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DUALIZING THE COARSE ASSEMBLY MAP

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Abstract We formulate and study a new coarse (co-)assembly map. It involves a modification of the Higson corona construction and produces a map dual in an appropriate sense to the standard coarse assembly map. The new assembly map is shown to be an isomorphism in many cases. For the underlying metric space of a group, the coarse co-assembly map is closely related to the existence of a dual Dirac morphism and thus to the Dirac dual Dirac method of attacking the Novikov conjecture.

Keywords: Novikov conjecture; coarse geometry; Higson compactification; coarse Baum–Connes conjecture; scalable space

AMS 2000 *Mathematics subject classification:* Primary 19K35; 46L80

1. Introduction

It is shown in [**7**] that a torsion free discrete group G with compact classifying space BG has a dual Dirac morphism (in the sense of [**17**]) if and only if a certain coarse co-assembly map

$$\mu^*\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G)) \to \mathrm{KX}^*(G)$$

is an isomorphism. The C^*-algebra $\mathfrak{c}(G)$ is called the stable Higson corona of G and up to isomorphism only depends on the coarse structure of G. The $\mathbb{Z}/2$-graded Abelian group $\mathrm{KX}^*(G)$ is called the coarse K-theory of G and also depends only on the coarse structure of G. Essentially the same result holds for torsion free, countable, discrete groups with finite-dimensional BG. In particular, for this class of groups, the existence of a dual Dirac morphism is a geometric invariant of G. In this article we introduce and study the map μ^* in detail. In particular, we

(1) examine the relationship between μ^* and the ordinary coarse Baum–Connes assembly map, and between $\mathfrak{c}(G)$ and compactifications of G;

(2) establish isomorphism of μ^* for scalable spaces;

(3) establish isomorphism of μ^* for groups which uniformly embed in Hilbert space.

The stable Higson corona $\mathfrak{c}(X)$ has better functoriality properties than the C^*-algebra $C^*(X)$ that figures in the usual coarse Baum–Connes assembly map (see [**13**, **19**, **20**, **24**–**27**]). The assignment $X \mapsto \mathfrak{c}(X)$ is functorial from the coarse category of coarse spaces to the category of C^*-algebras and C^*-algebra homomorphisms. The analogous statement for the coarse C^*-algebra $C^*(X)$ is true only after passing to K-theory. Moreover, the C^*-algebra $\mathfrak{c}(X)$ is designed to be closely related to certain bivariant Kasparov groups. This is the source of a homotopy invariance result, which implies our assertion for scalable spaces. Another advantage of the stable Higson corona and the coarse co-assembly map is their relationship with alternative approaches to the Novikov conjecture, namely, almost flat K-theory (see [**4**]) and the Lipschitz approach of [**5**].

The map μ^* is an isomorphism for any discrete group G that has a dual Dirac morphism, without any hypothesis on BG. This is how we are going to prove isomorphism of μ^* for groups that uniformly embed in a Hilbert space: we show that such groups have a dual Dirac morphism. Actually, already the existence of an approximate dual Dirac morphism implies that μ^* is an isomorphism. Using results of Kasparov and Skandalis [**16**], it follows that the coarse co-assembly map is an isomorphism for groups acting properly by isometries on bolic spaces. The usual coarse Baum–Connes conjecture for a group G is equivalent to the Baum–Connes conjecture with coefficients $\ell^\infty(G)$ [**25**]. Despite this, it is not known whether the existence of an action of G on a bolic space implies the coarse Baum–Connes conjecture for G. The existence of a dual Dirac morphism only implies split injectivity of the coarse Baum–Connes assembly map.

Given the above observations, we expect the coarse co-assembly map to become a useful tool in connection with the Novikov conjecture. However, at the moment we have no examples of groups for which our method proves the Novikov conjecture while others fail. We also remark that we do not know whether the map μ^* is an isomorphism for the standard counterexamples to the coarse Baum–Connes conjecture.

2. Coarse spaces

We begin by recalling the notion of a coarse space and some related terminology (see [**13**, **20**]). Then we introduce σ-coarse spaces, which are useful to deal with the Rips complex construction.

Let X be a set. We define the *diagonal* Δ_X, the *transpose* of $E \subseteq X \times X$, and the *composition* of $E_1, E_2 \subseteq X \times X$ by

$$\begin{aligned}\Delta_X &:= \{(x,x) \in X \times X \mid x \in X\},\\ E^{\mathrm{t}} &:= \{(y,x) \in X \times X \mid (x,y) \in E\},\\ E_1 \circ E_2 &:= \{(x,z) \in X \times X \mid (x,y) \in E_1 \text{ and } (y,z) \in E_2 \text{ for some } y \in X\}.\end{aligned}$$

Definition 2.1. A *coarse structure* on X is a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$—called *entourages* or *controlled subsets*—which satisfy the following axioms:

(1) if $E \in \mathcal{E}$ and $E' \subseteq E$, then $E' \in \mathcal{E}$ as well;

(2) if $E_1, E_2 \in \mathcal{E}$, then $E_1 \cup E_2 \in \mathcal{E}$;

(3) if $E \in \mathcal{E}$, then $E^{\mathrm{t}} \in \mathcal{E}$;

(4) if $E_1, E_2 \in \mathcal{E}$, then $E_1 \circ E_2 \in \mathcal{E}$;

(5) $\Delta_X \in \mathcal{E}$;

(6) all finite subsets of $X \times X$ belong to $\mathcal{E}$.

A subset B of X is called *bounded* if $B \times B$ is an entourage. A collection of bounded subsets (B_i) of X is called *uniformly bounded* if $\bigcup B_i \times B_i$ is an entourage.

A topology and a coarse structure on X are called *compatible* if

(7) some neighbourhood of $\Delta_X \subseteq X \times X$ is an entourage;

(8) every bounded subset of X is relatively compact.

A *coarse space* is a locally compact topological space equipped with a compatible coarse structure.

Since the intersection of a family of coarse structures is again a coarse structure, we can define the *coarse structure generated by* any set of subsets of $X \times X$. We call a coarse structure *countably generated* if there is an increasing sequence of entourages (E_n) such that any entourage is contained in E_n for some $n \in \mathbb{N}$. If X is a coarse space and $Y \subseteq X$ is a closed subspace, then $\mathcal{E} \cap (Y \times Y)$ is a coarse structure on Y called the *subspace coarse structure*.

Let X be a coarse space. Then the closure of an entourage is again an entourage. Hence the coarse structure is already generated by the closed entourages. Using an entourage that is a neighbourhood of the diagonal, we can construct a uniformly bounded open cover of X. This open cover has a subordinate partition of unity because X is locally compact. We shall frequently use this fact.

Example 2.2. Let (X, d) be a metric space. The *metric coarse structure on* X is the countably generated coarse structure generated by the increasing sequence of entourages

$$E_R := \{(x, y) \in X \times X \mid d(x, y) \leqslant R\}, \quad R \in \mathbb{N}.$$

A subset $E \subseteq X \times X$ is an entourage if and only if $d\colon X \times X \to \mathbb{R}_+$ is bounded on E. Note that this coarse structure depends only on the quasi-isometry class of d. The metric d also defines a topology on X. This topology and the coarse structure are compatible and thus define a coarse space if and only if bounded subsets of X are relatively compact. We call (X, d) a *coarse metric space* if this is the case.

Conversely, one can show that any countably generated coarse structure on a set X can be obtained from some metric on X as above. However, if X also carries a topology, it is not clear whether one can find a metric that generates both the coarse structure and the topology.

Example 2.3. Any locally compact group G has a canonical coarse structure that is invariant under left translations. It is generated by the entourages

$$E_K := \{(g_1, g_2) \in G \times G \mid g_1^{-1} g_2 \in K\},$$

where K runs through the compact subsets of G. Together with the given locally compact topology on G, this turns G into a coarse space.

A *coarse map* $\phi\colon X \to Y$ between coarse spaces is a Borel map which maps entourages in X to entourages in Y and which is *proper* in the sense that inverse images of bounded sets are bounded. Two coarse maps $\phi, \psi\colon X \to Y$ are called *close* if $(\phi\times\psi)(\Delta_X) \subseteq Y\times Y$ is an entourage. The *coarse category of coarse spaces* is the category whose objects are the coarse spaces and whose morphisms are the equivalence classes of coarse maps, where we identify two maps if they are close. A coarse map is called a *coarse equivalence* if it is an isomorphism in this category. Two coarse spaces are called *coarsely equivalent* if they are isomorphic in this category.

Lemma 2.4. *Let X be a countably generated coarse space. Then there exists a countable discrete subset $Z \subseteq X$ such that the inclusion $Z \to X$ is a coarse equivalence. Here we equip Z with the subspace coarse structure and the discrete topology. Thus X is coarsely equivalent to a countably generated, discrete coarse space.*

Proof. We claim that any countably generated coarse space is σ-compact. To see this, fix a point $x_0 \in X$ and an increasing sequence (E_n) of entourages that defines the coarse structure. The sets $K_n := \{x \in X \mid (x, x_0) \in E_n\}$ are bounded and hence relatively compact. Their union is all of X because $\bigcup E_n = X \times X$. Thus X is σ-compact. Let (B_i) be a uniformly bounded open cover of X. By σ-compactness, we can choose a countable partition of unity (ρ_n) on X subordinate to this covering. Let $B_n' := \rho_n^{-1}((0,\infty))$. These sets form a countable, locally finite, uniformly bounded open covering of X. Choose a point x_n in each B_n'. The subset $Z := \{x_n\}$ has the required properties. □

We shall also use formal direct unions of coarse spaces, which we call σ-coarse spaces. Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of subsets of a set $\mathcal{X}$ with $\mathcal{X} = \bigcup X_n$ such that each X_n is a coarse space and the coarse structure and topology on X_m are the subspace coarse structure and topology from X_n for any $n \geqslant m$. Then we can equip $\mathcal{X}$ with the direct limit topology and with the coarse structure that is generated by the coarse structures of the subspaces X_n. This coarse structure is compatible with the topology, but the topology need not be locally compact. In this situation, we call the inductive system of coarse spaces (X_n) or, by abuse of notation, its direct limit $\mathcal{X}$ a *σ-coarse space*. We define σ-locally compact spaces similarly. Notice that the system (X_n) is part of the structure of $\mathcal{X}$ even if the direct limit topology on $\mathcal{X}$ is locally compact.

Example 2.5. Let (X, d) be a discrete metric space with the property that bounded subsets are finite. Let $P_n(X)$ denote the set of probability measures on X whose support has diameter at most n. This is a locally finite simplicial complex and hence a locally compact topological space. We equip $P_n(X)$ with the coarse structure generated by the increasing sequence of entourages

$$\{(\mu,\nu) \in P_n(X) \times P_n(X) \mid \operatorname{supp}\mu \times \operatorname{supp}\nu \subseteq E_R\}$$

for $R \in \mathbb{N}$, with E_R as in Example 2.2. This turns $P_n(X)$ into a coarse space for any $n \in \mathbb{N}$ and turns $\mathcal{P}_X := \bigcup P_n(X)$ into a σ-coarse space. We discuss this example in greater detail in § 4.

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Example 2.6. Let G be a second countable, locally compact group and let X be a *G-compact*, proper G-space. We equip X with the coarse structure that is generated by the G-invariant entourages

$$E_L := \bigcup_{g\in G} gL \times gL, \quad L \subseteq X \text{ compact.}$$

Since X is necessarily σ-compact, this coarse structure is countably generated. It is also compatible with the topology of X, so that X becomes a countably generated coarse space. For $X = G$ with the action by left translation, this reproduces the coarse structure of Example 2.3. For any $x \in X$, the orbit map $G \to X$, $g \mapsto g \cdot x$, is a coarse equivalence, and these maps for different points in X are close. Thus we obtain a canonical isomorphism $X \cong G$ in the coarse category of coarse spaces.

