteq S' \subseteq Y^2 \;\;\&\;\; Q = R' \cup S' \cup (X \times Y)), \tag{Q2}$$

though it is implied by it. There are sets Q that satisfy $(Q1)$ and have in them a pair (y, x) for some $x \in X$ and some $y \in Y$.

Consider the map P from the set of partial orders on X to the set of relationships on X defined by replacing $R' \subseteq X^2$ in the definition of $E\langle R, X\rangle$ by $R' \subseteq X^2$ *and with* R' *a partial order*. It is again trivial to show that P is one-to-one (for the same reason why E is one-to-one).

Let the definitions of shuffle sum $+$ and concatenation product $\cdot$ on relationships be modified by replacing $Q \subseteq (X \cup Y)^2$ by $Q \subseteq (X \cup Y)^2$ *with* Q *a partial order*. A shuffle of two partial orders need not be a partial order, but the concatenation of two partial orders is a partial order (see Proposition 2.3); so the modified definition of concatenation product amounts to the old definition for relationships $[U, X]$ such that U is a set of partial orders on X. We can now prove the following proposition.

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Proposition 4.2. $P(\langle R,X\rangle + \langle S,Y\rangle) = P\langle R,X\rangle + P\langle S,Y\rangle$.

To prove this, we proceed as for Proposition 4.1. Now, however, we also have the following proposition.

Proposition 4.3. $P(\langle R,X\rangle \cdot \langle S,Y\rangle) = P\langle R,X\rangle \cdot P\langle S,Y\rangle$.

Proof. It is enough to prove that for partial orders Q the condition $(Q1)$ implies $(Q2)$ (the converse is trivial). Suppose $(Q1)$, and let $R' = Q \cap X^2$ and $S' = Q \cap Y^2$. To show $(Q2)$ it is enough to show

$$Q = (Q \cap X^2) \cup (Q \cap Y^2) \cup (X \times Y).$$

The fact that the right-hand side of this equation is indeed a subset of Q follows easily from $(Q1)$. For the converse inclusion, it is enough to verify that for every x in X and every y in Y we cannot have (y,x) in Q. This follows from $X \times Y \subseteq Q$ together with the transitivity and irreflexivity of Q. □

A relation $\langle R,X\rangle$ is a *linear order* when it is a partial order (as in Section 2) and for every distinct x and y in X either $(x,y) \in R$ or $(y,x) \in R$. Consider now the map L from the set of partial orders on X to the set of relationships on X defined by replacing $R' \subseteq X^2$ in the definition of $E\langle R,X\rangle$ by $R' \subseteq X^2$ *and with* R' *a linear order*. To prove that L is one-to-one is now not so trivial, and we will need some preparation to do it.

Proposition 4.4. For a partial order $\langle R,X\rangle$ such that for some distinct x and y in X we have $(y,x) \notin R$, the transitive closure $\langle Tr(R \cup \{(x,y)\}), X\rangle$ is a partial order.

Proof. We show that this transitive closure is irreflexive. If for some z in X we had $(z,z) \in Tr(R \cup \{(x,y)\})$, then there would be a chain $u_1,\dots,u_n$ such that $u_1 = u_n = z$, and either $(u_i,u_{i+1}) \in R$ or $(u_i,u_{i+1}) = (x,y)$. For some i we must have $(u_i,u_{i+1}) = (x,y)$; otherwise R would not be irreflexive. Let u_k be the leftmost x in the chain, and let u_l be the rightmost y in the chain. Then we must have $(u_l,u_k) \in R$, which contradicts $(y,x) \notin R$. □

The fact that every finite partial order on X can be extended to a liner order on X can be shown by elementary means. (This is related to what is called *topological sorting* in algorithmic graph theory.) Using less elementary means, the same thing can also be shown for any, not necessarily finite, partial order (see Jech (1973, page 19). So, by combining this with Proposition 4.4, we obtain the following.

Proposition 4.5. For a partial order $\langle R,X\rangle$ such that for some distinct x and y in X, we have $(y,x) \notin R$, there is a linear order $\langle R',X\rangle$ such that $R \subseteq R'$ and $(x,y) \in R'$.

We can now prove that L is one-to-one, which amounts to the following proposition.

Proposition 4.6. For the partial orders $\langle R,X\rangle$ and $\langle S,X\rangle$, we have that $L\langle R,X\rangle = L\langle S,X\rangle$ implies $R = S$.

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Proof. Suppose $L\langle R,X\rangle = L\langle S,X\rangle$ and $(u,v) \in R$. We infer that for every linear order $S' \subseteq X^2$ such that $S \subseteq S'$, we have $(u,v) \in S'$. If $(u,v) \notin S$, we obtain a contradiction with the help of Proposition 4.5. □

We now modify the definitions of shuffle sum $+$ and concatenation product $\cdot$ on relationships by replacing $Q \subseteq (X \cup Y)^2$ by $Q \subseteq (X \cup Y)^2$ *with* Q *a linear order*. A shuffle of two linear orders need not be a linear order, but the concatenation of two linear orders is a linear order, so the definition of concatenation product modified in this way amounts to the old definition for relationships $[U,X]$ such that U is a set of linear orders on X.

