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Further study of a fourth-order elliptic equation with negative exponent

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We continue to study the nonlinear fourth-order problem $T\Delta u - D\Delta^2 u = \lambda/(L+u)^2$, $-L < u < 0$ in Ω, $u = 0$, $\Delta u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\lambda > 0$ is a parameter. When $N = 2$ and Ω is a convex domain, we know that there is $\lambda_c > 0$ such that for $\lambda \in (0, \lambda_c)$ the problem possesses at least two regular solutions. We will see that the convexity assumption on Ω can be removed, i.e. the main results are still true for a general bounded smooth domain Ω. The main technique in the proofs of this paper is the blow-up argument, and the main difficulty is the analysis of touch-down behaviour.

1. Introduction

We consider the structure of solutions to the problem

$$\left.\begin{aligned} T\Delta u - D\Delta^2 u &= \frac{\lambda}{(L+u)^2} && \text{in } \Omega, \\ -L < u &< 0 && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \qquad (P_\lambda)$$

where $\lambda > 0$ is a parameter, $T > 0$, $D > 0$ and $L > 0$ are fixed constants, and $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a bounded smooth domain.

Problem (P_λ) arises in the study of the deflection of charged plates in electrostatic actuators (see [5]). It is known from [5] that there exists $0 < \lambda_c < \infty$ such that, for $\lambda \in (0, \lambda_c)$, (P_λ) has a maximal regular solution u_λ, which can be obtained from an iterative scheme. (By a regular solution u_λ of (P_λ), we mean that $u_\lambda \in C^4(\Omega) \cap C^3(\bar{\Omega})$ satisfies (P_λ).)

When $N = 2$ and Ω is a convex domain, Guo and Wei [4] obtained two solutions of (P_λ) for $\lambda \in (0, \lambda_c)$: the maximal and a mountain-pass solution. To obtain such results, they showed that all the solutions of (P_λ) are regular for $\lambda \in (0, \lambda_c)$ by using the convexity of Ω and some good properties of Green's function in the two-dimensional case. In this paper, we will see that the convexity assumption on Ω can be removed. The main result of this paper is the following theorem.

THEOREM 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$. For $\lambda \in (0, \lambda_c]$, any solution of the problem (P_λ) is regular and the following hold.*

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(i) *For* $0 < \lambda < \lambda_c$, *problem* (P_λ) *admits two regular solutions: the maximal and a mountain-pass solution.*

(ii) *For* $\lambda = \lambda_c$, *problem* (P_λ) *admits a unique regular solution.*

(iii) *For* $\lambda > \lambda_c$, *problem* (P_λ) *admits no regular solution.*

To remove the convexity of Ω, we need some non-existence results for complete solutions of the equation

$$-D\Delta^2 W = W^{-2} \quad \text{in } \mathbb{R}^2. \tag{1.1}$$

We consider an equivalent form of (P_λ):

$$\left.\begin{aligned} -T\Delta v + D\Delta^2 v &= \frac{\lambda}{(L-v)^2} && \text{in } \Omega, \\ 0 < v &< L && \text{in } \Omega, \\ v = \Delta v &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{T_λ}$$

(T_λ is equivalent to (P_λ) by taking $u = -v$.) Note that $v \in C^4(\Omega) \cap C^3(\bar{\Omega})$ provided that v is a regular solution of (T_λ). Moreover, if u_λ is a maximal solution of (P_λ), then v_λ is a minimal solution of (T_λ). We also know from the strong maximum principle that if v_λ is a regular solution of (T_λ), then

$$\Delta v_\lambda < 0 \quad \text{in } \Omega. \tag{1.2}$$

In this paper, we use C to denote a universal constant.

2. Non-existence of entire solution of (1.1)

In this section we will show that (1.1) does not admit a smooth positive solution. In the following, we present the proof for a general form of (1.1):

$$-D\Delta^2 W = W^{-p} \quad \text{in } \mathbb{R}^2, \tag{2.1}$$

where $p > 0$.

Theorem 2.1. *If* $p > 0$, *(2.1) admits no classical positive solution.*

To prove this theorem, we first show the following lemmas.

Lemma 2.2 (Gilbarg and Trudinger [3]). *Assume that* $w \in C^2(\mathbb{R}^2)$ *satisfies* $\Delta w \geqslant 0$ *and* $w \leqslant C$ *in* $\mathbb{R}^2$. *Then* $w \equiv \text{const.}$ *in* $\mathbb{R}^2$.

It follows from lemma 2.2 that if $w \in C^2(\mathbb{R}^2)$ satisfies $\Delta w \leqslant 0$ and $w \geqslant C$ in $\mathbb{R}^2$, then $w \equiv C$ in $\mathbb{R}^2$. Note that $-w$ satisfies the assumptions in lemma 2.2.

Lemma 2.3. *Assume that* $W \in C^4(\mathbb{R}^2)$ *is a positive solution of (2.1). Then* $\Delta W > 0$ *in* $\mathbb{R}^2$.

Proof. We first claim that

$$\Delta W \geqslant 0 \quad \text{in } \mathbb{R}^2. \tag{2.2}$$

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On the contrary, there is a point $x_0 \in \mathbb{R}^2$ such that $\Delta W(x_0) < 0$. Defining

$$\bar{W}(r) = \frac{1}{2\pi r}\int_{\partial B_r(x_0)} W \,\mathrm{d}\sigma \quad \text{and} \quad \bar{Z}(r) = \frac{1}{2\pi r}\int_{\partial B_r(x_0)} \Delta W \,\mathrm{d}\sigma,$$

since the function s^{-p} is convex for $s \in (0,\infty)$, we see from Jensen's inequality that

$$\overline{W^{-p}} \geqslant \bar{W}^{-p}.$$

Then

$$\Delta\bar{W} - \bar{Z} = 0 \quad \text{and} \quad \Delta\bar{Z} + D^{-1}\bar{W}^{-p} \leqslant 0. \tag{2.3}$$

