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BOHR PHENOMENON FOR OPERATOR-VALUED FUNCTIONS

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Abstract We establish Bohr inequalities for operator-valued functions, which can be viewed as analogues of a couple of interesting results from scalar-valued settings. Some results of this paper are motivated by the classical flavour of Bohr inequality, while others are based on a generalized concept of the Bohr radius problem.

Keywords: Bohr radius; harmonic function; biholomorphic function

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1. Introduction

The following remarkable result was proved by Harald Bohr [9] in 1914.

Theorem A. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be holomorphic in the open unit disk $\mathbb{D}$ and $|f(z)|<1$ for all $z\in\mathbb{D}$, then*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \tag{1.1}$$

for all $z\in\mathbb{D}$ with $|z|=r\leq 1/6$.

This constant $r\leq 1/6$ was sharpened to $r\leq 1/3$ by Wiener, Riesz and Schur independently, and the inequality (1.1) is popularly known as the *Bohr inequality* nowadays. This theorem was an outcome of the investigation on the absolute convergence problem for Dirichlet series of the form $\sum a_n n^{-s}$, but presently it has become an independent area of research. The Bohr radius problem saw a surge of interest from many mathematicians after it found an application to the characterization problem of Banach algebras satisfying the von Neumann inequality [14]. A part of the subsequent research in this area is directed towards extending the Bohr phenomenon in a multidimensional framework and in more abstract settings (see, for example [3, 4, 8, 19, 23–25]). The Bohr phenomenon is shown to have connections with local Banach space theory (cf. [12]). Also, the Bohr inequality is studied in the settings of ordinary and vector-valued Dirichlet series (see f.i. [5, 13]).

We now give a brief overview of the approaches to extend the Bohr inequality in two different settings. One of them aims at investigating the Bohr radius problem from an operator theoretic perspective. To be more specific, the Bohr phenomenon has been established in [22, Theorem 2.1] using positivity methods for operator-valued holomorphic functions, i.e. holomorphic functions from $\mathbb{D}$ to $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on a complex Hilbert space $\mathcal{H}$. Suitable assumptions in terms of operator inequalities are made to replicate the scalar-valued cases. It may be mentioned here that the inequalities recorded in [22, Theorem 2.1] are operator-valued analogues of the classical Bohr inequality in Theorem A. In the present article, we prove Bohr inequalities of similar nature for harmonic functions from $\mathbb{D}$ to $\mathcal{B}(\mathcal{H})$. It is also worth mentioning that some other versions of operator-valued Bohr inequality for non-commutative harmonic functions are available in [24, 25].

Another aspect of the Bohr phenomenon thrives on considering the Bohr radius problem for a holomorphic map g from $\mathbb{D}$ into a domain $\Omega \subsetneq \mathbb{C}$ other than $\mathbb{D}$. The key idea to accomplish that is to identify g as a member of $S(f)$, $S(f)$ being the class of functions subordinate to f, while f is the covering map from $\mathbb{D}$ onto Ω satisfying $f(0) = g(0)$. Here we clarify that for two holomorphic functions g and f in $\mathbb{D}$, we say that g is subordinate to f if there exists a function ϕ, holomorphic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, satisfying $g = f \circ \phi$. Throughout this article, we denote g is subordinate to f by $g \prec f$. A suitable definition for the Bohr phenomenon of $g \in S(f)$ was given in [1] to serve the purpose stated above, which we briefly describe here. Let the Taylor expansions of f and g in a neighbourhood of the origin be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.2}$$

and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k, \tag{1.3}$$

respectively. We will say that $S(f)$ has the Bohr phenomenon if for any $g \in S(f)$, where f and g have the Taylor expansions of the form (1.2) and (1.3), respectively in $\mathbb{D}$, there is a r_0, $0 < r_0 \leq 1$ so that

$$\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega) \tag{1.4}$$

for $|z| = r < r_0$. Here $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of the domain $\Omega = f(\mathbb{D})$. To see that this definition is indeed a generalization of the classical Bohr phenomenon, we observe that whenever $\Omega = \mathbb{D}$; $d(f(0), \partial\Omega) = 1 - |f(0)|$, and in this case (1.4) reduces to (1.1). However, to the best of our knowledge, no attempt has been made so far to obtain operator-valued analogues of the Bohr phenomenon for complex-valued functions treated according to the aforesaid definition from [1]. Therefore, another goal of the present article is to find the same under appropriate considerations and necessary restrictions. More precisely, we will consider a function f from $\mathbb{D}$ to $\mathcal{B}(\mathcal{H})$, and prove the Bohr inequality when f is holomorphic and satisfies

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certain conditions which, when restricted to the scalar-valued case, coincide with the situation that f maps $\mathbb{D}$ into its exterior, i.e. $\overline{\mathbb{D}}^c = \{z \in \mathbb{C} : |z| > 1\}$. Also, we prove the Bohr phenomenon for any $g \in S(f)$, when f is a convex or starlike biholomorphic function. Here we clarify that, given two complex Banach spaces X and Y and a domain $D \subset X$, a holomorphic mapping $f : D \to Y$ is said to be *biholomorphic* on D if $f(D)$ is a domain in Y, and f^{-1} exists and is holomorphic on $f(D)$. A biholomorphic function f is said to be starlike on its domain D with respect to $z_0 \in D$ if $f(D)$ is a starlike domain with respect to $f(z_0)$, i.e. $(1-t)f(z_0) + tf(z) \in f(D)$ for all $z \in D$ and $t \in [0, 1]$, and f is called *starlike biholomorphic* on D if f is starlike with respect to $0 \in D$ and $f(0) = 0$. Now a biholomorphic function f defined in D is said to be *convex* if f is starlike with respect to all $z \in D$. In particular, here we will work with $D = \mathbb{D}$, $X = \mathbb{C}$ and $Y = \mathcal{B}(\mathcal{H})$. It may be noted that the definition of subordination and the class $S(f)$ for operator-valued holomorphic functions can be adopted from the scalar case without any change. Now we fix some notation for the rest of our discussions. For any $A \in \mathcal{B}(\mathcal{H})$, $\|A\|$ will always denote the operator norm of A, and A^* is the adjoint of A. The operators $\mathrm{Re}(A) := (A + A^*)/2$, $\mathrm{Im}(A) := (A - A^*)/2i$ and $|A| := (A^*A)^{1/2}$ bear their usual meaning, while $B^{1/2}$ denotes the unique positive square root of a positive operator B. Also, $\sigma(A)$ will be recognized as the spectrum of A, i.e. the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is non-invertible, I being the identity operator on $\mathcal{H}$.

