ON THE CHARACTER RINGS OF FINITE GROUPS

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Introduction. The characters of the representations of a finite group G over a field K of characteristic zero generate a ring $\mathfrak{o}_K(G)$ of functions on G, the *K-character ring of* G, which is readily seen to be $\mathbf{Z}\phi_1 + \ldots + \mathbf{Z}\phi_n$, where $\mathbf{Z}$ is the ring of rational integers and $\phi_1, \ldots, \phi_n$ are the characters of the different irreducible representations of G over K. The theorem that every irreducible representation of G over an algebraically closed field Ω of characteristic zero is equivalent to a representation of G over the subfield of Ω which is generated by the g_0th roots of unity (g_0 the exponent of G) was proved by Brauer **(4)** via the theorems that

(1) $\mathfrak{o}_\Omega(G)$ is additively generated by the induced characters of representations of elementary subgroups of G, and

(2) the irreducible representations over Ω of any elementary group are induced by one-dimensional subgroup representations **(3)**.

Brauer's original proof of (1), being rather complicated in some of its details, was simplified considerably by Roquette **(5)** with the aid of suitable $\mathfrak{p}$-adic extensions of $\mathfrak{o}_\Omega(G)$. In turn, Witt **(7)** extended the use of Roquette's method to generalize (1) to arbitrary fields, or, rather, to finite-dimensional division algebras over an arbitrary field. The case for arbitrary fields was again treated by Berman **(2)**, who used similar techniques, and by Solomon **(6)**, whose approach is more like that in **(4)**.

The main object of this note is to obtain a theorem analogous to (1) which is still strong enough to be of use as a step in proving that every irreducible representation of G over a field containing all g_0th roots of unity is absolutely irreducible, but whose proof is substantially simpler than those mentioned. This will be done by means of a direct determination of the maximal ideals of $\mathfrak{o}_K(G)$ for arbitrary K, which dispenses with $\mathfrak{p}$-adic extensions as well as other tools previously employed. Moreover, it will be shown that the knowledge of the maximal ideals of $\mathfrak{o}_K(G)$ can also be used to prove Witt's generalization of (1).

The desired description of the maximal ideals of $\mathfrak{o}_K(G)$ actually requires only very few of the properties of the rings $\mathfrak{o}_K(G)$, and the material presented here has been arranged accordingly. Thus, the first section of this paper deals with the maximal ideals of certain rings of functions which are defined on a finite set and take their values in a suitable integral domain. The second section brings the application of the result obtained to the rings $\mathfrak{o}_K(G)$, leading

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to the theorems referred to above. The paper closes with some remarks pointing out further consequences of the method used here.

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1. Maximal ideals. Let X be a finite set, I an integral domain of characteristic zero, which is finitely generated as an additive group, and $\mathfrak{o}$ a ring of functions $\phi: X \to I$ such that (i) the constant function ϵ whose value is $1 \in I$ belongs to $\mathfrak{o}$ and (ii) $\mathfrak{o}$ is additively generated by finitely many functions $\phi_1, \ldots, \phi_n$ which are linearly independent over I. Then, the maximal ideals of $\mathfrak{o}$ are described as follows.

PROPOSITION 1. *For any $c \in X$ and any maximal ideal $\mathfrak{p}$ in I,*

$$\mathfrak{a}(c, \mathfrak{p}) = \{\phi \mid \phi \in \mathfrak{o}, \phi(c) \in \mathfrak{p}\}$$

is a maximal ideal of $\mathfrak{o}$, and each maximal ideal of $\mathfrak{o}$ is of this type.

Proof. Clearly, any $\mathfrak{a}(c, \mathfrak{p})$ is a *prime* ideal of $\mathfrak{o}$, proper because $\epsilon \notin \mathfrak{a}(c, \mathfrak{p})$. Moreover, $p\mathfrak{o} \subseteq \mathfrak{a}(c, \mathfrak{p})$ for the prime number $p \in \mathfrak{p}$, and since $\mathfrak{o}/p\mathfrak{o}$ is finite, the integral domain $\mathfrak{o}/\mathfrak{a}(c, \mathfrak{p})$ is also finite. Thus, $\mathfrak{o}/\mathfrak{a}(c, \mathfrak{p})$ is, in fact, a field and therefore $\mathfrak{a}(c, \mathfrak{p})$ is a maximal ideal.

