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On recursive operations over logic LTS[†]

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Recently, in order to mix algebraic and logic styles of specification in a uniform framework, the notion of a logic labelled transition system (Logic LTS or LLTS for short) has been introduced and explored. A variety of constructors over LLTS, including usual process-algebraic operators, logic connectives (*conjunction* and *disjunction*) and standard temporal modalities (*always* and *unless*), have been given. However, no attempt has been made so far to develop the general theory concerning (nested) recursive operations over LLTS and a few fundamental problems are still open. This paper intends to study this issue in a pure process-algebraic style. A few fundamental properties, including precongruence and the uniqueness of consistent solutions of equations, will be established.

1. Introduction

Algebra and logic are two dominant approaches for the specification, verification and systematic development of reactive and concurrent systems. They take different standpoints for looking at specifications and verifications, and offer complementary advantages (Peled 2001).

Logical approaches (Pnueli 1977) devote themselves to specifying and verifying abstract properties of systems. In such frameworks, the most common reasonable properties of concurrent systems, such as safety, liveness, etc., can be formulated in terms of logic formulas without resorting to operational details and verification is a deductive or model-checking activity (Clarke *et al.* 2000). However, due to their global perspective and abstract nature, logical approaches often give little support for modular design and compositional reasoning (Peled 2001).

Algebraic approaches put attention to behavioural aspects of systems, which have tended to use formalisms in algebraic styles. These formalisms are referred to as process algebras or process calculi (Bergstra and Klop 1984; Hoare 1985; Milner 1989a). In such paradigm, a specification and its implementation usually are formulated in terms of expressions (terms) built from a number of operators, and the underlying semantics is often defined operationally. The verification amounts to comparing terms, which is

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often referred to as *implementation verification* or *equivalence checking* (Aceto *et al.* 2012). Algebraic approaches often support compositional construction and reasoning, which bring us advantages in developing systems, such as, supporting modular design and verification, avoiding verifying the whole system from scratch when its parts are modified, allowing reusability of proofs and so on (Andersen *et al.* 1994). Thus such approaches offer significant support for rigorous systematic development of reactive and concurrent systems. However, since algebraic approaches specify a system by means of prescribing in detail how the system should behave, it is often difficult for people to describe abstract properties of systems in this paradigm.

To take advantage of these two approaches when designing systems, so-called heterogeneous specifications have been proposed, which uniformly integrate these two specification styles. Based on Büchi automata and labelled transition system (LTS) augmented with a predicate, a semantic framework for heterogenous system design is given in Cleaveland and Lüttgen (2000, 2002). In this framework, not only usual operational operators but also logic connectives are considered, and the must-testing preorder presented in Nicola and Hennessy (1983) is adopted to capture refinement relations. Unfortunately, this setting does not support compositional reasoning since must-testing preorder is not a precongruence in this situation. Moreover, the logic connective conjunction in this framework lacks the desired property that r is an implementation of a given specification $p \wedge q$ if and only if r implements both p and q.

Recently, Lüttgen and Vogler have introduced the notion of a Logic LTS (LLTS), which combines operational and logic styles of specification in one unified framework (Lüttgen and Vogler 2007, 2010, 2011). In addition to usual operational constructors, e.g., CSP-style parallel composition, hiding and so on, logic connectives (conjunction and disjunction) and standard modal operators (*always* and *unless*) are also integrated into this framework. Moreover the drawbacks in Cleaveland and Lüttgen (2000, 2002) mentioned above have been remedied by adopting the ready-tree semantics (Lüttgen and Vogler 2007). To support compositional reasoning in the presence of the parallel constructor, a variant of the usual notion of ready simulation is employed, which has been shown to be the largest precongruence satisfying some desired properties (Lüttgen and Vogler 2010).

Along the direction suggested by Lüttgen and Vogler (2010), a process calculus called CLL is presented in Zhang *et al.* (2011), which reconstructs their settings in a pure process-algebraic style. Moreover, a sound and ground-complete proof system for CLL is provided. In effect, it gives an axiomatization of ready simulation in the presence of logic operators. However, CLL is lack of the capability of describing infinite behaviour, which is important for representing reactive systems.

It is well known that recursive operations are widely used in representing objects with infinite behaviour (see for instance Bergstra *et al.* (2001)). However, to the best of our knowledge, as yet no attempt has been made to develop a general theory concerning recursive operations over LLTS and a few fundamental problems are still open. Since LLTS involves consideration of inconsistencies, it is far from straightforward to re-establish existent results concerning recursive operations in this framework. A solid effort is required, especially for handling inconsistencies. This paper intends to explore recursive operations over LLTS in a pure process-algebraic style. To this end, a process calculus

CLL_R will be given in this paper, which is obtained by enriching CLL with recursive operations. Following Baeten and Bravetti (2008), expressions with the form like $\langle X|E\rangle$ will be used to denote recursions. We will find that usual SOS rules associated with recursive operations are insufficient to capture recursions in CLL_R. The main theoretical results obtained in this paper include:

— It is shown that the ready simulation relation presented in Lüttgen and Vogler (2010) is precongruent w.r.t all operators in CLL_R.
— Under the hypothesis that X is strongly guarded and does not occur in the scope of any conjunction in t, it is shown that, modulo $=_{RS}$ recalled in the next section, there exists at most one consistent solution of any given equation $X =_{RS} t$. Moreover the process $\langle X|X = t\rangle$ is indeed the unique consistent solution whenever consistent solutions exist.

The remainder of this paper is organized as follows. The next section recalls some related notions. Section 3 introduces SOS rules of CLL_R. In Section 4, the existence and uniqueness of stable transition model for CLL_R is demonstrated, and a few of basic properties of the LTS associated with CLL_R are given. In Section 5, a number of preliminary properties of unfolding and transitions are considered. In Section 6, we shall show that the variant of ready simulation presented by Lüttgen and Vogler is precongruent in the presence of (nested) recursive operations. In Section 7, a theorem on the uniqueness of consistent solutions of equations is obtained. Finally, a brief discussion is given in Section 8.

2. Preliminaries

This section will set up notations and briefly recall the notions of LLTS and ready simulation presented in Lüttgen and Vogler (2010, 2011).

Let Act be the set of visible actions ranged over by letters a, b, etc., and let Act_τ denote $Act \cup \{\tau\}$ ranged over by α and β, where τ represents invisible actions. AnLTS with a predicate is a quadruple $(P, Act_\tau, \longrightarrow, F)$, where P is a set of states, $\longrightarrow \subseteq P \times Act_\tau \times P$ is a transition relation and $F \subseteq P$.

As usual, we write $p \stackrel{\alpha}{\longrightarrow}$ (or, $p \stackrel{\alpha}{\not\longrightarrow}$) if $\exists q \in P.p \stackrel{\alpha}{\longrightarrow} q$ ($\nexists q \in P.p \stackrel{\alpha}{\longrightarrow} q$ resp.). Given a state p, the ready set $\{\alpha \in Act_\tau | p \stackrel{\alpha}{\longrightarrow}\}$ of p is denoted by $\mathcal{I}(p)$. A state p is stable if $p \stackrel{\tau}{\not\longrightarrow}$. We also list some useful decorated transition relations: $p \stackrel{\alpha}{\longrightarrow}_F q$ iff $p \stackrel{\alpha}{\longrightarrow} q$ and $p, q \notin F$; $p \stackrel{\epsilon}{\Longrightarrow} q$ iff $p(\stackrel{\tau}{\longrightarrow})^* q$, where $(\stackrel{\tau}{\longrightarrow})^*$ is the transitive reflexive closure of $\stackrel{\tau}{\longrightarrow}$; $p \stackrel{\alpha}{\Longrightarrow} q$ iff $\exists r, s \in P.p \stackrel{\epsilon}{\Longrightarrow} r \stackrel{\alpha}{\longrightarrow} s \stackrel{\epsilon}{\Longrightarrow} q$; $p \stackrel{\gamma}{\Longrightarrow} |q$ iff $p \stackrel{\gamma}{\Longrightarrow} q \stackrel{\tau}{\not\longrightarrow}$ with $\gamma \in Act_\tau \cup \{\epsilon\}$; $p \stackrel{\epsilon}{\Longrightarrow}_F q$ iff there exists a sequence of τ-transitions from p to q such that all states along this sequence, including p and q, are not in F; the decorated transition $p \stackrel{\alpha}{\Longrightarrow}_F q$ may be defined similarly; $p \stackrel{\epsilon}{\Longrightarrow}_F |q$ (or, $p \stackrel{\alpha}{\Longrightarrow}_F |q$) iff $p \stackrel{\epsilon}{\Longrightarrow}_F q$ ($p \stackrel{\alpha}{\Longrightarrow}_F q$ resp.) and q is stable.

Remark 2.1. Notice that some notations above are slightly different from ones adopted by Lüttgen and Vogler. The notation $p \stackrel{\epsilon}{\Longrightarrow}|q$ (or, $p \stackrel{\alpha}{\Longrightarrow}|q$) in Lüttgen and Vogler (2010, 2011) has the same meaning as $p \stackrel{\epsilon}{\Longrightarrow}_F |q$ ($p \stackrel{\alpha}{\Longrightarrow}_F |q$ resp.) in this paper, while $p \stackrel{\epsilon}{\Longrightarrow} |q$ in this paper does not involve any requirement on F-predicate.

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Definition 2.1 (Lüttgen and Vogler 2010). An LTS $(P, Act_\tau, \longrightarrow, F)$ is an LLTS if, for each $p \in P$,
(LTS1) $p \in F$ if $\exists \alpha \in \mathcal{I}(p) \forall q \in P (p \stackrel{\alpha}{\longrightarrow} q$ implies $q \in F)$;
(LTS2) $p \in F$ if $\nexists q \in P . p \stackrel{\epsilon}{\Longrightarrow}_F | q$.

Moreover, an LTS $(P, Act_\tau, \longrightarrow, F)$ is τ-pure if, for each $p \in P$, $p \stackrel{\tau}{\longrightarrow}$ implies $\nexists a \in Act.\ p \stackrel{a}{\longrightarrow}$.

Any state p in a τ-pure LTS represents either an external or internal choice between its outgoing transitions. The predicate F is used to denote the set of all inconsistent states. Intuitively, an inconsistent state represents empty behaviour that cannot be implemented (Lüttgen and Vogler 2011). In the sequel, we shall use the phrase '*inconsistency predicate*' to refer to F. Compared with usual LTSs, it is one distinguishing feature of LLTS that it involves consideration of inconsistencies. Roughly speaking, the main motivation behind such consideration lies in dealing with inconsistencies caused by conjunctive composition. In classical process-algebraic frameworks, the composition between a process and its environment is captured by some parallel operator. The conjunction operator presented in Lüttgen and Vogler (2007) offers another pattern to compose a process and its environment. In this setting, some logical consideration is involved. For example, consider the process $a.0$ and its environment $b.0$, the usual synchronous composition of them is equivalent to the inactive process 0. In contrast, the conjunction composition of them is marked as inconsistent since a run of a process cannot begin with both a and b (in other words, this conjunction composition cannot be implemented nontrivially). Lüttgen and Vogler have proved that this conjunction setting (w.r.t $\sqsubseteq_{RS}$ recalled below) indeed satisfies the expected boolean laws. For more intuitive ideas and motivation about inconsistency, the reader may refer to Lüttgen and Vogler (2007, 2010). The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence (i.e., infinite sequences of τ-transitions) should be viewed as catastrophic.

In Lüttgen and Vogler (2010, 2011), the notion of ready simulation below is adopted to capture the refinement relation, which is a variant of the usual notion of weak ready simulation (Bloom *et al.* 1995; Larsen and Skou 1991). It has been proven that such kind of ready simulation is the largest precongruence w.r.t parallel composition and conjunction which satisfies the desired property that an inconsistent specification can only be refined by inconsistent ones (see Theorem 21 in Lüttgen and Vogler (2010)).

Definition 2.2 (ready simulation on LLTS). Let $(P, Act_\tau, \longrightarrow, F)$ be a LLTS. A relation $\mathcal{R} \subseteq P \times P$ is a stable ready simulation relation, if for any $(p, q) \in \mathcal{R}$ and $a \in Act$
(RS1) both p and q are stable;
(RS2) $p \notin F$ implies $q \notin F$;
(RS3) $p \stackrel{a}{\Longrightarrow}_F | p'$ implies $\exists q'. q \stackrel{a}{\Longrightarrow}_F | q'$ and $(p', q') \in \mathcal{R}$;
(RS4) $p \notin F$ implies $\mathcal{I}(p) = \mathcal{I}(q)$.
We say that p is stable ready simulated by q, in symbols $p \underset{\sim}{\sqsubset}_{RS} q$, if there exists a stable ready simulation relation $\mathcal{R}$ with $(p, q) \in \mathcal{R}$. Further, p is ready simulated by q, written $p \sqsubseteq_{RS} q$, if $\forall p' (p \stackrel{\epsilon}{\Longrightarrow}_F | p'$ implies $\exists q' (q \stackrel{\epsilon}{\Longrightarrow}_F | q'$ and $p' \underset{\sim}{\sqsubset}_{RS} q'))$. The kernels of $\underset{\sim}{\sqsubset}_{RS}$ and

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$\sqsubseteq_{RS}$ are denoted by $\approx_{RS}$ and $=_{RS}$ respectively. It is easy to see that $\underset{\sim RS}{\sqsubset}$ is a stable ready simulation relation and both $\underset{\sim RS}{\sqsubset}$ and $\sqsubseteq_{RS}$ are preorder.

3. Syntax and SOS rules of CLL$_R$

Following Baeten and Bravetti (2008), this paper adopts the notation $\langle X|E\rangle$ to denote recursive operations, which encompasses both the CCS operator $recX.t$ and standard way of expressing recursion in ACP. Let V_{AR} be an infinite set of variables. The terms in CLL$_R$ are defined by BNF:

$$t ::= 0 \mid \bot \mid (\alpha.t) \mid (t\Box t) \mid (t \wedge t) \mid (t \vee t) \mid (t \parallel_A t) \mid X \mid \langle Z|E\rangle$$

where $X \in V_{AR}$, $\alpha \in Act_\tau$, $A \subseteq Act$ and recursive specification $E = E(V)$ with $V \subseteq V_{AR}$ is a set of equations $\{X = t|X \in V\}$ and Z is a variable in V that acts as the initial variable.

As usual, 0 encodes deadlock. The prefix $\alpha.t$ has a single capability, expressed by α; the process t cannot proceed until α has been exercised. $\Box$ is an external choice operator. $\parallel_A$ is a CSP-style parallel operator, $t_1 \parallel_A t_2$ represents a process that behaves as t_1 in parallel with t_2 under the synchronization set A. $\bot$ represents an inconsistent process with empty behaviour. $\vee$ and $\wedge$ are logical operators, which are intended for describing logical combinations of processes.

In the sequel, we often denote $\langle X|\{X = t_X\}\rangle$ briefly by $\langle X|X = t_X\rangle$. Given a term $\langle X|E\rangle$ and variable Y, the phrase 'Y occurs in $\langle X|E\rangle$' means that Y occurs in t_Z for some $Z = t_Z \in E$. Moreover the scope of a recursive operation $\langle X|E\rangle$ exactly consists of all t_Z with $Z = t_Z \in E$. An occurrence of a variable X in a given t is free if it does not occur in the scope of any recursive operation $\langle Y|E\rangle$ with $E = E(V)$ and $X \in V$. A variable X in term t is a free variable if all occurrences of X in t are free, otherwise X is a recursive variable in t.

Convention 3.1. Throughout this paper, as usual, we make the assumption that recursive variables are distinct from each other. That is, for any two recursive specifications $E(V_1)$ and $E'(V_2)$ we have $V_1 \cap V_2 = \varnothing$. Moreover we will tacitly restrict our attention to terms where no recursive variable has free occurrences. For example we will not consider terms such as $X\Box\langle X|X = a.X\rangle$ because this term could be replaced by the clear term $X\Box\langle Y|Y = a.Y\rangle$ with the same meaning.

In the following, given a term t, we use $FV(t)$ to denote the set of all free variables of t. As usual, a term t is a *process* if it is closed, that is $FV(t) = \varnothing$. The set of all processes of CLL$_R$ is denoted by $T(\Sigma_{\mathrm{CLL}_R})$. Unless noted otherwise we use p, q, r to represent processes. We shall always use $t_1 \equiv t_2$ to mean that expressions t_1 and t_2 are syntactically identical. In particular, $\langle Y|E\rangle \equiv \langle Y'|E'\rangle$ means that $Y \equiv Y'$ and for any Z and t_Z, $Z = t_Z \in E$ iff $Z = t_Z \in E'$.

Definition 3.1. For any recursive specification $E(V)$ and term t, we define $\langle t|E\rangle$ to be $t\{\langle X|E\rangle/X : X \in V\}$, that is, $\langle t|E\rangle$ is obtained from t by simultaneously replacing all free occurrences of each $X(\in V)$ by $\langle X|E\rangle$.