2. Main results

A function $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is harmonic if and only if

$$f(z) = \sum_{n=0}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n^* \overline{z}^n, \tag{2.1}$$

where $A_n, B_n \in \mathcal{B}(\mathcal{H})$ for all $n \in \mathbb{N} \cup \{0\}$, and the series converges absolutely and locally uniformly in $\mathbb{D}$ (see, for example [17, Sec. 2.4, p. 352]). Bohr inequalities for complex-valued harmonic functions have already been obtained in [1, Theorem 2]. The aim of the first theorem is to derive inequalities of similar nature for operator-valued harmonic functions. For this purpose, we need to establish the following analogue of the Cauchy–Schwarz inequality.

Lemma 1. *Let $\{H_n\}_{n=0}^{\infty}$ be a sequence in $\mathcal{B}(\mathcal{H})$ such that $\sum_{n=0}^{\infty} |H_n|^2 \in \mathcal{B}(\mathcal{H})$. Then for any fixed $r \in [0, 1)$, and for any fixed non-negative integer k,*

$$\sum_{n=k}^{\infty} |H_n| r^n \le \frac{r^k}{\sqrt{1-r^2}} \left(\sum_{n=k}^{\infty} |H_n|^2 \right)^{1/2}. \tag{2.2}$$

Proof. For any fixed $m \in \mathbb{N}$ such that $m > k$, and for any $x \in \mathcal{H}$, it is immediately seen that

$$\left\langle \left(\sum_{n=k}^{m} |H_n| r^n \right)^2 x, x \right\rangle = \left\| \sum_{n=k}^{m} r^n |H_n| x \right\|^2 \le \left(\sum_{n=k}^{m} r^n \|H_n x\| \right)^2 \tag{2.3}$$

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for any fixed $r \in [0, 1)$. Now, a use of the Cauchy–Schwarz inequality on the right-hand side of the inequality (2.3) yields

$$\left\langle \left(\sum_{n=k}^{m} |H_n| r^n\right)^2 x, x \right\rangle \leq \left(\sum_{n=k}^{m} \|H_n x\|^2\right) \left(\sum_{n=k}^{m} r^{2n}\right), \tag{2.4}$$

which implies that

$$\left\langle \left(\sum_{n=k}^{m} |H_n| r^n\right)^2 x, x \right\rangle \leq \left\langle \sum_{n=k}^{m} |H_n|^2 x, x \right\rangle \frac{r^{2k}}{1 - r^2}. \tag{2.5}$$

Letting $m \to \infty$ in (2.5) we get

$$\left(\sum_{n=k}^{\infty} |H_n| r^n\right)^2 \leq \frac{r^{2k}}{1 - r^2} \left(\sum_{n=k}^{\infty} |H_n|^2\right), \tag{2.6}$$

from which (2.2) will follow (cf. [11, p. 244, Ex. 12]). □

We now state the first theorem of this article after all these preparations.

Theorem 1. *Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be a harmonic function with an expansion (2.1) such that $\|f(z)\| \leq 1$ for each $z \in \mathbb{D}$. Then*

(i) $|\mathrm{Re}(e^{i\mu} A_0)| + \sum_{n=1}^{\infty} |e^{i\mu} A_n + e^{-i\mu} B_n| r^n \leq (\sqrt{1 + 3r^2}/\sqrt{1 - r^2}) I$ *for* $|z| = r \in [0, 1)$, *and for any $\mu \in \mathbb{R}$. In particular, if we assume in addition that $e^{i\mu} A_n + e^{-i\mu} B_n$ is normal for each $n \in \mathbb{N}$, then the quantity in the right-hand side of the above inequality can be replaced by $(\sqrt{1 + r^2}/\sqrt{1 - r^2}) I$.*

(ii) $\sum_{n=1}^{\infty} \|e^{i\mu} A_n + e^{-i\mu} B_n\| r^n \leq \|I - \mathrm{Re}(e^{i\mu} A_0)\|$ *for $|z| = r \leq 1/5$, and for any $\mu \in \mathbb{R}$. Moreover, if we take $e^{i\mu} A_n + e^{-i\mu} B_n$ to be normal for each $n \in \mathbb{N}$, then the above inequality will hold for $r \leq 1/3$ instead of $r \leq 1/5$.*

(iii) $\sum_{n=1}^{\infty} |A_n| r^n + \sum_{n=1}^{\infty} |B_n^*| r^n \leq (1/2) I$ *for $|z| = r \leq 1/3$.*

Proof. (i) It is easy to observe that for each $z \in \mathbb{D}$, and for any $\mu \in \mathbb{R}$,

$$|\mathrm{Re}(e^{i\mu} f(z))|^2 + |\mathrm{Im}(e^{i\mu} f(z))|^2 = (1/2)(f(z) f(z)^* + f(z)^* f(z)).$$

We here note that for any $A \in \mathcal{B}(\mathcal{H})$, $\langle |A|^2 x, x \rangle = \|Ax\|^2 \leq \|A\|^2 \langle x, x \rangle$ for any $x \in \mathcal{H}$, i.e. $|A|^2 \leq \|A\|^2 I$. Using this fact, and that $\|A\| = \|A^*\|$ for any $A \in \mathcal{B}(\mathcal{H})$, we