Since $\Delta\bar{Z} = (1/r)(r\bar{Z}')' \leqslant 0$, we see that $(r\bar{Z}')' \leqslant 0$. Therefore, $\bar{Z}'(r) \leqslant 0$ (note that $\lim_{r\to 0} r\bar{Z}'(r) = 0$). This implies that $\bar{Z}(r) \leqslant \bar{Z}(0) < 0$ for all $r > 0$ (note that $\Delta W(x_0) < 0$). We then obtain that $\Delta\bar{W} \leqslant \bar{Z}(0)$. Therefore,

$$\bar{W}'(r) \leqslant \tfrac{1}{2} r\bar{Z}(0). \tag{2.4}$$

This implies

$$\bar{W}(r) - \bar{W}(0) \leqslant \tfrac{1}{4}\bar{Z}(0)r^2.$$

Since $\Delta W(x_0) < 0$, we see that $\bar{Z}(0) = \Delta W(x_0) < 0$. We then obtain that $\bar{W}(r) < 0$ provided that r is sufficiently large. This is clearly impossible since we have assumed that W is positive everywhere. Our claim (2.2) holds.

Now we claim

$$\Delta W > 0 \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

On the contrary, there is $x_1 \in \mathbb{R}^2$ such that $\Delta W(x_1) = 0$. This implies that x_1 is a minimum point of ΔW. Thus, $\Delta(\Delta W)(x_1) \geqslant 0$. This contradicts $\Delta^2 W(x_1) = -(1/D)W^{-p}(x_1) < 0$ and our claim (2.5) holds. □

Proof of theorem 2.1. Suppose that $W > 0$ is a classical solution of (2.1). It follows from lemma 2.3 that

$$\Delta W > 0 \quad \text{in } \mathbb{R}^2.$$

Since $\Delta(\Delta W) \leqslant 0$ in $\mathbb{R}^2$, we see from lemma 2.2 and the comments after it that

$$\Delta W \equiv \text{const.} \quad \text{in } \mathbb{R}^2.$$

This is clearly impossible since $\Delta(\Delta W) = -(1/D)W^{-p}$. This completes the proof of theorem 2.1. □

Remark 2.4. Arguments similar to those in the proof of theorem 2.1 imply that the equation

$$-D\Delta^2 Z = f(Z) \quad \text{in } \mathbb{R}^2$$

does not admit a classical lower-bound solution Z satisfying $f(Z) \geqslant 0$ and $f(Z) \not\equiv 0$ in $\mathbb{R}^2$ and $f(s)$ is a convex function of $s \in (\inf_\Omega Z, \sup_\Omega Z)$.

In the following we consider the equation

$$D\Delta^2 W = h(W) \quad \text{in } \mathbb{R}^2, \tag{2.6}$$

where

$$h(s) = \begin{cases} 0 & \text{for } 0 < s \leqslant L, \\ \dfrac{s^2}{L} - L & \text{for } s > L. \end{cases}$$

We have the following theorem.

THEOREM 2.5. *Equation (2.6) admits no bounded classical positive solution W satisfying $\sup_{\mathbb{R}^2} W > L$. Equation (2.6) admits only the constant positive solution $W \equiv \sup_{\mathbb{R}^2} W$ provided that $\sup_{\mathbb{R}^2} W \leqslant L$.*

Proof. We use the notation as above. The proof is divided into two steps:

(i) $\sup_{\mathbb{R}^2} W > L$,

(ii) $\sup_{\mathbb{R}^2} W \leqslant L$.

For case (i), we show

$$\Delta W \leqslant 0 \quad \text{in } \mathbb{R}^2,$$

which implies that $\Delta W \equiv$ const. by lemma 2.2. But this contradicts $\Delta^2 W = h(W) \not\equiv 0$.

Assume that there is a point $x_0 \in \mathbb{R}^2$ such that $\Delta W(x_0) > 0$. Then

$$\Delta \bar{W} = \bar{Z}, \qquad D\Delta \bar{Z} \geqslant h(\bar{W}), \qquad \bar{Z}(0) > 0$$

(note that h is a convex function). Thus, we have $(r\bar{Z}')' \geqslant 0$ and hence $\bar{Z}'(r) \geqslant 0$. Therefore,

$$\bar{Z}(r) \geqslant \bar{Z}(0) \quad \text{for } r > 0.$$

This implies

$$(r\bar{W}'(r))' \geqslant r\bar{Z}(0)$$

and

$$\bar{W}(r) - \bar{W}(0) \geqslant \tfrac{1}{4}\bar{Z}(0)r^2.$$

This and the fact $\bar{Z}(0) > 0$ derive a contradiction since W is bounded. Thus $\Delta W \leqslant 0$. This completes the proof of (i).

For case (ii), we see that $\Delta^2 W \equiv 0$ in $\mathbb{R}^2$. Since the proof above implies $\Delta W \leqslant 0$, we see that $\Delta W \equiv C$ in $\mathbb{R}^2$ by lemma 2.2. Noting that $\Delta W \leqslant 0$, we have $C \leqslant 0$ and $\bar{W}(r) = \bar{W}(0) + \frac{1}{4}Cr^2$. We can obtain $C = 0$ since $W > 0$. Thus, $\Delta W \equiv 0$ in $\mathbb{R}^2$ and

$$W \equiv \sup_{\mathbb{R}^2} W \quad \text{in } \mathbb{R}^2.$$

This completes the proof of (ii). □

3. Proof of theorem 1.1

In this section, we present the proof of theorem 1.1. Instead of proving theorem 1.1, we show the following theorem.

THEOREM 3.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$. For $\lambda \in (0, \lambda_c]$, any solution of the problem (T_λ) is regular and the following hold.*

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(i) *For* $0 < \lambda < \lambda_{\mathrm{c}}$*, problem* (T_λ) *admits two regular solutions: the minimal and a mountain-pass solution.*

(ii) *For* $\lambda = \lambda_{\mathrm{c}}$*, problem* (T_λ) *admits a unique regular solution.*

(iii) *For* $\lambda > \lambda_{\mathrm{c}}$*, problem* (T_λ) *admits no regular solution.*

The proof of (ii) and (iii) is known from [4]. We only need to show (i).

