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ITERATION OF ANALYTIC MULTIFUNCTIONS

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Abstract. It is shown that iteration of analytic set-valued functions can be used to generate composite Julia sets in $\mathbf{C}^N$. Then it is shown that the composite Julia sets generated by a finite family of regular polynomial mappings of degree at least 2 in $\mathbf{C}^N$, depend analytically on the generating polynomials, in the sense of the theory of analytic set-valued functions. It is also proved that every pluriregular set can be approximated by composite Julia sets. Finally, iteration of infinitely many polynomial mappings is used to give examples of pluriregular sets which are not composite Julia sets and on which Markov's inequality fails.

§1. Introduction

Let $P : \mathbf{C}^N \to \mathbf{C}^N$ be a polynomial mapping of degree d and let $P = P_d + \cdots + P_0$ be the expansion of P into homogeneous polynomial mappings, where P_j is homogeneous of degree j. We say that P is *regular* if $P_d^{-1}(0) = \{0\}$. This class of mappings was originally studied in [5] (see also [8]) in connection with invariance of L-regularity of sets and recently has re-emerged in the context of complex dynamics.

The symbol $\mathcal{L} = \mathcal{L}(\mathbf{C}^N)$ denotes the *Lelong class* of plurisubharmonic functions: $u \in \mathcal{L}$ if and only if $u \in PSH(\mathbf{C}^N)$ and

$$\sup_{z\in\mathbf{C}^N} \big(u(z) - \log^+ \|z\|\big) < \infty.$$

If $E \subset \mathbf{C}^N$, we define its *pluricomplex Green function* V_E by the formula

$$V_E(z) = \sup\{u(z) \ : \ u \in \mathcal{L} \text{ and } u \le 0 \text{ on } E\}, \qquad z \in \mathbf{C}^N,$$

and we say that E is *L-regular* or *pluriregular* if V_E is continuous at every point of $\overline{E}$. In particular, if E is compact, pluriregularity of E implies that the function V_E is continuous. Furthermore, if E is compact, then both E and its polynomially convex hull $\hat{E}$ have the same pluricomplex Green

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function. (For more details see e.g. [8].) By $\mathcal{R}$ we will denote the family of all compact polynomially convex L-regular subsets of $\mathbf{C}^N$. It can be shown (see [9]), that the formula

$$\Gamma(E,F) = \max\{\|V_E\|_F, \|V_F\|_E\}, \qquad E,F \in \mathcal{R},$$

defines a metric on $\mathcal{R}$ and that this metric space is complete. (If ϕ is a complex-valued function on a set S, then the symbol $\|\phi\|_S$ denotes the supremum of $|\phi|$ on S.) Note that $\Gamma(E,F)$ is well defined for all pluriregular compact sets E, F and $\Gamma(E,F) = \Gamma(\hat{E},\hat{F})$.

Let $P_1,\dots,P_k$ be regular polynomial mappings of degree at least 2. We know that $P_j^{-1}(E) \in \mathcal{R}$ for any $E \in \mathcal{R}$ and that

$$V_{P_j^{-1}(E)} = \frac{1}{\deg P_j}(V_E \circ P_j), \qquad j = 1,\dots,k.$$

(See [5] or [8] for more information.) Consequently (see [9]), the mapping

$$E \mapsto \left(\bigcup_{j=1}^{k} P_j^{-1}(E)\right)^{\wedge},$$

where $\wedge$ denotes the polynomial hull, is a contraction of the complete space $(\mathcal{R},\Gamma)$. The unique fixed point of this contraction is denoted by $K^+[P_1,\dots,P_k]$ and is called the *composite Julia set* of the mappings $P_1,\dots,$ P_k.

The internal structure of the set $K^+[P_1,\dots,P_k]$ can be described in terms of orbits. Define

$$\Sigma_k = \{\sigma = (\sigma_1,\sigma_2,\dots) \ : \ \sigma_j \in \{1,\dots,k\}\}.$$

If $z \in \mathbf{C}^N$ and $\sigma \in \Sigma_k$, we define the σ*-orbit* of z as the sequence $(P_{\sigma_n} \circ \cdots \circ P_{\sigma_1})(z)$, where $n \geq 1$. Let $S_\sigma^+[P_1,\dots,P_k]$ be the set of all points in $\mathbf{C}^N$ whose σ-orbits are bounded and let

$$S^+[P_1,\dots,P_k] = \bigcup_{\sigma\in\Sigma_k} S_\sigma^+[P_1,\dots,P_k].$$

It is not difficult to check that the set $S^+[P_1,\dots,P_k]$ is compact (see [12]). It was shown in [10] that the composite Julia set $K^+[P_1,\dots,P_k]$ is equal to the polynomially convex hull of $S^+[P_1,\dots,P_k]$. In the case of a single

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regular mapping P, the set $K^+[P] = S^+[P]$ is simply the filled-in Julia set of P.

In fact, the above results remain true for a more general class of proper polynomial mappings (see [10] for details) and, as we will show below, they can be generalized in a satisfactory way to the case of set-valued analytic mappings.

If (X,d) is a metric space then the symbol $\mathcal{K}(X)$ will denote the set of all (non-empty) compact subsets of X endowed with the Hausdorff metric

$$\chi_d(E,F) = \max\{\|\delta_E\|_F, \|\delta_F\|_E\},$$

where $\delta_L(x) = \operatorname{dist}(x,L)$ for $x \in X$ and $L \in \mathcal{K}(X)$. Let Ω_1 and Ω_2 be open subsets of $\mathbf{C}^{N_1}$ and $\mathbf{C}^{N_2}$, respectively. According to Słodkowski [21] a mapping $f : \Omega_1 \to \mathcal{K}(\Omega_2)$ is said to be an *analytic multifunction* or an *analytic set-valued function* if the following two conditions are satisfied:

- f is upper semicontinuous, i.e., for any $z_0 \in \Omega_1$ the function $z \mapsto \|\delta_{f(z_0)}\|_{f(z)}$ is upper semicontinuous in the usual sense;
- if $z_0 \in \Omega_1$ and u is a plurisubharmonic function in a neighbourhood of the set $\{z_0\} \times f(z_0)$, then the function

$$z \mapsto \max u\,(\{z\} \times f(z))$$

is plurisubharmonic in a neighbourhood of z_0.

The family of all such functions will be denoted by $\mathcal{AMF}(\Omega_1,\Omega_2)$. We say that $f \in \mathcal{AMF}(\Omega_1,\Omega_2)$ is *open at a point* $a \in \Omega_1$ if $f(a)$ is contained in the interior of $f(U)$ for some neighbourhood U of a. By $|f|$ we will denote the non-negative plurisubharmonic function $z \mapsto \max\{|w| : w \in f(z)\}$. If $S \subset \Omega_1$, then $f(S) = \bigcup_{z\in S} f(z)$.

Note that a more restrictive definition of analyticity also exists [22] and multifunctions such as above are then called *weakly analytic*. (For further information about analytic multifunctions see also [17], [18], [23], [6], [7]).