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Table 1. *Operational rules.*

$(Ra_1)\dfrac{-}{\alpha.x_1 \xrightarrow{\alpha} x_1}$	$(Ra_2)\dfrac{x_1 \xrightarrow{a} y_1, x_2 \not\xrightarrow{\tau}}{x_1 \Box x_2 \xrightarrow{a} y_1}$	$(Ra_3)\dfrac{x_1 \not\xrightarrow{\tau}, x_2 \xrightarrow{a} y_2}{x_1 \Box x_2 \xrightarrow{a} y_2}$
$(Ra_4)\dfrac{x_1 \xrightarrow{\tau} y_1}{x_1 \Box x_2 \xrightarrow{\tau} y_1 \Box x_2}$	$(Ra_5)\dfrac{x_2 \xrightarrow{\tau} y_2}{x_1 \Box x_2 \xrightarrow{\tau} x_1 \Box y_2}$	$(Ra_6)\dfrac{x_1 \xrightarrow{a} y_1, x_2 \xrightarrow{a} y_2}{x_1 \wedge x_2 \xrightarrow{a} y_1 \wedge y_2}$
$(Ra_7)\dfrac{x_1 \xrightarrow{\tau} y_1}{x_1 \wedge x_2 \xrightarrow{\tau} y_1 \wedge x_2}$	$(Ra_8)\dfrac{x_2 \xrightarrow{\tau} y_2}{x_1 \wedge x_2 \xrightarrow{\tau} x_1 \wedge y_2}$	$(Ra_9)\dfrac{-}{x_1 \vee x_2 \xrightarrow{\tau} x_1}$
$(Ra_{10})\dfrac{-}{x_1 \vee x_2 \xrightarrow{\tau} x_2}$	$(Ra_{11})\dfrac{x_1 \xrightarrow{\tau} y_1}{x_1 \parallel_A x_2 \xrightarrow{\tau} y_1 \parallel_A x_2}$	$(Ra_{12})\dfrac{x_2 \xrightarrow{\tau} y_2}{x_1 \parallel_A x_2 \xrightarrow{\tau} x_1 \parallel_A y_2}$
$(Ra_{13})\dfrac{x_1 \xrightarrow{a} y_1, x_2 \not\xrightarrow{\tau}}{x_1 \parallel_A x_2 \xrightarrow{a} y_1 \parallel_A x_2}(a \notin A)$	$(Ra_{14})\dfrac{x_1 \not\xrightarrow{\tau}, x_2 \xrightarrow{a} y_2}{x_1 \parallel_A x_2 \xrightarrow{a} x_1 \parallel_A y_2}(a \notin A)$	
$(Ra_{15})\dfrac{x_1 \xrightarrow{a} y_1, x_2 \xrightarrow{a} y_2}{x_1 \parallel_A x_2 \xrightarrow{a} y_1 \parallel_A y_2}(a \in A)$	$(Ra_{16})\dfrac{\langle t_X \vert E\rangle \xrightarrow{\alpha} y}{\langle X \vert E\rangle \xrightarrow{\alpha} y}(X = t_X \in E)$	

For example, consider $t \equiv X \Box a.\langle Y | Y = X \Box Y\rangle$ and $E(\{X\}) = \{X = t_X\}$ then $\langle t|E\rangle \equiv \langle X|X = t_X\rangle \Box a.\langle Y|Y = \langle X|X = t_X\rangle \Box Y\rangle$. In particular, for any recursive specification $E(V)$ and $t \equiv X$, $\langle t|E\rangle \equiv \langle X|E\rangle$ whenever $X \in V$ and $\langle t|E\rangle \equiv X$ if $X \notin V$.

As usual, an occurrence of X in t is strongly (or, weakly) guarded if such occurrence is within some subexpression $a.t_1$ with $a \in Act$ ($\tau.t_1$ or $t_1 \vee t_2$ resp.). A variable X is strongly (or, weakly) guarded in t if each occurrence of X is strongly (weakly resp.) guarded. Notice that, since the first move of $r \vee s$ is a τ-transition (see Table 1), which is independent of r and s, any occurrence of X in $r \vee s$ is treated as being weakly guarded. A recursive specification $E(V)$ is guarded if for each $X \in V$ and $Z = t_Z \in E(V)$, each occurrence of X in t_Z is (weakly or strongly) guarded.

Convention 3.2. It is well known that unguarded processes cause many problems in many aspects of the theory (Milner 1983) and unguarded recursion is incompatible with negative rules (Bloom 1994). As usual, we assume that all recursive specifications considered in the remainder of this paper are guarded.

We now provide SOS rules to specify the behaviour of processes (i.e., closed terms) formally. All SOS rules are divided into two parts: operational and predicate rules.

Operational rules $Ra_i (1 \leqslant i \leqslant 16)$ are listed in Table 1, where $a \in Act$, $\alpha \in Act_\tau$ and $A \subseteq Act$. Negative premises in Rules Ra_2, Ra_3, Ra_{13} and Ra_{14} give τ-transition precedence over visible transitions, which guarantees that the transition model of CLL_R is τ-pure. Rules Ra_9 and Ra_{10} illustrate that the operational aspect of $t_1 \vee t_2$ is same as internal choice in usual process calculus. Rule Ra_6 reflects that conjunction operator is a synchronous product for visible transitions. The operational rules of the other operators are as usual.

Table 2. *Predicate rules.*

$(Rp_1)\dfrac{-}{\bot F}$	$(Rp_2)\dfrac{x_1F}{\alpha.x_1F}$	$(Rp_3)\dfrac{x_1F,x_2F}{x_1\vee x_2F}$
$(Rp_4)\dfrac{x_1F}{x_1\Box x_2F}$	$(Rp_5)\dfrac{x_2F}{x_1\Box x_2F}$	$(Rp_6)\dfrac{x_1F}{x_1\parallel_A x_2F}$
$(Rp_7)\dfrac{x_2F}{x_1\parallel_A x_2F}$	$(Rp_8)\dfrac{x_1F}{x_1\wedge x_2F}$	$(Rp_9)\dfrac{x_2F}{x_1\wedge x_2F}$
$(Rp_{10})\dfrac{x_1\stackrel{a}{\longrightarrow}y_1,x_2\stackrel{a}{\not\longrightarrow},x_1\wedge x_2\stackrel{\tau}{\not\longrightarrow}}{x_1\wedge x_2F}$	$(Rp_{11})\dfrac{x_1\stackrel{a}{\not\longrightarrow},x_2\stackrel{a}{\longrightarrow}y_2,x_1\wedge x_2\stackrel{\tau}{\not\longrightarrow}}{x_1\wedge x_2F}$	
$(Rp_{12})\dfrac{x_1\wedge x_2\stackrel{\alpha}{\longrightarrow}z,\{yF:x_1\wedge x_2\stackrel{\alpha}{\longrightarrow}y\}}{x_1\wedge x_2F}$	$(Rp_{13})\dfrac{\{yF:x_1\wedge x_2\stackrel{\epsilon}{\Longrightarrow}\|y\}}{x_1\wedge x_2F}$	
$(Rp_{14})\dfrac{\langle t_X\|E\rangle F}{\langle X\|E\rangle F}(X=t_X\in E)$	$(Rp_{15})\dfrac{\{yF:\langle X\|E\rangle\stackrel{\epsilon}{\Longrightarrow}\|y\}}{\langle X\|E\rangle F}$	

Predicate rules in Table 2 specify the inconsistency predicate F. Rule Rp_1 says that $\bot$ is inconsistent. Hence $\bot$ cannot be implemented. While 0 is consistent, which is an implementable process. Thus 0 and $\bot$ represent different processes. Rule Rp_3 reflects that if both two disjunctive parts are inconsistent then so is the disjunction. Rules Rp_4-Rp_9 describe the system design strategy that if one part is inconsistent, then so is the whole composition. Rules Rp_{10} and Rp_{11} reveal that a stable conjunction is inconsistent if its conjuncts have distinct ready sets.

Rules Rp_{13} and Rp_{15} are used to capture (LTS2) in Definition 2.1, which are the abbreviation of the rules with the format

$$\frac{\{yF:\exists y_0,y_1,\ldots,y_n(z\equiv y_0\stackrel{\tau}{\longrightarrow}y_1\stackrel{\tau}{\longrightarrow}\cdots\stackrel{\tau}{\longrightarrow}y_n\equiv y\text{ and }y\stackrel{\tau}{\not\longrightarrow})\}}{zF}$$

with $z\equiv x_1\wedge x_2$ or $\langle X|E\rangle$. Intuitively, these two rules say that if all stable τ-descendants of z are inconsistent, then z itself is inconsistent. Notice that, especially for readers who are familiar with notations used in Lüttgen and Vogler (2010), the transition relation $\stackrel{\epsilon}{\Longrightarrow}|$ occurring in these two rules does not involve any requirement on consistency (see Remark 2.1 and notations above it).

Since the behaviour of any process in CLL is finite, each process can reach a stable state, and Rules Rp_1-Rp_{12} suffice to capture the inconsistency predicate F. In particular, these rules guarantee that the LTS associated with CLL satisfies (LTS1) and (LTS2) in Definition 2.1 (Zhang *et al.* 2011). However, for CLL_R, Rules Rp_1-Rp_{12} are insufficient even if the usual rule for recursive operations (i.e., Rp_{14}) is added. For instance, consider processes $q\equiv\langle X|X=\tau.X\rangle$ and $p\equiv\langle X|X=X\vee 0\rangle\wedge a.0$, it is not difficult to see that neither qF nor pF can be inferred by using only Rules Rp_1-Rp_{12} and Rp_{14}, however, both p and q should be inconsistent due to (LTS2). Fortunately, an inference of pF (or, qF) is at hand by admitting Rule Rp_{13} (Rp_{15} resp.).

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In the remainder of this paper, we use $\mathcal{P}_{\mathrm{CLL}_R}$ to denote the transition system specification $(\Sigma_{\mathrm{CLL}_R}, Act_\tau, \{F\}, \mathbb{R}_{\mathrm{CLL}_R})$ for CLL_R, where Σ_{CLL_R} is the set of all operators of CLL_R and $\mathbb{R}_{\mathrm{CLL}_R} = \{Ra_1, \ldots, Ra_{16}\} \cup \{Rp_1, \ldots, Rp_{15}\}$.

4. Stable transition model of $\mathcal{P}_{\mathrm{CLL}_R}$

For process calculi involving only positive SOS rules (e.g., CCS (Milner 1989a), π-calculus (Milner *et al.* 1992)), every transition $p \xrightarrow{\alpha} q$ is justified by an inference, which is a well-founded tree whose root is the transition itself. Therefore, it is appropriate to prove properties of transitions by induction on the depth of inference trees. In the framework of transition system specifications (TSSs), Bol and Groote have observed that such proof method is also powerful for some process calculi whose SOS rules contain negative premises. However, in the presence of negative premises, inference rules applied in proof trees are not SOS rules themselves but their stripped version (Bol and Groote 1996).

Several times in the remainder of this paper we shall need to prove properties of transitions $\xrightarrow{\alpha}$ and inconsistency predicate by induction on the depth of proof trees. This section will give the stripped version of $\mathcal{P}_{\mathrm{CLL}_R}$ and provide a few basic properties of the LTS associated with $\mathcal{P}_{\mathrm{CLL}_R}$. Here we assume that the notions of transition model, stable model, TSS and stratification of a TSS (see for instance Bol and Groote (1996)) are already familiar to the reader.

We begin with illustrating the existence and uniqueness of the stable model of $\mathcal{P}_{\mathrm{CLL}_R}$. By well-known results obtained in Bol and Groote (1996) and Groote (1993), in order to demonstrate that $\mathcal{P}_{\mathrm{CLL}_R}$ has a unique stable model, it is sufficient to give a stratification function of $\mathcal{P}_{\mathrm{CLL}_R}$. To this end, a few preliminary notions are introduced. Given a term t, the degree of t, denoted by $|t|$, is inductively defined as:

— $|0| = |\bot| = |\langle X|E\rangle| \triangleq 1$; $|\alpha.t| \triangleq |t| + 1$ with $\alpha \in Act_\tau$;
— $|t_1 \odot t_2| \triangleq |t_1| + |t_2| + 1$ for each $\odot \in \{\wedge, \Box, \vee, \|_A\}$.

Since it does not always hold that $|\langle t_X|E\rangle| \leqslant |\langle X|E\rangle|$, we cannot afford a stratification by using only this notion in the presence of Rule Ra_{16}. Fortunately, thanks to Convention 3.2, the function G defined below will bring us a measurement such that $G(\langle t_X|E\rangle) \leqslant G(\langle X|E\rangle)$. The function $G : T(\Sigma_{\mathrm{CLL}_R}) \longrightarrow \mathbb{N}$ is defined by:

— $G(\langle X|E\rangle) \triangleq 1$; $G(t_1 \odot t_2) \triangleq G(t_1) + G(t_2)$ for each $\odot \in \{\wedge, \Box, \|_A\}$;
— $G(0) = G(\bot) = G(\alpha.t) = G(t_1 \vee t_2) \triangleq 0$ with $\alpha \in Act_\tau$.

Clearly, given a term t, $G(t)$ is the number of unguarded recursive operations occurring in t. Further, the function $S_{\mathcal{P}_{\mathrm{CLL}_R}} : T(\Sigma_{\mathrm{CLL}_R}) \times Act_\tau \times T(\Sigma_{\mathrm{CLL}_R}) \cup T(\Sigma_{\mathrm{CLL}_R}) \times \{F\} \longrightarrow \omega \times 2 + 1$ is given below, where ω is the initial limit ordinal,

— $S_{\mathcal{P}_{\mathrm{CLL}_R}}(t \xrightarrow{\alpha} t') \triangleq G(t) \times \omega + |t|$;
— $S_{\mathcal{P}_{\mathrm{CLL}_R}}(tF) \triangleq \omega \times 2$.

It is trivial to check that $S_{\mathcal{P}_{\mathrm{CLL}_R}}$ is a stratification of $\mathcal{P}_{\mathrm{CLL}_R}$. Here we only consider Rule Ra_{16} as an example. It follows from Convention 3.2 that $G(\langle t_X|E\rangle) = 0$ and $G(\langle X|E\rangle) = 1$, which implies $S_{\mathcal{P}_{\mathrm{CLL}_R}}(\langle t_X|E\rangle \xrightarrow{\alpha} y) < S_{\mathcal{P}_{\mathrm{CLL}_R}}(\langle X|E\rangle \xrightarrow{\alpha} y)$ for any $y \in T(\Sigma_{\mathrm{CLL}_R})$, as desired.

Consequently, $\mathcal{P}_{\mathrm{CLL}_R}$ has a unique stable transition model. From now on, we use M_{CLL_R} to denote such stable model. In detail, M_{CLL_R} consists of all positive literals of the form $t \stackrel{\alpha}{\longrightarrow} t'$ or tF which are provable in $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R})$, where $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R})$ is the stripped version of $\mathcal{P}_{\mathrm{CLL}_R}$, that is, it is the positive TSS $(\Sigma_{\mathrm{CLL}_R}, Act_\tau, \{F\}, Strip(\mathbb{R}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}))$ with

$$Strip(\mathbb{R}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \triangleq \left\{ \frac{pprem(r)}{conc(r)} \middle| \quad r \in \mathbb{R}_{\mathrm{CLL}_R ground} \text{ and } M_{\mathrm{CLL}_R} \models nprem(r) \right\},$$

where $\mathbb{R}_{\mathrm{CLL}_R ground}$ denotes the set of all ground instances of rules in $\mathbb{R}_{\mathrm{CLL}_R}$, $nprem(r)$ (or, $pprem(r)$) is the set of negative (positive resp.) premises of r, $conc(r)$ is the conclusion of r and $M_{\mathrm{CLL}_R} \models nprem(r)$ means that for each $t \stackrel{\alpha}{\not\longrightarrow} \in nprem(r)$, $t \stackrel{\alpha}{\longrightarrow} s \notin M_{\mathrm{CLL}_R}$ for any $s \in T(\Sigma_{\mathrm{CLL}_R})$.

Definition 4.1. The LTS associated with CLL_R, in symbols $LTS(\mathrm{CLL}_R)$, is the quadruple $(T(\Sigma_{\mathrm{CLL}_R}), Act_\tau, \longrightarrow_{\mathrm{CLL}_R}, F_{\mathrm{CLL}_R})$, where $p \stackrel{\alpha}{\longrightarrow}_{\mathrm{CLL}_R} p'$ iff $p \stackrel{\alpha}{\longrightarrow} p' \in M_{\mathrm{CLL}_R}$, and $p \in F_{\mathrm{CLL}_R}$ iff $pF \in M_{\mathrm{CLL}_R}$.

Therefore, $p \stackrel{\alpha}{\longrightarrow}_{\mathrm{CLL}_R} p'$ (or, $p \in F_{\mathrm{CLL}_R}$) iff $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \stackrel{\alpha}{\longrightarrow} p'$ (pF resp.) for any p, p' and $\alpha \in Act_\tau$. This allows us to proceed by induction on depths of inferences in $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R})$ when demonstrating propositions concerning $\longrightarrow_{\mathrm{CLL}_R}$ and F_{CLL_R}.

Convention 4.1. To simplify notation, we shall omit the subscript in labelled transition relations $\stackrel{\alpha}{\longrightarrow}_{\mathrm{CLL}_R}$. Thus, the notation $\stackrel{\alpha}{\longrightarrow}$ has double utility: predicate symbols in the TSS $\mathcal{P}_{\mathrm{CLL}_R}$ and labelled transition relations on processes in $LTS(\mathrm{CLL}_R)$. Similarly, the notation F_{CLL_R} will be abbreviated by F. Hence the symbol F is overloaded, predicate symbol in the TSS $\mathcal{P}_{\mathrm{CLL}_R}$ and the set of all inconsistent processes within $LTS(\mathrm{CLL}_R)$.

In the following, we intend to provide a number of simple properties of $LTS(\mathrm{CLL}_R)$. In particular, we will show that $LTS(\mathrm{CLL}_R)$ is a τ-pure LLTS. We begin with a list of elementary facts.

Lemma 4.1. Let p and q be any two processes. Then

1. $p \vee q \in F$ iff $p, q \in F$.
2. $\alpha.p \in F$ iff $p \in F$ for each $\alpha \in Act_\tau$.
3. $p \odot q \in F$ iff either $p \in F$ or $q \in F$ with $\odot \in \{\Box, \|_A\}$.
4. Either $p \in F$ or $q \in F$ implies $p \wedge q \in F$.
5. $0 \notin F$ and $\bot \in F$.
6. $\langle X | X = \tau.X \rangle \in F$.
7. If $\forall q(p \stackrel{\epsilon}{\Longrightarrow} |q$ implies $q \in F)$ then $p \in F$.
8. $\langle X | E \rangle \in F$ iff $\langle t_X | E \rangle \in F$ for each X with $X = t_X \in E$.