Conversely, let $\mathfrak{m}$ be any maximal ideal in $\mathfrak{o}$. Then, the field $\mathfrak{o}/\mathfrak{m}$ is finitely generated under addition in view of (ii) and hence has non-zero characteristic p since any subgroup of a finitely generated additive group is again finitely generated, whereas the additive group of all rationals is not. Now let $\mathfrak{p}$ be any maximal ideal in I containing p. Choose $x_1, \ldots, x_n \in X$ such that $\delta = \det(\phi_i(x_j)) \neq 0$, and take h to be the natural number with $\delta \in \mathfrak{p}^{h-1}$ but $\delta \notin \mathfrak{p}^h$. Here, such $x_1, \ldots, x_n$ exist by the linear independence over I of the $\phi_1, \ldots, \phi_n$, and the choice of h is possible since $\delta \neq 0$ and

$$\bigcap_m \mathfrak{p}^m = 0.$$

Next, consider any $\phi \in \bigcap \mathfrak{a}(x, \mathfrak{p})$ $(x \in X)$, and put $\phi^h = \sum c_i \phi_i$ with suitable $c_i \in \mathbf{Z}$. Then, one has

$$\sum c_i \phi_i(x_j) = \phi^h(x_j) = \phi(x_j)^h \in \mathfrak{p}^h, \qquad \text{for each } j = 1, \ldots, n$$

and by using the adjugate (α_{jk}) of the matrix $(\phi_i(x_j))$ one obtains

$$c_k \delta = \sum \phi^h(x_j) \alpha_{jk} \in \mathfrak{p}^h \qquad \text{for } k = 1, \ldots, n.$$

Now, if some c_k were not in $\mathfrak{p}$, there would exist $\beta \in I$ and $\pi \in \mathfrak{p}$ with $1 = \beta c_k + \pi$, which would lead to $\delta = \beta c_k \delta + \pi \delta \in \mathfrak{p}^h$, contradicting the choice of h. Therefore, one has $c_k \in \mathfrak{p} \cap \mathbf{Z} = \mathbf{Z}p$, i.e. $c_k = b_k p$ with suitable $b_k \in \mathbf{Z}$ for each k. It now follows that $\phi^h = p \sum b_i \phi_i \in p\mathfrak{o}$; thus $\phi^h \in \mathfrak{m}$ by $p\mathfrak{o} \subseteq \mathfrak{m}$ (p was the characteristic of $\mathfrak{o}/\mathfrak{m}$), and finally $\phi \in \mathfrak{m}$ since $\mathfrak{m}$ is prime. This shows that

$$\bigcap_{x \in X} \mathfrak{a}(x, \mathfrak{p}) \subseteq \mathfrak{m},$$

and, again, by the primeness of $\mathfrak{m}$, one must have $\mathfrak{a}(c, \mathfrak{p}) \subseteq \mathfrak{m}$ for some $c \in X$. The maximality of $\mathfrak{a}(c, \mathfrak{p})$ finally leads to $\mathfrak{m} = \mathfrak{a}(c, \mathfrak{p})$.

Remark 1. The condition that the domain I be finitely generated under addition is not fully needed in the above proof. It is enough to assume I to be noetherian and such that no prime number has an inverse in I, and then to restrict the ideals $\mathfrak{p} \subseteq I$ in Proposition 1 to those for which $I/\mathfrak{p}$ has prime characteristic.

Remark 2. The arguments used in the above proof also lead to the determination of *all prime ideals* of the ring $\mathfrak{o}$. In fact, *any ideal*

$$\mathfrak{a}(c) = \{\phi \mid \phi \in \mathfrak{o}, \phi(c) = 0\}$$

is a non-maximal prime ideal of $\mathfrak{o}$, *and every such ideal of* $\mathfrak{o}$ *is of this type.* It is obvious that these $\mathfrak{a}(c)$ are prime ideals, non-maximal since $\mathfrak{a}(c)$ is not of the type $\mathfrak{a}(c, \mathfrak{p})$. Conversely, let $\mathfrak{r} \subseteq \mathfrak{o}$ be any prime ideal. If the integral domain $\mathfrak{o}/\mathfrak{r}$ has prime characteristic, then, as above, $\mathfrak{o}/\mathfrak{r}$ is finite, hence a field, and thus $\mathfrak{r}$ maximal. Now, let $\mathfrak{o}/\mathfrak{r}$ have characteristic zero. First, one has $\mathfrak{a}(c) \subseteq \mathfrak{r}$ for some c since

$$\bigcap_{c \in X} \mathfrak{a}(c) = 0 \subseteq \mathfrak{r}.$$

Now, $\mathfrak{o}/\mathfrak{a}(c)$ is also an integral domain of characteristic zero, finitely generated as an additive group, and therefore any one of its non-zero prime ideals is maximal. It follows that the prime ideal $\mathfrak{r}/\mathfrak{a}(c)$ of $\mathfrak{o}/\mathfrak{a}(c)$ is either maximal or zero, and since maximality here would imply the maximality of $\mathfrak{r}$ in $\mathfrak{o}$, one has $\mathfrak{r} = \mathfrak{a}(c)$.

