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COVERINGS AND COVERINGS

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Abstract

Coverings of certain simplicial complexes constructed from subgroups of a group G are related to covering groups of G.

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Simplicial complexes constructed from subgroups of a group have been prominent in group theory in recent years. One such space is the commuting complex [1] of a finite group. Fix a prime p and a finite group G. We construct the corresponding complex as follows: its vertices (0-simplices) are the subgroups of order p in G, and a subset of r subgroups of order p form an $(r-1)$-simplex if and only if the r subgroups commute pairwise. This complex is related to other simplicial complexes constructed from groups, for example, the Brown complex, which is built up from all non-identity p-subgroups of G using the containment relation. Indeed, these two complexes are of the same homotopy type [1].

One can also consider subcomplexes of the commuting complex. Let K be a normal subset of subgroups of order p in G and let $|K|$ be the corresponding complex, so its vertices are the subgroups of K and simplices are determined by the relation of pairwise commuting as above. S. Bouc [2] has discovered some startling properties of such subcomplexes in the case of the symmetric groups. In particular, he has shown that when $p = 2$, G is the symmetric group S_7, and K consists of the transpositions, then the fundamental group of $|K|$ is of order 3. Moreover, he has suggested that this should be related to the Schur multiplier of the alternating group A_7 and the triple cover of A_7. We shall see that this is so and prove a general theorem which explains the connection.

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In fact, we shall show that group extensions, in particular covering groups, are related to the fundamental groups via covering spaces; our title refers to this double use of the word 'covering'. Let us now fix some more notation. Let N be a normal p'-subgroup of G. Let $G^* = G/N$ and let K^* be the image of K in G^*, the set of all subgroups of order p which are images of subgroups in K. Therefore, there is a natural map of the simplicial complex $|K|$ onto the simplicial complex $|K^*|$ and this is a simplicial map since the relation of commuting is preserved by homomorphic images. Our main result can now be stated.

THEOREM. *Assume that K generates G and that $|K^*|$ is connected. The natural map of $|K|$ to $|K^*|$ is a covering map if, and only if, the following two conditions are fulfilled:*

1) *whenever X and Y are in K then $C_N(X) = C_N(Y)$;*
2) *N admits no proper supplement in G.*

Here, all notions, like covering, from the theory of simplicial complexes are the standard ones [3]. The last statement means that if $G = H.N$, where H is a subgroup of G then $H = G$. For example, take G to be the group which has a normal subgroup N of order 3 with quotient S_7 such that the extension of N by A_7 is a covering, in Schur's sense. Set $p = 2$ and take K to be inverse image in G of the transpositions. Then the conditions are easily seen to be fulfilled and the fundamental group of $|K^*|$ is of order divisible by 3. Thus, we have answered Bouc's question. M. Ronan informs us that the same argument can be made with the non-split extension of a group of order 3 by the Fischer group Fi_{24}.

We begin the proof of this result with two preliminary ones.

LEMMA 1. *If X is in K, then any simplex of K^* containing X^* is the image of a simplex of K containing X.*

PROOF. Set $X = X_0$ and let $X_1, \dots, X_d$ be in K such that $X_0^*, \dots, X_d^*$ are the distinct vertices of a d-simplex containing $X_0^* = X^*$. Therefore, $H = \langle N, X_0, \dots, X_d\rangle$ is a p-nilpotent group with abelian Sylow p-subgroup. Let P be a Sylow p-subgroup of H containing X. Since P covers H/N there are elements $m_1, \dots, m_d$ of N such that the d conjugates $Y_i = m_i^{-1}X_im_i$ are contained in P and so commute elementwise. Hence, $Y_0 = X, Y_1, \dots, Y_d$ are the vertices of the desired d-simplex.

Let us keep the notation of this last proof to use in the next lemma.