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obtain $|\mathrm{Re}(e^{i\mu}f(z))|^2 + |\mathrm{Im}(e^{i\mu}f(z))|^2 \leq \|f(z)\|^2 I$, and therefore

$$|\mathrm{Re}(e^{i\mu}f(z))|^2 \leq I. \tag{2.7}$$

Now

$$\mathrm{Re}(e^{i\mu}f(z)) = \mathrm{Re}(e^{i\mu}A_0) + (1/2)\sum_{n=1}^{\infty}(P_n z^n + P_n^* \overline{z}^n), \tag{2.8}$$

where $P_n = e^{i\mu}A_n + e^{-i\mu}B_n$. Now from (2.7) we can write, for any $z = re^{i\theta} \in \mathbb{D}$ and for any $x \in \mathcal{H}$,

$$\langle(\mathrm{Re}(e^{i\mu}f(re^{i\theta})))^*(\mathrm{Re}(e^{i\mu}f(re^{i\theta})))x, x\rangle \leq \langle x, x\rangle.$$

We plug the expression (2.8) in and fix $r \in [0, 1)$ in the above inequality, and thereafter integrating both sides of this inequality over θ from 0 to 2π we get

$$\langle|\mathrm{Re}(e^{i\mu}A_0)|^2 x, x\rangle + (1/4)\sum_{n=1}^{\infty}(\langle P_n^* P_n x, x\rangle + \langle P_n P_n^* x, x\rangle)r^{2n} \leq \langle Ix, x\rangle.$$

Therefore, we conclude

$$|\mathrm{Re}(e^{i\mu}A_0)|^2 + (1/4)\sum_{n=1}^{\infty}(|P_n|^2 + |P_n^*|^2) \leq I,$$

which implies

$$\sum_{n=1}^{\infty}|P_n|^2 \leq 4(I - |\mathrm{Re}(e^{i\mu}A_0)|^2). \tag{2.9}$$

Hence, a direct use of Lemma 1 (with $H_n = P_n$, $k = 1$) gives

$$|\mathrm{Re}(e^{i\mu}A_0)| + \sum_{n=1}^{\infty}|e^{i\mu}A_n + e^{-i\mu}B_n|r^n \leq T + \frac{2r}{\sqrt{1-r^2}}(I - T^2)^{1/2}, \tag{2.10}$$

where $T = |\mathrm{Re}(e^{i\mu}A_0)|$. The first half of part (i) of our theorem will now follow from a computation similar to the proof of [22, Theorem 2.1, part 4], applied to (2.10). For the sake of completeness, we include brief details of the calculation. Considering the real valued function $\psi(x) = x + (2r/\sqrt{1-r^2})\sqrt{1-x^2}$ on the interval $[0, 1]$, we see that ψ attains its maximum at $x_0 = \sqrt{1-r^2}/\sqrt{1+3r^2}$, and that $\psi(x) \leq \psi(x_0) = \sqrt{1+3r^2}/\sqrt{1-r^2}$ for any $x \in [0, 1]$. This validates our first assertion. Further, if we assume that P_n is normal for each $n \in \mathbb{N}$, then $|P_n|^2 = |P_n^*|^2$, which implies that the inequality (2.9) can be improved to $\sum_{n=1}^{\infty}|P_n|^2 \leq 2(I - |\mathrm{Re}(e^{i\mu}A_0)|^2)$. Rest of the proof can be completed by following the similar lines of computation as we did for the previous one.

(ii) In order to establish the second part of this theorem, we first observe that if $K(z) = e^{i\mu}A_0 + \sum_{n=1}^{\infty}P_n z^n$, then (2.8) implies that $\mathrm{Re}(K(z)) = \mathrm{Re}(e^{i\mu}f(z))$, and hence from (2.7), we get $\|\mathrm{Re}(K(z))\| \leq 1$. Now considering $\hat{K}(z) = \langle K(z)x, x\rangle$ for any

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fixed $x \in \mathcal{H}$ with $\|x\| = 1$, it is easily seen that $|\mathrm{Re}(\hat{K}(z))| \leq 1$. Therefore, $\hat{K}$ is holomorphic in $\mathbb{D}$ with an expansion

$$\hat{K}(z) = \langle e^{i\mu} A_0 x, x\rangle + \sum_{n=1}^{\infty} \langle P_n x, x\rangle z^n,$$

which maps $\mathbb{D}$ into the vertical strip $|\mathrm{Re}(z)| \leq 1$. As a consequence,

$$|\langle P_n x, x\rangle| \leq 2(1 - |\mathrm{Re}(\langle e^{i\mu} A_0 x, x\rangle)|)$$

for all $n \in \mathbb{N}$ (see, f.i. [1, Lemma 3]). Further, using the triangle inequality, we obtain

$$|\langle P_n x, x\rangle| \leq 2|\langle (I - \mathrm{Re}(e^{i\mu} A_0))x, x\rangle|. \tag{2.11}$$

Taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ on both sides of the inequality (2.11), we get

$$\sup_{\|x\|=1} |\langle P_n x, x\rangle| \leq 2\|I - \mathrm{Re}(e^{i\mu} A_0)\|,$$

and replacing A by P_n in [15, Theorem 1.2], we have

$$\sup_{\|x\|=1} |\langle P_n x, x\rangle| \geq (1/2)\|P_n\|.$$

Combining the above two results, we obtain

$$\|P_n\| \leq 4\|I - \mathrm{Re}(e^{i\mu} A_0)\| \tag{2.12}$$

for all $n \in \mathbb{N}$. The first half of part (ii) now follows from (2.12). Now it is known that $\|P_n\| = \sup\{|\langle P_n x, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$ whenever P_n is normal (cf. [15, p. 266], and replace A by P_n). Hence from (2.11) we obtain $\|P_n\| \leq 2\|I - \mathrm{Re}(e^{i\mu} A_0)\|$, which will prove the second assertion of part (ii).