Let $\underline{E}G$ denote a second countable, not necessarily locally compact model for the classifying space for proper actions of G as in [**1**]. We can write $\underline{E}G$ as an increasing union of a sequence of G-compact, G-invariant, closed subsets $X_n \subseteq \underline{E}G$. Turning each X_n into a coarse space as above, we turn $\underline{E}G$ into a σ-coarse space.

In the above two examples, the maps $X_n \to X_{n+1}$ are coarse equivalences. This happens in all examples that we need, and the general case is more difficult. Therefore, *we restrict attention in the following to σ-coarse spaces for which the maps $X_n \to X_{n+1}$ are coarse equivalences.*

3. Functions of vanishing variation

Let X be a coarse space and let D be a C^*-algebra. We define the C^*-algebras $\bar{\mathfrak{c}}(X, D)$ and $\mathfrak{c}(X, D)$ and discuss their relationship to the Higson compactification and the Higson corona and to admissible compactifications. Then we investigate their functoriality properties.

Definition 3.1. Let $\mathcal{X}$ be a σ-coarse space and let Y be a metric space. Let $f\colon \mathcal{X} \to Y$ be a Borel map (that is, $f|_{X_n}$ is a Borel map for all $n \in \mathbb{N}$). For an entourage $E \subseteq X_n \times X_n$, $n \in \mathbb{N}$, we define

$$\operatorname{Var}_E\colon X_n \to [0,\infty), \qquad \operatorname{Var}_E f(x) := \sup\{d(f(x), f(y)) \mid (x,y) \in E\}.$$

We say that f has *vanishing variation* if Var_E vanishes at ∞ for any such E, that is, for any $\varepsilon > 0$ the set of $x \in X_n$ with $\operatorname{Var}_E f(x) \geqslant \varepsilon$ is bounded.

If the coarse structure comes from a metric d on X, we also let

$$\operatorname{Var}_R f(x) := \sup\{d(f(x), f(y)) \mid d(x,y) \leqslant R\}$$

for $R \in \mathbb{R}_+$. This is the variation function associated to the entourage E_R defined in Example 2.2. Hence we can also use the functions $\operatorname{Var}_R f$ to define vanishing variation.

Definition 3.2. For any coarse space X and any C^*-algebra D, we let $\bar{\mathfrak{c}}(X, D)$ be the C^*-algebra of bounded, continuous functions of vanishing variation $X \to D \otimes \mathbb{K}$.

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Here $\mathbb{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. The quotient $\mathfrak{c}(X, D) := \bar{\mathfrak{c}}(X, D)/C_0(X, D \otimes \mathbb{K})$ is called the *stable Higson corona of X with coefficients D*.

Note 3.3. When $D = \mathbb{C}$ we abbreviate $\bar{\mathfrak{c}}(X, D)$ and $\mathfrak{c}(X, D)$ to $\bar{\mathfrak{c}}(X)$ and $\mathfrak{c}(X)$, respectively, and call $\mathfrak{c}(X)$ the *stable Higson corona of X*.

The reason for our terminology is the analogy with the Higson corona constructed in [**11**]. Let X be a coarse metric space. The *Higson compactification* ηX of X is the maximal ideal space of the C^*-algebra of continuous, bounded functions $X \to \mathbb{C}$ of vanishing variation. The *Higson corona* of X is $\partial_\eta X := \eta X \setminus X$. By construction, $M_n(\mathbb{C}) \otimes C(\eta X) = C(\eta X, M_n)$ is the C^*-algebra of bounded, continuous functions $X \to M_n(\mathbb{C})$ of vanishing variation and $C(\partial_\eta X, M_n(\mathbb{C})) = C(\eta X, M_n(\mathbb{C}))/C_0(X, M_n(\mathbb{C}))$. Of course, these C^*-algebras are contained in $\bar{\mathfrak{c}}(X)$ and $\mathfrak{c}(X)$, respectively. Since $\mathbb{K} = \varinjlim M_n(\mathbb{C})$, we also obtain canonical embeddings

$$\mathbb{K} \otimes C(\eta X) \cong C(\eta X, \mathbb{K}) \subseteq \bar{\mathfrak{c}}(X), \qquad \mathbb{K} \otimes C(\partial_\eta X) \cong C(\partial_\eta X, \mathbb{K}) \subseteq \mathfrak{c}(X).$$

Similarly, we obtain embeddings

$$C(\eta X, D \otimes \mathbb{K}) \subseteq \bar{\mathfrak{c}}(X, D), \qquad C(\partial_\eta X, D \otimes \mathbb{K}) \subseteq \mathfrak{c}(X, D)$$

for any C^*-algebra D. It turns out that $\bar{\mathfrak{c}}(X)$ is strictly larger than $C(\eta X, \mathbb{K})$. If $f \in C(\eta X, \mathbb{K})$, then $f(X) \subseteq f(\eta X)$ must be a relatively compact subset of $\mathbb{K}$. Conversely, one can show that a continuous function $X \to \mathbb{K}$ of vanishing variation with relatively compact range belongs to $C(\eta X, \mathbb{K})$. However, functions in $\bar{\mathfrak{c}}(X)$ need not have relatively compact range.

It is often preferable to replace the Higson compactification by smaller ones that are metrizable. This is the purpose of the following definition.

Definition 3.4 ([12]). Let X be a metric space and let $i\colon X \to Z$ be a metrizable compactification of X. We call Z *admissible* if there is a metric on Z generating the topology on Z for which the inclusion $i\colon X \to Z$ has vanishing variation.

Example 3.5. The following are examples of admissible compactifications:

(1) the one-point compactification of an arbitrary metric space;

(2) the hyperbolic compactification of a metric space that is hyperbolic in the sense of Gromov;

(3) the visibility compactification of a complete, simply connected, non-positively curved manifold.

Proposition 3.6. *Let X be a metric space and let $i\colon X \to Z$ be an admissible compactification of X. Then there are canonical injective $*$-homomorphisms*

$$C(Z, D \otimes \mathbb{K}) \to C(\eta X, D \otimes \mathbb{K}) \to \bar{\mathfrak{c}}(X, D),$$
$$C(Z \setminus X, D \otimes \mathbb{K}) \to C(\partial_\eta X, D \otimes \mathbb{K}) \to \mathfrak{c}(X, D)$$

for any C^*-algebra D.

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Any class in $\mathrm{K}_*(C(\partial_\eta X, D \otimes \mathbb{K}))$ *is the image of a class in* $\mathrm{K}_*(C(Z \setminus X, D \otimes \mathbb{K}))$ *for some admissible compactification* Z.

Proof. The admissible compactifications of X are exactly the metrizable quotients of the Higson compactification ηX. Thus the C^*-algebras $C(Z)$ for admissible compactifications of X are exactly the separable C^*-subalgebras of $C(\eta X)$. This implies the corresponding assertion for $C(Z \setminus X)$ and $C(\partial_\eta X)$ and also for tensor products with $D \otimes \mathbb{K}$. The assertion about K-theory follows because any C^*-algebra is the inductive limit of its separable C^*-subalgebras and K-theory commutes with inductive limits. □

Even on the level of K-theory, $C(\eta X, \mathbb{K})$ and $\bar{\mathfrak{c}}(X)$ are drastically different. The induced map $\mathrm{K}_*(C(\eta X, \mathbb{K})) \to \mathrm{K}_*(\bar{\mathfrak{c}}(X))$ is uncountable-to-one already in rather simple examples, as we shall see in § 5. The map $\mathrm{K}_*(C(\eta X, \mathbb{K})) \to \mathrm{K}_*(\bar{\mathfrak{c}}(X))$ may also fail to be surjective. For instance, this happens for the well-spaced ray (see Example 6.5 below). However, we do not know of a uniformly contractible example with this property. If the map $\mathrm{K}_*(C(\eta X, \mathbb{K})) \to \mathrm{K}_*(\bar{\mathfrak{c}}(X))$ is surjective, then Proposition 3.6 yields that any class in $\mathrm{K}_*(\bar{\mathfrak{c}}(X))$ can already be realized on some admissible compactification.

Now we turn to the functoriality of the algebras $\bar{\mathfrak{c}}(X, D)$ and $\mathfrak{c}(X, D)$ with respect to the coarse space X. The functoriality in the coefficient algebra D is analysed in § 7. It is evident that $\bar{\mathfrak{c}}(X, D)$ and $\mathfrak{c}(X, D)$ and the extension

$$0 \to C_0(X, D \otimes \mathbb{K}) \to \bar{\mathfrak{c}}(X, D) \to \mathfrak{c}(X, D) \to 0$$

are functorial for *continuous* coarse maps $X \to X'$. Of course, the morphisms in the category of C^*-algebras are the $*$-homomorphisms. We can drop the continuity hypothesis for $\mathfrak{c}(X, D)$.

Proposition 3.7. *Let* D *be a* C^**-algebra and let* X *and* Y *be coarse spaces. A coarse map* $\phi\colon X \to Y$ *induces a* $*$*-homomorphism* $\phi^*\colon \mathfrak{c}(Y, D) \to \mathfrak{c}(X, D)$*. Close maps induce the same* $*$*-homomorphism* $\mathfrak{c}(Y, D) \to \mathfrak{c}(X, D)$*. Thus the assignment* $X \mapsto \mathfrak{c}(X, D)$ *is a contravariant functor from the coarse category of coarse spaces to the category of* C^**-algebras.*

Proof. We identify $\mathfrak{c}(X, D)$ with another C^*-algebra that is evidently functorial for Borel maps. Let $B_0(X, D \otimes \mathbb{K})$ be the C^*-algebra of bounded Borel functions $X \to D \otimes \mathbb{K}$ that vanish at infinity. Let $\bar{\mathfrak{b}}(X, D)$ consist of bounded Borel functions $X \to D \otimes \mathbb{K}$ with vanishing variation and let $\mathfrak{b}(X, D) := \bar{\mathfrak{b}}(X, D)/B_0(X, D \otimes \mathbb{K})$. It is evident that $B_0(X, D \otimes \mathbb{K})$ and $\bar{\mathfrak{b}}(X, D)$ and hence $\mathfrak{b}(X, D)$ are functorial for coarse maps. Moreover, if $\phi, \phi'\colon X \to Y$ are close and $f \in \mathfrak{b}(Y, D)$, then $f \circ \phi - f \circ \phi'$ vanishes at infinity. Hence ϕ and ϕ' induce the same map $\mathfrak{b}(Y, D) \to \mathfrak{b}(X, D)$.