2. Character rings. Let G be a finite group of exponent g_0 and K a field of characteristic zero. Then, the K-character ring $\mathfrak{o}_K(G)$ of G defined in the Introduction satisfies the hypotheses concerning the ring $\mathfrak{o}$ in the previous section: the functions $\phi \in \mathfrak{o}_K(G)$ map G into the integral domain I generated by $\mathbf{Z}$ and the g_0th roots of unity in the algebraic closure Ω of K, the unit character ϵ belongs to $\mathfrak{o}_K(G)$, and the irreducible K-characters $\phi_1, \ldots, \phi_n$ of G are linearly independent over I, the latter since the linear extensions $\tilde{\phi}_i$ of the ϕ_i to functions on the group ring $K[G]$ with values in Ω satisfy the conditions $\tilde{\phi}_i(e_j) = \delta_{ij} d_j$, where e_j are the unit elements of the n simple components of $K[G]$ and the d_j are the dimensions of the corresponding irreducible representation modules. It follows now that the maximal ideals of $\mathfrak{o}_K(G)$ are determined according to Proposition 1. However, in the present situation, a more specific statement holds:

PROPOSITION 2. *The maximal ideals of* $\mathfrak{o}_K(G)$ *are all of the type* $\mathfrak{a}(c, \mathfrak{p})$ *where* c *is* p*-regular for the prime number* $p \in \mathfrak{p}$.

Proof. Recall that p-regularity means that p does not divide the order m of c. If this is not the case, let $m = m_0 p^k$ with natural numbers m_0 and k

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such that $p \nmid m_0$, and, accordingly, $1 = um_0 + vp^k$ with suitable $u, v \in \mathbf{Z}$. Then, for $s = c^{vp^k}$ and $t = c^{um_0}$, one has $c = st$, p does not divide the order of s, and the order of t is a power of p. Now the restriction of each character ϕ_i to the cyclic subgroup (c) of G is an integral linear combination of the irreducible Ω-characters $\xi_1, \ldots, \xi_m$ of (c). Since the orders of the roots of unity $\xi_j(t)$ are powers of p one sees that $\xi_j(t) - 1 \in \mathfrak{p}$; hence

$$\xi_j(c) - \xi_j(s) = \xi_j(s)(\xi_j(t) - 1) \in \mathfrak{p},$$

and thus $\phi_i(c) - \phi_i(s) \in \mathfrak{p}$, which extends to $\phi(c) - \phi(s) \in \mathfrak{p}$ for any $\phi \in \mathfrak{o}_K(G)$. This clearly implies that $\phi(c) \in \mathfrak{p}$ if and only if $\phi(s) \in \mathfrak{p}$ for any $\phi \in \mathfrak{o}_K(G)$, i.e. $\mathfrak{a}(c, \mathfrak{p}) = \mathfrak{a}(s, \mathfrak{p})$.

Remark. The second comment on Proposition 1 gives a description of all prime ideals of $\mathfrak{o}_K(G)$. This is also contained in **(1)**, but the proof there makes use of results which, in the present setting, follow rather than precede the determination of the maximal ideals of $\mathfrak{o}_K(G)$.

Next, the process of inducing functions on the whole group G by means of functions on a subgroup $H \subseteq G$ has to be mentioned. If ψ is any function on H (with values in a ring, say) then let ψ^* be the function on G defined by the formula

$$\psi^*(s) = \sum_r \psi(r^{-1}sr) \qquad (r^{-1}sr \in H, r \in R),$$

where R is a set of representatives for the cosets xH, $x \in G$. If ψ is the character of a representation of H, then ψ^* is the character of the corresponding induced representation of G. In general, the mapping $\psi \to \psi^*$ is additive, and if ϕ is defined on G and constant on the conjugate classes, then $\phi \cdot \psi^* = (\phi|H \cdot \psi)^*$, where $\phi|H$ denotes the restriction of ϕ to H. It follows from this that $(\mathfrak{o}_K(G)|H)^* = \{(\phi|H)^* | \phi \in \mathfrak{o}_K(G)\}$ is an ideal of $\mathfrak{o}_K(G)$, namely the principal ideal generated by ϵ_H^*, ϵ_H being the unit character of H.