Let $\mathcal{H} = H^2(\Omega) \cap H_0^1(\Omega)$ be the function space obtained by taking the completion under the norm of $H^2(\Omega) \cap H_0^1(\Omega)$ (i.e. $\|\psi\| = (\int_\Omega [T|\nabla\psi|^2 + D|\Delta\psi|^2]\,\mathrm{d}x)^{1/2}$) for the set of smooth functions that satisfy the boundary condition $\psi = \Delta\psi = 0$ on $\partial\Omega$. We first obtain the following lemma.

LEMMA 3.2. *For any fixed* $\lambda > 0$*, if* $v \in \mathcal{H}$ *is a positive solution of* (T_λ)*, then there exists* $0 < \tau < L$ *depending on* λ *but independent of* v *and locally independent of* λ *such that* $v \leqslant L - \tau$ *in* Ω*. This also implies that all the positive solutions of* (T_λ) *in* $\mathcal{H}$ *are regular.*

Proof. The embedding theorem implies that $v \in C^\alpha(\bar{\Omega})$ for any $0 < \alpha < 1$. Then there is $x_\lambda \in \Omega$ such that $v(x_\lambda) = \max_\Omega v$.

Suppose that there are $\lambda_0 > 0$ and sequences $\{\lambda_i\}$ and $\{v_i\} \equiv \{v_{\lambda_i}\}$ with $\max_\Omega v_i = L - \varepsilon_i$ such that $\lambda_i \to \lambda_0$, $\varepsilon_i \to 0$ as $i \to \infty$. Then $v_i \in C^4(\Omega) \cap C^3(\bar{\Omega})$. Denoting $x_i = x_{\lambda_i}$ and choosing subsequences if necessary, we consider two cases:

(i) $\lambda_i^{1/4}\varepsilon_i^{-3/4}\,\mathrm{dist}(x_i, \partial\Omega) \to \infty$ as $i \to \infty$;

(ii) $\lambda_i^{1/4}\varepsilon_i^{-3/4}\,\mathrm{dist}(x_i, \partial\Omega) \leqslant M$ for any i.

For the first case, making the transformation $w_i = L - v_i$, we see that w_i with $\min_\Omega w_i = \varepsilon_i$ satisfies the problem

$$T\Delta w_i - D\Delta^2 w_i = \lambda_i w_i^{-2} \quad \text{in } \Omega, \quad w_i = L, \quad \Delta w_i = 0 \quad \text{on } \partial\Omega.$$

Note that $w_i(x_i) = \min_\Omega w_i$. Setting $\tilde{w}_i(y) = w_i/\varepsilon_i$ and $y = \lambda_i^{1/4}\varepsilon_i^{-3/4}(x - x_i)$, we see that $\tilde{w}_i$ with $\tilde{w}_i(0) = \min_{\Omega_i} \tilde{w}_i = 1$ and $\tilde{w}_i$ satisfies the problem

$$\lambda_i^{-1/2}\varepsilon_i^{3/2} T\Delta_y \tilde{w}_i - D\Delta_y^2 \tilde{w}_i = \tilde{w}_i^{-2} \quad \text{in } \Omega_i, \quad \tilde{w}_i = \frac{L}{\varepsilon_i}, \quad \Delta_y \tilde{w}_i = 0 \quad \text{on } \partial\Omega_i, \tag{3.1}$$

where $\Omega_i = \{y = \lambda_i^{1/4}\varepsilon_i^{-3/4}(x - x_i) \colon x \in \Omega\}$. Note that $\tilde{w}_i \geqslant 1$ and $\tilde{w}_i^{-2} \leqslant 1$ in Ω_i. It follows from the regularity of the operator $T\Delta - D\Delta^2$ that $\tilde{w}_i \to W$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ as $i \to \infty$, where $W \in C^4(\mathbb{R}^2)$ with $W(0) = 1$ and $W \geqslant 1$ in $\mathbb{R}^2$ satisfies the equation

$$-D\Delta^2 W = W^{-2} \quad \text{in } \mathbb{R}^2, \qquad W(0) = 1. \tag{3.2}$$

It is known from theorem 2.1 that this is impossible.

For the second case, we denote $\eta_i \in \partial\Omega$ such that $\mathrm{dist}(x_i, \eta_i) = \mathrm{dist}(x_i, \partial\Omega)$. We see that $\eta_i \to \eta_0 \in \partial\Omega$ as $i \to \infty$ (we can choose subsequences if necessary). We also see that

$$-T\Delta v_i + D\Delta^2 v_i = \lambda_i \varepsilon_i^{-3}\left(\frac{L - v_i}{\varepsilon_i^{3/2}}\right)^{-2} \quad \text{in } \Omega, \quad v_i = \Delta v_i = 0 \quad \text{on } \partial\Omega. \tag{3.3}$$

Noticing that $(L-v_i)/\varepsilon_i \geqslant 1$, we see that

$$\frac{L-v_i}{\varepsilon_i^{3/2}} = \varepsilon_i^{-1/2}\frac{L-v_i}{\varepsilon_i} \geqslant 1 \quad \text{and} \quad \frac{L-v_i}{\varepsilon_i^{3/2}} \to +\infty \quad \text{in } \Omega \text{ as } i \to \infty.$$