Lemma 4.2. For any process p with $\tau \in \mathcal{I}(p)$, if $p \in F$ then $\forall q(p \stackrel{\tau}{\longrightarrow} q$ implies $q \in F)$.

Proof. Suppose $p \stackrel{\tau}{\longrightarrow} q$. We may prove $q \in F$ by induction on the depth of the inference of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \stackrel{\tau}{\longrightarrow} q$. The induction is easy to carry out by distinguishing several cases based on the form of p. □

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Theorem 4.1. $LTS(\mathrm{CLL}_R)$ is a τ-pure LLTS.

Proof. **(τ-purity)** Suppose $p \stackrel{\tau}{\longrightarrow}$. Hence $p \stackrel{\tau}{\longrightarrow} q$ for some q. Then it would be established by proving that $p \stackrel{a}{\not\longrightarrow}$ for any $a \in Act$. It is straightforward by induction on the depth of the inference of $p \stackrel{\tau}{\longrightarrow} q$. **(LTS1)** Suppose $\alpha \in \mathcal{I}(p)$ and $\forall r(p \stackrel{\alpha}{\longrightarrow} r$ implies $r \in F)$. Then $p \stackrel{\alpha}{\longrightarrow} q$ for some q. To complete the proof, it suffices to show $p \in F$. It proceeds by induction on the depth of the inference of $p \stackrel{\alpha}{\longrightarrow} q$. The induction carries out by distinguishing several cases based on the form of p. It is left to the reader. **(LTS2)** It immediately follows from Lemmas 4.1(7) and 4.2. □

Remark 4.1. It is worth pointing out that Lemma 4.2 does not always hold for LLTS. In fact, the property '*$p \in F$ implies $q \in F$ for each τ-derivative q of p*' is logically independent of Definition 2.1. It is SOS rules adopted in this paper that bring such additional property. Hence this paper restricts itself to specific LLTSs, which makes reasoning about inconsistency a bit easier than in the general LLTS setting.

A simple observation on proof trees for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \wedge qF$ is given below, which will be used in establishing a fundamental property of conjunctive compositions.

Lemma 4.3. For any finite sequence $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow}, \ldots, \stackrel{\tau}{\longrightarrow} p_i \wedge q_i \stackrel{\tau}{\longrightarrow}, \ldots, \stackrel{\tau}{\longrightarrow} |p_n \wedge q_n (n \geqslant 0)$, if $p_i \wedge q_i \in F$ and $p_i, q_i \notin F$ for each $i \leqslant n$, then the inference of $p_0 \wedge q_0 F$ essentially depends on $p_n \wedge q_n F$, that is, each proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_0 \wedge q_0 F$ has a subtree with root $p_n \wedge q_n F$, in particular, such subtree is proper if $n \geqslant 1$.

Proof. We prove the statement by induction on n. For the inductive basis $n = 0$, it holds trivially due to $p_0 \wedge q_0 \equiv p_n \wedge q_n$. For the inductive step, assume that $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow} p_1 \wedge q_1 (\stackrel{\tau}{\longrightarrow})^k | p_{k+1} \wedge q_{k+1}$. Let $\mathcal{T}$ be any proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_0 \wedge q_0 F$. Since $p_0, q_0 \notin F$ and $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow}$, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{p_0 \wedge q_0 \stackrel{\alpha}{\longrightarrow} r', \{rF : p_0 \wedge q_0 \stackrel{\alpha}{\longrightarrow} r\}}{p_0 \wedge q_0 F} \text{ or } \frac{\{rF : p_0 \wedge q_0 \stackrel{\epsilon}{\Longrightarrow} |r\}}{p_0 \wedge q_0 F}.$$

For the first alternative, since $LTS(\mathrm{CLL}_R)$ is τ-pure, we have $\alpha = \tau$. Then it follows from $p_0 \wedge q_0 \stackrel{\tau}{\longrightarrow} p_1 \wedge q_1$ that, in the proof tree $\mathcal{T}$, one of nodes directly above the root is labelled with $p_1 \wedge q_1 F$. Thus, by IH, $\mathcal{T}$ has a proper subtree with root $p_{k+1} \wedge q_{k+1} F$.

For the second alternative, since $p_0 \wedge q_0 \stackrel{\epsilon}{\Longrightarrow} |p_{k+1} \wedge q_{k+1}$, one of nodes directly above the root of $\mathcal{T}$ is labelled with $p_{k+1} \wedge q_{k+1} F$, as desired. □

The next three results has been obtained for CLL in a pure process-algebraic style in Zhang *et al.* (2011), where the proof essentially depends on the fact that, for any p within CLL and $\alpha \in Act_\tau$, p is of more complex structure than its α-derivatives. Unfortunately, such property does not always hold for CLL_R. Here we give another proof based on the well foundness of proof trees.

Lemma 4.4. If $p_1 \sqsubseteq_{RS} p_2$, $p_1 \sqsubseteq_{RS} p_3$ and $p_1 \notin F$ then $p_2 \wedge p_3 \notin F$.

Proof. Let $\Omega = \{q \wedge r : p \sqsubseteq_{RS} q, p \sqsubseteq_{RS} r$ and $p \notin F\}$. Clearly, it suffices to prove that $F \cap \Omega = \varnothing$. Conversely, suppose that $F \cap \Omega \neq \varnothing$. In the following, we intend to prove

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that, for each $t \in \Omega$, any proof tree of tF has a proper subtree with root $t'F$ for some $t' \in \Omega$. This contradicts the requirement on proof trees that they are well founded, and hence a contradiction arises at this point, as desired. So, to complete the proof, it suffices to show:

Claim. For any $s \in \Omega$, each proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash sF$ has a proper subtree with root $s'F$ for some $s' \in \Omega$.

Suppose $q \wedge r \in \Omega$. Then $p \sqsubset_{\sim RS} q$, $p \sqsubset_{\sim RS} r$ and $p \notin F$ for some p. Thus it follows that

$$q \notin F, r \notin F \text{ and } \mathcal{I}(p) = \mathcal{I}(q) = \mathcal{I}(r). \tag{4.4.1}$$

Let $\mathcal{T}$ be any proof tree of $q \wedge rF$. By (4.4.1), the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\{sF : q \wedge r \stackrel{\epsilon}{\Longrightarrow} |s\}}{q \wedge rF} \text{ or } \frac{q \wedge r \stackrel{\alpha}{\longrightarrow} s', \{sF : q \wedge r \stackrel{\alpha}{\longrightarrow} s\}}{q \wedge rF}.$$

Since both q and r are stable, so is $q \wedge r$. Then, for the first alternative, the label of the node directly above the root of $\mathcal{T}$ is $q \wedge rF$ itself, as desired.

Next we consider the second alternative. In this case, $\tau \neq \alpha \in \mathcal{I}(q \wedge r)$ and

$$\forall s(q \wedge r \stackrel{\alpha}{\longrightarrow} s \text{ implies } s \in F). \tag{4.4.2}$$

Hence $\alpha \in \mathcal{I}(q) \cap \mathcal{I}(r)$. Then $\alpha \in \mathcal{I}(p)$ due to (4.4.1). Further, since $p \notin F$, by Theorem 4.1, we get

$$p \stackrel{\alpha}{\longrightarrow}_F p' \stackrel{\epsilon}{\Longrightarrow}_F |p'' \text{ for some } p' \text{ and } p''. \tag{4.4.3}$$

Then it immediately follows from $p \sqsubset_{\sim RS} q$ and $p \sqsubset_{\sim RS} r$ that

$$q \stackrel{\alpha}{\longrightarrow}_F q' \stackrel{\epsilon}{\Longrightarrow}_F |q'' \text{ and } p'' \sqsubset_{\sim RS} q'' \text{ for some } q', q'', \text{and} \tag{4.4.4}$$

$$r \stackrel{\alpha}{\longrightarrow}_F r' \stackrel{\epsilon}{\Longrightarrow}_F |r'' \text{ and } p'' \sqsubset_{\sim RS} r'' \text{ for some } r', r''. \tag{4.4.5}$$

So, $q \wedge r \stackrel{\alpha}{\longrightarrow} q' \wedge r'$. Then $q' \wedge r' \in F$ by (4.4.2). Moreover, we obtain $q' \equiv q_0 \stackrel{\tau}{\longrightarrow}_F, \dots, \stackrel{\tau}{\longrightarrow}_F |q_n \equiv q''$ for some $q_i (0 \leqslant i \leqslant n)$, and $r' \equiv r_0 \stackrel{\tau}{\longrightarrow}_F, \dots, \stackrel{\tau}{\longrightarrow}_F |r_m \equiv r''$ for some $r_j (0 \leqslant j \leqslant m)$. Then

$$q' \wedge r' \equiv q_0 \wedge r_0 \stackrel{\tau}{\longrightarrow}, \dots, \stackrel{\tau}{\longrightarrow} q_n \wedge r_0 \stackrel{\tau}{\longrightarrow} q_n \wedge r_1, \dots, \stackrel{\tau}{\longrightarrow} |q_n \wedge r_m \equiv q'' \wedge r''. \tag{4.4.6}$$

By Lemma 4.2, it follows from $q' \wedge r' \in F$ that

$$q_i \wedge r_j \in F \text{ for each } q_i \wedge r_j \text{ occurring in (4.4.6).} \tag{4.4.7}$$

It follows from (4.4.3)–(4.4.5) that $q_n \wedge r_m \equiv q'' \wedge r'' \in \Omega$. Moreover, since one of nodes directly above the root of $\mathcal{T}$ is labelled with $q' \wedge r'F$, by (4.4.6), (4.4.7) and Lemma 4.3, it follows from $q_i \notin F (0 \leqslant i \leqslant n)$ and $r_j \notin F (0 \leqslant j \leqslant m)$ that $\mathcal{T}$ has a proper subtree with root $q_n \wedge r_m F$. □

Remark 4.2. The preceding proof is our first example of reasoning about consistency via proof trees for the F-predicate. We shall see many other proofs of this type later. The soundness of this reasoning manner relies on the fact that the predicate F is interpreted as the set of all terms p such that $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash pF$. It is not difficult to see that this fact is equivalent to the statement that F_{CLL_R} is the least set closed under positive rules with the form like $\frac{pprem(r)}{conc(r)}$, where r is any ground instance of rules in Table 2 such that $M_{\mathrm{CLL}_R} \models nprem(r)$. Based on the notion of so-called witnesses, Lüttgen and Vogler employ another manner to reason about consistency of states in the frameworks of LLTS. Such manner depends on the fact that for each operator $\triangle$ over LLTSs considered in Lüttgen and Vogler (2010), the inconsistent set $F_\triangle$ is defined as the least set of states satisfying analogous closure properties. These two reasoning manners are the same in spirit and are actually the two sides of the same coin: they both rely on the fact that F is the least set closed under certain rules but apply it in different style.

Lemma 4.5. If $p \underset{\sim}{\sqsubset}_{RS} q$ and $p \underset{\sim}{\sqsubset}_{RS} r$ then $p \underset{\sim}{\sqsubset}_{RS} q \wedge r$.

Proof. Set $\mathcal{R} = \{(p_1, p_2 \wedge p_3) : p_1 \underset{\sim}{\sqsubset}_{RS} p_2 \text{ and } p_1 \underset{\sim}{\sqsubset}_{RS} p_3\}$. It suffices to show that $\mathcal{R}$ is a stable ready simulation relation, which is almost immediate by using Lemma 4.4 to handle (RS2) and (RS3). □

We conclude this section with recalling a result obtained in Lüttgen and Vogler (2010) and Zhang *et al.* (2011) in different style, which reveals that $\sqsubseteq_{RS}$ is precongruent w.r.t the operators $\Box$, $\|_A$, $\vee$ and $\wedge$. Its proof is not much different from one given in Zhang *et al.* (2011). In particular, Lemma 4.5 is applied in the proof for the case $\odot = \wedge$.

Theorem 4.2.

1. For each $\odot \in \{\Box, \|_A, \wedge\}$, if $p \underset{\sim}{\sqsubset}_{RS} q$ and $s \underset{\sim}{\sqsubset}_{RS} r$ then $p \odot s \underset{\sim}{\sqsubset}_{RS} q \odot r$.
2. For each $\odot \in \{\Box, \|_A, \vee, \wedge\}$, if $p \sqsubseteq_{RS} q$ and $s \sqsubseteq_{RS} r$ then $p \odot s \sqsubseteq_{RS} q \odot r$.

5. Unfolding, context and transitions

As mentioned in Introduction, reasoning about the F-predicate is a crucial issue in this paper. The reasoning manner of Lemma 4.4 will be adopted often in the sequel. The main steps in such reasoning manner include: defining a set Ω of processes firstly, and then arguing that any proof tree of pF with $p \in \Omega$ contains a proper subtree with root qF for some $q \in \Omega$. Since the definition of Ω often depends on the formats of processes and SOS rules concerning F-predicate involve predicates $\xrightarrow{\alpha}$ and $\overset{\epsilon}{\Longrightarrow}|$, to carry out such reasoning, it is necessary to capture the connection between the formats of p and q for a given transition $p \xrightarrow{\alpha} q$ (or, $p \overset{\epsilon}{\Longrightarrow}|q$). This section will provide a detailed exposition of this. In Subsection 5.1, we will recall the notion of unfolding, which plays an important role when describing formats of derivatives in the presence of recursions. Subsection 5.2 will be concerned with capturing one-step transitions in terms of contexts and substitutions. A treatment of a more general case involving sequences of τ-transitions will be considered in Subsection 5.3.

5.1. *Unfolding*

In the presence of recursions, an α-derivative q of a given process p is not always a subterm of p. To describe the format of q explicitly, the notion of so-called unfolding is needed. This subsection will focus on this notion.

Definition 5.1. Let X be a free variable in a given term t. An occurrence of X in t is unfolded, if this occurrence does not occur in the scope of any recursive operation $\langle Y|E\rangle$. Moreover, X is unfolded if all occurrences of X in t are unfolded.

Definition 5.2 (Baeten and Bravetti 2008). A series of binary relations $\Rrightarrow_k$ over terms with $k < \omega$ is defined inductively as:

— $t \Rrightarrow_0 s$ if $t \equiv s$;
— $t \Rrightarrow_1 s$ if t has a subterm $\langle Y|E\rangle$ with $Y = t_Y \in E$ which is not in the scope of any recursive operation, and s is obtained from t by replacing this subterm by $\langle t_Y|E\rangle$;
— $t \Rrightarrow_{k+1} s$ if $t \Rrightarrow_k t'$ and $t' \Rrightarrow_1 s$ for some term t'.

Moreover $\Rrightarrow \triangleq \bigcup_{0\leqslant k<\omega} \Rrightarrow_k$. For any t and s, s is a multi-step unfolding of t if $t \Rrightarrow s$.

For instance, consider $t \equiv ((\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle\Box d.0)\Box Z$, we have

$$t \Rrightarrow_1 ((a.\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle\Box b.\langle Y|Y = c.Y\rangle)\Box d.0)\Box Z,$$

but it does not hold that $t \Rrightarrow_1 (\langle X|X = a.X\Box b.c.\langle Y|Y = c.Y\rangle\rangle\Box d.0)\Box Z$ because the subterm $\langle Y|Y = c.Y\rangle$ is in the scope of the recursive operation $\langle X|X = a.X\Box b.\langle Y|Y = c.Y\rangle\rangle$. As an immediate consequence of Definition 5.2, the simple result below provides an equivalent formulation of $\Rrightarrow_1$.

Lemma 5.1. $t_1 \Rrightarrow_1 t_2$ iff there exists a term s and variable X such that

$1_{\Rrightarrow}$. X is a unfolded variable in s,
$2_{\Rrightarrow}$. X occurs in s exactly once and
$3_{\Rrightarrow}$. $t_1 \equiv s\{\langle Y|E\rangle/X\}$ and $t_2 \equiv s\{\langle t_Y|E\rangle/X\}$ for some Y, E with $Y = t_Y \in E$.

A few trivial but useful results concerning $\Rrightarrow_n$ are listed in the next lemma. With the help of Lemma 5.1 and Convention 3.2, its proof is straightforward by induction on n.

Lemma 5.2. For any terms t, s and $X \in FV(t)$, if $t \Rrightarrow_n s$ then,

1. if X is unfolded in t then so it is in s and the number of occurrences of X in s is equal to that in t;
2. the number of unguarded occurrences of X in s is not more than that in t;
3. if X is (strongly) guarded in t then so it is in s;
4. $FV(s) \subseteq FV(t)$;
5. if X occurs in the scope of conjunction in s (that is, there exists a subterm $t_1 \wedge t_2$ of s such that X occurs in either t_1 or t_2) then so does it in t.

Notice that the clause (2) in the above lemma does not always hold for guarded occurrences. For example, consider $t \equiv \langle X|X = a.X \wedge b.Y\rangle$, we have $t \Rrightarrow_1 a.\langle X|X = a.X \wedge b.Y\rangle \wedge b.Y$, and Y guardedly occurs in the latter twice but occurs in t only once.

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Clearly, the clause (2) strongly depends on Convention 3.2. Moreover the clause (4) cannot be strengthened to '$FV(s) = FV(t)$'. Consider $t \equiv \langle X_1 | \{X_1 = a.0, X_2 = b.X_1 \Box Y\} \rangle$ and $t \Rrightarrow_1 a.0$, then we have $FV(t) = \{Y\}$ and $FV(a.0) = \emptyset$.

The next result depends on Convention 3.2, which asserts that, for any term t, its all unguarded occurrences of free variables may become unfolded through unfolding t enough times. For instance, consider $t \equiv \langle Z | Z = a.Z \Box \langle Y | Y = b.Y \Box X \rangle \rangle$, we have $t \Rrightarrow_1 a.t \Box \langle Y | Y = b.Y \Box X \rangle \Rrightarrow_1 a.t \Box (b.\langle Y | Y = b.Y \Box X \rangle \Box X)$. The proof of this result has been given in Zhang *et al.* (2013).

Lemma 5.3. For any term t, there exists a term s such that $t \Rrightarrow s$ and each unguarded occurrence of any free variable in s is unfolded.