(iii) Finally, since $\|f(z)\| \leq 1$ if and only if $|f(z)|^2 \leq I$, using methods similar to the proof of part (i) we are able to deduce

$$|A_0|^2 + \sum_{n=1}^{\infty} (|A_n|^2 + |B_n^*|^2) \leq I. \tag{2.13}$$

Observing that $|A_n|^2 + |B_n^*|^2 \geq (1/2)(|A_n| + |B_n^*|)^2$ for all $n \in \mathbb{N}$, (2.13) yields

$$\sum_{n=1}^{\infty} (|A_n| + |B_n^*|)^2 \leq 2I. \tag{2.14}$$

Therefore, applying Lemma 1 (letting $H_n = |A_n| + |B_n^*|$, $k = 1$) and (2.14) together, we get

$$\sum_{n=1}^{\infty} |A_n| r^n + \sum_{n=1}^{\infty} |B_n^*| r^n \leq \left(\frac{r\sqrt{2}}{\sqrt{1 - r^2}} \right) I, \tag{2.15}$$

from which part (iii) will directly follow. □

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Remark. The following observations are made in connection with the above theorem.

(i) Under the assumption that $e^{i\mu}A_n + e^{-i\mu}B_n$ ($\mu \in \mathbb{R}$) is normal for each $n \in \mathbb{N}$, from part (ii) of Theorem 1, we can write

$$\|\mathrm{Re}(e^{i\mu}A_0)\| + \sum_{n=1}^{\infty} \|e^{i\mu}A_n + e^{-i\mu}B_n\| r^n \leq \|\mathrm{Re}(e^{i\mu}A_0)\| + \|I - \mathrm{Re}(e^{i\mu}A_0)\|$$

for $r \leq 1/3$. When restricted to the scalar-valued case, this inequality reduces to

$$|\mathrm{Re}(e^{i\mu}A_0)| + \sum_{n=1}^{\infty} |e^{i\mu}A_n + e^{-i\mu}B_n| r^n \leq |\mathrm{Re}(e^{i\mu}A_0)| + |1 - \mathrm{Re}(e^{i\mu}A_0)| \tag{2.16}$$

for $r \leq 1/3$, where the coefficients A_0, A_n, B_n are complex numbers. Now since without loss of generality, we may consider $\mathrm{Re}(e^{i\mu}A_0) \geq 0$, therefore, the second part of [1, Theorem 2] follows directly from (2.16).

(ii) Part (iii) of Theorem 1 can be thought of as an operator-valued analogue of the very recent result from [18, p. 867, Sec. 4.4] which improves the first part of [1, Theorem 2].

(iii) If we set $B_n = 0$ for all $n \in \mathbb{N}$ in (2.1), i.e. f is taken to be a holomorphic function from $\mathbb{D}$ to $\mathcal{B}(\mathcal{H})$ with an expansion $f(z) = \sum_{n=0}^{\infty} A_n z^n$, then (2.13) takes the form $\sum_{n=0}^{\infty} |A_n|^2 \leq I$. Therefore, an application of Lemma 1 (with $H_n = A_n$, $k = 0$) yields

$$\sum_{n=0}^{\infty} |A_n| r^n \leq \left(\frac{1}{\sqrt{1-r^2}}\right) I, \quad r \in [0, 1). \tag{2.17}$$

We observe that $1/\sqrt{1-r^2} \leq (1 + r^2/(1-r)^2)^{1/2}$, and therefore (2.17) is an improvement over the inequality recorded in [22, Remark 2.2]. Moreover, from the scalar valued results (compare [10, Theorem 1.1]), we observe that the quantity $1/\sqrt{1-r^2}$ in inequality (2.17) is the 'best possible', in the sense that for the function $f(z) = ((z - 1/\sqrt{2})/(1 - z/\sqrt{2}))I, z \in \mathbb{D}$, equality occurs in (2.17) at $r = 1/\sqrt{2}$.

In the next result, we establish an operator-valued analogue of Bohr's inequality for holomorphic mappings from $\mathbb{D}$ into the exterior of $\mathbb{D}$, i.e. $\overline{\mathbb{D}}^c = \{z \in \mathbb{C} : |z| > 1\}$ (cf. [2, Theorem 2.1]). In order to prove this, we now introduce the notions of the spherical and the Hausdorff distance. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. The spherical distance λ between two points $z_1, z_2 \in \hat{\mathbb{C}}$ is given by

$$\lambda(z_1, z_2) = \begin{cases} \dfrac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, & \text{if } z_1, z_2 \in \mathbb{C}, \\ \dfrac{1}{\sqrt{1+|z_1|^2}}, & \text{if } z_2 = \infty. \end{cases}$$

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Also, it is well known that the collection $\mathcal{C}$ of compact subsets of $\mathbb{C}$ is a metric space with respect to the Hausdorff distance d_h given by

$$d_h(A,B) = \max\left\{\sup_{x\in A}\operatorname{dist}(x,B),\ \sup_{x\in B}\operatorname{dist}(x,A)\right\},\quad A,B\in\mathcal{C},$$

where $\operatorname{dist}(p,E) := \inf\{|p-e| : e\in E\}$ for any $E\subset\mathbb{C}$ and for any $p\in\mathbb{C}$. Now since for any $A\in\mathcal{B}(\mathcal{H})$, $\sigma(A)\in\mathcal{C}$, we are able to consider the mapping $A\mapsto\sigma(A)$ from $\mathcal{B}(\mathcal{H})$ to the metric space$(\mathcal{C},d_h)$, which is continuous on the subset of normal operators, equipped with the operator norm (see f.i. [20]).