We also see from (1.2) that $\Delta v_i < 0$ in Ω. Making the transformations:

$$y = \lambda_i^{1/4}\varepsilon_i^{-3/4}(x-\eta_i), \qquad \hat{v}_i(y) = v_i,$$

we see that $\hat{v}_i$ satisfies the problem

$$-T\lambda_i^{-1/2}\varepsilon_i^{3/2}\Delta_y\hat{v}_i + D\Delta_y^2\hat{v}_i = \left(\frac{L-\hat{v}_i}{\varepsilon_i^{3/2}}\right)^{-2} \quad \text{in } \hat{\Omega}_i, \quad \hat{v}_i = \Delta_y\hat{v}_i = 0 \quad \text{on } \partial\hat{\Omega}_i, \tag{3.4}$$

where $\hat{\Omega}_i = \{y = \lambda_i^{1/4}\varepsilon_i^{-3/4}(x-\eta_i) \colon x \in \Omega\}$. Since $((L-\hat{v}_i)/\varepsilon_i^{3/2})^{-2} \leqslant 1$ in $\hat{\Omega}_i$, the regularity of Δ^2 implies that $|\Delta_y\hat{v}_i|$ is uniformly bounded. By the regularity of Δ^2 and a similar blow-up argument to that in [1], we see that $\hat{v}_i \to \hat{V}$ in $C^3_{\text{loc}}(\Gamma)$ as $i \to \infty$ (we can choose subsequences if necessary) and $\hat{V} \in C^4(\Gamma) \cap C^3(\Gamma \cup \partial\Gamma)$ with $\hat{V} \leqslant L$, $|\Delta\hat{V}|$ being bounded and $\Delta\hat{V} \leqslant 0$ in Γ satisfies the problem

$$D\Delta^2\hat{V} \equiv 0 \quad \text{in } \Gamma, \quad \hat{V} = \Delta\hat{V} = 0 \quad \text{on } \partial\Gamma, \tag{3.5}$$

where $\Gamma = \{y = (y_1, y_2) \in \mathbb{R}^2 \colon y_1 > 0\}$ and there is an $\eta \in \Gamma$ with $\operatorname{dist}(0,\eta) \leqslant M$ such that $\hat{V}(\eta) = L$. Using Green's expression of the solution h of the problem $\Delta h = 0$ in Γ and $h = 0$ on $\partial\Gamma$, we easily see that $h \equiv 0$ in Γ. This implies that $\Delta\hat{V} \equiv 0$ in Γ and hence $\hat{V} \equiv 0$ in Γ. But this contradicts the fact that $\hat{V}(\eta) = L$. These contradictions complete the proof of this lemma. □

Proof of theorem 3.1. We modify the nonlinearity as in [4]. Since the nonlinearity $g(v) = 1/(L-v)^2$ is singular at $v = L$, we need to consider a regularized C^1 nonlinearity $g_\varepsilon(v)$, $0 < \varepsilon < L$, of the following form:

$$g_\varepsilon(v) = \begin{cases} \dfrac{1}{(L-v)^2}, & v \leqslant L-\varepsilon, \\ \dfrac{1}{\varepsilon^2} - \dfrac{L-\varepsilon}{\varepsilon^3} + \dfrac{1}{\varepsilon^3(L-\varepsilon)}v^2, & v > L-\varepsilon. \end{cases}$$

For $\lambda \in (0, \lambda_{\mathrm{c}})$, we study the regularized semilinear elliptic problem:

$$-T\Delta v + D\Delta^2 v = \lambda g_\varepsilon(v) \quad \text{in } \Omega, \quad v = \Delta v = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

From a variational viewpoint, the action functional associated to (3.6) is

$$J_{\varepsilon,\lambda}(v) = \frac{1}{2}\int_\Omega [T|\nabla v|^2 + D(\Delta v)^2]\,\mathrm{d}x - \lambda\int_\Omega G_\varepsilon(v)\,\mathrm{d}x, \quad v \in \mathcal{H},$$

where

$$G_\varepsilon(v) = \int_{-\infty}^{v} g_\varepsilon(s)\,\mathrm{d}s.$$

By arguments similar to those in the proof (i) of theorem 7.1 in [4] (see also lemmas 7.3 and 7.4 in [4]), we can obtain a mountain-pass solution $V_{\varepsilon,\lambda} \in \mathcal{H}$ of (3.6)

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such that

$$\|V_{\varepsilon,\lambda}\|_{\mathcal{H}} \leqslant C, \tag{3.7}$$

where $C > 0$ is independent of ε. The embedding $\mathcal{H} \hookrightarrow C^0(\bar{\Omega})$ implies $V_{\varepsilon,\lambda} \leqslant C$ in Ω. Moreover, since $V_{\varepsilon,\lambda} \in \mathcal{H}$ is a solution of (3.6), multiplying $V_{\varepsilon,\lambda}$ on both sides of (3.6), we see from (3.7) that

$$\int_{\Omega} g_{\varepsilon}(V_{\varepsilon,\lambda}) V_{\varepsilon,\lambda} \, \mathrm{d}x \leqslant \frac{C}{\lambda}, \tag{3.8}$$

where $C > 0$ is independent of ε.

We want to show that

$$V_{\varepsilon,\lambda} \leqslant L - \varepsilon \quad \text{in } \Omega. \tag{3.9}$$

This implies that $V_{\varepsilon,\lambda}$ is a solution of (T_{λ}).

We claim that if there is a sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$ as $i \to \infty$ such that $\max_{\Omega} V_{\varepsilon_i,\lambda} > L - \varepsilon_i$, then

$$\max_{\Omega} V_{\varepsilon_i,\lambda} \to L \quad \text{as } i \to \infty. \tag{3.10}$$

To show this claim, we denote $V_{\varepsilon_i,\lambda}(x_i) = \max_{\Omega} V_{\varepsilon_i,\lambda}$. Since $V_{\varepsilon_i,\lambda} \leqslant C$ in Ω, we see from the regularity of Δ^2 that $V_{\varepsilon_i,\lambda} \in C^4(\Omega) \cap C^3(\bar{\Omega})$. The maximum principle implies that

$$V_{\varepsilon_i,\lambda} > 0, \qquad \Delta V_{\varepsilon_i,\lambda} < 0 \quad \text{in } \Omega. \tag{3.11}$$

We first show

$$\lambda^{1/4} \varepsilon_i^{-3/4} \operatorname{dist}(x_i, \partial\Omega) \to \infty \quad \text{as } i \to \infty. \tag{3.12}$$

(We can choose subsequences if necessary.) On the contrary, we have that

$$\lambda^{1/4} \varepsilon_i^{-3/4} \operatorname{dist}(x_i, \partial\Omega) \leqslant M \quad \forall i. \tag{3.13}$$