If $u \in \mathcal{PSH}(\mathbf{C}^N)$, we define the *growth function* of u by the formula $\mathcal{M}(u,r) = \sup u\left(\bar{B}(0,r)\right)$ for all $r > 0$. (Here, and in other places, the symbol $\bar{B}(a,r)$ will denote the closed ball with centre at a and radius r.) The function $r \mapsto \mathcal{M}(u,e^r)$ is convex and increasing. We define the *order* $\varrho(u)$ of u as

$$\rho(u) = \limsup_{r\to\infty} \frac{\log \mathcal{M}(u^+,r)}{\log r}.$$

Because of Liouville's theorem the order of an entire plurisubharmonic function is always non-negative. Note that if $\alpha > 0$, then $\alpha\rho(e^u) = \rho(e^{\alpha u})$. Note also that if $\rho' > \rho(\exp(u))$, then

$$\sup_{z\in\mathbf{C}^N} \left(u(z) - \rho' \log^+ \|z\|\right) < \infty.$$

Consequently (see e.g. Proposition 5.2.1 in [8]), if $u \in \mathcal{PSH}(\mathbf{C}^N)$ and $\rho = \rho(\exp(u)) > 0$, then $\rho^{-1}u \in \mathcal{L}$.

Observe that if $P_1, \dots, P_k$ are regular polynomial mappings, then the set-valued function

$$z \mapsto P_1^{-1}(z) \cup \dots \cup P_k^{-1}(z)$$

is analytic, open (at every point) and the order of its modulus is $1/\min\{d_1, \dots, d_k\}$. Just as in the case of regular mappings, if one wants to iterate analytic multifunctions one has to ask about the set-valued analytic functions with the property that images of pluriregular sets are pluriregular. The following result provides an answer to this question thus generalizing earlier results by Plesniak, Nguyen Thanh Van, Sadullaev and the author (see [14], [4], [5], [13], [19]). Recall that if $E \subset \Omega \subset \mathbf{C}^N$, where Ω is open, then the *relative extremal function* $u_{E,\Omega}$ is defined by the formula

$$u_{E,\Omega}(z) = \sup\{u(z) \,:\, u \in PSH(\Omega), u \leq -1 \text{ on } E, u \leq 0 \text{ in } \Omega\}$$

for all $z \in \Omega$ (see e.g. [8]).

THEOREM 1. *Let $f : \Omega_1 \to \mathcal{K}(\Omega_2)$ be an analytic multifunction, where $\Omega_1 \subset \mathbf{C}^{N_1}$ and $\Omega_2 \subset \mathbf{C}^{N_2}$ are open sets, and let $E \subset \Omega_1$.*

(i) *If f is open almost everywhere in Ω_1, then*

$$\max u^*_{f(E),\Omega_2}(f(z)) \leq u^*_{E,\Omega_1}(z), \qquad z \in \Omega_1. \tag{1}$$

(ii) *If $\Omega_1 = \Omega_2 = \mathbf{C}^N$ and $|f|$ is of finite order, then*

$$\max V_{f(E)}(f(z)) \leq \rho(|f|)V_E(z), \qquad z \in \mathbf{C}^N. \tag{2}$$

(iii) *If $\Omega_1 = \Omega_2 = \mathbf{C}^N$, $|f|$ is of finite order, f is open almost everywhere, and E is compact, then*

$$\max V^*_{f(E)}(f(z)) \leq \rho(|f|)V^*_E(z), \qquad z \in \mathbf{C}^N. \tag{3}$$

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The above result allows us to iterate analytic multifunctions almost exactly like in the case of regular polynomials. The necessary terminology and the main result (Theorem 4) are deferred to Section 3.

Apart from the set-valued functions given by finite families of proper polynomial mappings we can have more general analytic multifunctions that are relevant in this context. Let $\mathcal{P}_d^*$ denote the family of all regular polynomial mappings $P : \mathbf{C}^N \to \mathbf{C}^N$ of degree d. The family of all polynomial mappings $P : \mathbf{C}^N \to \mathbf{C}^N$ of degree not greater than d will be denoted by $\mathcal{P}_d$. If $\nu(d)$ denotes the number of coefficients of a polynomial mapping (from $\mathbf{C}^N$ to $\mathbf{C}^N$) of degree d, then $\mathcal{P}_d^*$ can be regarded as an open subset of $\mathbf{C}^{\nu(d)} = \mathcal{P}_d$. In what follows, the polynomial map P regarded as an element of $\mathbf{C}^{\nu(d)}$ will also be denoted by P — the appropriate meaning will be determined by the context. In Section 4 we show that composite Julia sets depend analytically on the generating polynomials. This, in particular, provides further examples of families of multifunctions of the type described in Theorem 4. We prove the following.

THEOREM 2. *If $d_1, \dots, d_k \geq 2$ are integers and $\sigma \in \Sigma_k$, then the set-valued functions*

$$(P_1, \dots, P_k) \mapsto S_\sigma^+[P_1, \dots, P_k]$$

and

$$(P_1, \dots, P_k) \mapsto K^+[P_1, \dots, P_k],$$

are analytic multifunctions in $\mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^*$. If $H_1 \in \mathcal{P}_{d_1}^*, \dots, H_k \in \mathcal{P}_{d_k}^*$ are homogeneous, then the plurisubharmonic function

$$(P_1, \dots, P_k) \mapsto \sup \left\{ \|z\| \ : \ z \in K^+[H_1 + P_1, \dots, H_k + P_k] \right\}, \tag{4}$$

defined on $\mathcal{P}_{d_1-1} \times \cdots \times \mathcal{P}_{d_k-1}$, is of order one.

When $n = 1$ and $k = 1$ the analyticity of $P \mapsto K^+[P]$ follows from a result of Baribeau and Ransford [1] combined with the upper semicontinuity property shown by Douady [2].

In Section 5 we show that every pluriregular set can be approximated by composite Julia sets (generated by finite families of quadratic polynomials).

THEOREM 3. *The family of all composite Julia sets in $\mathbf{C}^N$ is a proper dense subset of the metric space $(\mathcal{R}, \Gamma)$.*

The fact that not all sets in $\mathcal{R}$ are composite Julia sets follows from an example given by Plesniak [15] and is related to two important properties of compact sets: Markov's inequality and the Hölder Continuity Property. We say that a compact set $E \subset \mathbf{C}^N$ has *Markov's Property* if there exist positive constants M and r such that for every polynomial $p : \mathbf{C}^N \to \mathbf{C}$

$$\|\mathrm{grad}\, p\|_E \leq M(\deg p)^r \|p\|_E.$$

We say that E has the *Hölder Continuity Property* if there exist positive constants C, κ such that

$$V_E(z) \leq C \mathrm{dist}(z, E)^\kappa, \qquad \text{if } \mathrm{dist}(z, E) \leq 1.$$

It follows from Cauchy's estimates that the Hölder Continuity Property implies Markov's Property. Plesniak has constructed an example of a Cantor set $K \subset \mathbf{R}$ which is regular but does not have Markov's property (see Propositions 1 and 2 in [15]). Consequently, V_K does not satisfy the Hölder Continuity Property and so K cannot be a composite Julia set (see [12]).