It is clear that

$$C_0(X, D \otimes \mathbb{K}) \subseteq B_0(X, D \otimes \mathbb{K}), \qquad \bar{\mathfrak{c}}(X, D) \subseteq \bar{\mathfrak{b}}(X, D).$$

Hence we get an induced $*$-homomorphism $j\colon \mathfrak{c}(X, D) \to \mathfrak{b}(X, D)$. We claim that this map is an isomorphism. Once this claim is established, we obtain the desired functoriality of $\mathfrak{c}(X,D)$. Injectivity and surjectivity of j are equivalent to

$$C_0(X,D\otimes\mathbb{K}) = B_0(X,D\otimes\mathbb{K})\cap\bar{\mathfrak{c}}(X,D),$$
$$\mathfrak{b}(X,D) = \mathfrak{c}(X,D) + B_0(X,D\otimes\mathbb{K}),$$

respectively. The first equation is evident. We prove the second one. Let $E\subseteq X\times X$ be an entourage that is a neighbourhood of the diagonal. We remarked after Definition 2.1 that there exists a uniformly bounded open cover (B_i) of X with $\bigcup B_i\times B_i\subseteq E$. Let (ρ_i) be a partition of unity subordinate to (B_i) and fix points $x_i\in B_i$. Take $f\in\mathfrak{b}(X,D)$ and define

$$Pf(x) := \sum \rho_i(x)f(x_i).$$

It is clear that Pf is continuous. Since f has vanishing variation, there exists a bounded set $\Sigma\subseteq X$ such that $\|f(x)-f(y)\|<\varepsilon$ for $(x,y)\in E$ and $x\notin\Sigma$. Hence

$$\|(f-Pf)(x)\|\leqslant\sum\rho_i(x)\|f(x)-f(x_i)\|\leqslant\sum\rho_i(x)\varepsilon=\varepsilon$$

for all $x\notin\Sigma$. This means that $f-Pf$ vanishes at infinity, that is, $f-Pf\in B_0(X,D\otimes\mathbb{K})$. It follows that $Pf\in\bar{\mathfrak{c}}(X,D)$. This finishes the proof. □

We are interested in the K-theory of the stable Higson corona $\mathfrak{c}(X,D)$. We can identify $D\otimes\mathbb{K}$ with the subalgebra of constant functions in $\mathfrak{c}(X,D)$. It is often convenient to neglect the part of the K-theory that arises from this embedding. This is the purpose of the following definition.

Definition 3.8. Let X be an unbounded coarse space and let D be a C^*-algebra. The *reduced* K-theory of $\bar{\mathfrak{c}}(X,D)$ and $\mathfrak{c}(X,D)$ is defined by

$$\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X,D)) := \mathrm{K}_*(\bar{\mathfrak{c}}(X,D))/\mathrm{range}[\mathrm{K}_*(D\otimes\mathbb{K})\to\mathrm{K}_*(\bar{\mathfrak{c}}(X,D))],$$
$$\tilde{\mathrm{K}}_*(\mathfrak{c}(X,D)) := \mathrm{K}_*(\mathfrak{c}(X,D))/\mathrm{range}[\mathrm{K}_*(D\otimes\mathbb{K})\to\mathrm{K}_*(\mathfrak{c}(X,D))].$$

Remark 3.9. If X is a bounded coarse space, then $\bar{\mathfrak{c}}(X,D)=C(X,D\otimes\mathbb{K})$ and $\mathfrak{c}(X,D)=0$. In this case, the above definition of $\tilde{\mathrm{K}}_*(\mathfrak{c}(X,D))$ is not appropriate and many things obviously fail. In order to get true results in this trivial case as well, we should define $\tilde{\mathrm{K}}_*(\mathfrak{c}(X,D)) := \mathrm{K}_*(\mathfrak{c}^{\mathrm{red}}(X,D))$ using the C^*-algebra $\mathfrak{c}^{\mathrm{red}}(X,D)$ introduced in Definition 5.4 below.

Lemma 3.10. *Let X be an unbounded coarse space. Then the inclusions $D\otimes\mathbb{K}\to\bar{\mathfrak{c}}(X,D)$ and $D\otimes\mathbb{K}\to\bar{\mathfrak{c}}(X,D)\to\mathfrak{c}(X,D)$ induce injective maps in* K*-theory.*

Proof. Let $j\colon C_0(X,D\otimes\mathbb{K})\to\bar{\mathfrak{c}}(X,D)$ and $\bar{\iota}\colon D\otimes\mathbb{K}\to\bar{\mathfrak{c}}(X,D)$ be the inclusions, let $\pi\colon\bar{\mathfrak{c}}(X,D)\to\mathfrak{c}(X,D)$ be the quotient map, and let $\iota:=\pi\circ\bar{\iota}\colon D\otimes\mathbb{K}\to\mathfrak{c}(X,D)$. To see that $\bar{\iota}_*$ is injective, consider the evaluation map $\mathrm{ev}_x\colon\bar{\mathfrak{c}}(X,D)\to D\otimes\mathbb{K}$ for $x\in X$. This map splits $\bar{\iota}$, from which the assertion follows.

To check that ι_* is injective, choose $a\in\mathrm{K}_*(D)$ with $\pi_*\bar{\iota}_*(a)=\iota_*(a)=0$. Hence $\bar{\iota}_*(a)=j_*(b)$ for some $b\in\mathrm{K}_*(C_0(X,D\otimes\mathbb{K}))$ by the K-theory long exact sequence. Let $\mathrm{ev}^0_x\colon C_0(X,D\otimes\mathbb{K})\to D\otimes\mathbb{K}$ denote the restriction of the evaluation map to $C_0(X,D\otimes\mathbb{K})$.

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Since X is unbounded and K-theory is compactly supported, we have $(\mathrm{ev}_x^0)_*(b) = 0$ for all $x \in X$ outside some compact set. Then $a = (\mathrm{ev}_x)_*\bar{\iota}_*(a) = (\mathrm{ev}_x)_*j_*(b) = (\mathrm{ev}_x^0)_*(b) = 0$, which concludes the proof. □

Remark 3.11. For any unbounded coarse space X and any C^*-algebra D, consider the long exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(C_0(X)\otimes D) & \longrightarrow & \mathrm{K}_0(\bar{\mathfrak{c}}(X,D)) & \xrightarrow{\pi_*} & \mathrm{K}_0(\mathfrak{c}(X,D)) \\ \partial\uparrow & & & & \downarrow\partial \\ \mathrm{K}_1(\mathfrak{c}(X,D)) & \xleftarrow{\pi_*} & \mathrm{K}_1(\bar{\mathfrak{c}}(X,D)) & \longleftarrow & \mathrm{K}_1(C_0(X)\otimes D) \end{array}$$

associated to the exact sequence of C^*-algebras

$$0 \to C_0(X) \otimes D \otimes \mathbb{K} \to \bar{\mathfrak{c}}(X, D) \to \mathfrak{c}(X, D) \to 0.$$

By construction, the map $\iota_*\colon \mathrm{K}_*(D \otimes \mathbb{K}) \to \mathrm{K}_*(\mathfrak{c}(X, D))$ factors through the map $\pi_*\colon \mathrm{K}_*(\bar{\mathfrak{c}}(X, D)) \to \mathrm{K}_*(\mathfrak{c}(X, D))$, whence $\partial \circ \iota_* = 0$. Lemma 3.10 shows that we get a long exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(C_0(X)\otimes D) & \longrightarrow & \tilde{\mathrm{K}}_0(\bar{\mathfrak{c}}(X,D)) & \xrightarrow{\pi_*} & \tilde{\mathrm{K}}_0(\mathfrak{c}(X,D)) \\ \partial\uparrow & & & & \downarrow\partial \\ \tilde{\mathrm{K}}_1(\mathfrak{c}(X,D)) & \xleftarrow{\pi_*} & \tilde{\mathrm{K}}_1(\bar{\mathfrak{c}}(X,D)) & \longleftarrow & \mathrm{K}_1(C_0(X)\otimes D). \end{array}$$

Finally, we extend the above definitions to the case of σ-coarse spaces. This is necessary to construct the coarse co-assembly map in the next section.

Let $\mathcal{X} = \bigcup X_n$ be a σ-coarse space and let D be a C^*-algebra. We let

$$C_0(\mathcal{X}, D) := \{f\colon \mathcal{X} \to D \mid f|_{X_n} \in C_0(X_n, D) \text{ for all } n \in \mathbb{N}\},$$
$$\bar{\mathfrak{c}}(\mathcal{X}, D) := \{f\colon \mathcal{X} \to D \otimes \mathbb{K} \mid f|_{X_n} \in \bar{\mathfrak{c}}(X_n, D) \text{ for all } n \in \mathbb{N}\}.$$

Both $C_0(\mathcal{X}, D)$ and $\bar{\mathfrak{c}}(\mathcal{X}, D)$ are σ–C^*-algebras in the terminology of [**18**] with respect to the sequence of C^*-seminorms

$$\|f\|_n := \sup\{\|f(x)\| \mid x \in X_n\}.$$

We evidently have

$$C_0(\mathcal{X}, D) = \varprojlim C_0(X_n, D \otimes \mathbb{K}), \qquad \bar{\mathfrak{c}}(\mathcal{X}, D) = \varprojlim \bar{\mathfrak{c}}(X_n, D \otimes \mathbb{K}),$$

where $\varprojlim$ denotes the projective limit in the category of σ–C^*-algebras.

Recall that we assumed the maps $X_n \to X_{n+1}$ to be coarse equivalences. Proposition 3.7 implies that the induced maps $\mathfrak{c}(X_{n+1}, D) \to \mathfrak{c}(X_n, D)$ are $*$-isomorphisms. Hence the inverse limit

$$\mathfrak{c}(\mathcal{X}, D) := \varprojlim \mathfrak{c}(X_n, D)$$

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is again a C^*-algebra: it is isomorphic to $\mathfrak{c}(X_m, D)$ for any $m \in \mathbb{N}$. The following lemma asserts that we also have a natural isomorphism

$$\mathfrak{c}(\mathcal{X}, D) \cong \bar{\mathfrak{c}}(\mathcal{X}, D)/C_0(\mathcal{X}, D \otimes \mathbb{K}).$$

Lemma 3.12. *The sequence of σ–C^*-algebras*

$$0 \to C_0(\mathcal{X}, D) \to \bar{\mathfrak{c}}(\mathcal{X}, D) \to \mathfrak{c}(\mathcal{X}, D) \to 0$$

is exact.

Proof of Lemma 3.12. The maps $\alpha_n \colon C_0(X_{n+1}, D\otimes\mathbb{K}) \to C_0(X_n, D\otimes\mathbb{K})$ associated to the inclusions $X_n \subseteq X_{n+1}$ are clearly surjective. The maps $\gamma_n \colon \mathfrak{c}(X_{n+1}, D) \to \mathfrak{c}(X_n, D)$ are surjective because they are isomorphisms. The Snake Lemma of homological algebra provides us with a long exact sequence

$$\cdots \to \operatorname{Coker} \alpha_n \to \operatorname{Coker} \beta_n \to \operatorname{Coker} \gamma_n \to 0.$$

Hence the maps $\beta_n \colon \bar{\mathfrak{c}}(X_{n+1}, D) \to \bar{\mathfrak{c}}(X_n, D)$ are surjective as well. Now the assertion follows from the following lemma from [**18**]. □

Lemma 3.13 ([18]). *Suppose that $\alpha_n \colon A_{n+1} \to A_n$, $n \in \mathbb{N}$, is a projective system of C^*-algebras with surjective maps α_n for all n. Let J_n be ideals in A_n such that the restriction of α_n to J_{n+1} maps J_{n+1} surjectively onto J_n. Then*

$$0 \to \varprojlim J_n \to \varprojlim A_n \to \varprojlim A_n/J_n \to 0$$

is an exact sequence of σ–C^*-algebras.

4. The coarse co-assembly map

We first define the coarse K-theory of X with coefficients D, which is the target of the coarse co-assembly map. Its definition is based on the Rips complex construction of Example 2.5. We reformulate it in terms of entourages and check carefully that we obtain a σ-coarse space. We require the coarse structure to be countably generated. Otherwise the construction below gives an uncountable system of coarse spaces, which we prefer to avoid. We begin with the case where X is discrete.