We can now prove the following propositions by proceeding as for Propositions 4.1 and 4.3.

Proposition 4.7. $L(\langle R,X\rangle + \langle S,Y\rangle) = L\langle R,X\rangle + L\langle S,Y\rangle$.

Proposition 4.8. $L(\langle R,X\rangle \cdot \langle S,Y\rangle) = L\langle R,X\rangle \cdot L\langle S,Y\rangle$.

By combining Proposition 3.1 with the fact that the maps E, P and L are one-to-one, we obtain new isomorphic representations of the structure made up of the diversified elements of **S** (see Section 3).

5. S-forests of graphs

In this section we deal with issues concerning the construction of graphs, which we summarised in the Introduction (see Section 1). This is the main and concluding section of our paper. We first define tree-like elements of the structure **S** of Section 3, and show that the structures corresponding to these elements by the isomorphism K are indeed tree-like relations in FTP.

Consider the set C of elements of **S** (see Section 3) defined inductively as follows:

— for every **S**-variable x, we have $[x] \in C$;
— if $[t], [s] \in C$, then $[t+s] \in C$;
— if $[t] \in C$ and x is an **S**-variable, then $[x \cdot t] \in C$.

An alternative definition of C is obtained by replacing the third clause with:

— if $[t] \in C$ and $+$ does not occur in the **S**-term s, then $[s \cdot t] \in C$.

Let an **S**-*forest* be a diversified element of C. An **S**-forest is, for example, $[((x \cdot y) \cdot z)+u]$. An **S**-*tree* is an **S**-forest that is not of the form $[t+s]$; for example, $[w \cdot (((x \cdot y) \cdot z)+(u+v))]$. Since $+$ and $\cdot$ are associative, there are some superfluous parentheses in these examples, which we will omit later. Note that $[x] = \{x\}$, and that every member of $[t_1 + t_2]$ is of the form $s_1 + s_2$, while every member of $[t_1 \cdot t_2]$ is of the form $s_1 \cdot s_2$.

We will say a partial order $\langle R,X\rangle$, for X a finite set of **S**-variables is an FTP-*forest* when for every $x,y,z \in X$, we have

$$((x,z) \in R \;\;\&\;\; (y,z) \in R) \Rightarrow (x = y \;\text{ or }\; (x,y) \in R \;\text{ or }\; (y,x) \in R).$$

It is easy to see that FTP-forests are trifunctional, and hence they are in FTP (see Section 3). We say that an FTP-forest $\langle R,X\rangle$ is an FTP-*tree* when there is an $x \in X$, called the *root*, such that for every $y \in X$ different from x we have $(x,y) \in R$. The root is

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unique. (Usually, our FTP-forests are called trees in set theory, and a tree, which need not be finite, is defined as a partial order such that for every element the set of its predecessors is well-ordered.)

The following four propositions are about the map K of Section 3.

Proposition 5.1. For $[t]$ an **S**-forest, $K[t]$ is an FTP-forest.

Proof. We proceed by induction on the length of t. If t is an **S**-variable, the result is trivial, because $K[t]$ is the empty relation, and if t is $t_1 + t_2$, it is trivial by the induction hypothesis.

Suppose t is $u \cdot t_1$, $(x, z) \in K[u \cdot t_1]$ and $(y, z) \in K[u \cdot t_1]$. Then if x and y are u, then $x = y$. If x is u and y is in t_1, then $(x, y) \in K[u \cdot t_1]$. If y is u and x is in t_1, then $(y, x) \in K[u \cdot t_1]$. If both x and y are in t_1, we apply the induction hypothesis. □

Proposition 5.2. For $[t]$ an **S**-tree, $K[t]$ is an FTP-tree.

Proof. If t is the **S**-variable x, then $K[t]$ is $\langle \emptyset, \{x\} \rangle$, which is an FTP-tree. If t is $[x \cdot t']$, then x is the root of $K[t]$. □

Proposition 5.3. For every FTP-forest $\langle R, X \rangle$ there is an **S**-forest $[t]$ such that $K[t] = \langle R, X \rangle$.

Proof. By Proposition 3.4, there is a diversified **S**-term t such that $K[t] = \langle R, X \rangle$. If t has a subterm of the form $(s + r) \cdot w$, then for an **S**-variable x in s, an **S**-variable y in r and an **S**-variable z in w, we have $(x, z) \in R$, $(y, z) \in R$, but $x \neq y$ and $(x, y) \notin R$ and $(y, x) \notin R$. So $\langle R, X \rangle$ is not an FTP-forest. □

Proposition 5.4. For every FTP-tree $\langle R, X \rangle$ there is an **S**-tree $[t]$ such that $K[t] = \langle R, X \rangle$.

Proof. We just note that the t in the preceding proof cannot be of the form $t_1 + t_2$ since otherwise $\langle R, X \rangle$ would not be an FTP-tree. □

So K establishes an isomorphism between **S**-forests and FTP-forests on the one hand, and **S**-trees and FTP-trees on the other.