In the last section of the paper we construct new examples of pluriregular sets without Markov's Property. The novelty element is the fact that these sets are obtained by polynomial iteration, except that we have to iterate infinitely many polynomials.

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§2. Invariance of pluriregularity

The goal of this section is to prove Theorem 1.

Proof. The inequality in (1), but without the upper semicontinuous regularization on the left side, is a direct consequence of the definition of the relative extremal functions. Consequently, if $v = u_{f(E),\Omega_2}$ and RHS stands for the right-hand side of (1), we can write that

$$g(w) := [z \mapsto \max v\,(f(z))]^*\,(a) \leq RHS(w), \qquad w \in \Omega_1.$$

Note that the functions g and $h(z) := \max v^*\,(f(z))$ are plurisubharmonic in Ω_1 and hence, to complete the proof of Theorem 1, it suffices to show

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that $h \le g$ almost everywhere in Ω_1. If f is open at $a \in \Omega_1$, then

$$\begin{aligned} h(a) &\le \lim_{r\to 0} \left(\sup\{v(z) : z \in f(B(a,r))\}\right) \\ &= \lim_{r\to 0} \left(\sup\{\max v(f(z)) : z \in B(a,r)\}\right) \\ &= g(a). \end{aligned}$$

Consider first the case when $|f|$ is bounded. Then, by Liouville's theorem for multifunctions (see [17]), the multifunction $z \mapsto \widehat{f(z)}$ is constant which implies that the left-hand side of (2) vanishes. Moreover f could not be open at any point because of compactness of $f(E)$, and so $\rho(|f|) > 0$ in (3). On the other hand, if $|f|$ is not bounded, then the convex non-decreasing function $h(r) = M(\log^+ |f|, e^r)$, $r > 0$, satisfies the inequality $h(s) < h(t)$ for some $s < t$. Let $g(r) = ar + b$ be an affine function such that $g \le h$ and $g(t) = h(t)$. Then

$$0 < a = \limsup_{r\to\infty} \frac{g(r)}{r} \le \limsup_{r\to\infty} \frac{h(r)}{r} = \rho(|f|).$$

If $\rho(|f|) > 0$, $u \in \mathcal{L}(\mathbf{C}^N)$, and $u \le 0$ on $f(E)$, then define

$$v(z) = \frac{1}{\rho(|f|)} \max u(f(z)), \qquad z \in \mathbf{C}^N.$$

Obviously $v \in \mathcal{L}(\mathbf{C}^N)$ and $v \le 0$ on E, and this gives (2).

Clearly (2) implies (3), but without the upper semicontinuous regularization on the left-hand side. If E is pluripolar, then the right-hand side is identically $+\infty$ an so we obtain (3). If E is not pluripolar, we can repeat the reasoning from the beginning of the proof but with $v = V_{f(E)}$ and RHS denoting the right-hand-side of (3). □

Corollary 1. *Let $f \in \mathcal{AMF}(\mathbf{C}^N, \mathbf{C}^N)$ be open almost everywhere in $\mathbf{C}^N$ and such that $|f|$ is of finite order. If $E \in \mathcal{K}(\mathbf{C}^N)$ is pluriregular, then so is $f(E)$.*

§3. Iteration of set-valued functions

When the set Σ_k (defined in the introduction) is endowed with the metric

$$d(\sigma, \tau) = \sum_{j=1}^{\infty} \frac{|\sigma_j - \tau_j|}{k^j},$$

it is often referred to as the *Cantor set of k symbols*. Let $\sigma = (\sigma_1, \sigma_2, \ldots) \in \Sigma_k$ and let $z_0 \in \mathbf{C}^N$. Let $f_j : \mathbf{C}^N \to 2^{\mathbf{C}^N}$ be set-valued functions for $j = 1, \ldots, k$. A sequence $\{z_j\}_{j\geq 0}$ of points in $\mathbf{C}^N$ is said to be a *backward σ-orbit* of z_0 (with respect to $\{f_1, \ldots, f_k\}$), if $z_j \in f_{\sigma_{j+1}}(z_{j+1})$ for $j = 0, 1, 2, \ldots$. Of course z_0 may have more than one backward σ-orbit. By $S_\sigma^-[f_1, \ldots, f_k]$ we will denote the set of points whose all backward σ-orbits are bounded.

If $g_j : \mathbf{C}^N \to \mathbf{C}^N$ are functions, where $j = 1, \ldots, k$, then by a *forward σ-orbit* of z_0 (with respect to $\{g_j\}_{j=1,\ldots,k}$) we mean the recursively defined sequence: $z_{j+1} = g_{\sigma_{j+1}}(z_j)$ for $j = 0, 1, \ldots$. By $S_\sigma^+[g_1, \ldots, g_k]$ we will denote the set of all points whose forward σ-orbits are bounded. Note that if the mappings g_j are surjective, then a backward σ-orbit of a point z_0 with respect to $\{z \mapsto g_j^{-1}(z)\}_{j=1,\ldots,k}$ is uniquely determined by that point and is exactly the same as the forward σ-orbit of z_0 with respect to $\{z \mapsto g_j(z)\}_{j=1,\ldots,k}$. If $k = 1$, we drop the letter σ from the above definitions.

Let $f : \mathbf{C}^N \to 2^{\mathbf{C}^N}$ be a set-valued function such that $f(\mathbf{C}^N) = \mathbf{C}^N$. We define the *multivalued inverse* of f to be the function $f^{-1} : \mathbf{C}^N \to 2^{\mathbf{C}^N}$ given by the formula

$$f^{-1}(w) = \{z \in \mathbf{C}^N : w \in f_j(z)\}, \qquad w \in \mathbf{C}^N.$$

Clearly $(f^{-1})^{-1} = f$ and

$$f^{-1}(S) = \{z \in \mathbf{C}^N : S \cap f(z) \neq \emptyset\}, \qquad S \in 2^{\mathbf{C}^N}.$$

Recall that an *iterated function system* on a complete metric space (X, d) is a family $\{f_1, f_2, \ldots, f_k\}$ of contractions of X (see [3]). Any such family generates the mapping $\mathcal{H} : \mathcal{K}(X) \to \mathcal{K}(X)$ given by the formula

$$\mathcal{H}(K) = f_1(K) \cup f_2(K) \cup \ldots \cup f_k(K), \qquad K \in \mathcal{K}(X).$$

Since the mapping $\mathcal{H}$ is a contraction of the complete metric space $(\mathcal{K}(X), \chi_d)$, it has a unique fixed point $\mathrm{Fix}\,(\mathcal{H})$. Furthermore, $\mathrm{Fix}\,(\mathcal{H})$ is the closure of the set of fixed points of the contractions $f_{\sigma_j} \circ \cdots \circ f_{\sigma_2} \circ f_{\sigma_1}$, where $\sigma_i \in \{1, 2, \ldots, k\}$ for $i = 1, 2, \ldots, j$ and $j = 1, 2, \ldots$ (see [3]). We will refer to $\mathcal{H}$ as the *Hutchinson mapping* associated with the iterated function system $\{f_1, \ldots f_k\}$. The set $\mathrm{Fix}(\mathcal{H})$ is sometimes called the attractor of the system $\{f_1, \ldots f_k\}$.