Writing the equation of $V_{\varepsilon_i,\lambda}$ as

$$-T\Delta V_{\varepsilon_i,\lambda} + D\Delta^2 V_{\varepsilon_i,\lambda} = \lambda \varepsilon_i^{-3}(\varepsilon_i^3 g_{\varepsilon}(V_{\varepsilon_i,\lambda})) \quad \text{in } \Omega, \quad V_{\varepsilon_i,\lambda} = \Delta V_{\varepsilon_i,\lambda} = 0 \quad \text{on } \partial\Omega, \tag{3.14}$$

and setting

$$y = \lambda^{1/4} \varepsilon_i^{-3/4}(x - \eta_i), \qquad \tilde{V}_{i,\lambda}(y) = V_{\varepsilon_i,\lambda}(x),$$

where $\eta_i \in \partial\Omega$ such that $\operatorname{dist}(x_i, \eta_i) = \operatorname{dist}(x_i, \partial\Omega)$, by blow-up arguments similar to those in the second case of the proof of lemma 3.2, we see that $\tilde{V}_{i,\lambda} \to \tilde{V}$ in $C^3_{\mathrm{loc}}(\Gamma)$ as $i \to \infty$. Moreover, $\tilde{V} \in C^4(\Gamma)$ with $L \leqslant \max_{\Gamma} \tilde{V} \leqslant C$, $\Delta\tilde{V} \leqslant 0$, and $|\Delta\tilde{V}|$ being bounded, satisfies

$$D\Delta^2 \tilde{V} = h(\tilde{V}) \quad \text{in } \Gamma, \quad \tilde{V} = \Delta\tilde{V} = 0 \quad \text{on } \partial\Gamma, \tag{3.15}$$

where

$$h(s) = \begin{cases} 0 & \text{for } 0 < s \leqslant L, \\ \dfrac{s^2}{L} - L & \text{for } s > L. \end{cases} \tag{3.16}$$

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Moreover, there is $\eta \in \Gamma$ with $\operatorname{dist}(0,\eta) \leqslant M$ such that $\tilde{V}(\eta) = \max_\Gamma \tilde{V}$. Note that $h \in C^1((0,\infty)\setminus\{L\})$ is a non-decreasing function. Let $\tilde{W} = -\Delta\tilde{V}$. Then (3.15) can be written as

$$\left.\begin{aligned} -\Delta\tilde{V} &= \tilde{W} && \text{in } \Gamma, \\ -\Delta\tilde{W} &= \frac{1}{D}h(\tilde{V}) && \text{in } \Gamma, \\ \tilde{V} = \tilde{W} &= 0 && \text{on } \partial\Gamma. \end{aligned}\right\} \tag{3.17}$$

There are two cases: (i) $\max_\Gamma \tilde{V} = L$; (ii) $\max_\Gamma \tilde{V} > L$.

For the first case, we see that $\tilde{V}$ satisfies

$$\Delta^2\tilde{V} = 0 \quad \text{in } \Gamma, \quad \tilde{V} = \Delta\tilde{V} = 0 \quad \text{on } \partial\Gamma.$$

It is easily seen that $\tilde{V} \equiv 0$ in Γ. This is clearly impossible.

For the second case, theorem 4 of [2] implies that

$$\frac{\partial\tilde{V}}{\partial y_1} > 0 \quad \text{and} \quad \frac{\partial\tilde{W}}{\partial y_1} > 0 \quad \text{for } y_1 > 0. \tag{3.18}$$

This contradicts $\tilde{V}(\eta) = \max_\Gamma \tilde{V}$. (Note that $h \in C^1((0,\infty)\setminus\{L\})$ here. Since h is Lipschitz continuous, arguments in the proof of theorem 4 of [2] still work for our case. The continuous differentiability assumption in [2] can be avoided if the equation system does not admit semitrivial solutions and the nonlinearities are C^1 near zero. This is true for our case here.) Thus, (3.12) holds.

Now, making the transformations

$$y = \lambda^{1/4}\varepsilon_i^{-3/4}(x - x_i), \qquad \hat{V}_{i,\lambda}(y) = V_{\varepsilon_i,\lambda}(x),$$

we see from (3.12) that $\hat{V}_{i,\lambda} \to \hat{V}$ in $C^3_{\text{loc}}(\mathbb{R}^2)$ as $i \to \infty$ (we can choose subsequences if necessary). We also know that $L \leqslant \hat{V}(0) = \max_{\mathbb{R}^2}\hat{V} \leqslant C$, $|\Delta\hat{V}| \leqslant C$, and $\hat{V}$ satisfies the equation

$$D\Delta^2\hat{V} = h(\hat{V}) \quad \text{in } \mathbb{R}^2, \tag{3.19}$$

where the function $h(s)$ is defined in (3.16). Arguments similar to those in case (i) of the proof of theorem 2.5 implies that $\hat{V}(0) = L$. It is also known from theorem 2.5 that $\hat{V} \equiv L$ in $\mathbb{R}^2$. This implies that

$$V_{\varepsilon_i,\lambda}(x_i) = \max_\Omega V_{\varepsilon_i,\lambda} \to L \quad \text{as } i \to \infty \tag{3.20}$$

and our claim (3.10) holds.

Now we show (3.9). On the contrary, there is a sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$ as $i \to \infty$ such that $\max_\Omega V_{\varepsilon_i,\lambda} > L - \varepsilon_i$. Thus, (3.20) holds. There are three cases to consider (we can choose subsequences if necessary):

(i) $V_{\varepsilon_i,\lambda}(x_i) = L + \xi_i$,

(ii) $V_{\varepsilon_i,\lambda}(x_i) = L$ for all i,

(iii) $V_{\varepsilon_i,\lambda}(x_i) = L - \xi_i$ with $\xi_i < \varepsilon_i$,

where $\xi_i > 0$ and $\xi_i \to 0$ as $i \to \infty$.