We fix an increasing sequence (E_n) of entourages such that any entourage is contained in some E_n. We assume that $E_0 = \Delta_X$ is the diagonal. Let $\mathcal{P}_X$ be the set of probability measures on X with finite support. This is a simplicial complex whose vertices are the Dirac measures on X. We give it the corresponding topology. Hence locally finite subcomplexes of $\mathcal{P}_X$ are locally compact. Let

$$P_n := \{\mu \in \mathcal{P}_X \mid \operatorname{supp}\mu \times \operatorname{supp}\mu \subseteq E_n\}.$$

In particular, $P_0 \cong X$. We have $\bigcup P_n = \mathcal{P}_X$ because any finite subset of $X \times X$ is contained in E_n for some n. Each P_n is a locally finite subcomplex of $\mathcal{P}_X$ and hence a locally

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compact topological space because bounded subsets of X are finite. We give $\mathcal{P}_X$ and its subspaces P_n the coarse structure $\mathcal{E}_n$ that is generated by the sequence of entourages

$$\{(\mu,\nu) \mid \operatorname{supp}\mu \times \operatorname{supp}\nu \subset E_m\}, \quad m \in \mathbb{N}.$$

The embeddings $X \cong P_0 \to P_n$ are coarse equivalences for all $n \in \mathbb{N}$. Thus $\mathcal{P}_X$ is a σ-coarse space.

The K-theory of the σ–C^*-algebra $C_0(\mathcal{P}_X, D)$ is going to be the coarse K-theory of X. In order to extend this definition to non-discrete coarse spaces, we must show that it is functorial on the coarse category of coarse spaces.

We first observe that the σ–C^*-algebra $C_0(\mathcal{P}_X, D)$ does not depend on the choice of the generating sequence (E_n). A function $f\colon \mathcal{P}_X \to D$ belongs to $C_0(\mathcal{P}_X, D)$ if and only if its restriction to P_n is C_0 for all $n \in \mathbb{N}$. If $E' \subseteq X \times X$ is any entourage, then $E' \subseteq E_n$ for some $n \in \mathbb{N}$. If we define $P_{E'}(X) \subseteq \mathcal{P}_X$ in the evident fashion, we obtain a subcomplex of $P_n(X)$. Thus the restriction of f to $P_{E'}(X)$ is C_0 for all entourages E'. Conversely, this condition implies easily that $f \in C_0(\mathcal{P}_X, D)$. Hence we can describe $C_0(\mathcal{P}_X, D)$ without using the generating sequence E_n. Similar arguments apply to $\bar{\mathfrak{c}}(\mathcal{P}_X, D)$ and, of course, to $\mathfrak{c}(\mathcal{P}_X, D)$. Actually, up to an appropriate notion of isomorphism of inductive systems, the σ-coarse space $\mathcal{P}_X$ is independent of the choice of (E_n).

To discuss the functoriality of $\mathcal{P}_X$, we define morphisms between σ-coarse spaces. Let $\mathcal{X} = \bigcup X_n$ and $\mathcal{Y} = \bigcup Y_n$ be σ-coarse spaces. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map with the property that for any $m \in \mathbb{N}$ there is $n = n(m) \in \mathbb{N}$ with $f(X_m) \subseteq Y_n$. We say that f is Borel, continuous, or coarse, respectively, if the restrictions $f|_{X_m}\colon X_m \to Y_{n(m)}$ have this property for all $m \in \mathbb{N}$. Here the choice of $n(m)$ is irrelevant because Y_n is a subspace of $Y_{n'}$ for all $n \leqslant n'$. Similarly, two coarse maps $\mathcal{X} \to \mathcal{Y}$ are called close if their restrictions to X_m are close for all $m \in \mathbb{N}$. It is clear that $C_0(\mathcal{X}, D)$ and $\bar{\mathfrak{c}}(\mathcal{X}, D)$ are functorial for continuous coarse maps.

Lemma 4.1. *Let X and Y be discrete, countably generated coarse spaces. Then a coarse map $X \to Y$ induces a continuous coarse map $\mathcal{P}_X \to \mathcal{P}_Y$.*

Let $\mathcal{X}$ be a σ-coarse space and let $\phi_0, \phi_1\colon \mathcal{X} \to \mathcal{P}_Y$ be two continuous coarse maps that are close. Then there exists a homotopy $\Phi\colon \mathcal{X} \times [0,1] \to \mathcal{P}_Y$ between ϕ_0 and ϕ_1 that is close to the constant homotopy $(x,t) \mapsto \phi_0(x)$. The homotopy Φ induces a homotopy $C_0(\mathcal{P}_Y) \to C([0,1]) \otimes C_0(\mathcal{X})$.

Proof. A coarse map $\phi\colon X \to Y$ induces a map $\phi_*\colon \mathcal{P}_X \to \mathcal{P}_Y$ by pushing forward probability measures. It is easy to see that ϕ_* is continuous and coarse.

We want to define $\Phi(x,t) := (1-t)\phi_0(x) + t\phi_1(x)$. It is clear that $\Phi(x,t)$ is a probability measure on Y with finite support for all $(x,t) \in \mathcal{X} \times [0,1]$. We claim that Φ has the required properties. Fix $n \in \mathbb{N}$ and an entourage $E \subseteq X_n \times X_n$. Since ϕ and ϕ' are close, there is an entourage $E' \subseteq Y \times Y$ such that

$$\phi_0 \times \phi_1(E) \subseteq \{(\mu,\nu) \mid \operatorname{supp}\mu \times \operatorname{supp}\nu \subseteq E'\}.$$

Let $E'' := E' \cup (E' \circ (E')^{\mathrm{t}})$. Since $\operatorname{supp}\Phi(x,t) \subseteq \operatorname{supp}\phi_0(x) \cup \operatorname{supp}\phi_1(x)$, we obtain $\Phi(x,t) \in P_{E''}$ and $\operatorname{supp}\phi_0(x) \times \operatorname{supp}\Phi(x,t) \subseteq E''$ for all $x \in X_n$, $t \in [0,1]$. That is, Φ is a coarse map that is close to the map $(x,t) \mapsto \phi_0(x)$. Continuity is easy to check. We also get an induced homotopy for the associated σ–C^*-algebras because $C_0(\mathcal{X}, D) \otimes C([0,1]) \cong C_0(\mathcal{X} \times [0,1], D)$. □

Since K-theory for σ–C^*-algebras is still homotopy invariant, we obtain the following corollary.

Corollary 4.2. *The assignment $X \mapsto \mathrm{K}_*(C_0(\mathcal{P}_X, D))$ is a functor from the coarse category of discrete, countably generated coarse spaces to the category of $\mathbb{Z}/2$-graded Abelian groups.*

Definition 4.3. Let X be a countably generated coarse space and let D be a C^*-algebra. Let $Z \subseteq X$ be a countably generated, discrete coarse space that is coarsely equivalent to X. This exists by Lemma 2.4. We let

$$\mathrm{KX}^*(X, D) := \mathrm{K}_*(C_0(\mathcal{P}_Z, D))$$

and call this the *coarse* K*-theory of X with coefficients D*.

Note 4.4. When $D = \mathbb{C}$ is trivial we simply write $\mathrm{KX}^*(X) := \mathrm{KX}^*(X, \mathbb{C})$ and refer to this as the *coarse* K*-theory of X*.

By construction, the discrete coarse space Z is uniquely determined up to coarse equivalence. Hence Corollary 4.2 yields that $\mathrm{KX}^*(X, D)$ is independent of the choice of Z and is a functor from the coarse category of countably generated coarse spaces to the category of $\mathbb{Z}/2$-graded Abelian groups. Since the homotopy type of $\mathcal{P}_Z$ is independent of the choice of Z, we also write $\mathcal{P}_X$ for this space.

Remark 4.5. Phillips shows in [**18**] that K-theory for σ–C^*-algebras can be computed by a Milnor $\varprojlim^1$-sequence. In our case, we obtain a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{K}_{*+1}(C_0(P_n(Z), D)) \to \mathrm{KX}^*(X, D) \to \varprojlim \mathrm{K}_*(C_0(P_n(Z), D)) \to 0.$$

For $D = \mathbb{C}$, this becomes a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{K}^{*+1}(P_n(Z)) \to \mathrm{KX}^*(X) \to \varprojlim \mathrm{K}^*(P_n(Z)) \to 0.$$

We are now in a position to define our coarse co-assembly map. Let D be a C^*-algebra and let X be a coarse space. By Lemma 3.12 the sequence

$$0 \to C_0(\mathcal{P}_X, D \otimes \mathbb{K}) \to \bar{\mathfrak{c}}(\mathcal{P}_X, D) \to \mathfrak{c}(\mathcal{P}_X, D) \cong \mathfrak{c}(X, D) \to 0 \tag{4.1}$$

is exact. In [**18**] it is shown that an exact sequence of σ–C^*-algebras induces a long exact sequence in K-theory. As in Remark 3.11, one shows that this remains exact if we use reduced K-theory everywhere (and assume X to be unbounded).

Definition 4.6. Let X be a countably generated, unbounded coarse space and let D be a C^*-algebra. The *coarse co-assembly map for X with coefficients D* is the map

$$\mu^*_{X,D} \colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(X, D)) \to \mathrm{KX}^*(X, D)$$

that is obtained from the connecting map of the exact sequence (4.1).

We conclude this section by noting that the coarse K-theory of X is equal to the usual K-theory of X if X is a uniformly contractible metric space of bounded geometry. An analogous assertion holds for the coarse K-homology.

Definition 4.7. A metric space (X, d) is *uniformly contractible* if for every $R > 0$ there exists $S \geqslant R$ such that for any $x \in X$, the inclusion $B_R(x) \to B_S(x)$ is nullhomotopic.

Theorem 4.8. *Let X be a uniformly contractible metric space of bounded geometry. There exists a canonical isomorphism $\mathrm{KX}^*(X, D) \cong \mathrm{K}_*(C_0(X, D))$ making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{K}_{*+1}(\mathfrak{c}(X,D)) & \xrightarrow{\mu^*_{X,D}} & \mathrm{KX}^*(X,D) \\ & \searrow^{\partial} \quad \nearrow_{\cong} & \\ & \mathrm{K}_*(C_0(X,D)). & \end{array}$$

Here ∂ is the boundary map associated to the exact sequence of C^*-algebras

$$0 \to C_0(X, D \otimes \mathbb{K}) \to \bar{\mathfrak{c}}(X, D) \to \mathfrak{c}(X, D) \to 0.$$

The proof uses the following lemma.

Lemma 4.9. *Let X be a uniformly contractible metric space of bounded geometry, and let $\phi\colon X \to X$ be a continuous coarse map which is close to the identity map $X \to X$. Then ϕ and* id *are homotopic, and the homotopy $F\colon X \times [0, 1] \to X$ can be chosen to be close to the coordinate projection $(x, t) \mapsto x$.*

Proof. Choose a uniformly bounded open cover $(U_i)_{i\in I}$ of X. For $\Sigma \subseteq I$ let $U_\Sigma = \bigcap_{i\in\Sigma} U_i$ and let Δ_Σ be the simplex

$$\Delta_\Sigma := \Big\{ (x_i) \in [0, 1]^\Sigma \;\Big|\; \sum x_i = 1 \Big\}.$$

By the bounded geometry assumption we may choose the cover so that $U_\Sigma = \emptyset$ whenever $|\Sigma| > N$ for some $N \in \mathbb{N}$. By induction on $n = |\Sigma| \geqslant 1$, we construct continuous maps $H_\Sigma\colon U_\Sigma \times \Delta_\Sigma \times [0, 1] \to X$ such that

(1) $H_\Sigma(x, p, 0) = x$ and $H_\Sigma(x, p, 1) = \phi(x)$ for all $x \in U_\Sigma$, $p \in \Delta_\Sigma$;

(2) $H_\Sigma(x, p, t) = H_{\mathrm{supp}(p)}(x, p, t)$ for all $x \in U_\Sigma$, $p \in \partial\Delta_\Sigma$, $t \in [0, 1]$;

(3) $d(H_\Sigma(x, p, t), x) \leqslant C_n$ for all x, p, t for some constant $C_n \geqslant 0$.