LEMMA 2. *The simplex containing X and satisfying the above conditions is unique if, and only if, for each $i = 1, \dots, d$, we have that $C_N(X)$ is contained in $C_N(Y_i)$.*

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PROOF. Suppose that, for the index j, the centralizer condition fails and that m is in $C_N(X)$ and is not contained in $C_N(Y_j)$. Then Y_j is distinct from Y_j^m and so

$$X = X^m, Y_1^m, \dots, Y_d^m$$

is another d-simplex containing X. Conversely, suppose that $X, Z_1, \dots, Z_d$ is another simplex containing X and each $Z_i^* = X_i^*$. Hence, we must have Z_j not equal to Y_j for some index j. Therefore, $\langle X, Y_j \rangle$ and $\langle X, Z_j \rangle$ are Sylow p-subgroups of $\langle N, X, Y_j \rangle$ so there is m in N which conjugates the first of these Sylow subgroups to the second. Thus, $X^m = X$ and $Y_j^m = Z_j$ which shows that $C_N(X)$ is not contained in $C_N(Y_j)$.

PROOF (OF THE THEOREM). First, suppose that the map of $|K|$ to $|K^*|$ is a covering map. In order to establish 1) it suffices, since $|K^*|$ is connected by hypothesis, to show that if X and Y are in K and commute then $C_N(X) = C_N(Y)$. For the covering property now implies that $|K|$ is also connected. However, if these are unequal, then we can assume (after perhaps interchanging X and Y) that there is m in $C_N(X)$ which is not contained in $C_N(Y)$. Thus, $\{X, Y\}$ and $\{X, Y^m\}$ are two edges mapping to $\{X^*, Y^*\}$, contradicting the path lifting property of a covering.

Next, assume that H is a proper supplement to N in G; we need to derive a contradiction in order to prove that 2) holds. Since N is a p'-group and H covers G/N it follows that H contains a Sylow p-subgroup of each subgroup of G containing N; in particular, if L consists of the subgroups of K which lie in H, then $L^* = K^*$. However, as K generates G and H is a proper subgroup, we have that L is a proper subset of K. Thus, choose X in L and Y in K with Y not in L. It suffices to prove that X and Y do not commute. For then, as this holds for any such choice of X and Y, we will have contradicted the connectivity of $|K|$. Hence, again arguing by contradiction, assume that X and Y do commute. Thus, X^* and Y^* are distinct as X does not commute with any of its conjugates in $\langle X, N \rangle$. Now we apply Lemma 1 to the group H and collection L of subgroups of order p. It follows, since $L^* = K^*$, that there is Z in L such that $Z^* = Y^*$ and X and Z commute. We now have the edges $\{X, Y\}$ and $\{X, Z\}$ both mapping to the edge $\{X^*, Y^*\}$ which contradicts the covering property of the map of $|K|$ to $|K^*|$.

We must now prove the converse, that the statements 1) and 2) imply the covering property. However, statement 1) gives us a uniqueness property directly from the second lemma, namely, that if X is in K, then any simplex of $|K^*|$ containing X^* is the image of a unique simplex of K containing X. Because of this property we need only prove that $|K|$ is connected. Indeed, we already have by this uniqueness property and statement 1) that each simplex of $|K^*|$ is the image of $|N : C_N(X)|$ disjoint simplices of $|K|$ for any subgroup X in K. This implies that the star neighborhood of a vertex of $|K^*|$ is similarly constituted and mapped. Since any point of $|K^*|$ has a neighborhood of this sort we have that the map of $|K|$ to $|K^*|$ is a covering [3].

Hence, once again let us argue to a contradiction; assume that M is a proper subset of K consisting of the vertices of a connected component of $|K|$. However, $|K^*|$ is connected, by hypothesis, so the first lemma implies that $M^* = K^*$. Thus, if H is the subgroup generated by the subgroups of M, then $G = H.N$. Therefore, we need only prove that H is a proper subgroup of G in order to reach the desired contradiction. To do this it suffices, since M is a proper subset of K, to show that M is a normal subset of H, as the equality $M^* = K^*$ implies that M contains a subgroup from each conjugacy class of subgroups in K. However, since M generates H, it is enough to show that if X is a subgroup of M, $X = \langle x \rangle$, then $M = M^x$. But M and M^x both contain X, so since M consists of the vertices of a connected component, we must have that M and M^x coincide.

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