Theorem 2. *Suppose $f:\mathbb{D}\to\mathcal{B}(\mathcal{H})$ is holomorphic with an expansion*

$$f(z) = \sum_{n=0}^{\infty} A_n z^n,\quad z\in\mathbb{D} \tag{2.18}$$

such that $|f(z)|>I$ for all $z\in\mathbb{D}$. Also, suppose $f(z)$ is normal for each $z\in\mathbb{D}$, $f(0)=A_0>0$ and $\sigma(f(z))$ does not separate 0 from ∞ for any $z\in\mathbb{D}$. Then

$$\lambda\left(\sum_{n=0}^{\infty}\|A_n\|r^n, \|A_0\|\right) \le \lambda\left(\|A_0\|,1\right) \tag{2.19}$$

for $|z|=r\le(2(\log\|A_0\|/\|\log A_0\|)-1)/(2(\log\|A_0\|/\|\log A_0\|)+1)$.

Proof. Since $|f(z)|>I$, we have $\langle|f(z)|x,x\rangle > \langle x,x\rangle$ for any $x\in\mathcal{H}\setminus\{0\}$, and for each $z\in\mathbb{D}$. A use of the Cauchy–Schwarz inequality exhibits that $\|f(z)x\|>\|x\|$, which further implies that $\|(f(z)-\lambda I)x\|>(1-|\lambda|)\|x\|$ for any $\lambda\in\mathbb{C}$, i.e. $f(z)-\lambda I$ is bounded below for any $\lambda\in\mathbb{D}$. As $f(z)$ is normal, $\sigma(f(z))\subset\mathbb{D}^c$ for each $z\in\mathbb{D}$. Since $\sigma(f(z))$ does not separate 0 from ∞, it is therefore possible to choose a holomorphic single-valued branch of complex logarithm on a simply connected domain Δ_z that contains $\sigma(f(z))$, but does not contain 0. As a consequence, we are able to define $\log f(z)$ as follows:

$$\log f(z) = \frac{1}{2\pi i}\int_{\Gamma}(\log\xi)(\xi I - f(z))^{-1}\mathrm{d}\xi,\quad z\in\mathbb{D}, \tag{2.20}$$

where Γ is a system of closed, positively oriented, rectifiable curves inside Δ_z which encloses $\sigma(f(z))$ (cf. [11, pp. 199–201]). Now it is also known that for each fixed $z\in\mathbb{D}$, $\log f(z)$ is normal, and $(\log f(z))^* = F(f(z)^*)$, where $F(z)=\overline{\log\overline{z}}$ (see f.i. [11, p. 205, Ex. 7, 8]). As $\exp z$ is an entire function and $\exp(\overline{\log\overline{z}})=z$, it follows that $\exp((\log f(z))^*) = f(z)^*$ (see [11, p. 205, Ex. 4]). As a consequence of these facts, we obtain $\exp(2\mathrm{Re}(\log f(z))) = f(z)^* f(z)$. It is easy to see that for any $x\in\mathcal{H}\setminus\{0\}$,

$$\langle\exp(2\mathrm{Re}(\log f(z)))x,x\rangle = \langle f(z)^* f(z)x,x\rangle = \|f(z)x\|^2 > \|x\|^2,$$

which, after an application of the Cauchy–Schwarz inequality asserts that

$$\|\exp(2\mathrm{Re}(\log f(z)))x\| > \|x\|.$$

Therefore, $\sigma(\exp(2\mathrm{Re}(\log f(z))))\subset\mathbb{D}^c$, and since the operator $\exp(2\mathrm{Re}(\log f(z)))$ is positive, we conclude that $\sigma(\exp(2\mathrm{Re}(\log f(z))))\subset[1,\infty)$. Now we know that

$\sigma(2\text{Re}(\log f(z))) \subset \mathbb{R}$, and hence $\exp(\sigma(2\text{Re}(\log f(z)))) \subset (0,\infty)$. As a result, choosing the principal branch of complex logarithm over the slit plane $\mathbb{C} \setminus (-\infty, 0]$, we get $\log(\exp(2\text{Re}(\log f(z)))) = 2\text{Re}(\log f(z))$. Now applying the spectral mapping theorem, we conclude that

$$\sigma(2\text{Re}(\log f(z))) = \log\left(\sigma(\exp(2\text{Re}(\log f(z))))\right) \subset [0,\infty).$$

As $2\text{Re}(\log f(z))$ is self adjoint, $2\text{Re}(\log f(z)) \geq 0$. Moreover, as $A_0 > 0$, $\sigma(A_0) \subset [1,\infty)$. Hence to define $\log A_0$ from (2.20), we choose, in particular, the principal branch of complex logarithm on the simply connected domain $\Delta_0 = \mathbb{C} \setminus (-\infty, 0]$ containing $\sigma(A_0^*) = \sigma(A_0)$. Now as $F(z) = \log z$ over $[1,\infty)$, $F(z) = \log z$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Therefore, $(\log A_0)^* = \log A_0^* = \log A_0$, which in turn gives $\log A_0 \geq 0$. Our aim is now to show that $\log f(z)$ is holomorphic at each $z \in \mathbb{D}$. As $f(z)$ is holomorphic, and therefore continuous on $\mathbb{D}$, $\lim_{h\to 0} \|f(z+h) - f(z)\| = 0$. Since $f(z)$ is also normal for each $z \in \mathbb{D}$, we have

$$\lim_{h\to 0} d_h(\sigma(f(z+h)), \sigma(f(z))) = 0.$$

Thus, we infer that for any $h \in \mathbb{C}$ with $|h|$ small enough, $\sigma(f(z+h))$ is enclosed by Γ again. As a result, we are able to show that the limit

$$\lim_{h\to 0} \frac{1}{2\pi i h} \int_\Gamma (\log \xi)\left((\xi I - f(z+h))^{-1} - (\xi I - f(z))^{-1}\right) \mathrm{d}\xi$$

exists and is equal to

$$\frac{1}{2\pi i} \int_\Gamma (\log \xi)(\xi I - f(z))^{-1} f'(z)(\xi I - f(z))^{-1} \mathrm{d}\xi,$$