If $|\Sigma| = 1$, then $\Sigma = \{i\}$ for some $i \in I$, $U_\Sigma = U_i$ and Δ_Σ is a point, denote it $\star$. The map H_Σ must satisfy $H_\Sigma(x, \star, 0) = x$ and $H_\Sigma(x, \star, 1) = \phi(x)$ for all $x \in U_i$. By the uniform contractibility, we can extend this to a continuous map on $U_i \times [0, 1]$ with the required properties. Now assume that H_Σ has been defined for $|\Sigma| < n$ and take $\Sigma \subseteq I$ with $|\Sigma| = n$. The previous induction step and our requirements determine H_Σ on $(U_\Sigma \times \partial\Delta_\Sigma \times [0, 1]) \cup (U_\Sigma \times \Delta_\Sigma \times \partial[0, 1])$. The uniform contractibility assumption of X allows us to extend this to $U_\Sigma \times \Delta_\Sigma \times [0, 1]$ as required.

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Finally, we choose a partition of unity $(\rho_i)_{i\in I}$ subordinate to the cover $(U_i)_{i\in I}$ and define $F\colon X\times[0,1]\to X$ as follows. For $x\in X$, let $\Sigma := \{i\in I \mid \rho_i(x)\neq 0\}$ and $F(x,t) := H_\Sigma(x,(\rho_i(x))_{i\in\Sigma},t)$. This defines a continuous map that is close to the identity map because $U_\Sigma=\emptyset$ for $|\Sigma|>N$. □

Proof of Theorem 4.8. Let $Z\subseteq X$ be a discrete subspace coarsely equivalent to X as in Lemma 2.4. For sufficiently large r, the balls of radius r centred at the points of Z cover X. Let $P_r(Z)$ be the Rips complex with parameter r as in Example 2.5. The natural maps $Z\to X$ and $Z\to P_r(Z)$ are coarse equivalences. Hence we obtain canonical coarse equivalences $X\to P_r(Z)$ and $P_r(Z)\to X$. We want to represent these morphisms in the coarse category by continuous coarse maps $F\colon X\to P_r(Z)$ and $G\colon P_r(Z)\to X$.

Let (ρ_i) be a partition of unity subordinate to the cover of X by r-balls centred at the points of Z. Choose $x_i\in Z$ close to $\operatorname{supp}\rho_i$ and define

$$F\colon X\to P_r(Z)\subseteq \mathcal{P}_Z,\qquad F(x):=\sum_i \rho_i(x)\delta_{x_i}.$$

This is a continuous coarse map, and its restriction to Z is close to the standard map $Z\to P_r(Z)$ as desired. Notice that this map exists for any countably generated coarse space X.

We define the maps $G\colon P_r(Z)\to X$ for any $r\geqslant 0$ by induction on skeletons. On the 0-skeleton Z, we let G be the inclusion $Z\to X$. Suppose that G has already been defined on the $(n-1)$-skeleton and let σ be an n-cell. Then the vertices of σ constitute a subset of $Z\subseteq X$ of diameter at most r. By our induction assumption, G maps the boundary of σ to a subset of X of diameter at most $C_{n-1}(r)$ for some constant depending only on r and $n-1$. By uniform contractibility, we can extend G to a map $\sigma\to X$ in such a way that $G(\sigma)$ has diameter at most $C_n(r)$ for some constant $C_n(r)$. Proceeding in this fashion, we construct a continuous coarse map $G\colon P_r(Z)\to X$ whose restriction to Z is the inclusion map.

The compositions $F\circ G$ and $G\circ F$ are continuous coarse maps which are close to the identity maps on $P_r(Z)$ and X, respectively. Lemmas 4.1 and 4.9 yield that $F\circ G$ and $G\circ F$ are homotopic to the identity maps on $\mathcal{P}_Z$ and X, respectively. Therefore, we get an isomorphism $\mathrm{K}_*(C_0(X,D))\cong \mathrm{K}_*(C_0(\mathcal{P}_Z,D))$ as desired. Since F and G extend to $*$-homomorphisms between $\bar{\mathfrak{c}}(X,D)$ and $\bar{\mathfrak{c}}(\mathcal{P}_X,D)$, the naturality of the boundary map in K-theory yields the commutative diagram in the statement of the theorem. □

5. A first vanishing theorem

We now calculate an example that illustrates the distinction between the stable and unstable Higson coronas. We begin by recalling the following result of [**6**].

Proposition 5.1. *Let $X=[0,\infty)$ be the ray with its (Euclidean) metric coarse structure. Then the reduced* K*-theory of the Higson compactification ηX of X is uncountable.*

This implies by the Five Lemma that the same is true for the Higson corona $\partial_\eta X$ of X. In contrast, we show that the reduced K-theory of the stable Higson corona of X is trivial. Since it involves no additional effort, we show the following more general result.

Theorem 5.2. *Let Y be an arbitrary coarse space and D a C^*-algebra. Let the ray $[0, \infty)$ be given its Euclidean coarse structure and let $X = Y \times [0, \infty)$ with the product coarse structure. Then $\tilde{\mathrm{K}}_*(\mathfrak{c}(X, D)) = 0$ for $* = 0, 1$.*

Remark 5.3. This result is consistent with the analogous assertion $\mathrm{K}_*(C^*(X)) = 0$ for the Roe C^*-algebras of such spaces.

For the purposes of this computation and for many others, it turns out to be much easier to work not with the reduced K-theory of the algebras $\bar{\mathfrak{c}}$ and $\mathfrak{c}$, but rather with the ordinary K-theory of modified (or reduced) versions of these algebras. Thus we introduce the following definition. If D is a C^*-algebra, we let $\mathcal{M}(D)$ be its multiplier algebra and $\mathcal{M}^s(D)$ be the multiplier algebra of $D \otimes \mathbb{K}$. We also define $\mathcal{Q}^s(D) := \mathcal{M}^s(D)/D \otimes \mathbb{K}$.

Definition 5.4. Let X be a coarse space and let D be a C^*-algebra. We let $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$ be the C^*-algebra of bounded continuous functions of vanishing variation $f\colon X \to \mathcal{M}^s(D)$ such that $f(x) - f(y) \in D \otimes \mathbb{K}$ for all $x, y \in X$. We let $\mathfrak{c}^{\mathrm{red}}(X, D)$ be the C^*-algebra $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)/C_0(X, D \otimes \mathbb{K})$.

Proposition 5.5. *For every unbounded metric space X and every C^*-algebra D, we have natural isomorphisms*

$$\mathrm{K}_*(\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)) \cong \tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D)),$$
$$\mathrm{K}_*(\mathfrak{c}^{\mathrm{red}}(X, D)) \cong \tilde{\mathrm{K}}_*(\mathfrak{c}(X, D)).$$

Proof. Choose any point $x \in X$, and consider the composition

$$\bar{\mathfrak{c}}^{\mathrm{red}}(X, D) \to \mathcal{M}^s(D) \to \mathcal{Q}^s(D),$$

where the first map is evaluation at $x \in X$ and the second is the quotient map. This map is surjective and its kernel is the unreduced algebra $\bar{\mathfrak{c}}(X, D)$. Therefore, it descends to a map on the quotient $\mathfrak{c}^{\mathrm{red}}(X, D)$. Applying the Bott periodicity isomorphisms $\mathrm{K}_*(\mathcal{Q}^s(D)) \cong \mathrm{K}_{*+1}(D)$, we obtain a long exact sequence

$$\begin{array}{ccccc}
\mathrm{K}_0(\bar{\mathfrak{c}}(X, D)) & \longrightarrow & \mathrm{K}_0(\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)) & \longrightarrow & \mathrm{K}_1(D) \\
{\scriptstyle \bar{\iota}_*}\uparrow & & & & \downarrow{\scriptstyle \bar{\iota}_*} \\
\mathrm{K}_0(D) & \longleftarrow & \mathrm{K}_1(\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)) & \longleftarrow & \mathrm{K}_1(\bar{\mathfrak{c}}(X, D)).
\end{array}$$

One can show that the boundary maps are induced by the inclusion $\bar{\iota}\colon D \otimes \mathbb{K} \to \bar{\mathfrak{c}}(X, D)$. Since the vertical maps are injective by Lemma 3.10, we obtain two exact sequences

$$0 \to \mathrm{K}_*(D) \xrightarrow{\bar{\iota}_*} \mathrm{K}_*(\bar{\mathfrak{c}}(X, D)) \to \mathrm{K}_*(\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)) \to 0$$

for $* = 0, 1$. This proves the first assertion. The second one is proved in the same fashion. □

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Proof of Theorem 5.2. We may replace $Y \times [0, \infty)$ by the coarsely equivalent space $Y \times \mathbb{N}$. Thus we let $X := Y \times \mathbb{N}$ with its product coarse structure. By Lemma 5.5 it suffices to calculate the K-theory of the algebras $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$. Implicit in the definition of $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$ is a Hilbert space. Let us integrate this Hilbert space temporarily into our notation by denoting $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$, built on the Hilbert space V, by $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D, V)$.

Now fix a Hilbert space H, and let $\tilde{H} := H \oplus H \oplus \cdots$. The C^*-algebras $\bar{\mathfrak{c}}(X, D, H)$ and $\bar{\mathfrak{c}}(X, D, \tilde{H})$ are (non-canonically) isomorphic. The inclusion $H \to \tilde{H}$ as the nth summand induces a $*$-homomorphism $i_n \colon \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, H) \to \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, \tilde{H})$ of the form

$$i_n(f)(x) = 0 \oplus \cdots \oplus 0 \oplus f(x) \oplus 0 \oplus 0 \oplus \cdots \in \mathbb{B}(D \otimes \tilde{H})$$

for $n \in \mathbb{N}$. Standard arguments yield that the maps $(i_n)_*$ all induce isomorphisms on K-theory and that $(i_n)_* = (i_m)_*$ for any $n, m \in \mathbb{N}$. Define a $*$-homomorphism

$$S \colon \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, H) \to \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, H), \qquad Sf(y, n) := f(y, n+1).$$

The variation condition implies that $\|Sf(y, n) - f(y, n)\| = \|f(y, n+1) - f(y, n)\| \to 0$ for $(y, n) \to \infty$ for all $f \in \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, H)$. That is, $Sf - f \in C_0(X, D \otimes \mathbb{K})$. Hence S induces the identity map $\mathfrak{c}^{\mathrm{red}}(X, D, H) \to \mathfrak{c}^{\mathrm{red}}(X, D, H)$.

We claim that $\tilde{S}f := \bigoplus_{n=0}^{\infty} i_n(S^n f)$, that is,

$$\tilde{S}f(y, n) = f(y, n) \oplus f(y, n+1) \oplus f(y, n+2) \oplus \cdots$$

defines a $*$-homomorphism $\tilde{S} \colon \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, H) \to \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, \tilde{H})$. Since f is bounded, $\tilde{S}f(x)$ is a bounded operator for all $x \in X$. We claim that $\tilde{S}f(x) - \tilde{S}f(x')$ is compact for all $x = (y, n)$, $x' = (y', n')$ in $Y \times \mathbb{N}$. The kth direct summand of $\tilde{S}f(x) - \tilde{S}f(x')$ is given by $f(y, n+k) - f(y', n'+k)$ and hence lies in $D \otimes \mathbb{K}(H)$. The sequence $((y, n+k), (y', n'+k))_{k \in \mathbb{N}}$ in $X \times X$ lies in an entourage and converges to ∞. Hence the vanishing variation of f implies

$$\lim_{k \to \infty} \|f(y, n+k) - f(y', n'+k)\| = 0.$$

Consequently, $\tilde{S}f(x) - \tilde{S}f(x')$ is a compact operator as claimed. The same reasoning shows that $\tilde{S}f$ satisfies the variation condition. Thus $\tilde{S}f \in \bar{\mathfrak{c}}^{\mathrm{red}}(X, D, \tilde{H})$.