5.2. *Contexts and transitions*

To show that $\sqsubseteq_{RS}$ is precongruent, i.e., $p \sqsubseteq_{RS} q$ implies $C_X\{p/X\} \sqsubseteq_{RS} C_X\{q/X\}$ for any context C_X, we are required to capture the connection between behaviours of $C_X\{p/X\}$ and $C_X\{q/X\}$. To this end, it is necessary to extract the pattern behind a given transition $C_X\{p/X\} \xrightarrow{\alpha} r$. This subsection intends to explore this issue. In particular, Lemmas 5.6 and 5.7 give general conclusions for this issue in cases $\alpha = \tau$ and $\alpha \in Act$ respectively. As simple applications of these results, Lemmas 5.8 and 5.9 consider two particular instances; moreover a few of useful properties concerning unfolding are also given in Lemmas 5.10, 5.12 and 5.13.

Definition 5.3 (context). A context $C_{\widetilde{X}}$ is a term whose free variables are in some n-tuple distinct variables $\widetilde{X} = (X_1, \ldots, X_n)$ with $n \geqslant 0$. Given $\widetilde{p} = (p_1, \ldots, p_n)$, the term $C_{\widetilde{X}}\{p_1/X_1, \ldots, p_n/X_n\}$ ($C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ for short) is obtained from $C_{\widetilde{X}}$ by replacing X_i by p_i for each $i \leqslant n$ simultaneously. In particular, we use $C_{\widetilde{X}}\{p/\widetilde{X}\}$ to denote the result of replacing all variables in $\widetilde{X}$ by p. A context $C_{\widetilde{X}}$ is stable if $C_{\widetilde{X}}\{0/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$.

In the remainder of this paper, whenever the expression $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ occurs, we always assume that $|\widetilde{p}| = |\widetilde{X}|$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ is subject to Convention 3.1 (recursive variables occurring in $\widetilde{p}$ may be renamed if it is necessary), where $|\widetilde{X}|$ is the length of the tuple $\widetilde{X}$.

Definition 5.4 (active). An occurrence of a free variable X in t is *active* if such occurrence is unguarded and unfolded. A free variable X in t is active if all its occurrences are active. A free variable X in t is *1-active* if X occurs in t exactly once and such occurrence is active.

For example, X is 1-active in $\langle Y | Y = a.Y \rangle \Box X$. Moreover, it is evident that, for any context $C_{\widetilde{X}}$, if there exists an active occurrence of some variable within $C_{\widetilde{X}}$, then $C_{\widetilde{X}}$ is not of the form $\alpha.B_{\widetilde{X}}$, $B_{\widetilde{X}} \vee D_{\widetilde{X}}$ and $\langle Y | E \rangle$. Applying this fact, the next two lemmas are almost immediate by induction on the structure of context.

Lemma 5.4. For any $C_{\widetilde{X}}$ with 1-active variable X_{i_0} and $\widetilde{p}$ with $p_{i_0} \xrightarrow{\tau} p'$, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} C_{\widetilde{X}}\{\widetilde{p}[p'/p_{i_0}]/\widetilde{X}\}$ where $\widetilde{p}[p'/p_{i_0}]$ is obtained from $\widetilde{p}$ by replacing p_{i_0} by p'.

Lemma 5.5. For any p and C_X with 1-active variable X, if $p \in F$ then $C_X\{p/X\} \in F$.

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Notice that Lemma 5.4 does not always hold for visible transitions. For instance, consider $C_X \equiv X\Box\tau.r$ and $p \equiv a.q$, although $p \xrightarrow{a} q$ and X is 1-active in C_X, it is false that $C_X\{p/X\} \xrightarrow{a}$.

To prove that $\sqsubseteq_{RS}$ is still precongruent in the presence of recursive operations, it is necessary to formally describe the contribution of $C_{\widetilde{X}}$ and $\widetilde{p}$ for a given transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\alpha} r$. In the following, we provide a few of results concerning this.

We begin with considering τ-transitions. Before giving the next lemma formally, we illustrate the intuition behind it by means of an example. Consider $C_X \equiv (a.0 \vee X)\Box X$, $B_X \equiv \langle Y|Y = X\Box b.Y\rangle\Box c.0$, $p \equiv \tau.0$ and $q \equiv d.0$, then we have two τ-transitions

$$C_X\{q/X\} \xrightarrow{\tau} a.0\Box d.0$$

and

$$B_X\{p/X\} \xrightarrow{\tau} (0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0.$$

It is not difficult to see that these two τ-transitions depend on the capability of C_X and p respectively. For the former, no matter what q' is, the corresponding τ-transition still exists for $C_X\{q'/X\}$. Moreover the target has the same pattern. That is, $C_X\{q'/X\} \xrightarrow{\tau} C'_X\{q'/X\}$ for any q', where $C'_X \equiv a.0\Box X$. To gain more intuition, we consider the proof tree of the second τ-transition:

$$\dfrac{\dfrac{\dfrac{\tau.0 \xrightarrow{\tau} 0}{\tau.0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle \xrightarrow{\tau} 0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle}}{\langle Y|Y = \tau.0\Box b.Y\rangle \xrightarrow{\tau} 0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle}}{B_X\{p/X\} \equiv \langle Y|Y = \tau.0\Box b.Y\rangle\Box c.0 \xrightarrow{\tau} (0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0}$$

It is evident that, although the free variable X occurs in B_X only once, the term $p(\equiv \tau.0)$ occurs twice in the unfolding $(\tau.0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0$ (of $B_X\{p/X\}$). It is the leftmost occurrence of $\tau.0$ that causes this τ-transition. For any p' with $p' \xrightarrow{\tau} p''$, we can get the proof tree of the τ-transition $B_X\{p'/X\} \xrightarrow{\tau} (p''\Box b.\langle Y|Y = p'\Box b.Y\rangle)\Box c.0$ by modifying the tree above in an obvious way. In order to illustrate that these two τ-derivatives $(0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0$ and $(p''\Box b.\langle Y|Y = p'\Box b.Y\rangle)\Box c.0$ have the same pattern, we may set $B'_{X,Z} \equiv (Z\Box b.\langle Y|Y = X\Box b.Y\rangle)\Box c.0$ with $Z \not\equiv X$. Clearly, $(0\Box b.\langle Y|Y = \tau.0\Box b.Y\rangle)\Box c.0 \equiv B'_{X,Z}\{\tau.0/X, 0/Z\}$ and $(p''\Box b.\langle Y|Y = p'\Box b.Y\rangle)\Box c.0 \equiv B'_{X,Z}\{p'/X, p''/Z\}$. Here the fresh variable Z is used to indicate the place where p''(or 0) is introduced.

We capture the preceding observation formally as follows, where two clauses concern themselves about τ-transitions exited by contexts and substitutions respectively; moreover some simple properties on contexts are also listed in (C-τ-3) which will be used in the sequel.

Lemma 5.6. For any $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$ then one of conclusions below holds.

1. There exists $C'_{\widetilde{X}}$ such that

   (C-τ-1) $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$;

   (C-τ-2) for any processes $\widetilde{q}$, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$;

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(C-τ-3) for each $X \in \widetilde{X}$,

(C-τ-3-i) if X is active in $C_{\widetilde{X}}$ then so it is in $C'_{\widetilde{X}}$ and the number of occurrences of X in $C'_{\widetilde{X}}$ is equal to that in $C_{\widetilde{X}}$;

(C-τ-3-ii) if X is unfolded in $C_{\widetilde{X}}$ then so it is in $C'_{\widetilde{X}}$ and the number of occurrences of X in $C'_{\widetilde{X}}$ is not more than that in $C_{\widetilde{X}}$;

(C-τ-3-iii) if X is strongly guarded in $C_{\widetilde{X}}$ then so it is in $C'_{\widetilde{X}}$;

(C-τ-3-iv) if X does not occur in the scope of any conjunction in $C_{\widetilde{X}}$ then neither does it in $C'_{\widetilde{X}}$.

2. There exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i \leqslant |\widetilde{X}|$ such that

(P-τ-1) $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$, in particular, if X_i is active in $C_{\widetilde{X}}$ then $C'_{\widetilde{X}} \equiv C_{\widetilde{X}}$;

(P-τ-2) $p_i \xrightarrow{\tau} p'$ and $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}$ for some p';

(P-τ-3) $C''_{\widetilde{X},Z}\{X_i/Z\} \equiv C'_{\widetilde{X}}$ and Z is 1-active in $C''_{\widetilde{X},Z}$;

(P-τ-4) $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\tau} C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$ for any processes $\widetilde{q}$ with $q_i \xrightarrow{\tau} q'$.

Proof. It proceeds by induction on the depth of the inference of $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} r$. It is easy to carry out based on the form of $C_{\widetilde{X}}$. In particular, Lemma 5.2 is used to handle the case $C_X \equiv \langle X|E\rangle$. The details may be found in Zhang *et al.* (2013). □

In the following, we intend to provide an analogue of Lemma 5.6 for visible transitions. To explain intuition behind the next result clearly, it is best to work with an example. Consider $C_{X_1,X_2} \equiv ((X_1 \wedge \langle Y|Y = a.Y\rangle)\Box a.b.0) \parallel_{\{b\}} (X_1 \wedge X_2)$, $p_1 \equiv a.0$ and $p_2 \equiv a.c.0$, we have three a-transitions

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} (0 \wedge \langle Y|Y = a.Y\rangle) \parallel_{\{b\}} (a.0 \wedge a.c.0),$$

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} b.0 \parallel_{\{b\}} (a.0 \wedge a.c.0),$$

and

$$C_{X_1,X_2}\{p_1/X_1, p_2/X_2\} \xrightarrow{a} ((a.0 \wedge \langle Y|Y = a.Y\rangle)\Box a.b.0) \parallel_{\{b\}} (0 \wedge c.0).$$

These visible transitions starting from $C_{X_1,X_2}\{p_1/X_1, p_2/X_2\}$ are activated by three distinct events. Clearly, both the context C_{X_1,X_2} and the substitution p_1 contribute to the first transition, while two latter transitions depend merely on the capability of C_{X_1,X_2} and $\widetilde{p_{1,2}}$ respectively. These three situations may be described uniformly in the lemma below. Here some additional properties on contexts are also listed in (CP-a-4).

Lemma 5.7. For any $a \in Act$, $C_{\widetilde{X}}$ and $\widetilde{p}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$ then there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \varnothing$ satisfying the conditions:

(CP-a-1) $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$;

(CP-a-2) for each $Y \in \widetilde{Y}$, Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$;

(CP-a-3) there exist $i_Y \leqslant |\widetilde{X}|$ for each $Y \in \widetilde{Y}$ such that

(CP-a-3-i) $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}}/\widetilde{Y}\} \equiv C'_{\widetilde{X}}$;

(CP-a-3-ii) for each $Y \in \widetilde{Y}$, $p_{i_Y} \xrightarrow{a} p'_Y$ for some p'_Y, and $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$;

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(CP-a-3-iii) for any $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{X}|$ and $\widetilde{q'}$ such that $|\widetilde{q'}| = |\widetilde{Y}|$ and $q_{i_Y} \xrightarrow{a} q'_Y$ for each $Y \in \widetilde{Y}$, if $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ is stable then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

(CP-a-4) for each $X \in \widetilde{X}$,

(CP-a-4-i) the number of occurrences of X in $C''_{\widetilde{X},\widetilde{Y}}$ is not more than that in $C'_{\widetilde{X},\widetilde{Y}}$;

(CP-a-4-ii) if X is active in $C'_{\widetilde{X},\widetilde{Y}}$ then so it is in $C''_{\widetilde{X},\widetilde{Y}}$;

(CP-a-4-iii) if X does not occur in the scope of any conjunction in $C_{\widetilde{X}}$ then neither does it in $C''_{\widetilde{X},\widetilde{Y}}$.

Proof. By induction on the depth of the inference of $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$. A full proof is given in Zhang *et al.* (2013). □

Intuitively, whenever all free variables occurring in $C_{\widetilde{X}}$ are guarded, any transition starting from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ must be performed by $C_{\widetilde{X}}$ itself. Formally, we have the following result whose proof is a simple application of Lemmas 5.2, 5.6 and 5.7.

Lemma 5.8. Let X be guarded in $C_{\widetilde{X}}$ for each $X \in \widetilde{X}$. If $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\alpha} r$ then there exists $B_{\widetilde{X}}$ such that $r \equiv B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{\alpha} B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $\widetilde{q}$.

The next lemma is another particular instance of Lemmas 5.6 and 5.7, which considers the case where the substitution is of the form $\langle Y|E\rangle$ or $\langle t_Y|E\rangle$. Its argument is splitted into two cases $\alpha = \tau$ or $\alpha \in Act$, and uses Lemmas 5.2, 5.6 and 5.7. Its full proof is also contained in Zhang *et al.* (2013).

Lemma 5.9. For any Y, E with $Y = t_Y \in E$ and context C_X with at most one occurrence of the unfolded variable X, if $C_X\{\langle Y|E\rangle/X\} \xrightarrow{\alpha} q$ then there exists B_X such that $q \equiv B_X\{\langle Y|E\rangle/X\}$, $C_X\{\langle t_Y|E\rangle/X\} \xrightarrow{\alpha} B_X\{\langle t_Y|E\rangle/X\}$, and X occurs in B_X at most once and such occurrence is unfolded. Moreover the statement still holds if we replace $\langle Y|E\rangle$ and $\langle t_Y|E\rangle$ by $\langle t_Y|E\rangle$ and $\langle Y|E\rangle$ respectively.

Based on the results obtained so far, we shall give a few further properties of unfolding. We first want to indicate some simple properties.

Lemma 5.10. The relation $\Rrightarrow$ satisfies the forward and backward conditions, that is, $p \Rrightarrow q$ implies that, for any $\rightsquigarrow \in \{\xrightarrow{\alpha}, \xRightarrow{\epsilon}, \xRightarrow{\epsilon}| : \alpha \in Act_\tau\}$,

1. if $p \rightsquigarrow p'$ then $q \rightsquigarrow q'$ and $p' \Rrightarrow q'$ for some q';
2. if $q \rightsquigarrow q'$ then $p \rightsquigarrow p'$ and $p' \Rrightarrow q'$ for some p'.

Proof. It follows from $p \Rrightarrow q$ that $p \Rrightarrow_n q$ for some n. In case $\rightsquigarrow = \xrightarrow{\alpha}$, the proof is straightforward by induction on n and using Lemmas 5.1 and 5.9. Moreover the arguments of other cases are also immediate by applying the conclusion for $\xrightarrow{\alpha}$ finitely often. □

In fact, it is to be expected that $p \Rrightarrow q$ implies $p =_{RS} q$. To verify it, we need to prove that $p \in F$ iff $q \in F$. The next lemma will serve as a stepping stone in proving this.

Lemma 5.11. For any Y, E with $Y = t_Y \in E$ and context C_X with at most one occurrence of the unfolded variable X, $C_X\{\langle Y|E\rangle/X\} \in F$ iff $C_X\{\langle t_Y|E\rangle/X\} \in F$.

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Proof. With the help of Lemma 5.9, both implications can be readily proved by induction on the depth of the inference of $C_X\{\langle Y|E\rangle/X\}F$ and $C_X\{\langle t_Y|E\rangle/X\}F$ respectively. □

By Lemmas 5.1 and 5.11, it is not difficult to get the following result, which asserts that the relation $\Rrightarrow$ preserves and respects the inconsistency.

Lemma 5.12. For any p, q, if $p \Rrightarrow q$, then $p \in F$ iff $q \in F$.

We now have the assertion of the equivalence of p and q modulo $=_{RS}$ whenever $p \Rrightarrow q$.

Lemma 5.13. If $p_1 \Rrightarrow p_2$ then $p_1 =_{RS} p_2$, in particular, $p_1 \approx_{RS} p_2$ whenever $p_1 \not\xrightarrow{\tau}$.

Proof. We only prove $p_1 \sqsubset_{\sim RS} p_2$ whenever $p_1 \not\xrightarrow{\tau}$. Set $\mathcal{R} = \{(p,q) : p \Rrightarrow q \text{ and } p \not\xrightarrow{\tau}\}$. It suffices to prove that $\mathcal{R}$ is a stable ready simulation relation. Let $(p,q) \in \mathcal{R}$. By Lemmas 5.10 and 5.12, it is evident that such pair satisfies (RS1), (RS2) and (RS4). For (RS3), suppose $p \overset{a}{\Longrightarrow}_F |p'$. Then $p \xrightarrow{a}_F p'' \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p''. By Lemmas 5.10 and 5.12, there exists q'' such that $q \xrightarrow{a}_F q''$ and $p'' \Rrightarrow q''$. Further, by Lemmas 4.2, 5.10 and 5.12, $p' \Rrightarrow q'$ and $q'' \overset{\epsilon}{\Longrightarrow}_F |q'$ for some q'. Moreover $(p', q') \in \mathcal{R}$, as desired. □

5.3. *τ-transition sequences and canonical evolution paths*

We have considered the pattern behind a given transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\alpha} r$. However, it is insufficient for the aim of this paper. The transitions with the form like $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} r$ play a central role when arguing consistency and behaviour of processes. Thus, this subsection will generalize Lemma 5.6 to the situation involving a sequence of τ-transitions (see Lemma 5.14). Moreover two kinds of canonical evolution paths for a given transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |r$ will be given in Lemmas 5.17 and 5.18.

Given a process $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, by Lemma 5.6, any τ-transition starting from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ may be caused by $C_{\widetilde{X}}$ itself or some p_i. Thus, for a sequence of τ-transitions, these two situations may occur alternately. Based on Lemma 5.6, we can capture this as follows.