THEOREM 4. *Suppose that $f_1, \ldots, f_k \in \mathcal{AMF}(\mathbf{C}^N, \mathbf{C}^N)$ are such that $\rho\,(|f_j|) < 1$ and that $f_j(\mathcal{R}) \subset \mathcal{R}$ for $j = 1, \ldots, k$.*

(i) *For any* $\sigma \in \Sigma_k$ *there exists* $R > 0$ *such that the sequence*

$$f_{\sigma_1}(f_{\sigma_2}(\dots(f_{\sigma_j}(\overline{B}(0,R)))\dots))$$

is decreasing (*with respect to inclusion*) *and*

$$S_\sigma^-[f_1,\dots,f_k] = \bigcap_{j=1}^{\infty} f_{\sigma_1}(f_{\sigma_2}(\dots(f_{\sigma_j}(\overline{B}(0,R)))\dots)).$$

(ii) *For any* $\sigma \in \Sigma_k$ *and any positive integer* m, *the mapping* $f_{\sigma_m} \circ f_{\sigma_{m-1}} \circ \dots \circ f_{\sigma_1} : \mathcal{R} \to \mathcal{R}$ *is a contraction with the contraction ratio equal to the product of the orders of the functions* $|f_{\sigma_m}|, |f_{\sigma_{m-1}}|, \dots, |f_{\sigma_1}|$ *and with the unique fixed point* $S^-[f_{\sigma_m} \circ \dots \circ f_{\sigma_1}] \in \mathcal{R}$.

(iii) *For any* $\sigma \in \Sigma_k$ *and* $E \in \mathcal{R}$, *the sequence* $\{f_{\sigma_1}(f_{\sigma_2}(\dots(f_{\sigma_j}(E))\dots))\}_{j\geq 1}$ *converges in* $(\mathcal{R},\Gamma)$ *to* $S_\sigma^-[f_1,\dots,f_k]$. *Moreover, for any set* $E \in \mathcal{R}$

$$\Gamma\left(S_\sigma^-[f_1,\dots,f_k], f_{\sigma_1}(f_{\sigma_2}(\dots(f_{\sigma_j}(E))\dots))\right) \leq \frac{\rho^j}{1-\rho} \max_{1\leq i\leq k}\{\Gamma(E, f_i(E))\}, \tag{5}$$

where $\rho = \max\{\rho(|f_j|) : j = 1,\dots,k\}$. *In particular, the mapping*

$$\sigma \mapsto S_\sigma^-[f_1,\dots,f_k]$$

is continuous.

(iv) *For each* $\sigma \in \Sigma_k$

$$S_\sigma^-[f_1,\dots,f_k] = \lim_{m\to\infty} S^-[f_{\sigma_m} \circ \dots \circ f_{\sigma_1}].$$

(v) *The family* $\{E \mapsto f_j(E)\}_{j=1,\dots,k}$ *is an iterated function system on* $(\mathcal{R},\Gamma)$ *whose attractor is the set*

$$\{S_\sigma^-[f_1,\dots,f_k] : \sigma \in \Sigma_k\} \in \mathrm{Comp}\,(\mathcal{R}).$$

(vi) *For every* $E \in \mathcal{R}$ *define* $\mathcal{F}(E)$ *to be the polynomially convex hull of set* $f_1(E) \cup \dots \cup f_k(E)$. *Then* $\mathcal{F}$ *is a contraction of* $(\mathcal{R},\Gamma)$ *and its unique fixed point is the polynomially convex hull of the set*

$$\bigcup_{\sigma\in\Sigma_k} S_\sigma^-[f_1,\dots,f_k].$$

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Proof. The first two statements follow directly from the definition of $\rho(|f_j|)$, Formula (2) and the fact that $(\mathcal{R}, \Gamma)$ is complete (see [9]).

To show the third one fix $\sigma \in \Sigma_k$ and $E \in \mathcal{R}$. Define

$$E_j = f_{\sigma_1}(f_{\sigma_2}(\ldots(f_{\sigma_j}(E))\ldots))$$

and

$$C = \max_{1 \le i \le k} \{\Gamma(E, f_i(E))\}.$$

First we will show that $\{E_j\}_{j \ge 1}$ is a Cauchy sequence in $(\mathcal{R}, \Gamma)$. Since the mappings $f_i : \mathcal{R} \to \mathcal{R}$ are contractions with the contraction ratio ρ, we have the following estimates.

$$\Gamma(E_{j+m}, E_j) \le \sum_{i=0}^{m-1} \Gamma(E_{j+i}, E_{j+i+1}) \le C \sum_{i=0}^{m-1} \rho^{j+i} = C \frac{\rho^j(1-\rho^m)}{1-\rho}. \tag{6}$$

This shows that $\{E_j\}_{j \ge 1}$ is a Cauchy sequence in the complete metric space $(\mathcal{R}, \Gamma)$. Note that the limit depends on σ but not on the choice of E. Indeed, if $F \in \mathcal{R}$ and $F_j = f_{\sigma_1}(f_{\sigma_2}(\ldots(f_{\sigma_j}(F))\ldots))$, then $\Gamma(E_j, F_j) \le \rho^j \Gamma(E, F) \to 0$ as $j \to \infty$. In particular, we can take the sequence from the first part of the theorem. Letting m go to ∞ in (6), we obtain (5).

The continuity of the mapping $\sigma \mapsto S_\sigma^-[f_1, \ldots, f_k]$ is a direct consequence of the following observation. If $\sigma, \tau \in \Sigma_k$ and $d(\sigma, \tau) < k^{-j}$, then $(\sigma_1, \ldots, \sigma_j) = (\tau_1, \ldots, \tau_j)$ and — according to (5) —

$$\begin{aligned}
&\Gamma(S_\sigma^-[f_1, \ldots, f_k], S_\tau^-[f_1, \ldots, f_k]) \\
&\le \Gamma\left(S_\sigma^-[f_1, \ldots, f_k], K_j\right) + \Gamma\left(S_\tau^-[f_1, \ldots, f_k], K_j\right) \\
&\le 2 \frac{\rho^j}{1-\rho} \max_{1 \le i \le k} \{\Gamma(E, f_i^{-1}(E)),
\end{aligned}$$

where $K_j = f_{\sigma_1}(f_{\sigma_2}(\ldots(f_{\sigma_j}(E))\ldots)) = f_{\tau_1}(f_{\tau_2}(\ldots(f_{\tau_j}(E))\ldots))$.