Now let $f \in \mathfrak{c}^{\mathrm{red}}(X, D, H)$ represent a class $[f]$ in either $\mathrm{K}_0(\mathfrak{c}^{\mathrm{red}}(X, D, H))$ or $\mathrm{K}_1(\mathfrak{c}^{\mathrm{red}}(X, D, H))$. We have to show that $[f] = 0$. Recall that S represents the identity map on $\mathfrak{c}^{\mathrm{red}}(X, D, H)$ and that $(i_n)_* = (i_m)_*$ for all n, m. Hence

$$[\tilde{S}f] = \left[\bigoplus_{n=0}^{\infty} i_n \circ S^n(f)\right] = (i_0)_*[f] + [\tilde{S}f].$$

Since $(i_0)_*$ is an isomorphism, we get $[f] = 0$ as desired. □

Remark 5.6. The proof of Theorem 5.2 exhibits the difference between $\mathfrak{c}(X)$ and $C(\eta X, \mathbb{K})$. If $f \colon \eta X \to \mathbb{K}$ is a continuous function, then f must satisfy the variation condition and, in addition, have compact range in $\mathbb{K}$. The function $\tilde{S}f$ need not have compact range and therefore can only be formed in the larger algebra $\mathfrak{c}(X)$.

Remark 5.7. Theorem 5.2, a Mayer–Vietoris argument and induction can be used to prove that the reduced K-theory for $\mathfrak{c}(\mathbb{R}^n)$ is given by

$$\tilde{\mathrm{K}}_i(\mathfrak{c}(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

We omit the argument because this also follows from our results on scalable spaces. This calculation shows clearly that the algebra $\mathfrak{c}(X)$ plays the role, at least K-theoretically, of a boundary of X.

6. Relationship with the coarse Baum–Connes assembly map

Let X be a coarse space and define $\mathcal{P}_X = \bigcup P_n$ as above. One can extend K-homology and even bivariant KK-theory from the category of C^*-algebras to the category of σ–C^*-algebras (see [**3**, **22**, **23**]). For K-homology, one obtains

$$\mathrm{K}^*(C_0(\mathcal{P}_X)) = \mathrm{K}^*(\varprojlim C_0(P_n)) \cong \varinjlim \mathrm{K}^*(C_0(P_n)) \cong \varinjlim \mathrm{K}_*(P_n).$$

The latter is, by definition, the *coarse* K*-homology of* X (see [**26**]). Thus we get a natural isomorphism

$$\mathrm{KX}_*(X) \cong \mathrm{K}^*(C_0(\mathcal{P}_X)).$$

The canonical pairing between the K-theory and K-homology for σ–C^*-algebras specializes to a natural pairing

$$\mathrm{KX}^*(X) \times \mathrm{KX}_*(X) \to \mathbb{Z}.$$

Let $C^*(X)$ be the C^*-algebra of the coarse space X (see [**13**, **26**]). The coarse Baum–Connes assembly map for X is a map

$$\mu\colon \mathrm{KX}_*(X) \to \mathrm{K}_*(C^*(X)).$$

The next theorem asserts that this map is dual to our coarse co-assembly map

$$\mu^*\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(X)) \to \mathrm{KX}^*(X).$$

Theorem 6.1. *Let X be an unbounded, countably generated coarse space X. Then there exists a natural pairing*

$$\tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(X)) \times \mathrm{K}_*(C^*(X)) \to \mathbb{Z}$$

compatible with the pairing $\mathrm{KX}^*(X) \times \mathrm{KX}_*(X) \to \mathbb{Z}$ *in the sense that*

$$\langle \mu(x), y\rangle = \langle x, \mu^*(y)\rangle \quad \text{for all } \; x \in \mathrm{KX}_*(X),\; y \in \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(X)).$$

Proof. Without loss of generality, we may assume that X is discrete. Let W be a separable Hilbert space, and form the ample Hilbert space $H_X = \ell^2(X) \otimes W$ over X (see [**13**]). We use H_X to construct $C^*(X)$. Thus $C^*(X)$ becomes the C^*-subalgebra of $\mathbb{B}(H_X)$ generated by the $*$-algebra of locally compact finite propagation operators

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on H_X. Let V be another separable Hilbert space and let $\mathbb{K} \cong \mathbb{K}(V)$ be represented on V in the obvious way. Let $(e_x)_{x\in X}$ be the canonical basis of $\ell^2(X)$. We represent $\bar{\mathfrak{c}}^{\mathrm{red}}(X)$ on $H_X \otimes V$ by the map $f \mapsto M_f$ with

$$M_f(e_x \otimes w \otimes v) = e_x \otimes w \otimes f(x)v,$$

for all $f \in \bar{\mathfrak{c}}^{\mathrm{red}}(X)$, viewed as a function $X \to \mathbb{K}(V)$. Represent $C^*(X)$ on $H_X \otimes V$ by $T \mapsto T \otimes 1_V$. The variation condition on $f \in \bar{\mathfrak{c}}^{\mathrm{red}}(X)$ and the definition of finite propagation imply easily that the commutator $[M_f, T\otimes 1_V]$ is compact for all $f \in \bar{\mathfrak{c}}^{\mathrm{red}}(X)$ and $T \in C^*(X)$. Hence we have defined a $*$-homomorphism from $\bar{\mathfrak{c}}^{\mathrm{red}}(X)$ into $\mathfrak{D}(C^*(X))$ in the notation of [**13**]. If $f \in C_0(X, \mathbb{K})$, then both $M_f \cdot (T \otimes 1_V)$ and $(T \otimes 1_V) \cdot M_f$ are compact. That is, $C_0(X, \mathbb{K})$ is mapped to $\mathfrak{D}(C^*(X)//C^*(X))$. Hence $\bar{\mathfrak{c}}^{\mathrm{red}}(X)$ is mapped to the relative dual

$$\mathfrak{D}_{\mathrm{red}}(C^*(X)) := \mathfrak{D}(C^*(X))/\mathfrak{D}(C^*(X)//C^*(X))$$

and we obtain a map $\mathrm{K}_*(\mathfrak{c}^{\mathrm{red}}(X)) \to \mathrm{K}_*(\mathfrak{D}_{\mathrm{red}}(C^*(X)))$. For every C^*-algebra A regardless of separability there is a canonical index pairing

$$\mathrm{K}_{*+1}(A) \times \mathrm{K}_*(\mathfrak{D}_{\mathrm{red}}(A)) \to \mathbb{Z}.$$

Since $\tilde{\mathrm{K}}(\mathfrak{c}(X)) \cong \mathrm{K}(\mathfrak{c}^{\mathrm{red}}(X))$, we obtain the required pairing.

We omit the details of the proof that μ and μ^* are compatible. First, one shows that it suffices to look at a fixed parameter in the Rips complex construction. The result then follows from the definition of μ given in terms of dual algebras (see [**13**]), our definition of μ^*, and the axioms for a Kasparov product. ☐

Corollary 6.2. *Let X be a uniformly contractible metric space of bounded geometry, endowed with the metric coarse structure. If the coarse co-assembly map for X is surjective, then the coarse assembly map is rationally injective.*

Proof. In this case, we can use X itself instead of the Rips complex by Theorem 4.8. Hence the pairing between $\mathrm{KX}_*(X) \otimes \mathbb{Q}$ and $\mathrm{KX}^*(X) \otimes \mathbb{Q}$ is non-degenerate. ☐

In particular, let X be the universal cover of a compact aspherical spin manifold. The surjectivity of μ^* for X implies rational injectivity of μ for X. This in turn implies that the manifold does not admit a metric of positive scalar curvature.

A natural question is whether or not rational surjectivity of μ_X can be detected by injectivity, or even bijectivity, of μ_X^*. There seems, however, little hope for this, as the following example illustrates. Let $X_1, X_2, \dots$ be a sequence of finite metric spaces and let $X = \bigsqcup X_n$ be their *coarse (uniform) disjoint union*, whose coarse structure is generated by entourages of the form $\bigsqcup_{i=1}^{\infty} E_{R,i}$, where $E_{R,i}$ is the entourage of diameter R in X_i.

Proposition 6.3. *Let (X_n) be a sequence of finite metric spaces and let $X = \bigsqcup X_n$ be the coarse disjoint union as above. Then the pairing*

$$\mathrm{K}_*(C^*(X)) \times \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(X)) \to \mathbb{Z}$$

is the zero map.

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That is, it is impossible in this example to detect elements of $\mathrm{K}_*(C^*(X))$ by pairing them with the K-theory of the stable Higson corona.

Proof. We use the ample Hilbert space

$$H_X = \ell^2(X) \otimes W \cong \bigoplus l^2(X_n) \otimes W$$

to realize $C^*(X)$. Let T be a finite propagation operator on X. Then T can be represented by a block diagonal operator $T = S \oplus T_N \oplus T_{N+1} \oplus \cdots$ with operators T_i on X_i of uniform finite propagation and an operator S that is supported on the bounded set $\bigcup_{i=1}^N X_i$. Note that S and the operators T_i are compact. Since the sum of $\mathbb{K}(H_X)$ and $\prod \mathbb{K}(H_{X_n})$ is a C^*-algebra, it contains $C^*(X)$. Thus any element α of $\mathrm{K}_0(C^*(X))$ is represented by a block diagonal projection P on H_X with compact blocks. To construct the pairing between $\mathrm{K}_0(C^*(X))$ and $\tilde{\mathrm{K}}_1(\mathfrak{c}(X))$, one forms the Hilbert space

$$H_X \otimes V \cong \bigoplus l^2(X_n) \otimes W \otimes V$$

as before.

Let $f \in \mathfrak{c}^{\mathrm{red}}(X)$ be a unitary element representing an element of $\tilde{\mathrm{K}}_1(\mathfrak{c}(X))$ and let $\bar{f}$ be a lifting of f to an element of $\bar{\mathfrak{c}}^{\mathrm{red}}(X)$. Hence $\bar{f}\bar{f}^* - 1$ and $\bar{f}^*\bar{f} - 1$ lie in $C_0(X, \mathbb{K})$. The pairing $\langle \alpha, [f] \rangle$ is given by the index of the Fredholm operator $PM_{\bar{f}}P + 1 - P$. This operator is also block diagonal, and its blocks are compact perturbations of 1. Hence the index vanishes. The same argument works for the pairing between $\mathrm{K}_1(C^*(X))$ and $\tilde{\mathrm{K}}_0(\mathfrak{c}(X))$. $\square$

Remark 6.4. Let E be a coarse disjoint union of graphs E_n. Let λ_n denote the lowest non-zero eigenvalue of the Laplacian on E_n. Assume that there exists a constant $c > 0$ such that $\lambda_n \geqslant c$ for all n. Thus the sequence $\{E_n\}$ is an *expanding sequence of graphs*. The coarse space E provides a counterexample to the coarse Baum–Connes conjecture (see [**14**]). Let P be the spectral projection for the Laplacian, which has been shown to not be in the range of the coarse Baum–Connes assembly map. The above argument shows that the class of $[P]$ in $\mathrm{K}_0(C^*(E))$ pairs trivially with $\tilde{\mathrm{K}}_0(\mathfrak{c}(E))$. More generally, if X is a coarse space, $i\colon E \to X$ is a coarse embedding and θ is any class in $\tilde{\mathrm{K}}_*(\mathfrak{c}(X))$, then $\langle i_*[P], \theta \rangle = \langle [P], i^*(\theta) \rangle = 0$ by functoriality of the pairings and by our discussion above. Hence such counterexamples cannot be detected by pairing with the stable Higson corona. In particular, this discussion applies to the groups containing an expanding sequence of graphs constructed by Gromov in [**8**].