Lemma 5.14. If $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} r$ then there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leqslant |\widetilde{X}|$ and p'_Y for $Y \in \widetilde{Y}$ such that

(MS-τ-1) $\widetilde{X} \cap \widetilde{Y} = \varnothing$ and Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ for each $Y \in \widetilde{Y}$;

(MS-τ-2) $p_{i_Y} \overset{\tau}{\Longrightarrow} p'_Y$ for each $Y \in \widetilde{Y}$ and $r \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$;

(MS-τ-3) for any $\widetilde{q}$ and $\widetilde{q'_Y}$ with $|\widetilde{q}| = |\widetilde{X}|$ and $Y \in \widetilde{Y}$,

(MS-τ-3-i) if $q_{i_Y} \overset{\epsilon}{\Longrightarrow} q'_Y$ for each $Y \in \widetilde{Y}$ then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow}\Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

(MS-τ-3-ii) if $q_{i_Y} \overset{\tau}{\Longrightarrow} q'_Y$ for each $Y \in \widetilde{Y}$ then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$;

(MS-τ-4) if $C_{\widetilde{X}}$ is stable then so is $C'_{\widetilde{X},\widetilde{Y}}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q_{i_Y}}/\widetilde{Y}\}$ for any $\widetilde{q}$;

(MS-τ-5) for each $X \in \widetilde{X}$, if X is strongly guarded in $C_{\widetilde{X}}$ then so it is in $C'_{\widetilde{X},\widetilde{Y}}$ and $X \not\equiv X_{i_Y}$ for each $Y \in \widetilde{Y}$;

(MS-τ-6) for each $X \in \widetilde{X}$ (or, $Y \in \widetilde{Y}$), if X (X_{i_Y} resp.) does not occur in the scope of any conjunction in $C_{\widetilde{X}}$ then neither does X (Y resp.) in $C'_{\widetilde{X},\widetilde{Y}}$;

(MS-τ-7) if r is stable then so are $C'_{\widetilde{X},\widetilde{Y}}$ and p'_Y for each $Y \in \widetilde{Y}$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^n r (n \geqslant 0)$. We proceed by induction on n. For the inductive base $n = 0$, the conclusion holds trivially by taking $C'_{\widetilde{X},\widetilde{Y}} \triangleq C_{\widetilde{X}}$ with $\widetilde{Y} = \varnothing$.

For the inductive step, assume $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k s \stackrel{\tau}{\longrightarrow} r$ for some s. For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k s$, by IH, there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leqslant |\widetilde{X}|$ and p'_Y for $Y \in \widetilde{Y}$ that realize (MS-τ-l) ($1 \leqslant l \leqslant 7$). In particular, we have $s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$ due to (MS-τ-2). Then, for the transition $s \stackrel{\tau}{\longrightarrow} r$, either the clause (1) or (2) in Lemma 5.6 holds. The argument splits into two cases.

Case 1. For the transition $s \stackrel{\tau}{\longrightarrow} r$, the clause (1) in Lemma 5.6 holds.

That is, for the transition $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \equiv s \stackrel{\tau}{\longrightarrow} r$, there exists $C''_{\widetilde{X},\widetilde{Y}}$ satisfying (C-τ-1,2,3) in Lemma 5.6. We shall check that $C''_{\widetilde{X},\widetilde{Y}}$, $\widetilde{i_Y}$ and $\widetilde{p'_Y}$ realize (MS-τ-1) - (MS-τ-7) w.r.t $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^{k+1} r$.

Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-1,5,6), it follows that $C''_{\widetilde{X},\widetilde{Y}}$ and $\widetilde{i_Y}$ realize (MS-τ-1), (MS-τ-5) and (MS-τ-6) due to (C-τ-3-i), (C-τ-3-iii) and (C-τ-3-iv) respectively. Moreover, as $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-1) it follows immediately that (MS-τ-2) holds. Since $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-2), by Lemma 5.6, $C'_{\widetilde{X},\widetilde{Y}}$ is not stable. Then neither is $C_{\widetilde{X}}$ because $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-4). Thus $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-4) trivially. Next we verify (MS-τ-3). Let $\widetilde{q}$ be any processes with $|\widetilde{q}| = |\widetilde{X}|$ and $q_{i_Y} \stackrel{\epsilon}{\Longrightarrow} q'_Y$ for each $Y \in \widetilde{Y}$.

(MS-τ-3-i) Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-i), $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ for some t. Moreover we have $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ due to (C-τ-2). Then it follows from Lemma 5.10 that $t \stackrel{\tau}{\longrightarrow} t' \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ for some t'. Therefore $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \stackrel{\tau}{\longrightarrow} t' \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$, as desired.

(MS-τ-3-ii) Straightforward as $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-ii) and $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (C-τ-2).

(MS-τ-7) Suppose $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \stackrel{\tau}{\not\longrightarrow}$. Then, since $C''_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-1), by Lemmas 5.4 and 5.6, it is easy to see that both $C''_{\widetilde{X},\widetilde{Y}}$ and $\widetilde{p'_Y}$ are stable.

Case 2. For the transition $s \stackrel{\tau}{\longrightarrow} r$, the clause (2) in Lemma 5.6 holds.

Then there exist $i_0 \leqslant |\widetilde{X}| + |\widetilde{Y}|$, $C''_{\widetilde{X},\widetilde{Y}} (\equiv C''_{X_1,\ldots,X_{|\widetilde{X}|},Y_{|\widetilde{X}|+1},\ldots,Y_{|\widetilde{X}|+|\widetilde{Y}|}})$ and $C'''_{\widetilde{X},\widetilde{Y},Z} (\equiv C'''_{X_1,\ldots,X_{|\widetilde{X}|},Y_{|\widetilde{X}|+1},\ldots,Y_{|\widetilde{X}|+|\widetilde{Y}|},Z})$ with $Z \notin \widetilde{X} \cup \widetilde{Y}$ satisfying (P-τ-1) - (P-τ-4). By (P-τ-3),

$$C''_{\widetilde{X},\widetilde{Y}} \equiv \begin{cases} C'''_{\widetilde{X},\widetilde{Y},Z}\{X_{i_0}/Z\}, \text{ if } 1 \leqslant i_0 \leqslant |\widetilde{X}|; \\ \\ C'''_{\widetilde{X},\widetilde{Y},Z}\{Y_{i_0}/Z\}, \text{ if } |\widetilde{X}| + 1 \leqslant i_0 \leqslant |\widetilde{X}| + |\widetilde{Y}|. \end{cases}$$

In case $|\widetilde{X}| + 1 \leqslant i_0 \leqslant |\widetilde{X}| + |\widetilde{Y}|$, by (P-$\tau$-2), there exists p' such that $p'_{Y_{i_0}} \stackrel{\tau}{\longrightarrow} p'$ and $r \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}, p'/Z\}$. Moreover, since Y_{i_0} is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$, by (P-$\tau$-1), we have $C'_{\widetilde{X},\widetilde{Y}} \equiv C''_{\widetilde{X},\widetilde{Y}}$. Further, since Z is 1-active in $C'''_{\widetilde{X},\widetilde{Y},Z}$ and $C''_{\widetilde{X},\widetilde{Y}} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{Y_{i_0}/Z\}$, it is easy to see that Y_{i_0} does not occur in $C'''_{\widetilde{X},\widetilde{Y},Z}$. Hence $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}[p'/p'_{Y_{i_0}}]/\widetilde{Y}\}$

$\equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_Y}[p'/p'_{Y_{i_0}}]/\widetilde{Y}\}$. Then it is not difficult to check that $C'_{\widetilde{X},\widetilde{Y}}$, $\widetilde{p'_Y}[p'/p'_{Y_{i_0}}]$ and $\widetilde{i_Y}$ realize (MS-τ-l)($1 \leqslant l \leqslant 7$) w.r.t the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^{k+1}r$, as desired.

In case $1 \leqslant i_0 \leqslant |\widetilde{X}|$, by (P-$\tau$-2), there exists p'' such that $p_{i_0} \stackrel{\tau}{\longrightarrow} p''$ and $r \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{p}/\widetilde{X},\widetilde{p'_Y}/\widetilde{Y},p''/Z\}$. Set $i_Z \triangleq i_0$ and $p'_Z \triangleq p''$. In the following, we intend to verify that $C'''_{\widetilde{X},\widetilde{Y},Z}$, i_U ($U \in \widetilde{Y} \cup \{Z\}$) and $|\widetilde{Y}|+1$-tuple $\widetilde{p'_U}$ with $U \in \widetilde{Y} \cup \{Z\}$ realize (MS-τ-1) - (MS-τ-7) w.r.t $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^{k+1}r$.

(MS-τ-1) By (P-τ-1), we have $C'_{\widetilde{X},\widetilde{Y}} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}$. Moreover, since $C'_{\widetilde{X},\widetilde{Y}}$ satisfy (MS-τ-1), by Lemma 5.2(1), Y is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$ for each $Y \in \widetilde{Y}$. Further, by (P-τ-3), each $Y (\in \widetilde{Y})$ and Z are 1-active in $C'''_{\widetilde{X},\widetilde{Y},Z}$. **(MS-$\tau$-2)** It is straightforward. **(MS-τ-4)** Assume $C_{\widetilde{X}}$ is stable. By (MS-τ-4), $C'_{\widetilde{X},\widetilde{Y}}$ is stable and for any $\widetilde{q}$, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y}\}$. Then, by Lemma 5.10, it follows from $C'_{\widetilde{X},\widetilde{Y}} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}$ (i.e., (P-τ-1)) and $C'''_{\widetilde{X},\widetilde{Y},Z}\{X_{i_Z}/Z\} \equiv C''_{\widetilde{X},\widetilde{Y}}$ ((P-τ-3)) that $C'''_{\widetilde{X},\widetilde{Y},Z}$ is stable and $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y}\} \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q_{i_Y}}/\widetilde{Y},q_{i_Z}/Z\}$. **(MS-$\tau$-5,6)** By Lemma 5.2(3)(5), they immediately follow from the fact that $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-5,6) and $C'''_{\widetilde{X},\widetilde{Y},Z}$ satisfies (P-τ-1,3). **(MS-τ-7)** Immediately follows from (MS-τ-1), (MS-τ-2) and Lemmas 5.4 and 5.6. In the following, we check (MS-τ-3). Let $\widetilde{q}$ be any processes with $|\widetilde{q}| = |\widetilde{X}|$.

(MS-τ-3-i) Suppose $q_{i_U} \stackrel{\epsilon}{\Longrightarrow} q'_U$ for each $U \in \widetilde{Y} \cup \{Z\}$. Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-i), we have $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\}$ for some t. It follows from $q_{i_Z} \stackrel{\epsilon}{\Longrightarrow} q'_Z$ that $q_{i_Z}(\stackrel{\tau}{\longrightarrow})^m q'_Z$ for some $m \geqslant 0$. We shall distinguish two cases based on m.

In case $m = 0$, we get $q_{i_Z} \equiv q'_Z$. Since $C'''_{\widetilde{X},\widetilde{Y},Z}$ satisfies (P-τ-1) and (P-τ-3), we have

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\} \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}.$$

Therefore, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \Rrightarrow C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\} \equiv C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$.

In case $m > 0$, i.e., $q_{i_Z} \stackrel{\tau}{\longrightarrow} q'' \stackrel{\epsilon}{\Longrightarrow} q'_Z$ for some q'', by (P-τ-4), we obtain

$$C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q''/Z\}.$$

Moreover, since Z is 1-active, $C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q''/Z\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$ by Lemma 5.4. Then, by Lemma 5.10, it follows from $t \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\}$ that there exist t' such that $t \stackrel{\epsilon}{\Longrightarrow} t' \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$. Consequently, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} t \stackrel{\epsilon}{\Longrightarrow} t' \Rrightarrow C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$.

(MS-τ-3-ii) Suppose $q_{i_U} \stackrel{\tau}{\Longrightarrow} q'_U$ for each $U \in \widetilde{Y} \cup \{Z\}$. Since $C'_{\widetilde{X},\widetilde{Y}}$ satisfies (MS-τ-3-ii), we have $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\}$. Moreover, $q_{i_Z} \stackrel{\tau}{\longrightarrow} q'' \stackrel{\epsilon}{\Longrightarrow} q'_Z$ for some q'' because of $q_{i_Z} \stackrel{\tau}{\Longrightarrow} q'_Z$. Hence, $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q''/Z\}$ by (P-τ-4). Further, since Z is 1-active, $C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q''/Z\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$ by Lemma 5.4. Consequently, $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'''_{\widetilde{X},\widetilde{Y},Z}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Y}/\widetilde{Y},q'_Z/Z\}$, as desired. □

Lemma 5.15. For any $\widetilde{p}$ and stable context $C_{\widetilde{X}}$, if, for each $i \leqslant |\widetilde{X}|$, $p_i \overset{\epsilon}{\Longrightarrow} |p'_i$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |q$ for some q.

Proof. By Lemmas 5.3 and 5.2(4), $C_{\widetilde{X}} \Rrightarrow C'_{\widetilde{X}}$ for some $C'_{\widetilde{X}}$ such that each unguarded occurrence of any free variable in $C'_{\widetilde{X}}$ is unfolded. Moreover, since $C_{\widetilde{X}}$ is stable, so is $C'_{\widetilde{X}}$ by $C_{\widetilde{X}}\{0/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{0/\widetilde{X}\}$ and Lemma 5.10.

Let $C'_{\widetilde{X},\widetilde{Y}}$ be the context obtained from $C'_{\widetilde{X}}$ by replacing simultaneously all unguarded and unfolded occurrences of free variables in $\widetilde{X}$ by distinct and fresh variables $\widetilde{Y}$. Here distinct occurrences are replaced by distinct variables. Clearly, we have

1. for each $Y \in \widetilde{Y}$, there exists exactly one $i_Y \leqslant |\widetilde{X}|$ such that $C'_{\widetilde{X}} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{X_{i_Y}}/\widetilde{Y}\}$,
2. all variables in $\widetilde{Y}$ are 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ and
3. $C'_{\widetilde{X},\widetilde{Y}}$ is stable.

Then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}$, and by Lemma 5.4, we obtain $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_{i_Y}}/\widetilde{Y}\}$. Further, since $C'_{\widetilde{X},\widetilde{Y}}$ and $\widetilde{p'_{i_Y}}$ are stable and $\widetilde{Y}$ contains all unguarded occurrences of variables in $C'_{\widetilde{X},\widetilde{Y}}$, $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_{i_Y}}/\widetilde{Y}\} \overset{\tau}{\not\longrightarrow}$ by Lemma 5.6. So, by Lemma 5.10, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |q \Rrightarrow C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_{i_Y}}/\widetilde{Y}\}$ for some q. □

Now we turn to considering 'canonical' paths. Given $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |r$, in general there exist more than one evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to r. Since each τ-transition in CLL_R is activated by a single process, a natural conjecture arises at this point that there exist some 'canonical' evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to r in which the context $C_{\widetilde{X}}$ first evolves itself into a stable context then p_i evolves. Clearly, if such canonical path indeed exists, by Lemma 4.2, each process along this path is not in F whenever $r \notin F$. A weak version of this conjecture will be verified in Lemma 5.17. To this end, a preliminary result is given below. Its proof is a tedious but straightforward induction on t_1, and hence is omitted.

Lemma 5.16. Let t_1, t_2 be two terms and $\widetilde{X}$ a tuple of variables such that any recursive variable occurring in $t_i (i = 1, 2)$ is not in $\widetilde{X}$, and let $\widetilde{a_X.0}$ be a tuple of processes, where a_X is a fresh visible action for each $X \in \widetilde{X}$. Then

1. if $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \equiv t_2\{\widetilde{a_X.0}/\widetilde{X}\}$ then $t_1 \equiv t_2$;
2. if $t_1\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{a_X.0}/\widetilde{X}\}$ then $t_1\{\widetilde{r}/\widetilde{X}\} \Rrightarrow_1 t_2\{\widetilde{r}/\widetilde{X}\}$ for any $\widetilde{r}$.

Having disposed of this preliminary step, we can now verify a weak version of the conjecture mentioned above, which is sufficient for the aim of this paper.

Lemma 5.17. If $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |r$ then there exists a stable context $D_{\widetilde{X}}$ such that

1. $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |r' \Rrightarrow r$ for some r' and
2. $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{X}|$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\overset{\tau}{\longrightarrow})^n|r$. It proceeds by induction on n. For the inductive base $n = 0$, it follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv r \overset{\tau}{\not\longrightarrow}$ that $C_{\widetilde{X}}$ is stable by Lemma 5.6. Then it is straightforward to verify that $C_{\widetilde{X}}$ itself is exactly what we seek. For the inductive step,

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assume $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} t(\xrightarrow{\tau})^k|r$ for some t. Then, for $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{\tau} t$, either the clause (1) or (2) in Lemma 5.6 holds. The first alternative is easy to handle and is thus omitted. Next we consider the second alternative. In this situation, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leqslant |\widetilde{X}|$ that satisfy (P-τ-1) – (P-τ-4). By (P-τ-2), we have

$$t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \text{ for some } p' \text{ with } p_{i_0} \xrightarrow{\tau} p'.$$

Then, for $C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}(\xrightarrow{\tau})^k|r$, by IH, there exists a stable context $D'_{\widetilde{X},Z}$ such that

$$C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \overset{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \overset{\epsilon}{\Longrightarrow} |r' \Rrightarrow r \text{ for some } r' \tag{5.17.1}$$

and for any q' and $\widetilde{q}$, we have

$$C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\} \overset{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}. \tag{5.17.2}$$

In particular, we have $C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \overset{\epsilon}{\Longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\}$ where distinct visible actions $\widetilde{a_X}$ and a_Z are fresh. For this transition, applying Lemma 5.6 finitely often (notice that, in this procedure, since $\widetilde{a_X.0}$ and $a_Z.0$ are stable, the clause (2) in Lemma 5.6 is always false), then by clause (1) in Lemma 5.6, we get the sequence

$$C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \equiv C^0_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \xrightarrow{\tau} C^1_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \xrightarrow{\tau} \cdots \xrightarrow{\tau} C^n_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\} \equiv D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_Z.0/Z\}.$$

Here, $n \geqslant 0$ and $C^i_{\widetilde{X},Z}$ satisfies (C-τ-1,2,3) for each $1 \leqslant i \leqslant n$. Moreover, since Z is 1-active in $C''_{\widetilde{X},Z}$, by (C-τ-3-i), so is Z in $C^n_{\widetilde{X},Z}$. We also have $C^n_{\widetilde{X},Z} \equiv D'_{\widetilde{X},Z}$ by Lemma 5.16. Hence we can conclude that

$$Z \text{ is 1-active in } D'_{\widetilde{X},Z}. \tag{5.17.3}$$

Since $C'_{\widetilde{X}}$ and $C''_{\widetilde{X},Z}$ satisfy (P-τ-1) and (P-τ-3), for any $\widetilde{s}$, we get

$$C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \equiv C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\}. \tag{5.17.4}$$

In order to complete the proof, it suffices to find a stable context $D_{\widetilde{X}}$ satisfying conditions (1) and (2). In the following, we shall use $\widetilde{a_X.0}$ again to obtain such context.