Now, consider the set $\mathfrak{H}$ of all subgroups $H \subseteq G$ such that $H = (c)P$ where, for some prime number p, c is p-regular and P a p-Sylow subgroup of the normalizer $N(c)$ of the cyclic subgroup (c). $\mathfrak{H}$ plays the following important role for the ring $\mathfrak{o}_K(G)$:

PROPOSITION 3. $\mathfrak{o}_K(G) = \sum(\mathfrak{o}_K(G) \mid H)^*$ $(H \in \mathfrak{H})$, *i.e. the ideal generated by the induced characters* ϵ_H^*, $H \in \mathfrak{H}$, *is the whole ring* $\mathfrak{o}_K(G)$.

Proof. Assume that the ideal in question is proper. Then, there exists a maximal ideal $\mathfrak{a}(c, \mathfrak{p})$ containing all ϵ_H^*, and by Proposition 2 the element c may be taken p-regular for the prime number $p \in \mathfrak{p}$. Next, let $H = (c)P$ with a p-Sylow subgroup P of $N(c)$. By definition, one has

$$\epsilon_H^*(c) = \sum_r 1 \qquad (r^{-1}c\,r \in (c)P, r \in R),$$

R being a set of representatives for the cosets xH, $x \in G$. If $r \in R$ contributes to this sum, then one has $r^{-1}cr = c^h t$ with suitable integer h and $t \in P$. Now

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$t^{-1}ct = c^m$ for some integer m and $\operatorname{ord}(t) = p^k$ for some natural number k, and therefore

$$(r^{-1}cr)^{p^k} = (c^h t)^{p^k} = c^{h(m+m^2+\ldots+m^{p^k})};$$

since p does not divide the order of c, this leads to $r^{-1}cr \in (c)$ and hence to $r^{-1}(c)r = (c)$, i.e. $r \in N(c)$. Of course, if $r \in N(c)$, then obviously $r^{-1}cr \in (c)P$, and it follows that

$$\epsilon_H^*(c) = (N(c)\colon (c)P).$$

However, P being a p-Sylow subgroup of $N(c)$, p does not divide this index, and hence $\epsilon_H^*(c) \notin \mathfrak{p}$, which contradicts the assumption that $\epsilon_H^* \in \mathfrak{a}(c, \mathfrak{p})$ for *all* $H \in \mathfrak{H}$. This proves the proposition.

As an immediate consequence, based on the obvious inclusion

$$\mathfrak{o}_K(G) \mid H \subseteq \mathfrak{o}_K(H)$$

for any subgroup H, one has the following corollary.

COROLLARY 1. $\mathfrak{o}_K(G) = \sum \mathfrak{o}_K(H)^* \ (H \in \mathfrak{H})$.

Proposition 3 readily leads to a similar statement about the ideals of $\mathfrak{o}_K(G)$:

$$\mathfrak{a} = \sum_{H \in \mathfrak{H}} (\mathfrak{a}|H)^* \qquad \text{for any ideal } \mathfrak{a} \subseteq \mathfrak{o}_K(G).$$

A particular ideal, considered by Witt **(7)**, may be described as follows. Let Δ be a division ring with centre K, of finite dimension over K. Then, any representation Φ of G over Δ determines an associated representation Φ_K over K obtained by considering the representation module for Φ as a K-module, and the character of Φ is defined to be the character of Φ_K. Calling the characters obtained in this fashion the Δ-characters of G, one can make the following remarks **(7)**. The set of all Δ-characters is closed under addition, the product of a Δ-character with any K-character is a Δ-character, and the Δ-characters of a subgroup of G induce Δ-characters on G. As a result of this, the differences of Δ-characters of G form an ideal $\mathfrak{a}_\Delta(G)$ of $\mathfrak{o}_K(G)$, and for any subgroup H of G one has $\mathfrak{a}_\Delta(H)^* \subseteq \mathfrak{a}_\Delta(G)$. Hence, the comment at the beginning of this paragraph and the fact that $\mathfrak{a}_\Delta \mid H \subseteq \mathfrak{a}_\Delta(H)$ for any subgroup H of G lead to the following analogue of Satz 6 in **(7)**.