Example 6.5. Consider now the special case $X = \bigsqcup X_n$ where each X_n is just a point (the *well-spaced ray*). Again the pairing between $\mathrm{K}_*(C^*(X))$ and $\tilde{\mathrm{K}}_*(\mathfrak{c}(X))$ vanishes. In this case, both the ordinary coarse assembly map μ and the coarse co-assembly map μ^* are isomorphisms. The assembly map is treated in [**24**], the co-assembly map can be treated easily using $\bar{\mathfrak{c}}(X, D) = \ell^\infty(\mathbb{N}, D \otimes \mathbb{K})$.

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7. Homotopy invariance in the coefficient algebra

In order to prove that the coarse co-assembly map $\mu^*_{X,D}$ is an isomorphism for scalable spaces, we investigate the properties of $\tilde{\mathrm{K}}_*(\mathfrak{c}(X,D))$ as a functor of D. A $*$-homomorphism $f\colon D_1 \to D_2$ induces compatible $*$-homomorphisms

$$\begin{aligned} f\otimes 1\colon C_0(X, D_1\otimes \mathbb{K}) &\to C_0(X, D_2\otimes \mathbb{K}),\\ \bar{f}_X\colon \bar{\mathfrak{c}}(X,D_1) &\to \bar{\mathfrak{c}}(X,D_2),\\ f_X\colon \mathfrak{c}(X,D_1) &\to \mathfrak{c}(X,D_2). \end{aligned}$$

Thus the assignments $D\mapsto \bar{\mathfrak{c}}(X,D), \mathfrak{c}(X,D)$ are functors, which we denote by $\bar{\mathfrak{c}}(X,\cdot)$ and $\mathfrak{c}(X,\cdot)$, respectively. We also have functors $\bar{\mathfrak{c}}^{\mathrm{red}}(X,\cdot)$ and $\mathfrak{c}^{\mathrm{red}}(X,\cdot)$.

Proposition 7.1. *The functors* $\mathrm{K}\circ\bar{\mathfrak{c}}(X,\cdot)$, $\mathrm{K}\circ\mathfrak{c}(X,\cdot)$, $\mathrm{K}\circ\mathfrak{c}^{\mathrm{red}}(X,\cdot)$ *and* $\mathrm{K}\circ\bar{\mathfrak{c}}^{\mathrm{red}}(X,\cdot)$ *are stable, split exact functors from the category of C^*-algebras and C^*-algebra homomorphisms to the category of Abelian groups and Abelian group homomorphisms; here* K *denotes the* K*-theory functor.*

Proof. If f is a completely bounded linear map $D_1\to D_2$, then it induces completely bounded linear maps $\bar{f}_X$ and f_X as above. Hence the functors $\bar{\mathfrak{c}}$ and $\mathfrak{c}$ map extensions of C^*-algebras with completely bounded sections again to such extensions. This yields the split exactness of $\mathrm{K}\circ\bar{\mathfrak{c}}(X,\cdot)$ and $\mathrm{K}\circ\mathfrak{c}(X,\cdot)$. The split exactness of $\mathrm{K}\circ\bar{\mathfrak{c}}^{\mathrm{red}}(X,\cdot)$ and $\mathrm{K}\circ\mathfrak{c}^{\mathrm{red}}(X,\cdot)$ now follows from Lemma 3.10. □

Now we use a deep result of Higson [**10**].

Theorem 7.2 (Higson). *Let F be a functor from the category of C^*-algebras to the category of Abelian groups that is split exact and stable. Then F is homotopy invariant. Moreover, F gives rise to a functor on the Kasparov category* KK*, that is, if D_1, D_2 are separable C^*-algebras, then the functoriality of F can be extended to natural maps* $\mathrm{KK}(D_1,D_2)\otimes F(D_1)\to F(D_2)$.

Proposition 7.1 allows us to apply this theorem to our functors $\tilde{\mathrm{K}}\circ\mathfrak{c}(X,\cdot) = \mathrm{K}\circ\mathfrak{c}^{\mathrm{red}}(X,\cdot)$, $\tilde{\mathrm{K}}\circ\bar{\mathfrak{c}}(X,\cdot) = \mathrm{K}\circ\bar{\mathfrak{c}}^{\mathrm{red}}(X,\cdot)$, $\mathrm{K}\circ\mathfrak{c}(X,\cdot)$ and $\mathrm{K}\circ\bar{\mathfrak{c}}(X,\cdot)$.

Theorem 7.3. *The functors* $\tilde{\mathrm{K}}\circ\mathfrak{c}(X,\cdot)$, $\tilde{\mathrm{K}}\circ\bar{\mathfrak{c}}(X,\cdot)$, $\mathrm{K}\circ\mathfrak{c}(X,\cdot)$ *and* $\mathrm{K}\circ\bar{\mathfrak{c}}(X,\cdot)$ *are homotopy invariant. They descend to the Kasparov category* KK.

8. Maps of weakly vanishing variation and scalable spaces

In this section we apply the above ideas to prove that the map $\mu^*_{X,D}$ is an isomorphism for scalable, uniformly contractible metric spaces. In fact we prove that $\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X,D)) = 0$ for any scalable space X and any C^*-algebra D. We begin by discussing a type of functoriality of the algebras $\bar{\mathfrak{c}}(X,D)$ in the X-variable.

Definition 8.1. Let X be a coarse space, let Y be a coarse metric space, and let $f\colon X \to Y$ be a continuous map. We say that f has *weakly vanishing variation*, or is WVV, if f maps entourages to entourages and if

$$f^{-1}(K) \cap (\operatorname{Var}_E f)^{-1}([\varepsilon, \infty))$$

is bounded for all $\varepsilon > 0$, all entourages $E \subseteq X \times X$ and all bounded subsets $K \subseteq Y$.

This allows us to treat coarse maps and maps of vanishing variation simultaneously because of the following obvious lemma.

Lemma 8.2. *Let X, Y and f be as in Definition 8.1. If f is coarse or if f has vanishing variation, then f has weakly vanishing variation.*

Lemma 8.3. *Let X be a coarse space, Y a coarse metric space, and Z a metric space, and let $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ be continuous maps. If ψ has vanishing variation and ϕ has weakly vanishing variation, then $\psi \circ \phi\colon X \to Z$ has vanishing variation.*

Proof. Let $\varepsilon > 0$ and let E be an entourage in X. We must show that there exists a bounded subset $K \subseteq X$ such that $\operatorname{Var}_E(\psi \circ \phi)(x) < \varepsilon$ for $x \in X \setminus K$. Since $\phi(E)$ is an entourage in Y, there is $R > 0$ with $d(\phi(x), \phi(x')) < R$ for $(x, x') \in E$. Choose a bounded subset $L \subseteq Y$ such that $\operatorname{Var}_R \psi(y) < \varepsilon$ for $y \in Y \setminus L$. Since ψ has vanishing variation and is continuous, it is uniformly continuous. Hence there is $\delta > 0$ such $d(\psi(y), \psi(y')) < \varepsilon$ whenever $d(y, y') < \delta$. Now let $K := \phi^{-1}(L) \cap (\operatorname{Var}_E \phi)^{-1}([\delta, \infty))$. Then K is bounded since ϕ has weakly vanishing variation. Choose $(x, x') \in E$ with $x \in X \setminus K$. Then $\phi(x) \in Y \setminus L$ or $\operatorname{Var}_E \phi(x) < \delta$. Suppose that $\phi(x) \in Y \setminus L$. Since also $d(\phi(x), \phi(x')) < R$, we get $d(\psi\phi(x), \psi\phi(x')) < \varepsilon$ by choice of L. If $\operatorname{Var}_E \phi(x) < \delta$, then $d(\phi(x), \phi(x')) < \delta$ and thus $d(\psi\phi(x), \psi\phi(x')) < \varepsilon$. That is, $\psi\phi$ has vanishing variation. □

Corollary 8.4. *Let X be a coarse space and let Y be a coarse metric space. Then a continuous map $\phi\colon X \to Y$ of weakly vanishing variation induces $*$-homomorphisms $\bar{\mathfrak{c}}(Y, D) \to \bar{\mathfrak{c}}(X, D)$ and $\bar{\mathfrak{c}}^{\mathrm{red}}(Y, D) \to \bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$ by the formula $f \mapsto f \circ \phi$.*

Hence $\mathrm{K}(\bar{\mathfrak{c}}(\,\cdot\,, D))$ and $\tilde{\mathrm{K}}(\bar{\mathfrak{c}}(\,\cdot\,, D))$ are functorial for WVV maps. However, since WVV maps need not be proper, they need *not* act on $\mathrm{K}(C_0(\,\cdot\,, D))$, $\mathrm{K}(\mathfrak{c}(\,\cdot\,, D))$ or $\mathrm{K}(\mathfrak{c}^{\mathrm{red}}(\,\cdot\,, D))$.

Let X be a coarse space, let Y be a coarse metric space, and let $\phi, \phi'\colon X \to Y$ be WVV maps. Let $\Phi\colon X \times [0,1] \to Y$ be a continuous map with $\Phi(x, 0) = \phi(x)$ and $\Phi(x, 1) = \phi'(x)$ for all $x \in X$. For an entourage E in X, we define

$$\operatorname{Var}^1_E \Phi\colon X \times [0,1] \to [0, \infty), \qquad (x, t) \mapsto \sup\{d(\Phi(x, t), \Phi(y, t)) \mid (x, y) \in E\}.$$

Definition 8.5. We say that $\Phi\colon X \times [0,1] \to Y$ is a *WVV homotopy* between ϕ and ϕ' and call ϕ and ϕ' *WVV homotopic* if the function Var^1_E is bounded for any entourage E and the set $\Phi^{-1}(K) \cap (\operatorname{Var}^1_E \Phi)^{-1}([\varepsilon, \infty))$ is bounded in $X \times [0,1]$ for all $\varepsilon > 0$, all entourages $E \subseteq X \times X$ and all bounded subsets $K \subseteq Y$. We call Y *WVV contractible* if the identity on Y is WVV homotopic to a constant map.

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Proposition 8.6. *Let X and Y be as above, let $\Phi\colon X\times[0,1]\to Y$ be a WVV homotopy and let D be a C^*-algebra. Then $f\mapsto f\circ\Phi$ defines $*$-homomorphisms*

$$\bar{\mathfrak{c}}(Y,D)\to\bar{\mathfrak{c}}(X,D\otimes I)\quad\text{and}\quad\bar{\mathfrak{c}}^{\mathrm{red}}(Y,D)\to\bar{\mathfrak{c}}^{\mathrm{red}}(X,D\otimes I).$$

Proof. Continuity of Φ and f imply immediately that $t\mapsto f(\Phi(x,t))$ indeed lies in $D\otimes I=C([0,1],D)$ for every $x\in X$ and that the map $f\circ\Phi\colon X\to D\otimes I$ is continuous. It is evidently bounded. It remains to check the variation condition. Choose an entourage E in X and $\varepsilon>0$. We have to find a bounded subset $K\subseteq X$ such that $\|f(\Phi(x,t))-f(\Phi(x',t))\|<\varepsilon$ for $(x,x')\in E$, $t\in[0,1]$ with $x\notin K$.