It is clear that periodic sequences are dense in Σ_k. Moreover, if σ is obtained as a periodic repetition of the block $(\sigma_1, \ldots, \sigma_j)$, then $S_\sigma^-[f_1, \ldots, f_k] = S^-[f_{\sigma_j} \circ \cdots \circ f_{\sigma_1}]$. Therefore the third part of the theorem follows from the continuity of the mapping $\sigma \mapsto S_\sigma^-[f_1, \ldots, f_k]$.

The next statement is a direct consequence of Hutchinson's result mentioned above.

Now we will verify the last part of the theorem. Let $\mathcal{H} : \mathcal{K}(\mathcal{R}) \to \mathcal{K}(\mathcal{R})$ be the Hutchinson mapping associated with the iteration function system $\{E \mapsto f_j(E)\}_{j=1,\dots,k}$. We know that

$$\begin{aligned}
\mathcal{H}^1(\{\mathrm{Fix}(\mathcal{F})\}) &= \{f_{\sigma_1}(\mathrm{Fix}(\mathcal{F})) : \sigma_1 \in \{1,\dots,k\}\}\\
\mathcal{H}^2(\{\mathrm{Fix}(\mathcal{F})\}) &= \{(f_{\sigma_1} \circ f_{\sigma_2})(\mathrm{Fix}(\mathcal{F})) : \sigma_1, \sigma_2 \in \{1,\dots,k\}\}\\
\dots\dots\dots\dots &\quad \dots\\
\mathcal{H}^p(\{\mathrm{Fix}(\mathcal{F})\}) &= \big\{(f_{\sigma_1} \circ \dots \circ f_{\sigma_p})(\mathrm{Fix}(\mathcal{F})) : \sigma_1, \dots, \sigma_p \in \{1,\dots,k\}\big\}\,.
\end{aligned}$$

Note that if $E_1, E_2, \dots, E_m \in \mathcal{R}$ are such that

$$\mathrm{Fix}(\mathcal{F}) = \left(\bigcup_{j=1}^{m} E_j\right)^{\wedge},$$

then

$$\mathrm{Fix}(\mathcal{F}) = \mathcal{F}\left(\mathrm{Fix}(\mathcal{F})\right) = \left(\bigcup_{i=1}^{k}\bigcup_{j=1}^{m} f_i(E_j)\right)^{\wedge}.$$

Therefore, by induction,

$$\left(\bigcup \mathcal{H}^p(\{\mathrm{Fix}(\mathcal{F})\})\right)^{\wedge} = \mathrm{Fix}(\mathcal{F}), \qquad p = 1, 2, \dots.$$

Furthermore

$$\lim_{p\to\infty} \chi_\Gamma\left(\mathcal{H}^p(\{\mathrm{Fix}(\mathcal{F})\}), \mathrm{Fix}(\mathcal{H})\right) = 0.$$

Take $\varepsilon > 0$. Choose p so that

$$\chi_\Gamma\left(\mathcal{H}^p(\{\mathrm{Fix}(\mathcal{F})\}), \mathrm{Fix}(\mathcal{H})\right) < \varepsilon.$$

The set $\mathcal{H}^p(\{\mathrm{Fix}(\mathcal{F})\})$ is finite and its elements, say $B_1, \dots, B_l \in \mathcal{R}$, satisfy the equality $\mathrm{Fix}(\mathcal{F}) = (B_1 \cup B_2 \cup \dots \cup B_l)^{\wedge}$ (see above). It follows from the definition of the Hausdorff metric that the Γ-distance of these sets from $\mathrm{Fix}(\mathcal{H})$ is less than ε. Therefore we can find $A_1, \dots, A_l \in \mathrm{Fix}(\mathcal{H})$ such that $\Gamma(A_j, B_j) < \varepsilon$ for $j = 1, \dots, l$. In view of Corollary 2 in [9], $\Gamma(\mathrm{Fix}(\mathcal{F}), \bigcup A_j) < \varepsilon$. But since ε can be arbitrarily small and

$$\bigcup A_j \subset \bigcup \mathrm{Fix}(\mathcal{H}) \subset \mathrm{Fix}(\mathcal{F}),$$

we can conclude that

$$\Gamma\left(\bigcup \mathrm{Fix}(\mathcal{H}), \mathrm{Fix}(\mathcal{F})\right) = 0.$$

In other words, the two sets have the same polynomially convex hull as required. ∎

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§4. Analyticity of families of composite Julia sets

The proof of the Theorem 2 will be based on several lemmas.

Let $P : \mathbf{C}^N \to \mathbf{C}^N$ be a regular polynomial mapping. We say that $r > 0$ is an *escape radius* for P if for any point $z \in \mathbf{C}^N$ the inequality $\|z\| > r$ implies that the forward orbit of z under P escapes to infinity, that is $\lim_{n\to\infty} \|P^n(z)\| = \infty$. First we show an escape radius formula for regular polynomial mappings. The formula is a direct generalization of its well-known one variable counterpart.

For any $P \in \mathcal{P}_d^*$ we define the *floor* of P as the number

$$\lfloor P \rfloor = \inf_{\|z\|=1} \|P_d(z)\| > 0,$$

where P_d is the homogeneous part of P of degree d.

Lemma 1. *Let*

$$P = P_d + P_{d-1} + \cdots + P_0 : \mathbf{C}^N \longrightarrow \mathbf{C}^N$$

be a regular polynomial mapping, where P_j denotes the homogeneous part of P of degree j and $d \geq 2$. Then the formula

$$r(P) = \frac{1 + \lfloor P_d \rfloor + \|P_{d-1}\| + \cdots + \|P_0\|}{\lfloor P_d \rfloor}$$

gives an escape radius of P which depends continuously on P. Furthermore

$$\|P(z)\| \geq \frac{1 + \lfloor P_d \rfloor}{r(P)} \|z\|^d \quad \text{if} \quad \|z\| \geq r(P).$$

In particular, if $R \geq r(P)$, then $P^{-1}\left(\overline{B}(0, (1 + \lfloor P_d \rfloor)R)\right) \subset \overline{B}(0, R)$.

Proof. If $\|z\| \geq r(P)$, then (by the triangle inequality and the definition of $r(P)$) we have:

$$\frac{\|P(z)\|}{\|z\|^d} \geq \lfloor P_d \rfloor - \frac{\|P_{d-1}\| + \cdots + \|P_0\|}{r(P)} = \frac{1 + \lfloor P_d \rfloor}{r(P)}.$$

In particular, if $\|z\| > R \geq r(P)$, then, since $d \geq 2$, we have $\|P(z)\| > (1 + \lfloor P_d \rfloor)R$ and thus $\|P^n(z)\| > (1 + \lfloor P_d \rfloor)^n R \to \infty$ as $n \to \infty$. ∎

The next lemma comes from [2].

LEMMA 2. *Let X and Y be metrizable topological spaces. Assume that X is locally compact and $f : X \to \mathcal{K}(Y)$ is a set-valued mapping. Define the graph of f as the set*

$$\mathrm{Graph}(f) = \{(x, y) \in X \times Y \;:\; y \in f(x)\}.$$

The following conditions are equivalent.