COROLLARY 2.

$$\mathfrak{a}_\Delta(G) = \sum_{H \in \mathfrak{H}} \mathfrak{a}_\Delta(H)^*.$$

Remark 1. Although Corollary 1 refers to somewhat more complicated subgroups of G than (1) since the products $(c)P$ need not be direct here, it is still good enough for a proof of the absolute irreducibility of the irreducible representations of G over a field containing all g_0th roots of unity. This follows from Witt's generalization of Blichfeldt's theorem **(7)** according to which the

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irreducible representations over an algebraically closed field of a group G with abelian normal subgroup $A \subseteq G$ such that the factors in the principal series of G/A are cyclic are all induced by one-dimensional subgroup representations. The abelian normal subgroup in the present case is, of course, (c).

Remark 2. For analogous reasons, Corollary 2 is sufficient to obtain Satz 7 of **(7)**.

It will now be shown how (1) and its generalization referred to in the Introduction may also be proved with the aid of Proposition 2. The argument is contained essentially in Section 6 of **(7)**, and it will be enough to indicate briefly how it applies here.

Let $g_0 = p^k g_1$, with natural numbers k and g_1 such that $p \nmid g_1$, $K^* = K(\zeta)$ with a primitive g_1th root of unity ζ, J_K the set of those natural numbers i for which $\zeta \to \zeta^i$ determines an automorphism of K^*/K, and $\mathfrak{E}_K$ the set of all subgroups $E \subseteq G$ such that $E = (c)P$ where, for some prime p, c is a p-regular element of G and P a p-Sylow subgroup of the normalizer NC_K of the set $C_K = \{c^i \mid i \in J_K\}$. Clearly, for $K^* = K$ one has $C_K = \{c\}$, and $E = (c)P$ is then the direct product of (c) and P, i.e. E is elementary in the sense of Brauer **(3)**. On the other hand, if K^* has the greatest possible dimension over K, then $\mathfrak{E}_K = \mathfrak{H}$, the set of subgroups used in Proposition 3.

PROPOSITION 4.

$$\mathfrak{o}_K(G) = \sum_{E \in \mathfrak{E}_K} \mathfrak{o}_K(E)^*.$$

Proof. Suppose the ideal

$$\mathfrak{e} = \sum_{E \in \mathfrak{E}_K} \mathfrak{o}_K(E)^*$$

is proper. Then $\mathfrak{e} \subseteq \mathfrak{a}(c, \mathfrak{p})$ with some p-regular $c \in G$ and some prime ideal $\mathfrak{p} \subseteq I$ such that $p \in \mathfrak{p}$. Now, consider any $E = (c)P \in \mathfrak{E}_K$.

According to **(7)**, the linear combination

$$\sum_{\xi, s \in C_K} \xi(s^{-1})\xi,$$

ranging over all *absolutely irreducible* characters of (c), is of the form $\sum \alpha_i \eta_i$ with K-characters η_i of (c) and coefficients $\alpha_i \in I$ since

$$\sum_{s \in C_K} \xi^i(s^{-1}) = \sum_{s \in C_K} \xi(s^{-i}) = \sum_{s \in C_K} \xi(s^{-1})$$

for each $i \in J_K$. Moreover, by the way E is defined in relation to the automorphisms of K^*/K, *every K-character of (c) is the restriction of a K-character of E* (**7**, Section 6, Hilfssatz; also **6**). With extensions χ_i of the η_i to E, which therefore exists, one now considers the function $\sum \alpha_i \eta_i^*$ on G. Since $t^{-1}ct \in E$ holds if and only if $t^{-1}ct \in (c)$ by the same argument as in the proof of Proposition 3, one has, with a set R of representatives for the cosets xE,

$$\begin{aligned}\sum_i \alpha_i \chi_i^*(c) &= \sum_{i,r} \alpha_i \eta_i(r^{-1}cr) && (r \in R, r^{-1}cr \in (c)) \\ &= \sum_{\xi,s,r} \xi(s^{-1})\xi(r^{-1}cr) && (s \in C_K; r \in R, r^{-1}cr \in (c)) \\ &= \sum_{\xi,s,r} \xi(s^{-1})\xi(r^{-1}cr) && (s \in C_K; r \in R, r^{-1}cr \in C_K) \\ &= (NC_K : E) \operatorname{ord} c,\end{aligned}$$

where the second equation follows from the definition of the α_i and η_i, and the last two from the orthogonality relations for the absolutely irreducible characters ξ of (c).