We now move on to graphs and their construction. After the following definitions, we will give a series of examples.

A *graph* is a symmetric and irreflexive relation $\langle G, X \rangle$ whose domain X is finite and non-empty (these are, of course, undirected graphs; see Harary (1969, Chapter 2)). We will now inductively define a map T from the set of graphs $\langle G, X \rangle$, such that X is a set of **S**-variables, to the power set of the set of **S**-forests; that is, $T\langle G, X \rangle$, which abbreviates $T(\langle G, X \rangle)$, is a set of **S**-forests:

— If $X = \{x\}$, then $T\langle G, X \rangle = \{[x]\}$.

(for the following two clauses, we suppose that there are at least two **S**-variables in X)

— If $\langle G, X \rangle$ is connected, then

$$T\langle G, X \rangle = \{[x \cdot t] \mid x \in X \;\;\&\;\; [t] \in T\langle G{-}x, X{-}\{x\}\rangle\}.$$

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— If $\langle G, X\rangle$ is not connected, and is hence of the form $\langle G_1, X_1\rangle + \langle G_2, X_2\rangle$ for $\langle G_1, X_1\rangle$ and $\langle G_2, X_2\rangle$ graphs (that is, for X_1 and X_2 non-empty), then

$$T\langle G, X\rangle = \{[t_1 + t_2] \mid [t_1] \in T\langle G_1, X_1\rangle \;\;\&\;\; [t_2] \in T\langle G_2, X_2\rangle\}.$$

It is not difficult to prove that for every graph $\langle G, X\rangle$, and x and y distinct elements of X, we have $(x, y) \in G$ if and only if for every $[t]$ in $T\langle G, X\rangle$ the **S**-term t has no subterm $t_1 + t_2$ with x in one of t_1 and t_2, and y in the other. (For the left to right direction, we proceed by a straightforward induction on the cardinality $|X|$ of X, where, in the induction step, when $|X| > 1$, we have that $[t]$ is either of the form $[z \cdot s]$ or $[s_1 + s_2]$, with the induction hypothesis applying to s, and s_1 or s_2. For the right to left direction, we suppose that $(x, y) \notin G$; then we build a $[t]$ such that t has a subterm $t_1 + t_2$ with x in t_1 and y in t_2 by leaving x and y for the end. Officially, this is again an induction on $|X|$, with the basis being when $X = \{x, y\}$.) From this we can infer immediately that the map T is one-to-one.

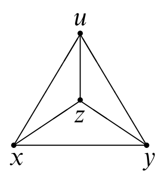
Note that if $\langle G, X\rangle$ is connected, the **S**-forests in $T\langle G, X\rangle$ are **S**-trees. These **S**-trees are in one-to-one correspondence with what Devadoss (2009, Section 2) calls maximal (n–1)-tubings of $\langle G, X\rangle$, where n is the cardinality of X, provided one adapts this notion by relying on the modified notion of tubing of Devadoss and Forcey (2008, Section 2)[†]. The connection between **S**-trees and tubings is explained in detail in Došen and Petrić (2011) – see, in particular, Section 3 and Appendix A.

However, the tubings of graphs $\langle G, X\rangle$ that are not connected do not correspond exactly to the **S**-forests in $T\langle G, X\rangle$[‡].

5.1. *Connected graph examples*

We will now give some examples of $T\langle G, X\rangle$ for a number of connected graphs $\langle G, X\rangle$.

Example 5.1.1. If $\langle G, X\rangle$ is the connected graph



we find twenty four **S**-trees in $T\langle G, X\rangle$, which are obtained from the twenty four permutations of the four **S**-variables x, y, z and u by inserting $\cdot$.

[†] The original notion of tubing was introduced in Carr and Devadoss (2006, Section 2) – *cf.*, Došen and Petrić (2011, Appendix A).

[‡] In the approach with tubings, graphs that are not connected are identified with connected hypergraphs, which always have as a member the set of all the vertices; see Došen and Petrić (2011) for details, which should be compared with Forcey and Springfield (2010, Remark 3.6 and Lemma 3.7).

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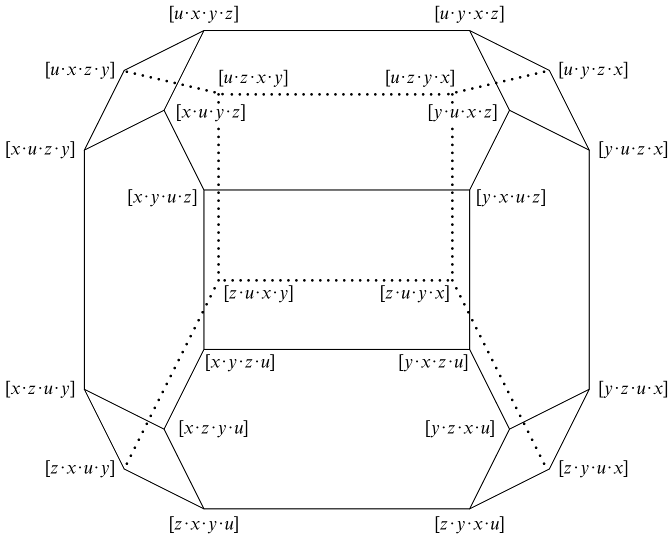
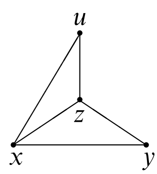
These **S**-trees naturally label the vertices of the three-dimensional permutohedron:



In this permutohedron, and in the other examples later, there is an edge between the vertices labelled by $[t]$ and $[s]$ when there is a linear order in $L(K[t])$ and another one in $L(K[s])$ that only differ from each other by a transposition of immediate neighbours. (We will discuss this further after the examples.)

Example 5.1.2. If $\langle G, X\rangle$ is the connected graph



obtained from the graph in the preceding example by omitting the edge $\{y, u\}$, we find twenty two **S**-trees in $T\langle G, X\rangle$ that label the vertices of the following polyhedron, which is obtained from the three-dimensional permutohedron by collapsing the two vertices $[x \cdot z \cdot y \cdot u]$ and $[x \cdot z \cdot u \cdot y]$ into the single vertex labelled $[x \cdot z \cdot (y + u)]$, and the two vertices $[z \cdot x \cdot y \cdot u]$ and $[z \cdot x \cdot u \cdot y]$ into the single vertex labelled $[z \cdot x \cdot (y + u)]$:

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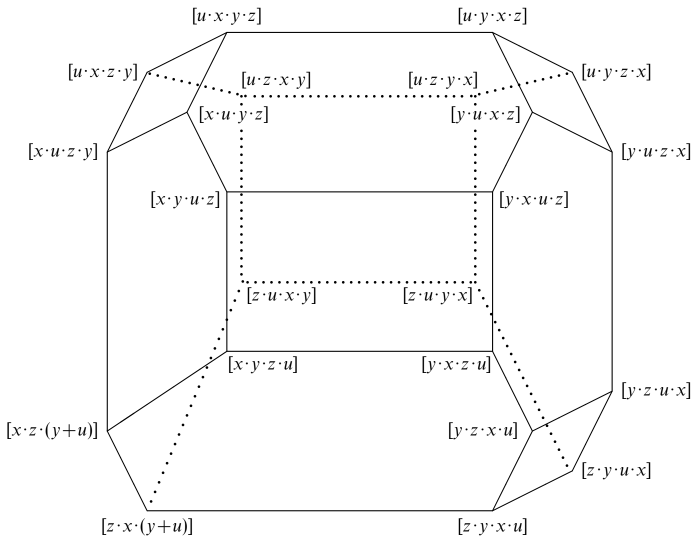
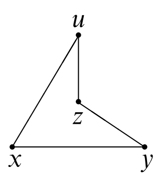


We will call this polyhedron the *hemicyclohedron* – the reason for this name will be explained in the next example. (We will not prove in this paper that the hemicyclohedron, conceived as an abstract polytope, can be realised, or that other such polyhedra discussed later can be – we deal with these issues in Došen and Petrić (2011).)

The hemicyclohedron may be found in Forcey and Springfield (2010, Figure 10). We will not investigate here how hemicyclohedra of dimension higher than 3 could be defined, or whether such a definition should be expected at all. The generalisation, unlike the case for permutohedra, cyclohedra, associahedra and astrohedra (see the examples below), is not obvious. It will also not be obvious for the hemiassociahedron (see Example 5.1.4).

Example 5.1.3. If $\langle G, X \rangle$ is the connected graph



obtained from the graph in the preceding example by omitting the edge $\{x, z\}$, we find the twenty **S**-trees in $T\langle G, X \rangle$ that label the vertices of the three-dimensional cyclohedron – see Stasheff (1997, Section 4) and Carr and Devadoss (2006, Corollary 2.7):