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For the first case, we show

$$\frac{\varepsilon_i}{\xi_i} \not\to 0 \quad \text{as } i \to \infty. \tag{3.21}$$

(We can choose subsequences if necessary.) On the contrary, there is a sequence $\{\varepsilon_i/\xi_i\}$ such that $\varepsilon_i/\xi_i \to 0$ as $i \to \infty$. Set $Z_i = L - V_i$, where $\{V_i\} \equiv \{V_{\varepsilon_i,\lambda}\}$. Then $Z_i(x_i) := \min_\Omega Z_i = -\xi_i$ and Z_i satisfies

$$T\Delta Z_i - D\Delta^2 Z_i = \lambda k_i(Z_i) \quad \text{in } \Omega, \quad Z_i = L, \Delta Z_i = 0 \quad \text{on } \partial\Omega,$$

where

$$k_i(Z_i) = \begin{cases} \dfrac{1}{Z_i^2}, & Z_i \geqslant \varepsilon_i, \\ \dfrac{1}{\varepsilon_i^2} + \dfrac{2(\varepsilon_i - Z_i)}{\varepsilon_i^3} + \dfrac{(\varepsilon_i - Z_i)^2}{\varepsilon_i^3(L - \varepsilon_i)}, & Z_i < \varepsilon_i. \end{cases}$$

Making the transformations $\tilde{Z}_i(y) = Z_i/\xi_i$ and $y = \varepsilon_i^{-3/4}(x - x_i)$, we can see that $\tilde{Z}_i(0) = \min_\Omega \tilde{Z}_i = -1$ and $\tilde{Z}_i$ satisfies the problem

$$\varepsilon_i^{3/2} T\Delta_y \tilde{Z}_i - D\Delta_y^2 \tilde{Z}_i = \lambda \tilde{k}_i(\tilde{Z}_i) \quad \text{in } \tilde{\Omega}_i, \quad \tilde{Z}_i = L/\xi_i, \quad \Delta_y \tilde{Z}_i = 0 \quad \text{on } \partial\tilde{\Omega}_i, \tag{3.22}$$

where $\tilde{\Omega}_i = \{y = \varepsilon_i^{-3/4}(x - x_i) \colon x \in \Omega\}$ and

$$\tilde{k}_i(\tilde{Z}_i) = \begin{cases} \dfrac{1}{\tilde{Z}_i^2}\left(\dfrac{\varepsilon_i}{\xi_i}\right)^3, & \tilde{Z}_i \geqslant \dfrac{\varepsilon_i}{\xi_i}, \\ 3\left(\dfrac{\varepsilon_i}{\xi_i}\right) - 2\tilde{Z}_i + \dfrac{\varepsilon_i^2}{\xi_i(L - \varepsilon_i)} - 2\left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\tilde{Z}_i + \left(\dfrac{\xi_i}{L - \varepsilon_i}\right)\tilde{Z}_i^2, & \tilde{Z}_i < \dfrac{\varepsilon_i}{\xi_i}. \end{cases}$$

Since $\varepsilon_i/\xi_i \to 0$ as $i \to \infty$, we see that $\{|\tilde{k}_i(\tilde{Z}_i)|\}$ is bounded. Thus, it follows from the regularity of Δ^2 that $\tilde{Z}_i \to \tilde{Z}$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ (note that (3.12) holds) with $\tilde{Z}(0) = \min_{\mathbb{R}^2} \tilde{Z} = -1$, and $\tilde{Z}$ satisfies the equation

$$-D\Delta^2 \tilde{Z} = \lambda \tilde{k}(\tilde{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\tilde{k}(\tilde{Z}) = \begin{cases} 0, & \tilde{Z} \geqslant 0, \\ -2\tilde{Z}, & \tilde{Z} < 0. \end{cases}$$

It is known from remark 2.4 that $\tilde{Z}$ does not exist. This contradiction implies that (3.21) holds.

Now we show that

$$\frac{\varepsilon_i}{\xi_i} \not\to \infty \quad \text{as } i \to \infty. \tag{3.23}$$

On the contrary, making the transformations $\hat{Z}_i(y) = Z_i/\varepsilon_i$ and $y = \varepsilon_i^{-3/4}(x - x_i)$, we see that $\hat{Z}_i(0) = \min_\Omega \hat{Z}_i = -\xi_i/\varepsilon_i$ ($\to 0$ as $i \to \infty$) and $\hat{Z}_i$ satisfies the problem

$$\varepsilon_i^{3/2} T\Delta_y \hat{Z}_i - D\Delta_y^2 \hat{Z}_i = \lambda \hat{k}_i(\hat{Z}_i) \quad \text{in } \hat{\Omega}_i, \quad \hat{Z}_i = L/\varepsilon_i, \quad \Delta_y \hat{Z}_i = 0 \quad \text{on } \partial\hat{\Omega}_i, \tag{3.24}$$

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where $\hat{\Omega}_i = \{y = \varepsilon_i^{-3/4}(x - x_i) \colon x \in \Omega\}$ and

$$\hat{k}_i(\hat{Z}_i) = \begin{cases} \dfrac{1}{\hat{Z}_i^2}, & \hat{Z}_i \geqslant 1, \\ 3 - 2\hat{Z}_i + \dfrac{\varepsilon_i}{L - \varepsilon_i} - 2\left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i + \left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i^2, & \hat{Z}_i < 1. \end{cases}$$

It is easily seen that $\{|\hat{k}_i(\hat{Z}_i)|\}$ is bounded. Therefore, the regularity of Δ^2 implies that $\hat{Z}_i \to \hat{Z}$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ as $i \to \infty$ with $\hat{Z}(0) = \min_{\mathbb{R}^2} \hat{Z} = 0$, and $\hat{Z}$ satisfies the equation

$$-D\Delta^2 \hat{Z} = \lambda \hat{k}(\hat{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\hat{k}(\hat{Z}) = \begin{cases} \hat{Z}^{-2}, & \hat{Z} \geqslant 1, \\ 3 - 2\hat{Z}, & \hat{Z} < 1. \end{cases}$$

We know from remark 2.4 that $\hat{Z}$ does not exist. This contradiction implies that (3.23) holds.