thereby proving that $\log f(z)$ is holomorphic in $\mathbb{D}$. In view of the above discussion, there exist a Hilbert space $\mathcal{K}$, a unitary operator U on $\mathcal{K}$ and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that $2\log A_0 = V^*V$ and $C_n = V^*U^nV$ for all $n \geq 1$, where $\log f(z) = \log A_0 + \sum_{n=1}^\infty C_n z^n$, $z \in \mathbb{D}$ (see f.i. [21, Ex. 3.15, 3.16, 4.14]). Hence for any $z \in \mathbb{D}$, we have

$$2\log f(z) = V^*(I + zU)(I - zU)^{-1}V,$$

which immediately gives

$$f(z) = \exp\left((1/2)V^*(I + zU)(I - zU)^{-1}V\right). \tag{2.21}$$

From (2.21), it can be observed that all the 'A_n's are the combinations of U, V and V^*, associated with non-negative real constants only. Therefore, a use of the triangle inequality will provide the upper bounds for '$\|A_n\|$'s, which are the combinations of $\|U\| = 1$, $\|V\| = \|V^*\|$, associated with the same constants. Hence after appropriate rearrangement,

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we find that

$$\sum_{n=0}^{\infty} \|A_n\| r^n \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\|V\|^2}{2} \frac{1+r}{1-r} \right)^n = \exp\left(\frac{\|V\|^2}{2} \frac{1+r}{1-r} \right) \tag{2.22}$$

for any $|z| = r$. As $\|V\|^2 = 2\|\log A_0\|$, we therefore get

$$\sum_{n=0}^{\infty} \|A_n\| r^n \leq \exp(2 \log \|A_0\|) = \|A_0\|^2, \tag{2.23}$$

whenever $r \leq r_0 := (2(\log \|A_0\| / \|\log A_0\|) - 1)/(2(\log \|A_0\| / \|\log A_0\|) + 1)$. Now if α, β, γ are non-negative real numbers satisfying $\gamma \leq \alpha \leq \beta$, then it is easily seen that

$$(\alpha - \gamma)^2(1+\beta^2) - (\beta - \gamma)^2(1+\alpha^2) = (\alpha - \beta)((\alpha - \gamma) + (\beta - \gamma) \\ + \alpha\gamma(\beta - \gamma) + \beta\gamma(\alpha - \gamma)) \leq 0.$$

As a consequence, $(\alpha - \gamma)/\sqrt{1+\alpha^2} \leq (\beta - \gamma)/\sqrt{1+\beta^2}$, which readily gives

$$\lambda(\alpha, \gamma) \leq \lambda(\beta, \gamma). \tag{2.24}$$

Setting $\alpha = \sum_{n=0}^{\infty} \|A_n\| r^n$, $\beta = \|A_0\|^2$ and $\gamma = \|A_0\|$, we observe that $\gamma \leq \alpha \leq \beta$ if $r \leq r_0$, and therefore from (2.24), we get

$$\lambda \left(\sum_{n=0}^{\infty} \|A_n\| r^n, \|A_0\| \right) \leq \lambda \left(\|A_0\|^2, \|A_0\| \right)$$

for $r \leq r_0$. A little computation using the AM–GM inequality yields

$$\lambda(\|A_0\|^2, \|A_0\|) \leq \|A_0\|(\|A_0\| - 1)/(\sqrt{1+\|A_0\|^2}\sqrt{2}\|A_0\|) \\ = (\|A_0\| - 1)/(\sqrt{2}\sqrt{1+\|A_0\|^2}) = \lambda(\|A_0\|, 1).$$

It is now clear that an application of the above inequality upon the right-hand side of the previous one will complete the proof. □

Remark. It does not seem plausible that we can get a uniform bound on $|z|$ which is not dependent on A_0 and will still imply (2.19). Nevertheless, if f is taken to be scalar valued, then since it is always possible to assume that $f(0) > 0$, the quantity $(2(\log \|A_0\| / \|\log A_0\|) - 1)/(2(\log \|A_0\| / \|\log A_0\|) + 1)$ converts to the constant $1/3$, and $\lambda(\|A_0\|, 1) = \lambda(A_0, \partial\Omega)$, A_0 being an element of $\mathbb{C}$ and $\partial\Omega$ being the boundary of $\overline{\mathbb{D}}^c$. Therefore, Theorem 2 provides an operator-valued analogue of [2, Theorem 2.1]. It is interesting to note that here one has to consider the spherical distance between complex numbers to obtain the Bohr inequality instead of the Euclidean distance used in (1.4).

We will now discuss the operator-valued analogues of the Bohr radius problem for the subordination classes of functions which belong to well-known subclasses of scalar-valued univalent functions. We therefore consider f to be biholomorphic for our purpose. Now it is possible to carry out further investigation if we restrict f to some subclass

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of biholomorphic functions. In particular, we intend to establish Bohr inequalities for $g \in S(f)$ where $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ is a convex or starlike biholomorphic function. Apart from the definitions given in the introduction, the reader is urged to glance through [16] for a rich exposition of Banach space valued starlike and convex biholomorphic functions. For our purpose, we suppose that $g \in S(f)$ has an expansion

$$g(z) = \sum_{k=0}^{\infty} B_k z^k, \quad z \in \mathbb{D}. \tag{2.25}$$

Also, we mention that for any scalar-valued univalent function F defined on $\mathbb{D}$, the Euclidean distance between $F(0)$ and the boundary $\partial\Omega$ of $\Omega = F(\mathbb{D})$ is given by $d(F(0), \partial\Omega) = \liminf_{|z|\to 1-} |F(z) - F(0)|$, which will be used frequently in our forthcoming discussions.