Since continuous vanishing variation functions are uniformly continuous, we can find $\delta>0$ such that $\|f(y)-f(y')\|<\varepsilon$ if $d(y,y')<\delta$. Since Var^1_E is bounded, there is $R>0$ such that $d(\Phi(x,t),\Phi(x',t))\leqslant R$ for all $(x,x')\in E$, $t\in[0,1]$. Since f has vanishing variation, we can find a bounded subset $L\subseteq Y$ such that $\mathrm{Var}_R f(y)<\varepsilon$ for $y\in Y\setminus L$. Let $K\subseteq X$ denote the projection to X of the bounded subset

$$\Phi^{-1}(L)\cap(\mathrm{Var}^1_E\Phi)^{-1}([\delta,\infty))\subset X\times[0,1].$$

If $x\in X\setminus K$, $t\in[0,1]$, then $\Phi(x,t)\in Y\setminus L$ or $\mathrm{Var}^1_E\Phi(x,t)<\delta$. Suppose first that $\Phi(x,t)\in Y\setminus L$. Since $(x,x')\in E$, we have $d(\Phi(x,t),\Phi(x',t))\leqslant R$. The choice of L yields $\|f(\Phi(x,t))-f(\Phi(x',t))\|<\varepsilon$ as desired. If $\mathrm{Var}^1_E\Phi(x,t)<\delta$, then $d(\Phi(x,t),\Phi(x',t))<\delta$ and hence $\|f(\Phi(x,t))-f(\Phi(x',t))\|<\varepsilon$ as well by the choice of δ. □

Remark 8.7. The notion of WVV homotopy is motivated by the following example. Let X be a complete Riemannian manifold of non-positive curvature. Fix a point $x_0\in X$ and let $\exp\colon T_{x_0}(X)\to X$ be the exponential map at x_0. It is well known that exp is a diffeomorphism satisfying $d(\exp v,\exp w)\geqslant d(v,w)$ for all tangent vectors $v,w\in T_{x_0}(X)$. Let log denote the inverse of exp. Define

$$\Phi\colon X\times[0,1]\to X,\qquad \Phi(x,t):=\exp(t\log x).$$

Then it is easy to check that Φ is a WVV homotopy between the identity map $X\to X$ and the constant map x_0. That is, X is WVV contractible.

More generally, we can make the following definition.

Definition 8.8. Let X be a coarse metric space. We call X *scalable* if there is a continuous map $r\colon X\times[0,1]\to X$, $(x,t)\mapsto r_t(x)$ such that

(1) $r_1(x)=x$;

(2) the map $X\times[\varepsilon,1]\to X$, $(x,t)\mapsto r_t(x)$ is proper for all $\varepsilon>0$;

(3) the maps r_t are uniformly Lipschitz and satisfy

$$\lim_{t\to 0}\sup_{x\neq x'\in X}\frac{d(r_t(x),r_t(x'))}{d(x,x')}=0.$$

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The following proposition follows immediately from the definition.

Proposition 8.9. *Scalable spaces are WVV contractible.*

Corollary 8.10. *If X is a scalable space, then $\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D)) = 0$ for every C^*-algebra D. If X is also uniformly contractible and has bounded geometry, then μ^* is an isomorphism.*

Proof. Since X is scalable, the identity map and a constant map on X are WVV homotopic. Proposition 8.6 and Theorem 7.3 yield that they induce the same map on $\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D))$. Since we divided out the contribution of the constant functions in the reduced K-theory, we get $\tilde{\mathrm{K}}_*(\bar{\mathfrak{c}}(X, D)) \cong 0$. By Theorem 4.8 and the K-theory long exact sequence, this is equivalent to μ^* being an isomorphism. □

We remark that Higson and Roe show in [**13**] that the coarse Baum–Connes conjecture is an isomorphism for scalable spaces.

Corollary 8.11. *Let π be the fundamental group of a compact manifold of non-positive curvature. Then the coarse co-assembly map $\mu^*_{\pi,D}\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(\pi, D)) \to \mathrm{KX}^*(\pi, D)$ is an isomorphism for every coefficient C^*-algebra D.*

Similar results hold for groups G that admit cocompact, isometric, proper actions on CAT(0) spaces.

9. Groups which uniformly embed in Hilbert space

We have introduced the co-assembly map in [**7**] because of its close relationship to the existence of a dual Dirac morphism in the group case. In this section, we use the notation of [**7**, **17**] concerning Dirac morphisms, dual Dirac morphisms and γ-elements. We give a few explanations in the proof of Proposition 9.9.

Theorem 9.1 ([7]). *If a discrete group G has a dual Dirac morphism, then*

$$\mu^*_{G,D}\colon \tilde{\mathrm{K}}_{*+1}(\mathfrak{c}(G, D)) \to \mathrm{KX}^*(G, D)$$

is an isomorphism for every C^*-algebra D.

We are going to use this fact to prove that the coarse co-assembly map is an isomorphism for groups that embed uniformly in a Hilbert space. This method only applies to coarse spaces that are quasi-isometric to a group. It would be nice to have a more direct proof that μ^* is an isomorphism that applies to all coarse spaces that uniformly embed.

Theorem 9.2. *Let G be a countable discrete group that embeds uniformly in a Hilbert space. Then G possesses a dual Dirac morphism. Hence the coarse co-assembly map for G is an isomorphism.*

Yu has shown the analogous result for the coarse assembly map for all coarse spaces with bounded geometry [**27**].

Remark 9.3. It has been pointed out to us that Skandalis and Tu are aware of Theorem 9.2; their work is independent of ours.

By a theorem of Higson, Guentner and Weinberger [**9**], every countable subgroup of either $\mathrm{GL}_n(k)$ for some field k or of an almost connected Lie group admits a uniform embedding in Hilbert space. Consequently, one obtains the following extension of Kasparov's results in [**15**].

Corollary 9.4. *If G is a countable subgroup either of $\mathrm{GL}_n(k)$ for some field k or of an almost connected Lie group, then G possesses a dual Dirac morphism.*

The proof of Theorem 9.2 is a consequence of various results of Higson, Skandalis, Tu and Yu (see [**12**, **20**]). Higson shows in [**12**] that if G is a discrete group admitting a topologically amenable action on a compact metrizable space, then the Novikov conjecture holds for G. But the argument manifestly also applies to the potentially larger class of groups admitting an a-T-menable action on a compact space.

Definition 9.5 (see [20]). Let G be a discrete group and let X be a compact G-space. We call the action of G on X *a-T-menable* if there exists a proper, continuous, real-valued function ψ on $X \times G$ satisfying

(1) $\psi(x, e) = 0$ for all $x \in X$;

(2) $\psi(x, g) = \psi(g^{-1}x, g^{-1})$ for all $x \in X$, $g \in G$;

(3) $\sum_{i,j=1}^{n} t_i t_j \psi(g_i^{-1}x, g_i^{-1}g_j) \leqslant 0$ for all $x \in X$, $t_1, \dots, t_n \in \mathbb{R}$, $g_1, \dots, g_n \in G$ for which $\sum t_i = 0$.

Such a function ψ is called a *negative type function*.

The existence of a negative type function implies that the groupoid $G \ltimes X$ possesses an affine isometric action on a continuous field of Hilbert spaces over X. The above definition is relevant because of the following results of [**20**, **21**].

Theorem 9.6 ([20]). *Let G be a discrete group. If G admits a uniform embedding in a Hilbert space, then G admits an a-T-menable action on a second countable compact space.*

Theorem 9.7 ([21]). *Let G be a discrete group and let X be a locally compact G-space. If G acts a-T-menably on X, then the transformation groupoid $G \ltimes X$ has a dual Dirac morphism and we have $\gamma = 1$.*

We are not going to need the fact that $\gamma = 1$.

If X is any compact space, let $\mathrm{Prob}(X)$ denote the collection of probability measures on X, equipped with the weak* topology. This is again a compact space. The space $\mathrm{Prob}(X)$ is convex and hence contractible. Furthermore, it is equivariantly contractible with respect to any action of a compact group on $\mathrm{Prob}(X)$.

Lemma 9.8 ([20]). *Let X be a compact, second countable space on which a discrete group G acts a-T-menably. Then G acts a-T-menably on $\mathrm{Prob}(X)$ as well.*

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Proposition 9.9. *Let G be a second countable locally compact group and let X be a second countable compact G-space. Suppose that X is H-equivariantly contractible for all compact subgroups $H \subseteq G$ and that the groupoid $G \ltimes X$ has a dual Dirac morphism. Then G has a dual Dirac morphism as well.*

Proof. Let $\mathsf{D} \in \mathrm{KK}^{G\ltimes X}(\mathsf{P}, C(X))$ be a *Dirac morphism* for $G \ltimes X$ in the sense of [**17**]. This means two things. First, D is a *weak equivalence*, that is, for any compact subgroup $H \subseteq G$, restriction to H maps D to an invertible morphism in $\mathrm{KK}^{H\ltimes X}(\mathsf{P}, C(X))$. Second, P belongs to the localizing subcategory of $\mathrm{KK}^{G\ltimes X}$ that is generated by compactly induced algebras. This subcategory contains all proper $G \ltimes X$–C^*-algebras. It is shown in [**17**] that a Dirac morphism for $G \ltimes X$ always exists. A *dual Dirac morphism* is an element $\eta \in \mathrm{KK}^{G\ltimes X}(C(X), \mathsf{P})$ such that $\eta \circ \mathsf{D} = \mathrm{id}_{\mathsf{P}}$. It is shown in [**17**, Theorem 8.2] that a dual Dirac morphism exists whenever the Dirac dual Dirac method in the sense of proper actions applies. In particular, it exists in the situation of Theorem 9.7.

Since X is compact, $C(X)$ contains the constant functions. This defines a G-equivariant $*$-homomorphism $j\colon \mathbb{C} \to C(X)$. The contractibility hypothesis on X insures that j is invertible in KK^H for any compact subgroup H. That is, j is a weak equivalence. Proposition 4.4 in [**17**] yields that j induces an isomorphism

$$\mathrm{KK}^G(\mathsf{P}, \mathbb{C}) \cong \mathrm{KK}^G(\mathsf{P}, C(X)).$$

Thus we obtain $\mathsf{D}' \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ with $j_*(\mathsf{D}') = F(\mathsf{D})$, where $F\colon \mathrm{KK}^{G\ltimes X} \to \mathrm{KK}^G$ is the functor that forgets the X-structure. It is clear that $F(\mathsf{D})$ is still a weak equivalence. Since both $F(\mathsf{D})$ and j are weak equivalences and $j \circ \mathsf{D}' = F(\mathsf{D})$, it follows that D' is a weak equivalence. This is a general fact about localization of triangulated categories. Thus $\mathsf{D}' \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ is a Dirac morphism for G. Now let $\eta' := F(\eta) \circ j$. Then $\eta' \circ \mathsf{D}' = F(\eta \mathsf{D}) = \mathrm{id}_{\mathsf{P}}$ by construction. Thus η' is a dual Dirac morphism. □

Theorem 9.2 now follows by combining the above results. If G uniformly embeds in a Hilbert space, then it admits an a-T-menable action on a second countable compact space X by Theorem 9.6. Lemma 9.8 allows us to assume that X is H-equivariantly contractible for any compact subgroup $H \subseteq G$. The transformation group $G \ltimes X$ has a dual Dirac morphism by Theorem 9.7. Hence so has G by Proposition 9.9.

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