Equations (3.21) and (3.23) imply that there exists $0 < A_1 < \infty$ such that

$$\frac{\varepsilon_i}{\xi_i} \to A_1 \quad \text{as } i \to \infty. \tag{3.25}$$

(We can choose subsequences if necessary.) Making the transformations $\utilde{Z}_i(y) = Z_i/\xi_i$ and $y = \xi_i^{-3/4}(x - x_i)$, we see that $\utilde{Z}_i(0) = \min_{\Omega} \utilde{Z}_i = -1$ and $\utilde{Z}_i$ satisfies the problem

$$\xi_i^{3/2} T \Delta_y \utilde{Z}_i - D\Delta_y^2 \utilde{Z}_i = \lambda \utilde{k}_i(\utilde{Z}_i) \quad \text{in } \utilde{\Omega}_i, \quad \utilde{Z}_i = L/\xi_i, \quad \Delta_y \utilde{Z}_i = 0 \quad \text{on } \utilde{\Omega}_i, \tag{3.26}$$

where $\utilde{\Omega}_i = \{y = \xi_i^{-3/4}(x - x_i) \colon x \in \Omega\}$ and

$$\utilde{k}_i(\utilde{Z}_i) = \begin{cases} \dfrac{1}{\utilde{Z}_i^2}, & \utilde{Z}_i \geqslant \dfrac{\varepsilon_i}{\xi_i}, \\ 3\left(\dfrac{\xi_i}{\varepsilon_i}\right)^2 - 2\left(\dfrac{\xi_i}{\varepsilon_i}\right)^3 \utilde{Z}_i + \dfrac{\xi_i^2}{\varepsilon_i(L - \varepsilon_i)} & \\ \qquad -2\left(\dfrac{\xi_i^3}{\varepsilon_i^2(L - \varepsilon_i)}\right)\utilde{Z}_i + \left(\dfrac{\xi_i^4}{\varepsilon_i^3(L - \varepsilon_i)}\right)\utilde{Z}_i^2, & \utilde{Z}_i < \dfrac{\varepsilon_i}{\xi_i}. \end{cases}$$

Note that $\{|\utilde{k}_i(\utilde{Z}_i)|\}$ is bounded and $\xi_i^{-3/4} \operatorname{dist}(x_i, \partial\Omega) \to \infty$ as $i \to \infty$ (see (3.12)). Thus, the regularity of Δ^2 implies that $\utilde{Z}_i \to \utilde{Z}$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ as $i \to \infty$ with $\utilde{Z}(0) = \min_{\mathbb{R}^2} \utilde{Z} = -1$, and $\utilde{Z}$ satisfies the equation

$$-D\Delta^2 \utilde{Z} = \lambda \utilde{k}(\utilde{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\utilde{k}(\utilde{Z}) = \begin{cases} \utilde{Z}^{-2}, & \utilde{Z} \geqslant A_1, \\ \dfrac{3}{A_1^2} - \dfrac{2}{A_1^3}\utilde{Z}, & \utilde{Z} < A_1. \end{cases}$$

We know from remark 2.4 that $\utilde{Z}$ does not exist. This contradiction implies that case (i) does not occur.

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For case (ii), we make the transformations $\hat{Z}_i(y) = Z_i/\varepsilon_i$ and $y = \varepsilon_i^{-3/4}(x - x_i)$. We see that $\hat{Z}_i(0) = \min_\Omega \hat{Z}_i = 0$ and $\hat{Z}_i$ satisfies the problem

$$\varepsilon_i^{3/2} T \Delta_y \hat{Z}_i - D\Delta_y^2 \hat{Z}_i = \lambda \hat{k}_i(\hat{Z}_i) \quad \text{in } \hat{\Omega}_i, \quad \hat{Z}_i = L/\varepsilon_i, \quad \Delta_y \hat{Z}_i = 0 \quad \text{on } \partial\hat{\Omega}_i, \tag{3.27}$$

where $\hat{\Omega}_i = \{y = \varepsilon_i^{-3/4}(x - x_i) : x \in \Omega\}$ and

$$\hat{k}_i(\hat{Z}_i) = \begin{cases} \dfrac{1}{\hat{Z}_i^2}, & \hat{Z}_i \geqslant 1, \\ 3 - 2\hat{Z}_i + \dfrac{\varepsilon_i}{L - \varepsilon_i} - 2\left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i + \left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i^2, & \hat{Z}_i < 1. \end{cases}$$

It is easily seen that $\{|\hat{k}_i(\hat{Z}_i)|\}$ is bounded. Therefore, the regularity of Δ^2 implies that $\hat{Z}_i \to \hat{Z}$ in $C^3_{\text{loc}}(\mathbb{R}^2)$ as $i \to \infty$ with $\hat{Z}(0) = \min_{\mathbb{R}^2} \hat{Z} = 0$, and $\hat{Z} \in C^4(\mathbb{R}^2)$ satisfies the equation

$$-D\Delta^2 \hat{Z} = \lambda \hat{k}(\hat{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\hat{k}(\hat{Z}) = \begin{cases} \hat{Z}^{-2}, & \hat{Z} \geqslant 1, \\ 3 - 2\hat{Z}, & \hat{Z} < 1. \end{cases}$$

We know from remark 2.4 that $\hat{Z}$ does not exist. This contradiction implies that case (ii) does not occur.