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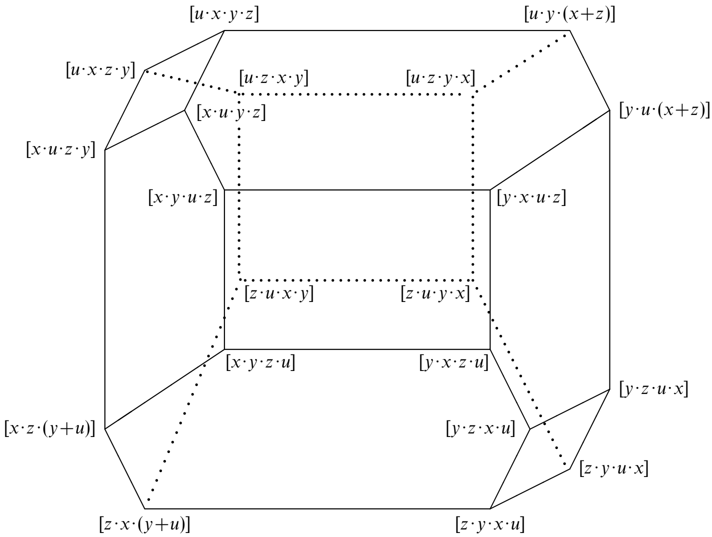
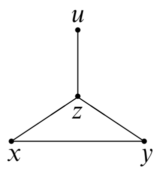
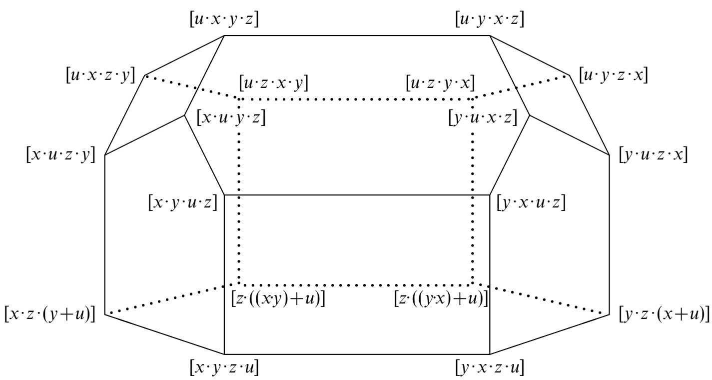


Something analogous to what happened in the lower left-hand corner of our picture of the three-dimensional permutohedron when obtaining the hemicyclohedron has now also happened in the upper right-hand corner – this explains why we use the name hemicyclohedron.

Example 5.1.4. If $\langle G, X\rangle$ is the connected graph



obtained from the graph in Example 5.1.2 by omitting the edge $\{x, u\}$, we find the eighteen **S**-trees in $T\langle G, X\rangle$ that label the vertices of the following polyhedron:



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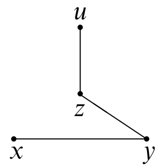
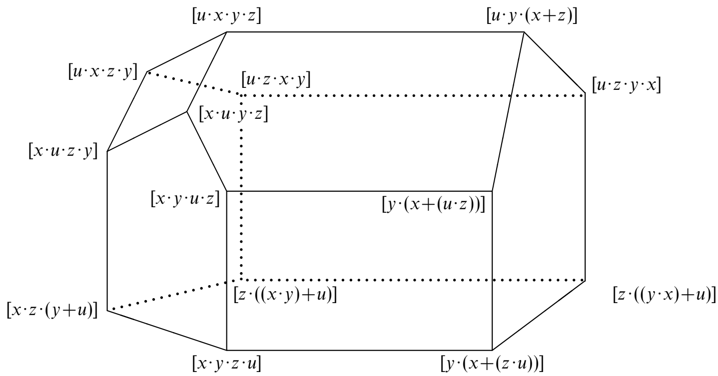
We will call this polyhedron (which is called X_4^a in Armstrong *et al.* (2009, Figure 17) and $P_{1,2}$ in Bloom (2011, Figure 6)) the *hemiassociahedron* – this name will be explained in the next example.

Example 5.1.5. If $\langle G, X\rangle$ is the connected graph

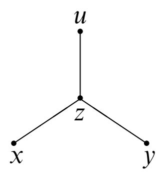


obtained from the graph in the preceding example by omitting the edge $\{x, z\}$, then we find the fourteen **S**-trees in $T\langle G, X\rangle$ that label the vertices of the three-dimensional associahedron:

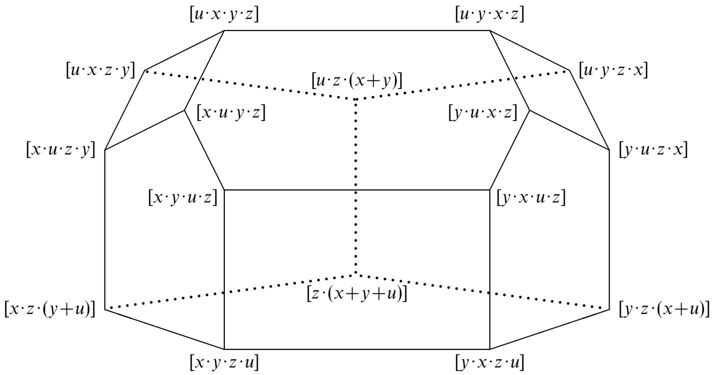


Tonks (1997) explained how this associahedron is obtained from the three-dimensional permutohedron by two perpendicular cuts. The previous polyhedron, the hemiassociahedron, is obtained by one such cut. This should also be clear from our picture of the associahedron, where one cut, which it shares with our hemiassociahedron, is at the basis, while the other is on the right-hand side. This explains why we use the name hemiassociahedron.