Since $\widetilde{a_X.0}$ and $D'_{\widetilde{X},Z}$ are stable, by (5.17.2), we get $C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \overset{\epsilon}{\Longrightarrow} |D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$. Moreover we have $C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \equiv C''_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$ by (5.17.4). Thus $C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$. Then, since $\widetilde{a_X.0}$ are stable, by Lemma 5.14, there exists a stable context $B_{\widetilde{X}}$ such that

$$B_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \equiv D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \tag{5.17.5}$$

and

$$C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} B_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \text{ for any } \widetilde{s}. \tag{5.17.6}$$

In addition, by (5.17.4) and Lemma 5.10, we have $C_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \overset{\epsilon}{\Longrightarrow} |t' \Rrightarrow D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\}$ for some t'. Further, since $\widetilde{a_X.0}$ are stable, by Lemma 5.14, there exists a stable context $D_{\widetilde{X}}$ such that

$$t' \equiv D_{\widetilde{X}}\{\widetilde{a_X.0}/\widetilde{X}\} \Rrightarrow D'_{\widetilde{X},Z}\{\widetilde{a_X.0}/\widetilde{X}, a_{X_{i_0}}.0/Z\} \tag{5.17.7}$$

and

$$C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \text{ for any } \widetilde{s}. \tag{5.17.8}$$

Notice that, (5.17.8) follows from (MS-τ-3-ii) with $\widetilde{Y} = \emptyset$. In the following, we intend to prove that $D_{\widetilde{X}}$ is what we seek. It immediately follows from (5.17.8) that $D_{\widetilde{X}}$ meets the requirement (2). We are left with the task of verifying that $D_{\widetilde{X}}$ satisfies the condition (1). So far, for any $\widetilde{s}$, we have the diagram below, where the first line follows from (5.17.4),

$$\begin{array}{ccccc}
C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \Rrightarrow & C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \equiv & C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\} \\
\Downarrow \epsilon \text{ by } (5.17.8) & & \Downarrow \epsilon \text{ by } (5.17.6) & & \Downarrow \epsilon \text{ by } (5.17.2) \\
D_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \Rrightarrow & B_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} & \equiv & D'_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s_{i_0}/Z\}.
\end{array}$$

Here the last line in the above follows from (5.17.7) and (5.17.5) using Lemma 5.16. Further, by Lemma 5.4 and $p_{i_0} \stackrel{\tau}{\longrightarrow} p'$, it follows from (5.17.1) and (5.17.3) that

$$B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p_{i_0}/Z\} \stackrel{\tau}{\longrightarrow} D'_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow} |r' \Rrightarrow r.$$

Finally, since $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, by Lemma 5.10, we get $D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r'' \Rrightarrow r' \Rrightarrow r$ for some r'', which, together with $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} D_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$, implies that the stable context $D_{\widetilde{X}}$ also meets the requirement (1), as desired. □

The result below asserts that there exist another ‘canonical’ evolution paths from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ to a given stable τ-descendant r. For these paths, an unstable p_i evolves first provided that such p_i is located in an active position.

Lemma 5.18. If $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$ and X_i is 1-active in $C_{\widetilde{X}}$ for some $i \leqslant |\widetilde{X}|$, then there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$.

Proof. Suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^n|q$ for some $n \geqslant 0$. We shall prove it by induction on n. For the inductive base $n = 0$, we have $p_i \stackrel{\tau}{\not\longrightarrow}$ by Lemma 5.4, and hence it holds trivially by taking $p' \equiv p_i$. For the inductive step $n = k + 1$, suppose $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r(\stackrel{\tau}{\longrightarrow})^k|q$ for some r. For $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r$, either the clause (1) or (2) in Lemma 5.6 holds. For the first alternative, there exists a context $C'_{\widetilde{X}}$ such that

1.1. X_i is 1-active in $C'_{\widetilde{X}}$ (by (C-τ-3-i)),
1.2. $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and
1.3. $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\}$ for any $\widetilde{s}$.

By (1.1), we can apply IH for the transition $r \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k|q$, and hence there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and $C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$. Moreover, since X_i is 1-active in $C_{\widetilde{X}}$ and $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$, we have $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\}$ by Lemma 5.4.

We also have $C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\}$ by (1.3). Therefore, $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$, as desired.

For the second alternative, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ and $i_0 \leqslant |\widetilde{X}|$ such that

2.1. Z is 1-active in $C''_{\widetilde{X},Z}$,

2.2. $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\}$ for some p'_{i_0} with $p_{i_0} \stackrel{\tau}{\longrightarrow} p'_{i_0}$ and

2.3. $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{s}/\widetilde{X}, s'/Z\}$ for any $\widetilde{s}$ and s' with $s_{i_0} \stackrel{\tau}{\longrightarrow} s'$.

In case $i_0 = i$, we have $C_{\widetilde{X}} \equiv C'_{\widetilde{X}}$ by (P-τ-1), and hence $r \equiv C_{\widetilde{X}}\{\widetilde{p}\,[p'_{i_0}/p_i]/\widetilde{X}\}$ by (2.2) and (P-τ-3). For the transition $r \equiv C_{\widetilde{X}}\{\widetilde{p}\,[p'_{i_0}/p_i]/\widetilde{X}\}(\stackrel{\tau}{\longrightarrow})^k|q$, by IH, there exists p'' such that $p'_{i_0} \stackrel{\epsilon}{\Longrightarrow} |p''$ and $C_{\widetilde{X}}\{\widetilde{p}\,[p'_{i_0}/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p''/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$. Hence $p_{i_0} \stackrel{\tau}{\longrightarrow} p'_{i_0} \stackrel{\epsilon}{\Longrightarrow} |p''$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'_{i_0}/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p''/p_i]/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |q$. Next we consider the other case $i_0 \neq i$. Then for the transition $r \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\}(\stackrel{\tau}{\longrightarrow})^k|q$, by IH, there exists p' such that $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$ and

$$C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} |q.$$

In addition, since X_i is 1-active in $C_{\widetilde{X}}$ and $p_i \stackrel{\epsilon}{\Longrightarrow} |p'$, by Lemma 5.4, we obtain $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\}$. Moreover $C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\}$ by (2.3). Thus $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X}}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{p}\,[p'/p_i]/\widetilde{X}, p'_{i_0}/Z\} \stackrel{\epsilon}{\Longrightarrow} |q$. □

6. Precongruence

This section intends to establish a fundamental property that $\sqsubseteq_{RS}$ is a precongruence. Its proof is far from trivial and requires a solid effort. As mentioned in Section 1, a distinguishing feature of LLTS is that it involves consideration of inconsistencies. It is the inconsistency predicate F that makes everything become quite troublesome. A crucial part in carrying out the proof is that we need to prove that $C_X\{q/X\} \in F$ implies $C_X\{p/X\} \in F$ whenever $p \sqsubseteq_{RS} q$. Its argument will be divided into two steps. First, we shall show that, for any stable process p, $C_X\{\tau.p/X\} \in F$ iff $C_X\{p/X\} \in F$ (see Lemmas 6.2 and 6.4). Second, we intend to prove that $C_X\{q/X\} \in F$ implies $C_X\{p/X\} \in F$ whenever p and q are uniform w.r.t stability and $p \sqsubseteq_{RS} q$ (see Lemma 6.6).

Definition 6.1 (uniform w.r.t stability). Two tuples $\widetilde{p}$ and $\widetilde{q}$ with $|\widetilde{q}| = |\widetilde{p}|$ are uniform w.r.t stability, in symbols $\widetilde{p} \bowtie \widetilde{q}$, if for each $i \leqslant |\widetilde{p}|$, p_i is stable iff q_i is stable.

Notation. For convenience, given tuples $\widetilde{p}$ and $\widetilde{q}$, for $R \in \{\sqsubseteq_{RS}, \underset{\sim RS}{\sqsubset}, \stackrel{\epsilon}{\Longrightarrow}|, \equiv\}$, the notation $\widetilde{p}R\widetilde{q}$ means that $|\widetilde{p}| = |\widetilde{q}|$ and p_iRq_i for each $i \leqslant |\widetilde{p}|$.

Lemma 6.1. For any $C_{\widetilde{X}}$, $\widetilde{p}$ and $\widetilde{q}$ with $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, if $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ are stable and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\longrightarrow}$ iff $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow}$ for any $a \in Act$.

Proof. We give the proof only for the implication from right to left, the same argument applies to the other implication. Assume $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow} q'$. Then there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and

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$C''_{\widetilde{X},\widetilde{Y}}$ with $\widetilde{X} \cap \widetilde{Y} = \varnothing$ that satisfy (CP-a-1) – (CP-a-4) in Lemma 5.7. Hence, due to (CP-a-1) and (CP-a-3-i), there exist $i_Y \leqslant |\widetilde{X}|(Y \in \widetilde{Y})$ such that for any $\widetilde{r}$ with $|\widetilde{r}| = |\widetilde{X}|$

$$C_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{r}/\widetilde{X}, \widetilde{r_{i_Y}}/\widetilde{Y}\}. \tag{6.1.1}$$

In particular, by Lemma 5.10, it follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\not\longrightarrow}$ that both $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}$ and $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q_{i_Y}}/\widetilde{Y}\}$ are stable. Then, for each $Y \in \widetilde{Y}$, both p_{i_Y} and q_{i_Y} are stable by Lemma 5.4 and (CP-a-2). Moreover, by (6.1.1) with $\widetilde{r} \equiv \widetilde{p}$ and Lemmas 5.12 and 5.5, we have $p_{i_Y} \notin F$ for each $Y \in \widetilde{Y}$ due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$. Therefore, for each $Y \in \widetilde{Y}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that $p_{i_Y} \sqsubset_{\sim RS} q_{i_Y}$, and $\mathcal{I}(p_{i_Y}) = \mathcal{I}(q_{i_Y})$ because of $p_{i_Y} \notin F$. Hence $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\longrightarrow}$ by (CP-a-3-iii). □

Convention 6.1. The arguments in the remainder of this paper often proceed by distinguishing several cases based on the last rule applied in an inference. For such argument, since rules associated with operations $\wedge$, $\vee$, $\|_A$ and $\Box$ are symmetric w.r.t their two operands, we shall consider only one of two symmetric rules and omit the other one.

In the following, we intend to show that $C_X\{p/X\} \in F$ iff $C_X\{\tau.p/X\} \in F$ for any stable p, which falls naturally into two parts: Lemmas 6.2 and 6.4.

Lemma 6.2. For any C_X and stable p, $C_X\{p/X\} \notin F$ implies $C_X\{\tau.p/X\} \notin F$.

Proof. Let p be any stable process. Set

$$\Omega \triangleq \{B_X\{\tau.p/X\} : B_X\{p/X\} \notin F \text{ and } B_X \text{ is a context}\}.$$

Similar to Lemma 4.4, it suffices to prove that for any $t \in \Omega$, each proof tree of tF has a proper subtree with root sF for some $s \in \Omega$. Suppose that $C_X\{\tau.p/X\} \in \Omega$ and $\mathcal{T}$ is any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{\tau.p/X\}F$. Hence $C_X\{p/X\} \notin F$. The rest of the proof runs by distinguishing cases based on C_X. Here we handle only two cases.
Case 1. $C_X \equiv \langle Y|E\rangle$. Then the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{\tau.p/X\}F}{\langle Y|E\rangle\{\tau.p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{\tau.p/X\}F}.$$

For the first alternative, since $C_X\{p/X\} \equiv \langle Y|E\rangle\{p/X\} \notin F$, by Lemma 4.1(8), we get $\langle t_Y|E\rangle\{p/X\} \notin F$. Hence $\langle t_Y|E\rangle\{\tau.p/X\} \in \Omega$. For the second alternative, since $C_X\{p/X\} \notin F$, we get $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. Moreover, by Lemma 5.14, it follows from $p \stackrel{\tau}{\not\longrightarrow}$ that there exists a stable context C'_X such that

$$q \equiv C'_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{\tau.p/X\}. \tag{6.2.1}$$

Further, by Lemma 5.15 and $\tau.p \stackrel{\tau}{\longrightarrow} |p$, we get

$$C'_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |s \text{ for some } s. \tag{6.2.2}$$

So, by Lemma 5.14 again, there exists $C''_{X,\widetilde{Z}}$ with $X \notin \widetilde{Z}$ such that $s \equiv C''_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\}$ and $C'_X\{p/X\} \Rrightarrow C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\}$. Thus, by Lemma 5.12, $C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \notin F$ because of

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$q \equiv C'_X\{p/X\} \notin F$. Set $C'''_X \triangleq C''_{X,\widetilde{Z}}\{p/\widetilde{Z}\}$. It follows from $C'''_X\{p/X\} \equiv C''_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \notin F$ that $s \equiv C'''_X\{\tau.p/X\} \in \Omega$. Moreover $\mathcal{T}$ contains a proper subtree with root sF due to (6.2.1) and (6.2.2).

Case 2. $C_X \equiv B_X \wedge D_X$.

Clearly, if the last rule applied in $\mathcal{T}$ is Rule Rp_8 then it is immediate that $B_X\{\tau.p/X\} \in \Omega$, as desired. Moreover, similar to the second alternative in Case 1, we can deal with the case where Rule Rp_{13} is the last rule employed in $\mathcal{T}$. We now turn to the other cases.

Case 2.1. $\frac{B_X\{\tau.p/X\} \xrightarrow{a} r}{C_X\{\tau.p/X\}F}$ with $D_X\{\tau.p/X\} \not\xrightarrow{a}$ and $C_X\{\tau.p/X\} \not\xrightarrow{\tau}$.

In this situation, $B_X\{\tau.p/X\}$, C_X and B_X are stable. Moreover, since p is stable, so is $B_X\{p/X\}$. Due to $C_X\{p/X\} \notin F$, we obtain $B_X\{p/X\} \notin F$. Then, by Lemma 6.1, it follows from $p =_{RS} \tau.p$ and $B_X\{\tau.p/X\} \xrightarrow{a}$ that

$$B_X\{p/X\} \xrightarrow{a} r_1 \text{ for some } r_1. \tag{6.2.3}$$

Similarly, it follows from $D_X\{\tau.p/X\} \not\xrightarrow{a}$ that

$$D_X\{p/X\} \not\xrightarrow{a}. \tag{6.2.4}$$

In addition, since $B_X \wedge D_X$ and p are stable, so is $B_X\{p/X\} \wedge D_X\{p/X\}$. Clearly, the rule $\frac{B_X\{p/X\} \xrightarrow{a} r_1}{C_X\{p/X\}F}$ is a ground instance of the rule $\frac{pprem(Rp_{10})}{conc(Rp_{10})}$; moreover it is in $Strip(R_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R})$ because of (6.2.4) and $C_X\{p/X\} \not\xrightarrow{\tau}$. So, we get $C_X\{p/X\} \equiv B_X\{p/X\} \wedge D_X\{p/X\} \in F$ by (6.2.3), which contradicts that $C_X\{\tau.p/X\} \in \Omega$. Hence this case is impossible.

Case 2.2. $\frac{C_X\{\tau.p/X\} \xrightarrow{\alpha} s, \{rF : C_X\{\tau.p/X\} \xrightarrow{\alpha} r\}}{C_X\{\tau.p/X\}F}$.

The argument splits into two cases based on α.

Case 2.2.1. $\alpha = \tau$. We distinguish two cases depending on whether C_X is stable.

Case 2.2.1.1. C_X is not stable.

Since $C_X\{p/X\} \notin F$, we have $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$ for some p'. Moreover, by Lemma 5.17, there exist p'' and stable C^*_X such that $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |p'' \Rrightarrow p'$ and $C_X\{t/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{t/X\}$ for any t. Further, since C_X is not stable and $p \not\xrightarrow{\tau}$, by Lemma 5.6, there exists C'_X such that

$$C_X\{p/X\} \xrightarrow{\tau} C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \xrightarrow{\tau} C'_X\{\tau.p/X\}.$$

Since $p' \notin F$ and $p'' \Rrightarrow p'$, by Lemma 5.12, we get $p'' \notin F$. Together with the transitions $C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |p''$, by Lemma 4.2, this implies $C'_X\{p/X\} \notin F$. Hence $C'_X\{\tau.p/X\} \in \Omega$, and $\mathcal{T}$ has a proper subtree with root $C'_X\{\tau.p/X\}F$.

Case 2.2.1.2. C_X is stable.

Due to $C_X\{\tau.p/X\} \xrightarrow{\tau} s$, either the clause (1) or (2) in Lemma 5.6 holds. Since C_X is stable, by (C-τ-2) in Lemma 5.6, it is easy to see that the clause (1) does not hold, and hence the clause (2) holds, that is, there exists $C'_{X,Z}$ with $X \neq Z$ such that

$$C_X\{\tau.p/X\} \xrightarrow{\tau} C'_{X,Z}\{\tau.p/X, p/Z\} \text{ and } C_X\{p/X\} \Rrightarrow C'_{X,Z}\{p/X, p/Z\}.$$

Set $C''_X \triangleq C'_{X,Z}\{p/Z\}$. Hence $\mathcal{T}$ has a proper subtree with root $C''_X\{\tau.p/X\}F$. Moreover, by Lemma 5.12 and $C_X\{p/X\} \notin F$, we have $C'_{X,Z}\{p/X, p/Z\} \notin F$. Thus $C''_X\{\tau.p/X\} \equiv C'_{X,Z}\{\tau.p/X, p/Z\} \in \Omega$, as desired.