For case (iii), we see that $\xi_i < \varepsilon_i$ for all i since $V_i(x_i) > L - \varepsilon_i$. We first show that

$$\frac{\varepsilon_i}{\xi_i} \not\to \infty \quad \text{as } i \to \infty. \tag{3.28}$$

(We can choose subsequences if necessary.) On the contrary, making the transformations $\hat{Z}_i(y) = Z_i/\varepsilon_i$ and $y = \varepsilon_i^{-3/4}(x - x_i)$, we see that $\hat{Z}_i(0) = \min_\Omega \hat{Z}_i = \xi_i/\varepsilon_i$ ($\to 0$ as $i \to \infty$) and $\hat{Z}_i$ satisfies the problem

$$\varepsilon_i^{3/2} T \Delta_y \hat{Z}_i - D\Delta_y^2 \hat{Z}_i = \lambda \hat{k}_i(\hat{Z}_i) \quad \text{in } \hat{\Omega}_i, \quad \hat{Z}_i = L/\varepsilon_i, \quad \Delta_y \hat{Z}_i = 0 \quad \text{on } \partial\hat{\Omega}_i, \tag{3.29}$$

where $\hat{\Omega}_i = \{y = \varepsilon_i^{-3/4}(x - x_i) : x \in \Omega\}$ and

$$\hat{k}_i(\hat{Z}_i) = \begin{cases} \dfrac{1}{\hat{Z}_i^2}, & \hat{Z}_i \geqslant 1, \\ 3 - 2\hat{Z}_i + \dfrac{\varepsilon_i}{L - \varepsilon_i} - 2\left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i + \left(\dfrac{\varepsilon_i}{L - \varepsilon_i}\right)\hat{Z}_i^2, & \hat{Z}_i < 1. \end{cases}$$

It is easily seen that $\{|\hat{k}_i(\hat{Z}_i)|\}$ is bounded. Therefore, the regularity of Δ^2 implies that $\hat{Z}_i \to \hat{Z}$ in $C^3_{\text{loc}}(\mathbb{R}^2)$ as $i \to \infty$ with $\hat{Z}(0) = \min_{\mathbb{R}^2} \hat{Z} = 0$, and $\hat{Z} \in C^4(\mathbb{R}^2)$ satisfies the equation

$$-D\Delta^2 \hat{Z} = \lambda \hat{k}(\hat{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\hat{k}(\hat{Z}) = \begin{cases} \hat{Z}^{-2}, & \hat{Z} \geqslant 1, \\ 3 - 2\hat{Z}, & \hat{Z} < 1. \end{cases}$$

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We know from remark 2.4 that $\hat{Z}$ does not exist. This contradiction implies that (3.28) does not hold. Therefore, there exists $A_2 \geqslant 1$ such that

$$\frac{\varepsilon_i}{\xi_i} \to A_2 \quad \text{as } i \to \infty. \tag{3.30}$$

(We can choose subsequences if necessary.) Making the transformations $\underline{Z}_i(y) = Z_i/\xi_i$ and $y = \xi^{-3/4}(x - x_i)$, we see that $\underline{Z}_i(0) = \min_{\Omega} \underline{Z}_i = 1$ and $\underline{Z}_i \geqslant 1$ satisfies the problem

$$\xi_i^{3/2} T\Delta_y \underline{Z}_i - D\Delta_y^2 \underline{Z}_i = \lambda \underline{k}_i(\underline{Z}_i) \quad \text{in } \Omega_i, \quad \underline{Z}_i = L/\xi_i, \quad \Delta_y \underline{Z}_i = 0 \quad \text{on } \Omega_i, \tag{3.31}$$

where $\Omega_i = \{y = \xi_i^{-3/4}(x - x_i) \colon x \in \Omega\}$ and

$$\underline{k}_i(\underline{Z}_i) = \begin{cases} \dfrac{1}{\underline{Z}_i^2}, & \underline{Z}_i \geqslant \dfrac{\varepsilon_i}{\xi_i}, \\ 3\left(\dfrac{\xi_i}{\varepsilon_i}\right)^2 - 2\left(\dfrac{\xi_i}{\varepsilon_i}\right)^3 \underline{Z}_i + \dfrac{\xi_i^2}{\varepsilon_i(L-\varepsilon_i)} & \\ \qquad -2\left(\dfrac{\xi_i^3}{\varepsilon_i^2(L-\varepsilon_i)}\right)\underline{Z}_i + \left(\dfrac{\xi_i^4}{\varepsilon_i^3(L-\varepsilon_i)}\right)\underline{Z}_i^2, & \underline{Z}_i < \dfrac{\varepsilon_i}{\xi_i}. \end{cases}$$

Note that $\{|\underline{k}_i(\underline{Z}_i)|\}$ is bounded and $\xi_i^{-3/4} \operatorname{dist}(x_i, \partial\Omega) \to \infty$ as $i \to \infty$ (see (3.12)). Thus, the regularity of Δ^2 implies that $\underline{Z}_i \to \underline{Z}$ in $C^3_{\text{loc}}(\mathbb{R}^2)$ as $i \to \infty$ with $\underline{Z}(0) = \min_{\mathbb{R}^2} \underline{Z} = 1$, and $\underline{Z}$ satisfies the equation

$$-D\Delta^2 \underline{Z} = \lambda \underline{k}(\underline{Z}) \quad \text{in } \mathbb{R}^2,$$

where

$$\underline{k}(\underline{Z}) = \underline{Z}^{-2}$$

provided $A_2 = 1$ and

$$\underline{k}(\underline{Z}) = \begin{cases} \underline{Z}^{-2}, & \underline{Z} \geqslant A_2, \\ \dfrac{3}{A_2^2} - \dfrac{2}{A_2^3}\underline{Z}, & \underline{Z} < A_2, \end{cases}$$

provided $A_2 > 1$. We know from theorem 2.1 and remark 2.4 that $\underline{Z}$ does not exist. This contradiction implies that case (iii) does not occur.

The above arguments imply that

$$\max_{\Omega} V_{\varepsilon,\lambda} \leqslant L - \varepsilon$$

for ε sufficiently small. Actually, arguments similar to those above imply that there exists $\delta > 0$ independent of ε such that

$$V_{\varepsilon,\lambda} \leqslant L - \delta \quad \text{in } \Omega,$$

for ε sufficiently small. This completes the proof of this theorem. □

Remark 3.3. We can also obtain the same asymptotic behaviour as in theorem 8.2 of [4] of the mountain-pass solutions as $\lambda \to 0^+$ by arguments similar to those in the proof of theorem 8.2 of [4].

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