- *The mapping f is upper semicontinuous.*
- $\mathrm{Graph}(f)$ *is closed in $X \times Y$ and the natural projection $\pi_X : \mathrm{Graph}(f) \to X$ is proper.*
- $\mathrm{Graph}(f)$ *is closed in $X \times Y$ and every point in X has a neighbourhood V such that $\pi_Y(\pi_X^{-1}(V))$ is relatively compact in Y, where $\pi_Y : \mathrm{Graph}(f) \to Y$ is the natural projection.*

LEMMA 3. *If $P : \mathbf{C}^N \to \mathbf{C}^N$ is a regular mapping and $a \in \mathbf{C}^N$, then $P \mapsto P^{-1}(a)$ is an analytic multifunction on $\mathcal{P}_d^*$.*

Proof. Without loss of generality we may suppose that $a = 0$. Then $P^{-1}(0) \subset \overline{B}(0, r(P))$ for every P. We use Lemma 2 with $X = \mathcal{P}_d^*$, $Y = \mathbf{C}^N$ and $f(P) = P^{-1}(0)$. The set $\mathrm{Graph}(f)$ is an analytic subvariety of $\mathcal{P}_d^* \times \mathbf{C}^N$. Moreover

$$\mathrm{Graph}(f) \subset \mathcal{P}_d^* \times \overline{B}(0, r(P)),$$

and thus the last condition in Lemma 2 is satisfied. Hence π_X is a branched covering and so for any $u \in \mathcal{L}$ the function

$$P \mapsto \sup u\left(\{P\} \times P^{-1}(0)\right)$$

is plurisubharmonic. This implies analyticity of f (see e.g. [7]). ∎

LEMMA 4. *If $P : \mathbf{C}^N \to \mathbf{C}^N$ is a regular mapping, then $P \mapsto P^{-1}(\overline{B}(0, 1))$ is an analytic multifunction on $\mathcal{P}_d^*$.*

Proof. If $u \in \mathcal{L}$, then

$$\sup u\left(\{P\} \times P^{-1}(\overline{B}(0,1))\right) = \sup_{\|a\| \le 1} \sup u\left(\{P\} \times P^{-1}(a)\right).$$

Consequently it suffices to show that

$$f : \mathcal{P}_d^* \longrightarrow \mathcal{K}(\mathbf{C}^N), \qquad P \mapsto P^{-1}(\overline{B}(0,1)),$$

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is upper semicontinuous. But this follows from Lemma 2 because

$$\operatorname{Graph}(f) \subset \mathcal{P}_d^* \times \overline{B}(0, r(P)).$$

∎

Proof of Theorem 2. Consider the following set-valued mappings:

$$\begin{aligned} f_{\sigma,n} : \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* &\longrightarrow \mathcal{R}, (P_1, \ldots, P_k) \mapsto (P_{\sigma_1} \circ \cdots \circ P_{\sigma_n})^{-1}\left(\overline{B}(0,1)\right); \\ f_\sigma : \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* &\longrightarrow \mathcal{R}, (P_1, \ldots, P_k) \mapsto S_\sigma^+[P_1, \ldots, P_k]; \\ f : \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* &\longrightarrow \mathcal{R}, (P_1, \ldots, P_k) \mapsto K^+[P_1, \ldots, P_k], \end{aligned}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots) \in \Sigma_k$. Lemma 4 tells us that $f_{\sigma,n}$ is analytic. In view of [10],

$$\lim_{n\to\infty} f_{\sigma,n}(P_1, \ldots P_k) = f_\sigma(P_1, \ldots P_k)$$

with respect to Γ. Observe that if $v \in \mathcal{L}$ and $E, F \in \mathcal{R}$, then

$$|\sup v(E) - \sup v(F)| \leq \Gamma(E, F).$$

As a consequence, analyticity of the function f_σ will follow if we show upper semicontinuity. We are going to use the last condition in Lemma 2. First observe that $\operatorname{Graph}(f_\sigma)$ is closed because it is the complement in $\mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* \times \mathbf{C}^N$ of the set

$$\bigcup_{n\geq 1} \{(P_1, \ldots, P_k, z) \,:\, \|(P_{\sigma_n} \circ \cdots \circ P_{\sigma_1})(z)\| > R\},$$

where $R \geq \max\{r(P_1), \ldots, r(P_k)\}$. Secondly

$$\operatorname{Graph}(f_\sigma) \subset \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* \times \overline{B}(0, R),$$

and thus Lemma 2 implies upper semicontinuity of f_σ.

Consider

$$g : \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* \longrightarrow \mathcal{K}(\mathbf{C}^N), \qquad (P_1, \ldots, P_k) \mapsto \bigcup_{\sigma \in \Sigma_k} S_\sigma^+[P_1, \ldots, P_k].$$

Using the same argument as in the proof of Proposition 4.4 in [12], we can show that $\operatorname{Graph}(g)$ is closed in $\mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* \times \mathbf{C}^N$. Since

$$\operatorname{Graph}(g) \subset \mathcal{P}_{d_1}^* \times \cdots \times \mathcal{P}_{d_k}^* \times \overline{B}(0, \max\{r(P_1), \ldots, r(P_k)\}),$$

Lemma 2 gives us upper semicontinuity of g. This, in turn, implies upper semicontinuity of f. (Given a neighbourhood V of $K = f(P_1, \ldots, P_k)$, there exists a Runge domain D such that $K \subset D \subset V$. Since g is upper semicontinuous, there is a neighbourhood U of $(P_1, \ldots, P_k)$ such that $g(Q) \in D$ for any $Q \in U$. Hence $f(Q) \in D$ for any $Q \in U$.)

Now it suffices to observe that if u is a function of class $\mathcal{L}$ on $\mathbf{C}^{\nu(d_1)+\ldots+\nu(d_n)+N}$ and $Q = (P_1, \ldots, P_k)$, then

$$\sup u\left(\{Q\} \times f(Q)\right) = \sup u\left(\{Q\} \times g(Q)\right) = \sup_{\sigma \in \Sigma_k} \sup u\left(\{Q\} \times f_\sigma(Q)\right).$$

Now we will check the last statement of Theorem 2. Recall that if $u \in PSH(\mathbf{C}^N)$ then the order of u is the number

$$\rho(u) = \limsup_{r \to \infty} \frac{\log\left(\sup\left\{\max\{0, u(z)\} \ : \ \|z\| \le r\right\}\right)}{\log r}.$$

Let u denote the function given by (4). The fact that the order of u is not greater than one follows directly from the escape radius formula because

$$K^+[H_1 + P_1, \ldots, H_k + P_k] \subset \overline{B}(0, \max\{r(H_1 + P_1), \ldots, r(H_k + P_k)\}).$$

Since $K^+[H_1 + P_1, \ldots, H_k + P_k]$ contains all $K^+[H_j + P_j] = S^+[H_j + P_j]$, it suffices to show equality for $k = 1$. So suppose a homogeneous polynomial mapping $H \in \mathcal{P}_d^*$ is fixed. Let $v \in \mathbf{C}^N$ be a vector of norm one. Then nv is a fixed point for the polynomial mapping $Q_n : z \mapsto H(z - nv) + nv$. So $nv \in K^+[Q_n]$ and

$$1 \le \limsup_{n \to \infty} \frac{\log\left(\sup\{\|z\| \ : \ z \in K^+[Q_n]\}\right)}{\log n} \le \rho(u),$$

as required. ∎

Example 1. Let $P_j, Q_j : \mathbf{C}^N \to \mathbf{C}^N$ be regular polynomial mappings of degree at least two and assume that $P_j(0) = 0$ for $j = 1, \ldots, k$. Define the analytic multifunctions $f_j : \mathbf{C}^N \to \mathcal{R}$ by the formula

$$c \mapsto Q_j^{-1}\left(K^+[P_j + c]\right), \qquad c \in \mathbf{C}^N, j = 1, 2, \ldots, k.$$

Then the functions $f_1, \ldots, f_k$ satisfy the assumptions of Theorem 4.