On the other hand, $\mathfrak{e} \subseteq \mathfrak{a}(c, \mathfrak{p})$ implies that $\chi^*(c) \in \mathfrak{p}$ for any K-character χ of E and hence

$$(NC_K : E) \operatorname{ord} c = \sum_i \alpha_i \chi_i^*(c) \in \mathfrak{p},$$

a contradiction because E contains a p-Sylow subgroup of NC_K and c is p-regular. It follows that $\mathfrak{e} = \mathfrak{o}_K(G)$.

Remark. Comparing Proposition 3 (or, more precisely, its first corollary) with Proposition 4, one can make the following observations. In either case, the *number* of subgroups of G which enter into the proof is the same; these are, first, the cyclic subgroups which ensure that the ideal of induced characters is not contained in any $\mathfrak{a}(c, \mathfrak{p})$ with $g_0 \notin \mathfrak{p}$, and, secondly, one subgroup for each $\mathfrak{a}(c, \mathfrak{p})$, where the prime number $p \in \mathfrak{p}$ divides g_0 and c is p-regular. However, the subgroups used to get outside the latter maximal ideals may differ in the two cases. Those that occur in the proof of Proposition 3 have the advantage of not depending on the field K but they may be larger and, therefore, more complicated than those used in the proof of Proposition 4. In any case, for each $\mathfrak{a}(c, \mathfrak{p})$ of the second type the two subgroups can always be chosen such that one is contained in the other: for $E = (c)P$ in $\mathfrak{E}_K$ one can take $H = (c)P'$ in $\mathfrak{H}$ such that the p-Sylow subgroup P' of $N(c)$ contains the p-Sylow subgroup P of NC_K, in view of $NC_K \subseteq N(c)$. Another noteworthy feature of the proof of Proposition 3 is that the orthogonality relations for characters are not required here and only the identity characters enter into it.

3. Conclusion. For a finite abelian group A, Proposition 1 leads to the following observation. The simple $\mathbf{Z}[A]$-modules are the additive groups $M(\chi, \mathfrak{p})$, images of $\mathbf{Z}[A]$ under the homomorphism $\nu_{\mathfrak{p}}\chi$, where χ is the linear extension to $\mathbf{Z}[A]$ of any absolutely irreducible character of A and $\nu_{\mathfrak{p}}$ is the natural homomorphism $I \to I/\mathfrak{p}$ for some maximal ideal $\mathfrak{p} \subseteq I$, with $a \in A$ acting on $M(\chi, \mathfrak{p})$ by $a(\nu_{\mathfrak{p}}\chi(x)) = \nu_{\mathfrak{p}}\chi(ax)$. This is an immediate consequence of the natural isomorphism $\mathbf{Z}[A] \to \mathfrak{o}_\Omega(A^*)$, A^* the character group of A, in view of which the maximal ideals of $\mathbf{Z}[A]$ are all of the type

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$$\mathfrak{a}(\chi, \mathfrak{p}) = \{x \mid x \in \mathbf{Z}[A], \chi(x) \in \mathfrak{p}\},$$

with χ and $\mathfrak{p}$ as above.

The methods employed in the previous sections can also be used to obtain the following analogue of Proposition 2. Let $\mathfrak{O}$ be the ring of functions on the finite group G with values in the domain I of the g_0th roots of unity which consists of all linear combinations, *with coefficients from* I, of the absolutely irreducible characters of G. Then, the maximal ideals of $\mathfrak{O}$ are again the ideals $\mathfrak{A}(c, \mathfrak{p}) = \{\phi \mid \phi \in \mathfrak{O}, \phi(c) \in \mathfrak{p}\}$ with $c \in G$ and $\mathfrak{p}$ any maximal ideal in I. As in the previous case, $\mathfrak{A}(a, \mathfrak{p}) = \mathfrak{A}(b, \mathfrak{p})$ if a and b are p-conjugate, i.e. have conjugate p-regular parts. Moreover, in this case one can also show the converse, using an argument from **(5)**, and therefore the different maximal ideals in $\mathfrak{O}$ associated with the same $\mathfrak{p} \subseteq I$ correspond to the different p-conjugate classes in G. From this one can easily deduce the results of **(5)** concerning the extension of $\mathfrak{o}_\Omega(G)$ by means of arbitrary coefficients from the ring $I_\mathfrak{p}$ of $\mathfrak{p}$-adic integers.

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