Example 5.1.6. If $\langle G, X\rangle$ is the connected graph

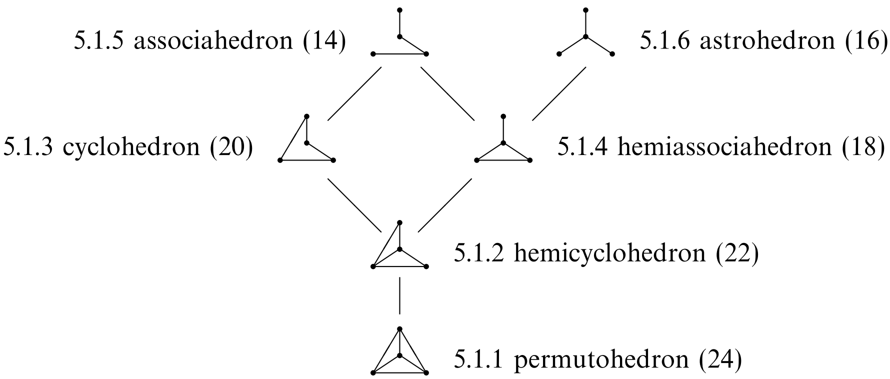


obtained from the graph in Example 5.1.4 by omitting the edge $\{x, y\}$, we find the sixteen **S**-trees in $T\langle G, X\rangle$ that label the vertices of the following polyhedron:



As it arises from a three-pointed star, we could call this polyhedron the three-dimensional *astrohedron*. It is called the stellohedron in Postnikov *et al.* (2008, Section 10.4) (see also Postnikov (2009, Section 8.4)) and $D4$ in Armstrong *et al.* (2009, Figure 17). 'Astrohedron' is just a translation of 'stellohedron' into Greek.

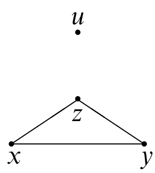
The following picture summarises the previous six examples (with the number of vertices in parentheses):



5.2. *Non-connected graph example*

We will now give an example for a graph that is not connected.

Example 5.2.1. If $\langle G, X\rangle$ is the graph



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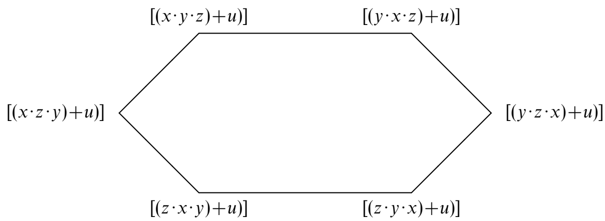
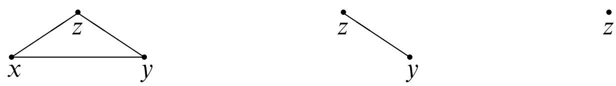
obtained from the graph in Example 5.1.4 by omitting the edge $\{z, u\}$, we find the six **S**-forests in $T\langle G, X\rangle$ that label the vertices of the following hexagon:



The **S**-forests in $T\langle G, X\rangle$ may be thought of as records of the history of the destruction of $\langle G, X\rangle$, which is a history of the construction of $\langle G, X\rangle$ in reverse order. This destruction of graphs is based on *vertex removal* (which can be found in Ulam's Conjecture – see Harary (1969, Chapter 2)). We read the **S**-forest from left to right, and interpret the occurrence of an **S**-variable that we encounter in this reading as the record of the removal of the vertex made of this **S**-variable and of the edges involving this vertex. The removal of vertices joined by $\cdot$ happened consecutively, while for those joined by $+$ it happened simultaneously in time. The commutativity of $+$ means that what is recorded on the two sides of $+$ happened simultaneously.

For example, the **S**-forest $[(x \cdot y \cdot z) + u]$ from Example 5.2.1 may be taken as a record of a destruction where, simultaneously, we remove on one side the vertices x, y and z, and on the other side the vertex u; the removal of x, y and z is done consecutively, so as to produce the 'film':



5.3. *Generalisation*

Our examples of collapsing depend on specific graphs $\langle G, X\rangle$, but we will show in this section that this is a general phenomenon, and not just found in our examples. The maps T and L for a given graph $\langle G, X\rangle$ with n vertices induce an equivalence relation on the set of vertices of the n–1-dimensional permutohedron, whose equivalence classes are described by $L(K[t])$ for $[t]$ in $T\langle G, X\rangle$. Moreover, the permutations corresponding to the members $L(K[t])$ are those assigned to a connected family of vertices of the permutohedron. For example, the four permutations that correspond to $[x \cdot y \cdot z \cdot u]$, $[x \cdot y \cdot u \cdot z]$, $[x \cdot u \cdot y \cdot z]$ and $[u \cdot x \cdot y \cdot z]$ are given by the linear orders in $L(K[(x \cdot y \cdot z) + u])$. The four vertices of the permutohedron labelled by these permutations make a connected family (see Example 5.1.1). We will first define precisely the required notions, and then prove three propositions, which establish all these statements.

For a linear order $\langle L, X\rangle$ of a finite set X, we say L is a *permutation* of X. Let Λ be a set of permutations of X. For L_1 and L_n, where $n \geqslant 2$, distinct members of Λ, we write $L_1 \sim_\Lambda L_n$ when there is a sequence $L_1 \ldots L_n$ such that $L_1, \ldots, L_n \in \Lambda$ and for every

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$i \in \{1, \dots, n-1\}$, for two distinct x and y in X, we have

$$L_{i+1} = (L_i - \{(x, y)\}) \cup \{(y, x)\}.$$

In other words, L_{i+1} only differs from L_i by a transposition of immediate neighbours. We say that Λ is *connected* when for every two distinct L and L' in Λ we have $L \sim_\Lambda L'$. Here are the three propositions we mentioned earlier.