Case 2.2.2. $\alpha \in Act$.

Then both C_X and $C_X\{p/X\}$ are stable. Moreover, since $C_X\{\tau.p/X\} \stackrel{\alpha}{\longrightarrow}$, $\tau.p =_{RS} p$ and $C_X\{p/X\} \notin F$, by Lemma 6.1, we get $C_X\{p/X\} \stackrel{\alpha}{\longrightarrow}$. Further, by Theorem 4.1, it follows from $C_X\{p/X\} \notin F$ that $C_X\{p/X\} \stackrel{\alpha}{\longrightarrow}_F q$ for some q. For this transition, by Lemma 5.7, there exist C'_X, $C'_{X,\widetilde{Z}}$ and $C''_{X,\widetilde{Z}}$ with $X \notin \widetilde{Z}$ that realize (CP-a-1,2,3,4). To complete the proof, we intend to prove that $\widetilde{Z} = \varnothing$. On the contrary, suppose $\widetilde{Z} \neq \varnothing$. Then, by (CP-$a$-2) and (CP-$a$-3-i), there exists an active occurrence of the variable X in C'_X. So, by Lemma 5.4, $C'_X\{\tau.p/X\} \stackrel{\tau}{\longrightarrow}$. Then, by Lemma 5.10, it follows from $C_X\{\tau.p/X\} \Rrightarrow C'_X\{\tau.p/X\}$ (i.e., (CP-a-1)) that $C_X\{\tau.p/X\} \stackrel{\tau}{\longrightarrow}$, which contradicts $C_X\{\tau.p/X\} \stackrel{\alpha}{\longrightarrow}$. Thus $\widetilde{Z} = \varnothing$, and hence $q \equiv C''_{X,\widetilde{Z}}\{p/X\}$ by (CP-a-3-ii). Since $C_X\{\tau.p/X\}$ is stable, by (CP-a-3-iii), we get $C_X\{\tau.p/X\} \stackrel{\alpha}{\longrightarrow} C''_{X,\widetilde{Z}}\{\tau.p/X\}$. Thus, $\mathcal{T}$ contains a proper subtree with root $C''_{X,\widetilde{Z}}\{\tau.p/X\}F$; moreover $C''_{X,\widetilde{Z}}\{\tau.p/X\} \in \Omega$ due to $C''_{X,\widetilde{Z}}\{p/X\} \equiv q \notin F$. □

To show the converse of the above result, the preliminary result below is given. Here, for any finite set S of processes, by virtue of the commutative and associative laws of external choice (Zhang *et al.* 2011), we may introduce the notion of a generalized external choice (denoted by $\underset{p\in S}{\Box} p$) by the standard method.

Lemma 6.3. Let t_1, t_2 be two terms and $\{X\} \cup \widetilde{Z}$ a tuple of variables such that none of recursive variable occurring in $t_i (i = 1, 2)$ is in $\{X\} \cup \widetilde{Z}$. Suppose that Z is active in t_1, t_2 for each $Z \in \widetilde{Z}$. Given $a \in Act$ and distinct fresh visible actions a_X and $\widetilde{a_Z}$, we put

$$T_a \triangleq \begin{cases} \underset{Z\in\widetilde{Z}}{\Box} a.a_Z.0 & \text{if } \widetilde{Z} \neq \varnothing; \\ a_X.0 & \text{otherwise.} \end{cases}$$

Then, for any p and $\widetilde{q}$,

1. $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ implies $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$;
2. $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ implies $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$.

Proof. **(1)** If $FV(t_1) \cap (\{X\} \cup \widetilde{Z}) = \varnothing$ then (1) holds trivially. In the following, we consider the other case $FV(t_1) \cap (\{X\} \cup \widetilde{Z}) \neq \varnothing$. It proceeds by induction on t_1. We give the proof only for the case $t_1 \equiv s_1 \Box s_2$, the others are left to the reader.

In this case, the topmost operator of $t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ is an external choice □. Clearly, such operator comes from either t_2 or T_a. For the former, we get $t_2 \equiv s'_1 \Box s'_2$ for some s'_1 and s'_2, and the proof is easy to carry out by using IH. Next we shall show that the latter case is impossible. In this situation, we have $t_2 \equiv X$ and $|\widetilde{Z}| > 1$ (otherwise, T_a does not contain any operator □ at all). Clearly, $a_Z.0$ is guarded in T_a for each $Z \in \widetilde{Z}$. Moreover, since each Z in $\widetilde{Z}$ is active in t_1 and $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv T_a$, we get $FV(t_1) \cap \widetilde{Z} = \varnothing$. Hence it follows from the assumption $FV(t_1) \cap (\{X\} \cup \widetilde{Z}) \neq \varnothing$ that $X \in FV(t_1)$. Then $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Box s_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \not\equiv T_a \equiv t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ due to $X \in FV(s_1) \cup FV(s_2) (= FV(t_1))$, a contradiction, as desired.

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(2) If $FV(t_1) \cap \widetilde{Z} = \varnothing$, it follows by Lemma 5.16. Next we consider the other case $FV(t_1) \cap \widetilde{Z} \neq \varnothing$. It proceeds by induction on t_1. Since Z is active in t_1 for each $Z \in \widetilde{Z}$, we get either $t_1 \equiv Z$ or $t_1 \equiv s_1 \odot s_2$ for some s_1 and s_2, where $Z \in \widetilde{Z}$ and $\odot \in \{\wedge, \|_A, \Box\}$. We give the proof only for the case $t_1 \equiv s_1 \Box s_2$, and the remaining cases are straightforward.

It follows from $t_1 \equiv s_1 \Box s_2$ that

$$t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Box s_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}.$$

So the topmost operator of $t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ is an external choice $\Box$ which comes from either T_a or t_2. Similar to Case $t_1 \equiv s_1 \Box s_2$ in item (1), we can conclude that there exist s_1', s_2' such that $t_2 \equiv s_1' \Box s_2'$. Moreover, it is easily seen that either $s_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ or $s_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ triggers the unfolding from $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ to $t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. W.l.o.g, we consider the first alternative. Then $s_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \Rrightarrow_1 s_1'\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ and $s_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\} \equiv s_2'\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$. Hence, by IH and item (1), for any p and $\widetilde{q}$, we have $s_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 s_1'\{p/X, \widetilde{q}/\widetilde{Z}\}$ and $s_2\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv s_2'\{p/X, \widetilde{q}/\widetilde{Z}\}$. Thus $t_1\{p/X, \widetilde{q}/\widetilde{Z}\} \equiv s_1\{p/X, \widetilde{q}/\widetilde{Z}\} \Box s_2\{p/X, \widetilde{q}/\widetilde{Z}\} \Rrightarrow_1 t_2\{p/X, \widetilde{q}/\widetilde{Z}\}$. □

The intuition, which is captured by the above result, is obvious. Since both T_a and $\widetilde{a_Z}$ contain fresh actions, none of them occurs in $t_i (i = 1, 2)$. We may take $t_i\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ to be the 'open' term obtained from t_i by renaming its holes using T_a (or $\widetilde{a_Z.0}$) instead of X ($\widetilde{Z}$ resp.). Based on this intuition, the clause (1) is nothing but the statement that the relation $\equiv$ is preserved under substituting. For the clause (2), since T_a and $\widetilde{a_Z.0}$ contain no recursive operators, the unfolding from $t_1\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ to $t_2\{T_a/X, \widetilde{a_Z.0}/\widetilde{Z}\}$ is excited by t_1. Then the clause (2) merely asserts that such unfolding is preserved after substituting. One might wonder why the term T_a is put in such form. It is really due to the application of Lemma 6.3. Roughly speaking, in the argument of the next lemma, we need a stable term Q satisfying the following conditions:

i. for each $Z \in \widetilde{Z}$, $Q \xrightarrow{a} Q_Z'$ for some Q_Z';
ii. $B_{X,\widetilde{Z}}\{Q/X, \widetilde{Q_Z'}/\widetilde{Z}\} \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{Q/X, Q/\widetilde{Y}, \widetilde{Q_Z'}/\widetilde{Z}\}$ implies $B_{X,\widetilde{Z}}\{\tau.p/X, \widetilde{p_Z'}/\widetilde{Z}\} \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p_Z'}/\widetilde{Z}\}$.

Clearly, T_a is of the most simple scheme so that (i) holds, while Lemma 6.3 just asserts that it also realizes (ii).

Lemma 6.4. For any C_X and stable process p, $C_X\{\tau.p/X\} \notin F$ implies $C_X\{p/X\} \notin F$.

Proof. Let p be any stable process. Set

$$\Omega \triangleq \{B_X\{p/X\} : B_X\{\tau.p/X\} \notin F \text{ and } B_X \text{ is a context}\}.$$

Assume $t \in \Omega$. Then $t \equiv C_X\{p/X\}$ for some C_X such that $C_X\{\tau.p/X\} \notin F$. Let $\mathcal{T}$ be any proof tree of $Strip(\mathcal{P}_{\text{CLL}_R}, M_{\text{CLL}_R}) \vdash C_X\{p/X\}F$. Similar to Lemma 6.2, it is sufficient to prove that $\mathcal{T}$ has a proper subtree with root sF for some $s \in \Omega$, which is a routine case analysis based on the last rule applied in $\mathcal{T}$. We treat only three cases.

Case 1. $\frac{\{rF : C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} |r\}}{C_X\{p/X\}F}$ with $C_X \equiv \langle Y|E \rangle$.

Since, $C_X\{\tau.p/X\} \notin F$, we get $C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |q$ for some q. By Lemma 5.14, for this transition, there exists a stable context $C'_{X,\widetilde{Z}}$ satisfying (MS-τ-1) – (MS-τ-7). In particular,

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since p and q are stable, by (MS-τ-2,7), we have $q \equiv C'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \notin F$. Moreover, since each $Z(\in \widetilde{Z})$ is 1-active in $C'_{X,\widetilde{Z}}$ (i.e., (MS-τ-1)) and $\tau.p \stackrel{\tau}{\longrightarrow} p$, by Lemma 5.4, we get $C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \equiv q \notin F$, which, by Lemma 4.2, implies

$$C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\} \notin F. \tag{6.4.1}$$

Let a_X be any fresh visible action. By (MS-τ-3-i), it follows from $a_X.0 \stackrel{\epsilon}{\Longrightarrow} |a_X.0$ that there exists s such that

$$C_X\{a_X.0/X\} \stackrel{\epsilon}{\Longrightarrow} s \Rrightarrow C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}. \tag{6.4.2}$$

Since $a_X.0$ and $C'_{X,\widetilde{Z}}$ are stable, so is $C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}$ by Lemma 5.6. Then, by Lemma 5.10, s is stable. Thus, for the transition in (6.4.2), by Lemma 5.14, there exists a stable context C^*_X such that

$$s \equiv C^*_X\{a_X.0/X\} \text{ and } C_X\{r/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{r/X\} \text{ for any } r. \tag{6.4.3}$$

Then, by Lemma 5.16, it follows from $s \equiv C^*_X\{a_X.0/X\} \Rrightarrow C'_{X,\widetilde{Z}}\{a_X.0/X, a_X.0/\widetilde{Z}\}$ that $C^*_X\{\tau.p/X\} \Rrightarrow C'_{X,\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Z}\}$. Hence $C^*_X\{\tau.p/X\} \notin F$ by (6.4.1) and Lemma 5.12, which implies $C^*_X\{p/X\} \in \Omega$. Moreover, since C^*_X and p are stable, so is $C^*_X\{p/X\}$, which implies $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |C^*_X\{p/X\}$ by (6.4.3). Therefore $\mathcal{T}$ has a proper subtree with root $C^*_X\{p/X\}F$.

Case 2. $\frac{B_X\{p/X\} \stackrel{a}{\longrightarrow} r}{C_X\{p/X\}F}$ with $C_X \equiv B_X \wedge D_X$, $D_X\{p/X\} \stackrel{a}{\not\longrightarrow}$ and $C_X\{p/X\} \stackrel{\tau}{\not\longrightarrow}$.

Clearly, in this situation, both B_X and D_X are stable. Since $C_X\{\tau.p/X\} \notin F$, we have $C_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q$ for some q. So, there exist s and t such that $q \equiv s \wedge t$, $B_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s$ and $D_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |t$. Then, for these two transitions, by Lemma 5.14, there exist $B'_{X,\widetilde{Y}}$ and $D'_{X,\widetilde{Z}}$ satisfying (MS-τ-1) – (MS-τ-7) respectively. In particular, since p, B_X and D_X are stable, by (MS-τ-2,4,7), we have

1. $s \equiv B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ and $B_X\{p/X\} \Rrightarrow B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$;
2. $t \equiv D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\}$ and $D_X\{p/X\} \Rrightarrow D'_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\}$.

By (1) and Lemma 5.10, it follows from $B_X\{p/X\} \stackrel{a}{\longrightarrow}$ that $B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \stackrel{a}{\longrightarrow}$. Then, since $B'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$ and $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ are stable, by Lemma 6.1, it follows from $\tau.p =_{RS} p$ and $s \equiv B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$ that $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \stackrel{a}{\longrightarrow} r_1$ for some r_1. Similarly, we also have $D'_{X,\widetilde{Z}}\{p/X, p/\widetilde{Z}\} \stackrel{a}{\not\longrightarrow}$ and then $D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \stackrel{a}{\not\longrightarrow}$. Clearly, the rule $\frac{B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \stackrel{a}{\longrightarrow} r_1}{B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \wedge D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\}F}$ is a ground instance of the rule $\frac{pprem(Rp_{10})}{conc(Rp_{10})}$; moreover it is in $Strip(R_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R})$ due to $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \wedge D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \stackrel{\tau}{\not\longrightarrow}$ and $D'_{X,\widetilde{Z}}\{\tau.p/X, p/\widetilde{Z}\} \stackrel{a}{\not\longrightarrow}$. Then $q \equiv s \wedge t \in F$ due to $B'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \stackrel{a}{\longrightarrow} r_1$, a contradiction. Thus this case is impossible.

Case 3. $\frac{C_X\{p/X\} \stackrel{\alpha}{\longrightarrow} r', \{rF : C_X\{p/X\} \stackrel{\alpha}{\longrightarrow} r\}}{C_X\{p/X\}F}$ with $C_X \equiv B_X \wedge D_X$.

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Since $C_X\{\tau.p/X\} \notin F$, we have

$$C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |q \text{ for some } q. \tag{6.4.4}$$

Next we distinguish two cases based on α.

Case 3.1. $\alpha = \tau$.

By (6.4.4) and Lemma 5.17, there exist t and stable context C_X^* such that $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C_X^*\{p/X\}$ and $C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} C_X^*\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |t \Rightarrow q \notin F$. Moreover, since $p \overset{\tau}{\not\longrightarrow}$ and $\tau \in \mathcal{I}(C_X\{p/X\})$, by Lemma 5.6, there exists a context C'_X such that $C_X\{p/X\} \overset{\tau}{\longrightarrow} C'_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C_X^*\{p/X\}$ and $C_X\{\tau.p/X\} \overset{\tau}{\longrightarrow} C'_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} C_X^*\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |t$. Further, by Lemma 5.12, it follows from $q \notin F$ and $t \Rightarrow q$ that $t \notin F$. Then $C'_X\{\tau.p/X\} \notin F$ by Lemma 4.2. Hence $C'_X\{p/X\} \in \Omega$ and one of nodes directly above the root of $\mathcal{T}$ is labelled with $C'_X\{p/X\}F$, as desired.

Case 3.2. $\alpha \in Act$.