Theorem 3. *Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be a convex biholomorphic function and $g \in S(f)$ with expansions (2.18) and (2.25), respectively. Then for $|z| = r \leq 1/(1 + 2\|A_1\|\|A_1^{-1}\|)$, we have*

$$\sum_{k=1}^{\infty} \|B_k\| r^k \leq \liminf_{|z|\to 1-} \|f(z) - f(0)\|. \tag{2.26}$$

Also, for $|z| = r \leq 1/3$ we have

$$\sum_{k=1}^{\infty} |B_k| r^k \leq (1/2)|A_1|. \tag{2.27}$$

Proof. We observe that the well-known argument used in proving [26, Theorem X] can be used in a similar fashion for $g \in S(f)$, where f is an operator-valued convex biholomorphic function. Thus, we have $B_k = \phi'(0) f'(0)$, $k \geq 1$ for some holomorphic map $\phi : \mathbb{D} \to \mathbb{D}$ with $\phi(0) = 0$. Therefore, we immediately see $\|B_k\| \leq \|A_1\|$ and hence the following inequality will hold:

$$\sum_{k=1}^{\infty} \|B_k\| r^k \leq (r/1 - r)\|A_1\|. \tag{2.28}$$

Now for any fixed $a \in \mathbb{D}$, we construct the familiar Koebe transform as follows:

$$G(z) = (1 - |a|^2)^{-1} (f'(a))^{-1} \left(f\left((z + a)(1 + \overline{a}z)^{-1} \right) - f(a) \right), z \in \mathbb{D}. \tag{2.29}$$

We see that $G(z)$ is convex biholomorphic with the normalization $G(0) = 0$ and $G'(0) = I$. From [16, Theorem 6.3.5], we get that G satisfies

$$zG''(z) + G'(z) = p(z)G'(z),$$

where $p : \mathbb{D} \to \mathbb{C}$ is holomorphic with $\mathrm{Re}(p(z)) > 0$ for all $z \in \mathbb{D}$ and $p(0) = 1$. Therefore, for any fixed $x \in \mathcal{H}$ with $\|x\| = 1$, the function $\hat{G} : \mathbb{D} \to \mathbb{C}$ defined by

$$\hat{G}(z) = \langle G(z)x, x \rangle$$

satisfies $\hat{G}(0) = \hat{G}'(0) - 1 = 0$ and $z\hat{G}''(z) + \hat{G}'(z) = p(z)\hat{G}'(z)$, which together imply that $\hat{G}(z)$ is a complex valued normalized convex univalent function (see

[16, Theorem 2.2.3]). As a consequence, $\liminf_{|z|\to 1-} |\hat{G}(z)| \geq 1/2$ (cf. [16, Theorem 2.2.9]), which, after an application of the Cauchy–Schwarz inequality for inner product yields

$$\liminf_{|z|\to 1-} \left\|(1-|a|^2)^{-1}(f'(a))^{-1}\left(f\left((z+a)(1+\overline{a}z)^{-1}\right)-f(a)\right)\right\| \geq 1/2. \tag{2.30}$$

Now inequality (2.30) will further give

$$\liminf_{|z|\to 1-} \left\|\left(f\left((z+a)(1+\overline{a}z)^{-1}\right)-f(a)\right)\right\| \geq (1-|a|^2)/(2\|(f'(a))^{-1}\|). \tag{2.31}$$

In particular, for $a=0$ we get

$$\liminf_{|z|\to 1-} \|f(z)-f(0)\| \geq 1/2\|A_1^{-1}\|. \tag{2.32}$$

From (2.28) and (2.32), a little computation reveals that (2.26) will hold if

$$(r/1-r)\|A_1\| \leq 1/2\|A_1^{-1}\|, \quad \text{or equivalently if } r \leq 1/(1+2\|A_1\|\|A_1^{-1}\|).$$

Now going back to the relation $B_k = \phi'(0)f'(0)$, it is readily seen that $|B_k| \leq |A_1|$ for any $k \geq 1$, and therefore

$$\sum_{k=1}^{\infty} |B_k| r^k \leq (r/1-r)|A_1|. \tag{2.33}$$

It is easy to see that for $r \leq 1/3$, (2.33) is converted to (2.27). □

Remark. We make the following observations related to Theorem 3.

(i) The quantity $1/(1+2\|A_1\|\|A_1^{-1}\|)$ in Theorem 3 will turn into 1/3 for scalar-valued functions, as whenever A_1 is a scalar, $\|A_1\|\|A_1^{-1}\| = 1$. Therefore, (2.26) gives an operator-valued analogue of the Bohr phenomenon for the subordinating family of a complex-valued convex univalent function defined on $\mathbb{D}$ (compare [1, Remark 1]).

(ii) The right-hand side of the inequality (2.27) can be further estimated to observe $(1/2)|A_1| \leq d(f(0), \partial\Omega)$ when scalar-valued functions are being considered (see [1, Lemma 3]), $\partial\Omega$ being the boundary of $\Omega = f(\mathbb{D})$. Due to this fact, it can be thought of as a generalization of the Bohr phenomenon mentioned in [1, Remark 1].

Before we proceed further, we prove the following lemma which will be required to establish the subsequent results.

Lemma 2. *Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be holomorphic and $g \in S(f)$ with expansions* (2.18) *and* (2.25), *respectively. Then for $|z| = r \leq 1/3$ we have*

(i) $\sum_{k=1}^{\infty} |B_k| r^k \leq (\sum_{n=1}^{\infty} \|A_n\| r^n) I$.

(ii) $\sum_{k=1}^{\infty} \|B_k\| r^k \leq \sum_{n=1}^{\infty} \|A_n\| r^n$.