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EXAMPLE 2. Fix $\sigma \in \Sigma_k$ and homogeneous polynomials $H_1 \in \mathcal{P}^*_{d_1}, \ldots, H_k \in \mathcal{P}^*_{d_k}$, where $d_1, \ldots, d_k \geq 2$. Define $g : \mathcal{P}_{d_1-1} \times \cdots \times \mathcal{P}_{d_k-1} \to \mathcal{R}$ by the formula:

$$g(P_1, \ldots, P_k) = S^+_\sigma[H_1 + P_1, \ldots, H_k + P_k].$$

We claim that

$$\begin{aligned} &V_{\mathrm{Graph}(g)}(P_1, \ldots, P_k, z) \\ &= V_{S^+_\sigma[H_1+P_1,\ldots,H_k+P_k]}(z) \\ &= \lim_{j\to\infty} \frac{1}{d_{\sigma_1} + \cdots + d_{\sigma_k}} \log^+ \left\| \left((H_{\sigma_j} + P_{\sigma_j}) \circ \cdots \circ (H_{\sigma_1} + P_{\sigma_1}) \right)(z) \right\|, \end{aligned}$$

for $(P_1, \ldots, P_k, z)$ from the complement of $\mathrm{Graph}(g)$ and that the convergence is locally uniform. Note that if we can show that the function $V_{S^+_\sigma[H_1+P_1,\ldots,H_k+P_k]}(z)$ is plurisubharmonic and of class $\mathcal{L}$, then the first of the two equalities will follow.

Assume that $R > \max\{r(H_1 + P_1), \ldots, r(H_k + P_k)\}$. Let

$$E_j[P_1, \ldots, P_k] = \left((H_{\sigma_j} + P_{\sigma_j}) \circ \cdots \circ (H_{\sigma_1} + P_{\sigma_1}) \right)^{-1} \left(\overline{B}(0, R) \right).$$

Then

$$\begin{aligned} &V_{E_j[P_1,\ldots,P_k]}(z) \\ &= \frac{\max\left\{0, \log \left\| \left((H_{\sigma_j} + P_{\sigma_j}) \circ \cdots \circ (H_{\sigma_1} + P_{\sigma_1}) \right)(z) \right\| - \log R \right\}}{d_{\sigma_1} + \ldots + d_{\sigma_k}}. \end{aligned}$$

Clearly the function $(P_1, \ldots, P_k, z) \mapsto V_{E_j[P_1,\ldots,P_k]}(z)$ is in $\mathcal{L}(\mathbf{C}^{\nu(d_1-1)+\cdots+\nu(d_k-1)+N})$. Moreover, it was shown in [10] (see also (5)), that if $\delta = \min\{d_1, \ldots, d_k\}$, then

$$\begin{aligned} &\Gamma(S^+_\sigma[H_1 + P_1, \ldots, H_k + P_k], E_j[P_1, \ldots, P_k]) \\ &\leq \frac{1}{\delta^{j-1}(\delta - 1)} \max_{1\leq i\leq k} \log \frac{\|H_i + P_i\|_{\overline{B}(0,R)}}{R}, \end{aligned}$$

which justifies our claim.

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§5. Density of families of composite Julia sets

In this section we show that every pluriregular set can be approximated in the sense of the metric Γ by composite Julia sets. In fact these sets will be generated by quadratic polynomials. In what follows $\overline{P}(a,r)$ will denote the closed polydisc with centre at a and radius r.

Proof of Theorem 3. Since all composite Julia sets have Markov's property [12] and there are pluriregular sets without Markov's property [15], what remains to be shown is density of Julia sets in $\mathcal{R}$.

Suppose that $E \in \mathcal{R}$ and $\epsilon > 0$. We may assume that the interior of E is non-empty and $E = \overline{\mathrm{int}(E)}$, for otherwise we may replace E by a finite union $E_1 \cup \ldots \cup E_l$ of closed polydiscs, with radii so small that

$$E \subset E_1 \cup \cdots \cup E_l \subset \{z \in \mathbf{C}^N \,:\, V_E(z) < \epsilon\}.$$

Then

$$\Gamma\left(E_1 \cup \cdots \cup E_l, E\right) = \Gamma\left((E_1 \cup \cdots \cup E_l)^{\wedge}, E\right) < \epsilon.$$

Let $\delta > 0$ be chosen so that the δ-dilation of E (with respect to the polydisc norm) is contained in $\{z \in \mathbf{C}^N \,:\, V_E(z) < \epsilon\}$. Let $a = (a_1, \ldots, a_N) \in \mathrm{int}(E)$ and let $R > 0$ be such that $E \subset \overline{P}(a,R)$. For every $b = (b_1, \ldots, b_N) \in E$ we define

$$P_b : \mathbf{C}^N \longrightarrow \mathbf{C}^N$$

by the formula

$$P_b(z_1, \ldots, z_N) = \left(\frac{R}{\delta^2}(z_1 - b_1)^2 + a_1, \ldots, \frac{R}{\delta^2}(z_N - b_N)^2 + a_N\right).$$

Then

$$b \in P_b^{-1}\left(\mathrm{int}(E)\right) \subset P_b^{-1}\left(\overline{P}(a,R)\right) = \overline{P}(b,\delta).$$

Since E is compact, we can find a finite number of points $b^1, b^2, \ldots, b^k$ such that

$$E \subset \bigcup_{j=1}^{k} \overline{P_{b^j}^{-1}\left(\mathrm{int}(E)\right)} \subset (\delta - \text{dilation of } E).$$

Hence

$$\Gamma\left(E, \bigcup_{j=1}^{k} \overline{P_{b^j}^{-1}\left(\mathrm{int}(E)\right)}\right) = \Gamma\left(E, \bigcup_{j=1}^{k} P_{b^j}^{-1}(E)\right) < \epsilon.$$

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Let $\mathcal{P}:\mathcal{R}\longrightarrow\mathcal{R}$ be defined by the formula

$$\mathcal{P}(F)=\left(\bigcup_{j=1}^{k}P_{b^j}^{-1}(F)\right)^{\wedge},\qquad F\in\mathcal{R}.$$

Clearly $\mathcal{P}$ is a contraction with respect to Γ with the contractivity factor $1/2$. Consequently we have

$$\begin{aligned}\Gamma\left(E,\mathcal{P}^n(E)\right)&\le\sum_{j=0}^{n-1}\Gamma\left(\mathcal{P}^j(E),\mathcal{P}^{j+1}(E)\right)\\&\le\Gamma(E,\mathcal{P}(E))\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{m-1}}\right)\\&\le 2\epsilon.\end{aligned}$$

This completes the proof of Theorem 3. ∎

§6. Pluriregular sets without Markov's property

Our construction of pluriregular sets on which Markov's estimate fails will be based on the following elementary observation.