Proposition 5.5. For every partial order $\langle R, X \rangle$ with X finite and $L\langle R, X \rangle = [\Lambda, X]$, the set of permutations Λ is connected.

Proof. If X is $\emptyset$ or a singleton, then $R = \emptyset$ and $\Lambda = \{\emptyset\}$, and this is connected by our definition. If the cardinality $|X|$ of X is at least 2, we proceed by induction on $|X|$.

For the basis, if $|X| = 2$, then the only interesting case is when $X = \{x, y\}$ and $R = \emptyset$, and in this case $\Lambda = \{\{(x, y)\}, \{(y, x)\}\}$, which is clearly connected.

If $|X| > 2$, let x be an element of X such that for every y in X we have $(y, x) \notin R$. Since X is finite, there must be such an x. Let L and L' be two different elements of Λ. We want to show that $L \sim_\Lambda L'$. Let

$$\begin{aligned} S^x &= \{(y, x) \mid (y, x) \in S\} \\ S_x &= \{(x, y) \mid (y, x) \in S\} \\ M &= (L - L^x) \cup L_x \\ M' &= (L' - L'^x) \cup L'_x. \end{aligned}$$

It is clear that the finite sequences that correspond to the permutations M and M' begin with x. We can then conclude that $L \sim_\Lambda M$ or $L = M$, and $L' \sim_\Lambda M'$ or $L' = M'$. By the induction hypothesis, we have $M - x \sim_\Lambda M' - x$ or $M - x = M' - x$. From all this, we get $L \sim_\Lambda L'$. □

Proposition 5.6. For every graph $\langle G, X \rangle$ with X a finite non-empty set of **S**-variables, and every permutation L of X, there is an **S**-forest $[t]$ in $T\langle G, X \rangle$ such that $L \in L(K[t])$.

Proof. We proceed by induction on the cardinality of X. If X is a singleton, then we just follow the definitions. Suppose for the induction step that X has at least two **S**-variables.

If $\langle G, X \rangle$ is connected, we let the sequence corresponding to the permutation L be $xy_1 \dots y_n$ for $n \geqslant 1$. By the induction hypothesis, there is an **S**-forest $s \in T\langle G - x, X - \{x\} \rangle$ such that the permutation L' corresponding to $y_1 \dots y_n$ belongs to $L(K[s])$. Then we have that $L \in L(K[x \cdot s])$.

Suppose $\langle G, X \rangle$ is not connected, and is of the form $\langle G_1, X_1 \rangle + \langle G_2, X_2 \rangle$ for $\langle G_1, X_1 \rangle$ and $\langle G_2, X_2 \rangle$ graphs (that is, for X_1 and X_2 non-empty). By the induction hypothesis, there are **S**-forests $s_1 \in T\langle G_1, X_1 \rangle$ and $s_2 \in T\langle G_2, X_2 \rangle$ such that for a permutation L_1 of X_1 and a permutation L_2 of X_2 we have $L_1 \in L(K[s_1])$ and $L_2 \in L(K[s_2])$, and $\langle L, X \rangle$ is a shuffle of $\langle L_1, X_1 \rangle$ and $\langle L_2, X_2 \rangle$. So we have $L \in L(K[s_1 + s_2])$. □

Proposition 5.7. For every graph $\langle G, X \rangle$ with X a finite non-empty set of **S**-variables, and every $[t]$ and $[t']$ in $T\langle G, X \rangle$, if $L(K[t])$ and $L(K[t'])$ are not disjoint, then $[t] = [t']$.

Proof. We proceed by induction on the cardinality of X. If X is a singleton $\{x\}$, then $T\langle\emptyset,\{x\}\rangle = \{[x]\}$, and $L(K[x]) = \{\langle\emptyset,\{x\}\rangle\}$, so the proposition holds trivially. Suppose for the induction step that X has at least two **S**-variables.

If $\langle G, X\rangle$ is connected, then every element of $T\langle G,X\rangle$ is of the form $[x\cdot s]$ for some x. Suppose that for some

$$[x\cdot s], [x'\cdot s'] \in T\langle G,X\rangle$$

we have

$$L(K[x\cdot s]) \cap L(K[x'\cdot s']) \neq \emptyset.$$

It follows that x is x', and since

$$[s], [s'] \in T\langle G{-}x, X{-}\{x\}\rangle$$

and

$$L(K[s]) \cap L(K[s']) \neq \emptyset,$$

by the induction hypothesis, we get $[s] = [s']$. Hence $[x\cdot s] = [x'\cdot s']$.