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Proof. Since $g \prec f$, there exists a function ϕ, holomorphic in $\mathbb{D}$, satisfying $\phi(0) = 0$ and $\phi(\mathbb{D}) \subset \mathbb{D}$ such that

$$g = f \circ \phi. \tag{2.34}$$

Since ϕ is holomorphic, the Taylor expansion of the tth power of ϕ, where $t \in \mathbb{N}$, can be written as

$$\phi^t(z) = \sum_{l=t}^{\infty} \alpha_l{}^{(t)} z^l. \tag{2.35}$$

Now we plug equality (2.35) into (2.34), and equating the coefficients for z^k from both sides we have, for any $k \geq 1$:

$$B_k = \sum_{n=1}^{k} \alpha_k^{(n)} A_n.$$

Now we see that

$$\begin{aligned}\sum_{k=1}^{m} |B_k| r^k &= \sum_{k=1}^{m} \left| \sum_{n=1}^{k} \alpha_k^{(n)} A_n \right| r^k \\ &\leq \left(\sum_{k=1}^{m} \left\| \sum_{n=1}^{k} \alpha_k^{(n)} A_n \right\| r^k \right) I \leq \left(\sum_{k=1}^{m} \sum_{n=1}^{k} |\alpha_k^{(n)}| \|A_n\| r^k \right) I.\end{aligned}$$

We observe that the rightmost term of the above inequality can be written as $(\sum_{n=1}^{m} \|A_n\| M_m^{(n)}(r))I$, where $M_m^{(n)}(r) := \sum_{k=n}^{m} |\alpha_k^{(n)}| r^k$. The proof of part (i) can now be completed by adopting the techniques similar to the proof of [6, Lemma 1] hereafter. Further, part (ii) can be proved by directly following the same line of computations as in the proof of [6, Lemma 1]. □

We now state and prove a theorem including the Bohr phenomenon for $S(f)$, where f is an operator- valued normalized starlike biholomorphic function. It may be mentioned that the known techniques to find out the coefficient bounds for functions subordinate to a complex-valued normalized starlike univalent function do not seem to be directly applicable in this situation, while a use of Lemma 2 will prove the following theorem.

Theorem 4. *Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be a normalized starlike biholomorphic function with an expansion $f(z) = zI + \sum_{n=2}^{\infty} A_n z^n$ and $g \in S(f)$ with an expansion (2.25). Then for $|z| = r \leq 3 - 2\sqrt{2}$ we have*

(i) $\sum_{k=1}^{\infty} \|B_k\| r^k \leq \liminf_{|z| \to 1-} \|f(z)\|$.

(ii) $\sum_{k=1}^{\infty} |B_k| r^k \leq (1/4)I$.

Proof. From [16, Theorem 6.2.6], it is seen that a starlike biholomorphic function $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ normalized by $f(0) = f'(0) - I = 0$ satisfies

$$zf'(z) = p(z)f(z), z \in \mathbb{D}, \tag{2.36}$$

where $p : \mathbb{D} \to \mathbb{C}$ is holomorphic with $\mathrm{Re}(p(z)) > 0$ for all $z \in \mathbb{D}$ and $p(0) = 1$. Here we mention that a holomorphic function $f : \mathbb{D} \to \mathbb{C}$, normalized by $f(0) = f'(0) - 1 = 0$ is starlike univalent if and only if (2.36) holds. Now a standard method based on induction (see f.i. [16, Theorem 2.2.16]) yields $\|A_n\| \leq n$ for all $n \geq 2$. As a consequence, $\sum_{n=1}^{\infty} \|A_n\| r^n \leq r/(1-r)^2$, where $A_1 = I$. Now let us define $G : \mathbb{D} \to \mathbb{C}$ by

$$G(z) = \langle f(z)x, x\rangle,$$

where $x \in \mathcal{H}$ with $\|x\| = 1$. It is easy to see that $G(0) = G'(0) - 1 = 0$. Therefore, following the similar lines of argument as in the proof of Theorem 3, (2.36) implies that G is a starlike univalent function. Hence $\liminf_{|z|\to 1-} |G(z)| \geq 1/4$ (see [16, Theorem 1.1.5], and observe that the Koebe function $k(z) = z/(1-z)^2$ which skips the value $-1/4$ is starlike univalent), and as a result the Cauchy–Schwarz inequality for inner product gives $\liminf_{|z|\to 1-} \|f(z)\| \geq 1/4$. From a direct calculation, we get that $\sum_{n=1}^{\infty} \|A_n\| r^n \leq 1/4$ for $|z| = r \leq 3 - 2\sqrt{2}$, which is less than $1/3$. By virtue of the Lemma 2, our proofs for both part (i) and (ii) will be complete. □

Remark. We end the article with the following observations.

(i) It is immediately seen that for a complex-valued function f, part (i) of the Theorem 4 converts to the Bohr inequality for $S(f)$, where f is a normalized starlike univalent function. Again, if f is a complex-valued normalized starlike univalent function defined on $\mathbb{D}$, then the right-hand side of the inequality in part (ii) is converted to $1/4$ which is known to be less than or equal to $d(f(0), \partial\Omega)$, $\partial\Omega$ being the boundary of $\Omega = f(\mathbb{D})$. This shows that part (ii) can also be considered as an operator-valued analogue of the Bohr phenomenon for $S(f)$. We note that the scalar-valued result is a direct consequence of [1, Theorem 1].

(ii) In view of Theorems 3 and 4, it is a natural question to ask if the inequality (2.26) holds for $|z| = r \leq r_0$ for some $r_0 > 0$, where f is any function in the entire family of biholomorphic functions from $\mathbb{D}$ to $\mathcal{B}(\mathcal{H})$ and $g \in S(f)$. The Bohr radius $1/(1 + 2\|A_1\|\|A_1^{-1}\|)$ determined in the first part of the Theorem 3 is not bounded below by a positive constant if we allow A_1 to be any invertible operator from $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ varying on the family of complex Hilbert spaces. Therefore, we remark that the answer of the aforesaid question could possibly be negative, even when f is convex biholomorphic, and that this can be an interesting problem for future research. However, a similar problem for Banach space valued holomorphic functions in $\mathbb{D}$ has already been settled (cf. [7, Theorem 1.2]), where the notion of the Bohr inequality is analogous to (1.1).

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