PROPOSITION 1. *Let $P_n:\mathbf{C}^N\to\mathbf{C}^N$ be a regular mapping of degree $d_n\ge 2$, for $n=1,2,\dots$. Let $K\in\mathcal{R}$. Define*

$$E_n=(P_n\circ\cdots\circ P_1)^{-1}(K)$$

for $n=1,2,\dots$. If

$$\sum_{n=1}^{\infty}\frac{\Gamma\left(P_{n+1}^{-1}(K),K\right)}{d_1d_2\cdot\ \cdots\ \cdot d_n}<\infty \tag{7}$$

then the sequence $\{E_n\}$ is convergent in $(\mathcal{R},\Gamma)$ to a set E. Any other choice of $K\in\mathcal{R}$ for which the condition (7) is satisfied, results in the same limit E. If we assume that $P_n^{-1}(K)\subset K$ for all n, then the sequence $\{E_n\}$ is decreasing and

$$E=\bigcap_{n\ge 1}E_n=\{z\in K:(P_n\circ\cdots\circ P_1)(z)\in K\ \textit{for all}\ n\ge 1\}. \tag{8}$$

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Proof. The mapping $P = (P_n \circ \cdots \circ P_1)$ is regular hence

$$\Gamma\left(E_{n+1}, E_n\right) = \Gamma\left(P^{-1}\left(P_{n+1}^{-1}(K)\right), P^{-1}(K)\right) \leq \frac{\Gamma\left(P_{n+1}^{-1}(K), K\right)}{d_n \dots d_1}.$$

Consequently, the first conclusion of the theorem follows from the completeness of $\mathcal{R}$.

If $K^1, K^2 \in \mathcal{R}$ satisfy (7) and $E_n^j = (P_n \circ \cdots \circ P_1)^{-1}(K^j)$ for $j = 1, 2$, then

$$\Gamma(E_n^1, E_n^2) \leq \frac{1}{d_1 \dots d_n} \Gamma(K^1, K^2),$$

and hence $\lim_{n\to\infty} E_n^1 = \lim_{n\to\infty} E_n^2$.

Assume now that $P_n^{-1}(K) \subset K$ for all n. Clearly $K \supset E_n \supset E_{n+1}$ and since $V_{E_n} \leq V_{E_{n+1}}$ we get the first equality in (8). The second equality follows from the first one and the definition of E_n. ∎

Two examples of pluriregular sets without Markov's property are given in the form of corollaries from the above proposition.

COROLLARY 2. *For a positive number λ the logistic function is defined as*

$$f_\lambda(z) = \lambda z(1 - z), \qquad z \in \mathbf{C}.$$

Suppose that (λ_n) is a sequence of numbers such that $\lambda_n > 4$,

$$\sum_{n=1}^{\infty} \frac{\log \lambda_n}{2^n} < \infty. \tag{9}$$

and

$$\limsup_{n\to\infty} (\lambda_1 \lambda_2 \dots \lambda_n)^{1/n} = \infty. \tag{10}$$

Let

$$E_n = \left(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}\right)^{-1}([0, 1]), \qquad n = 1, 2, \dots .$$

Then the sequence (E_n) converges in $(\mathcal{R}, \Gamma)$ to a set which does not have Markov's property.

Proof. Let $\lambda > 4$. If $x \in \mathbf{R}$, then (see e.g. [8])

$$V_{[-1,1]}(x) = \log\left|x + (x^2 - 1)^{1/2}\right|, \qquad x \in \mathbf{R} \setminus (-1, 1).$$

Hence

$$V_{[0,1]}(x) = \log \left| 2x - 1 + 2(x^2 - x)^{1/2} \right|, \qquad x \in \mathbf{R} \setminus (0,1).$$

Moreover the function $V_{[0,1]}$ is monotone in the intervals $(-\infty, 0]$,$[1,\infty)$ and its graph is symmetric with respect to the vertical line $x = 1/2$. Because of this, the function

$$V_{f_\lambda^{-1}([0,1])}(x) = \frac{1}{2} V_{[0,1]}\left(f_\lambda(x)\right)$$

attains is maximum in $[0,1]$ at the point $x = 1/2$. Consequently

$$\begin{aligned}
\Gamma\left(f_\lambda^{-1}([0,1]), [0,1]\right) &= \frac{1}{2} V_{[0,1]}\left(\frac{\lambda}{4}\right) \\
&= \frac{1}{2} \log \left(\frac{\lambda}{2} - 1 + \left(\lambda \left(\frac{\lambda}{4} - 1 \right) \right)^{1/2} \right) \\
&\leq \log \lambda.
\end{aligned}$$

Consequently Proposition 1, with $P_n = f_{\lambda_n}$ and $K = [0,1]$, implies the first conclusion.

Let $E = \lim_{n \to \infty} E_n$. To see that E does not have Markov's property we can proceed as follows. Define

$$q_n = f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}.$$

Note that 0 is a fixed point of f_λ and thus $0 \in E$. Moreover $q_n'(0) = \lambda_1 \lambda_2 \ldots \lambda_n$ and $\|q_n\|_E \leq 1$. If Markov's condition were satisfied, then in particular we would have

$$\lambda_1 \lambda_2 \ldots \lambda_n \leq M 2^{nr}$$

for some $M, r > 0$ and for all n. But this would contradict our assumption. $\square$

Corollary 3. *For a positive number* λ *define*

$$f_\lambda(z) = \lambda z^3 - (\lambda - 1), \qquad z \in \mathbf{C}.$$

Suppose that (λ_n) *is a sequence of numbers greater than 2 satisfying the conditions* (9) *and* (10). *Let*

$$E_n = (f_{\lambda_n} \circ \cdots \circ f_{\lambda_1})^{-1}(\overline{D}(0,1)), \qquad n = 1, 2, \ldots.$$

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Then the sequence (E_n) *converges in* $(\mathcal{R}, \Gamma)$ *to a set which does not have Markov's property.*

Proof. The proof goes along the same lines as the previous one.

Examples of sequences (λ_n) satisfying the conditions in the above corollaries are easy to find. We can take, for instance, $\lambda_n = n + 4$ or $\lambda_n = e^{n+1}$.

Due to a recent result of Kosek [12], mentioned in the introduction, the limit sets obtained above are not composite Julia sets.

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