Suppose $\langle G,X\rangle$ is not connected and is of the form $\langle G_1,X_1\rangle + \langle G_2,X_2\rangle$ for $\langle G_1,X_1\rangle$ and $\langle G_2,X_2\rangle$ graphs. Suppose also that $[t],[t'] \in T\langle G,X\rangle$. Then, using the associativity and commutativity of $+$, we may infer that $[t] = [t_1 + t_2]$ and $[t'] = [t'_1 + t'_2]$ for $[t_i],[t'_i] \in T\langle G_i,X_i\rangle$ and $i \in \{1,2\}$. Suppose for some $\langle L,X\rangle$ that

$$\langle L,X\rangle \in L(K[t]) \cap L(K[t']).$$

We can then infer that

$$\langle L\cap X_i^2, X_i\rangle \in L(K[t_i]) \cap L(K[t'_i]),$$

and hence, by the induction hypothesis, we get $[t_i] = [t'_i]$. Hence $[t] = [t']$. □

For every graph $\langle G,X\rangle$ with X a finite non-empty set of **S**-variables, from Propositions 5.6 and 5.7, we can infer that

$$\{L(K[t]) \mid [t] \in T\langle G,X\rangle\}$$

is a partition of

$$\{\langle L,X\rangle \mid L \text{ is a permutation of } X\}.$$

Moreover, we know by Proposition 5.5 that every member $L(K[t])$ of this partition is a connected set of permutations. Hence, the examples we have given in this section exhibit a general phenomenon.

6. Conclusion

We conclude this paper with some remarks on related work, and on developments of our approach we have made in further papers.

Our treatment of shuffles and concatenations may be connected to the algebras studied in Joni and Rota (1979) and elsewhere. However, the connection is not clear. Similar connections have been made in Forcey and Springfield (2010). The polyhedra of Section 5 have been previously investigated in a manner comparable to ours in Stasheff (1997,

Appendix B), Tonks (1997), Carr and Devadoss (2006), Devadoss (2009), Devadoss and Forcey (2008), Armstrong *et al.* (2009), Forcey and Springfield (2010) and Bloom (2011). Further references on antecedents of this approach may be found in Postnikov (2009), Postnikov *et al.* (2008) and Došen and Petrić (2011). The results given at the end of Section 5 in the current paper should be compared with Postnikov *et al.* (2008) (end of Section 8). We consider these polyhedra and other related polytopes in a separate study, Došen and Petrić (2011), as indicated below.

Some of the polyhedra of the preceding section stand for commuting diagrams that arise in various coherence questions in category theory, in the same way as a pentagon stands for Mac Lane's commuting diagram required for the coherence of monoidal categories – see Mac Lane (1998, Section VII.1). Coherence means that all diagrams of canonical arrows commute. We showed in Došen and Petrić (2006) how Mac Lane's pentagon arises through a collapsing of the same kind as we have described in the current paper from a hexagon involved in symmetric monoidal coherence, and this matter is related to the collapsing investigated in Tonks (1997) and Forcey and Springfield (2010).

Some similar coherence questions based on the conceptual apparatus introduced in Došen and Petrić (2006), which we deal with in Došen and Petrić (2010), involve some of the less familiar polyhedra that occur as examples in the current paper. The *hemiassociahedron* of Example 5.1.4 arises in the definition of a coherent notion of weak Cat-operad.

An operad (by which we mean non-symmetric operad) may be thought of as a partial algebra with a family of insertion operations (*viz.* Gerstenhaber's circle-i products), which satisfy two kinds of associativity, one of them involving commutativity. A Cat-operad is an operad enriched over the category Cat of small categories, in the same way as a 2-category with small hom-categories is a category enriched over Cat. The notion of a weak Cat-operad is to the notion of a Cat-operad what the notion of a bicategory is to the notion of a 2-category. The equations of operads, like the associativity of insertions, are replaced by isomorphisms in a category.

The goal of Došen and Petrić (2010) is to formulate conditions concerning these isomorphisms that ensure coherence, in the sense that all diagrams of canonical arrows commute. This is the sense in which the notion of a monoidal category, as mentioned above, and also the notion of a bicategory are coherent. Some of the coherence conditions for weak Cat-operads lead to the hemiassociahedron. The commuting diagrams assumed for this notion may be pasted to make the hemiassociahedron, as well as making the three-dimensional associahedron and permutohedron.

The vertices of the polytopes investigated in Došen and Petrić (2011) may be understood as constructions of hypergraphs, in the same way as the vertices of the polyhedra of Section 5 may be understood as constructions of graphs. Limit cases in the family of polytopes of Došen and Petrić (2011) are, at one end, simplices, and, at the other end, permutohedra. In between, notable members are the associahedra and cyclohedra. The polytopes in this family are investigated in Došen and Petrić (2011), both as abstract polytopes and as realised in Euclidean spaces of all finite dimensions. In the latter realisations, passing from simplices to permutohedra, via associahedra, cyclohedra and other interesting polytopes, involves truncating vertices, edges and other faces. With the help of hypergraphs, the results presented in Došen and Petrić (2011) reformulate, systematise and extend previously obtained results from the papers mentioned in the second paragraph of this section, as well as the results of the current paper for polytopes.

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