In this case, C_X is stable by Lemma 5.6. By (6.4.4) and Lemma 5.14, there exists a stable context $C'_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ that satisfies (MS-τ-1) – (MS-τ-7). Then we have $q \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ due to $p \overset{\tau}{\not\longrightarrow}$ and (MS-τ-2). Moreover, since C_X is stable, by (MS-τ-4), we have

$$C_X\{r/X\} \Rightarrow C'_{X,\widetilde{Y}}\{r/X, r/\widetilde{Y}\} \text{ for any } r. \tag{6.4.5}$$

Then, by $C_X\{p/X\} \overset{\alpha}{\longrightarrow}$ and Lemma 5.10, we get

$$C'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \overset{\alpha}{\longrightarrow}. \tag{6.4.6}$$

Further, by Lemma 6.1, we also have $C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \overset{\alpha}{\longrightarrow}$ because of $\tau.p =_{RS} p$ and $q \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$. Thus, by Theorem 4.1, we obtain $C'_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \overset{\alpha}{\longrightarrow}_F t$ for some t. For this transition, by Lemma 5.7, there exist $C''_{X,\widetilde{Y}}$, $C''_{X,\widetilde{Y},\widetilde{Z}}$ and $C'''_{X,\widetilde{Y},\widetilde{Z}}$ with $(\{X\} \cup \widetilde{Y}) \cap \widetilde{Z} = \varnothing$ that realize (CP-a-1,2,3,4). In particular, due to $\tau.p \overset{\alpha}{\not\longrightarrow}$ and (CP-a-3-ii), there exist $p'_Z (Z \in \widetilde{Z})$ such that

$$p \overset{\alpha}{\longrightarrow} p'_Z \text{ for each } Z \in \widetilde{Z} \text{ and } t \equiv C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \notin F. \tag{6.4.7}$$

Moreover, by (CP-a-3-iii), for any r, s and $s'_Z (Z \in \widetilde{Z})$ such that $s \overset{\alpha}{\longrightarrow} s'_Z$ for each $Z \in \widetilde{Z}$, we have

$$C'_{X,\widetilde{Y}}\{r/X, s/\widetilde{Y}\} \overset{\alpha}{\longrightarrow} C'''_{X,\widetilde{Y},\widetilde{Z}}\{r/X, s/\widetilde{Y}, \widetilde{s'_Z}/\widetilde{Z}\} \text{ whenever } C'_{X,\widetilde{Y}}\{r/X, s/\widetilde{Y}\} \text{ is stable.} \tag{6.4.8}$$

For each $Z \in \widetilde{Z} \cup \{X\}$, we fix a fresh and distinct visible action a_Z and set

$$T_\alpha \triangleq \begin{cases} \underset{Z \in \widetilde{Z}}{\Box} \alpha.a_Z.0, & \text{if } \widetilde{Z} \neq \varnothing; \\ a_X.0, & \text{otherwise.} \end{cases}$$

Since, T_α and $C'_{X,\widetilde{Y}}$ are stable, so is $C'_{X,\widetilde{Y}}\{T_\alpha/X, T_\alpha/\widetilde{Y}\}$ by Lemma 5.6. Then, by (6.4.8), we have $C'_{X,\widetilde{Y}}\{T_\alpha/X, T_\alpha/\widetilde{Y}\} \overset{\alpha}{\longrightarrow} C'''_{X,\widetilde{Y},\widetilde{Z}}\{T_\alpha/X, T_\alpha/\widetilde{Y}, \widetilde{a_Z.0}/\widetilde{Z}\}$. So, by Lemma 5.10, it follows

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from (6.4.5) that there exists t' such that

$$C_X\{T_\alpha/X\} \xrightarrow{\alpha} t' \text{ and } t' \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{T_\alpha/X, T_\alpha/\widetilde{Y}, \widetilde{a_Z.0}/\widetilde{Z}\}. \tag{6.4.9}$$

Then, by Lemma 5.7, it is not difficult to see that there exists a context $B_{X,\widetilde{Z}}$ that satisfies the conditions:

a. $t' \equiv B_{X,\widetilde{Z}}\{T_\alpha/X, \widetilde{a_Z.0}/\widetilde{Z}\}$;
b. none of a_Z with $Z \in \widetilde{Z}$ occurs in $B_{X,\widetilde{Z}}$;
c. for any s and $s'_Z(Z \in \widetilde{Z})$ such that $s \xrightarrow{\alpha} s'_Z$ for each $Z \in \widetilde{Z}$,

$$C_X\{s/X\} \xrightarrow{\alpha} B_{X,\widetilde{Z}}\{s/X, \widetilde{s'_Z}/\widetilde{Z}\} \text{ whenever } C_X\{s/X\} \text{ is stable.}$$

Now we obtain the diagram

$$\begin{array}{ccc} C_X\{p/X\} & \overset{\text{by (6.4.5)}}{\Rrightarrow} & C'_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \\ \downarrow \alpha \text{ by (c)} & & \downarrow \alpha \text{ by (6.4.6) and (6.4.8)} \\ B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\} & \overset{\text{by (6.4.9), } (a) \text{ and Lemma 6.3}}{\Rrightarrow} & C'''_{X,\widetilde{Y},\widetilde{Z}}\{p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\}. \end{array}$$

By Lemma 6.3, we also have

$$B_{X,\widetilde{Z}}\{\tau.p/X, \widetilde{p'_Z}/\widetilde{Z}\} \Rrightarrow C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\}. \tag{6.4.10}$$

For each $Y \in \widetilde{Y}$, since Y is 1-active in $C'_{X,\widetilde{Y}}$, by Lemma 5.2(1)(2) and $C'_{X,\widetilde{Y}} \Rrightarrow C''_{X,\widetilde{Y}}$ (i.e., (CP-a-1)), so it is in $C''_{X,\widetilde{Y}}$. Moreover, by (CP-a-4-i,ii), for each $Y \in \widetilde{Y} \cap FV(C'''_{X,\widetilde{Y},\widetilde{Z}})$, Y is 1-active in $C'''_{X,\widetilde{Y},\widetilde{Z}}$. Then, by Lemma 5.4, we have

$$C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \overset{\epsilon}{\Longrightarrow} C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\}$$

which, together with (6.4.7), implies $C'''_{X,\widetilde{Y},\widetilde{Z}}\{\tau.p/X, \tau.p/\widetilde{Y}, \widetilde{p'_Z}/\widetilde{Z}\} \notin F$ by Lemma 4.2. Hence, by Lemma 5.12, it follows from (6.4.10) that $B_{X,\widetilde{Z}}\{\tau.p/X, \widetilde{p'_Z}/\widetilde{Z}\} \notin F$. Thus, $B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\} \in \Omega$; moreover $\mathcal{T}$ has a proper subtree with root $B_{X,\widetilde{Z}}\{p/X, \widetilde{p'_Z}/\widetilde{Z}\}F$ due to (c) and (6.4.7). □

Hitherto we have completed the first step mentioned at the beginning of this section. We now turn to the second step. The argument of Lemma 6.6 will carry out by distinguishing several cases based on the form of $C_{\widetilde{X}}$. In particular, in case $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\widetilde{X}}$, a common reasoning pattern is adopted to deal with two subcases. To shorten the argument of Lemma 6.6, we extract this pattern and describe it first in a separate lemma as follows.

Lemma 6.5. Let $C_{\widetilde{X},\widetilde{Z}}$ be any context such that for each $Z \in \widetilde{Z}$, Z is active and occurs at most once. If $\widetilde{p}$, $\widetilde{q}$, $\widetilde{t}$, $\widetilde{s}$ and $\widetilde{r}$ are any processes such that (a) $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, (b) $\widetilde{p} \bowtie \widetilde{q}$, (c) $\widetilde{r} \overset{\epsilon}{\Longrightarrow} |\widetilde{t}$, (d) $\widetilde{s} \mathrel{\underset{\sim RS}{\sqsubset}} \widetilde{t}$ and (e) $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{s}/\widetilde{Z}\} \notin F$, then, for any proof tree $\mathcal{T}$ for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{r}/\widetilde{Z}\}F$, there exist $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, p''_Y and

q''_Y for $Y \in \widetilde{Y}$ such that (1) $\mathcal{T}$ has a subtree with root $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q''_Y}/\widetilde{Y}\}F$, (2) $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p''_Y}/\widetilde{Y}\} \notin F$ and (3) $\widetilde{p''_Y} \sqsubset_{\sim RS} \widetilde{q''_Y}$.

Proof. It proceeds by induction on the depth of $\mathcal{T}$. We distinguish several cases depending on the form of $C_{\widetilde{X},\widetilde{Z}}$. Here we give the proof only for two cases, the other cases are left to the reader.

Case 1. $C_{\widetilde{X},\widetilde{Z}}$ is closed or $C_{\widetilde{X},\widetilde{Z}} \equiv X_i$ or $C_{\widetilde{X},\widetilde{Z}} \equiv Z_j$ for some $i \leqslant |\widetilde{X}|$ and $j \leqslant |\widetilde{Z}|$.

It is straightforward to show that this lemma holds trivially for such case. As an example, we consider the case $C_{\widetilde{X},\widetilde{Z}} \equiv Z_j$. Since $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \equiv s_j \notin F$ and $\widetilde{s} \sqsubset_{\sim RS} \widetilde{t}$, we have $t_j \notin F$. Hence $r_j \overset{\epsilon}{\Longrightarrow}_F |t_j$ by Lemma 4.2. So $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \equiv r_j \notin F$. That is, there is no proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F$. Thus the conclusion holds trivially.

Case 2. $C_{\widetilde{X},\widetilde{Z}} \equiv B_{\widetilde{X},\widetilde{Z}} \wedge D_{\widetilde{X},\widetilde{Z}}$.

The argument splits into four cases based on the last rule applied in $\mathcal{T}$. For the case where the last rule is Rp_8, the proof is straightforward by applying IH. In the following, we deal with other cases.

Case 2.1. $\dfrac{B_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{a}{\longrightarrow}r'}{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$ with $D_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \overset{a}{\not\longrightarrow}$ and $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \overset{\tau}{\not\longrightarrow}$.

For any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$, since Z is active and $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \overset{\tau}{\not\longrightarrow}$, by Lemma 5.4, we have $r_Z \overset{\tau}{\not\longrightarrow}$, and hence $r_Z \equiv t_Z$ because of (c). So, $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \equiv C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}$. Hence $\mathcal{T}$ has the root labelled with $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}F$. Clearly, the conclusion holds by setting $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}} \triangleq C_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{Y} = \varnothing$.

Case 2.2. $\dfrac{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{\alpha}{\longrightarrow}s',\ \{rF : C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{\alpha}{\longrightarrow}r\}}{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.

If $\alpha \in Act$, the argument is similar to one of Case 2.1 and omitted. In the following, we handle the case $\alpha = \tau$. If $r_Z \overset{\tau}{\not\longrightarrow}$ for any $Z(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$, then the conclusion holds trivially by putting $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}} \triangleq C_{\widetilde{X},\widetilde{Z}}$ with $\widetilde{Y} = \varnothing$. Next we consider the other case where $r_{Z_0} \overset{\tau}{\longrightarrow}$ for some $Z_0(\in \widetilde{Z})$ occurring in $C_{\widetilde{X},\widetilde{Z}}$. Then $r_{Z_0} \overset{\tau}{\longrightarrow} r' \overset{\epsilon}{\Longrightarrow} |t_{Z_0}$ for some r' by (c); moreover Z_0 is 1-active in $C_{\widetilde{X},\widetilde{Z}}$. Thus $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \overset{\tau}{\longrightarrow} C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}[r'/r_{Z_0}]/\widetilde{Z}\}$ by Lemma 5.4. So, $\mathcal{T}$ has a proper subtree $\mathcal{T}'$ with root $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}[r'/r_{Z_0}]/\widetilde{Z}\}F$. Since $\widetilde{r}[r'/r_{Z_0}] \overset{\epsilon}{\Longrightarrow} |\widetilde{t}$ and $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \notin F$, by IH, $\mathcal{T}'$ has a subtree with root $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q''_Y}/\widetilde{Y}\}F$ for some $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, $\widetilde{p''_Y}$ and $\widetilde{q''_Y}$ such that $C^*_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p''_Y}/\widetilde{Y}\}\notin F$ and $\widetilde{p''_Y} \sqsubset_{\sim RS} \widetilde{q''_Y}$.

Case 2.3. $\dfrac{\{rF : C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}\overset{\epsilon}{\Longrightarrow}|r\}}{C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\}F}$.

It follows from $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \notin F$ that $C_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z}\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. Then, by Lemma 5.14, for such transition, there exist a stable context $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$ and $i_Y, p'''_Y(Y \in \widetilde{Y})$ that realize (MS-τ-1) – (MS-τ-7). In particular, since each $s(\in \widetilde{s})$ is stable, by (MS-τ-2,7), for each $Y \in \widetilde{Y}$, we have $i_Y \leqslant |\widetilde{X}|$ and $p_{i_Y} \overset{\tau}{\Longrightarrow} |p'''_Y$ for some p'''_Y, and $p' \equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\} \notin F$. Then, by Lemma 5.5, it follows from (MS-τ-1) that, for each

$Y \in \widetilde{Y}$, $p'''_Y \notin F$ and hence $p_{i_Y} \stackrel{\tau}{\Longrightarrow}_F |p'''_Y$ by Lemma 4.2. Further, since $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, there exist $q'''_Y(Y \in \widetilde{Y})$ such that $q_{i_Y} \stackrel{\tau}{\Longrightarrow}_F |q'''_Y$ and $p'''_Y \underset{\sim RS}{\sqsubset} q'''_Y$ for each $Y \in \widetilde{Y}$. Then, by (MS-τ-3-ii), it follows that

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}. \tag{6.5.1}$$

Moreover, since Z is active and occurs at most once in $C_{\widetilde{X},\widetilde{Z}}$ for each $Z \in \widetilde{Z}$, by Lemma 5.4, it follows from $\widetilde{r} \stackrel{\epsilon}{\Longrightarrow} \widetilde{t}$ that

$$C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{r}/\widetilde{Z}\} \stackrel{\epsilon}{\Longrightarrow} C_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z}\}. \tag{6.5.2}$$

Since $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{s} \underset{\sim RS}{\sqsubset} \widetilde{t}$ and $\widetilde{p'''_Y} \underset{\sim RS}{\sqsubset} \widetilde{q'''_Y}$, by $p' \equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\} \stackrel{\tau}{\not\longrightarrow}$ and Lemma 5.6, we can conclude that $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with root $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{q}/\widetilde{X},\widetilde{t}/\widetilde{Z},\widetilde{q'''_Y}/\widetilde{Y}\}F$ by (6.5.1) and (6.5.2); moreover we also have $p' \equiv C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}\{\widetilde{p}/\widetilde{X},\widetilde{s}/\widetilde{Z},\widetilde{p'''_Y}/\widetilde{Y}\} \notin F$ and $\widetilde{p'''_Y} \underset{\sim RS}{\sqsubset} \widetilde{q'''_Y}$. Consequently, $C'_{\widetilde{X},\widetilde{Z},\widetilde{Y}}$, $\widetilde{p'''_Y}$ and $\widetilde{q'''_Y}$ are what we seek. □

Lemma 6.6. If $\widetilde{r} \bowtie \widetilde{s}$ and $\widetilde{r} \sqsubseteq_{RS} \widetilde{s}$, then $C_{\widetilde{X}}\{\widetilde{r}/\widetilde{X}\} \notin F$ implies $C_{\widetilde{X}}\{\widetilde{s}/\widetilde{X}\} \notin F$.

Proof. Set $\Omega = \{B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} : \widetilde{p} \bowtie \widetilde{q},\ \widetilde{p} \sqsubseteq_{RS} \widetilde{q},\ B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and B_X is a context$\}$. Let $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$ and $\mathcal{T}$ be any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F$. Similar to Lemma 6.2, it suffices to show that $\mathcal{T}$ has a proper subtree with root sF for some $s \in \Omega$. The proof proceeds by distinguishing several cases based on the form of $C_{\widetilde{X}}$. We handle two nontrivial cases.

Case 1. $C_{\widetilde{X}} \equiv \langle Y|E\rangle$. Clearly, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F}{\langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{\widetilde{q}/\widetilde{X}\}F}.$$

For the first alternative, we have $\langle t_Y|E\rangle\{\widetilde{p}/\widetilde{X}\} \notin F$ because of $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, and hence $\langle t_Y|E\rangle\{\widetilde{q}/\widetilde{X}\} \in \Omega$. For the second alternative, due to $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we get $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_F |s$ for some s. For this transition, by Lemma 5.14, there exist $C'_{\widetilde{X},\widetilde{Z}}$, $i_Z \leqslant |\widetilde{X}|$ and p'_Z for $Z \in \widetilde{Z}$ that realize (MS-τ-1) – (MS-τ-7). Amongst them, by (MS-τ-2,7), we have

$$p_{i_Z} \stackrel{\tau}{\Longrightarrow} |p'_Z \text{ for each } Z \in \widetilde{Z} \text{ and } s \equiv C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_Z}/\widetilde{Z}\} \notin F. \tag{6.6.1}$$

Thus, for each $Z \in \widetilde{Z}$, by (MS-τ-1) and Lemma 5.5, it follows that $p'_Z \notin F$, and hence $p_{i_Z} \stackrel{\tau}{\Longrightarrow}_F |p'_Z$ by Lemma 4.2. Further, since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that

$$\text{for each } Z \in \widetilde{Z},\ q_{i_Z} \stackrel{\tau}{\Longrightarrow}_F |q'_Z \text{ and } p'_Z \underset{\sim RS}{\sqsubset} q'_Z \text{ for some } q'_Z. \tag{6.6.2}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Z}/\widetilde{Z}\}$ by (MS-τ-3-ii). In addition, since $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{p'_Z} \underset{\sim RS}{\sqsubset} \widetilde{q'_Z}$ and $s \equiv C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{p}/\widetilde{X},\widetilde{p'_Z}/\widetilde{Z}\} \stackrel{\tau}{\not\longrightarrow}$, by Lemma 5.6, we get $C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X},\widetilde{q'_Z}/\widetilde{Z}\} \stackrel{\tau}{\not\longrightarrow}$.

Therefore $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} |C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}$. Hence $\mathcal{T}$ has a proper subtree with root $C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\}F$; moreover $C'_{\widetilde{X},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Z}/\widetilde{Z}\} \in \Omega$ due to (6.6.1) and (6.6.2).

Case 2. $C_{\widetilde{X}} \equiv B_{\widetilde{X}} \wedge D_{\widetilde{X}}$.

By Lemmas 5.6 and 6.1, it is not difficult to show that Rule Rp_{10} cannot be applied in the last inferring step of $\mathcal{T}$. Hence the argument splits into three cases depending on the last rule ξ applied in $\mathcal{T}$. If ξ is Rule Rp_8, the proof is straightforward. In case $\xi = Rp_{13}$, the proof is similar to the second alternative in Case 1 and omitted. In the following, we handle the case $\xi = Rp_{12}$ by considering two subcases.

Case 2.1. $\frac{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r', \{rF : C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} r\}}{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F}$.

It follows from $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ that

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ for some } s. \tag{6.6.3}$$

Since $\widetilde{p} \bowtie \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow}$, by Lemma 5.6, we get $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow}$. Then, by (6.6.3), we have $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow}_F t \stackrel{\epsilon}{\Longrightarrow}_F |s$ for some t. For the τ-transition leading to t, either the clause (1) or (2) in Lemma 5.6 holds.

For the former, there exists $C'_{\widetilde{X}}$ such that $t \equiv C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$. Hence $C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F$ is one of premises of the last inferring step in $\mathcal{T}$. Moreover it is evident that $C'_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \in \Omega$.

For the latter, there exist $C'_{\widetilde{X}}$, $C''_{\widetilde{X},Z}$ with $Z \notin \widetilde{X}$ and $i_0 \leqslant |\widetilde{X}|$ that realize (P-τ-1,2,3,4). In particular, by (P-τ-2), we have $t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\}$ for some p' with $p_{i_0} \stackrel{\tau}{\longrightarrow} p'$. Further, since $t \stackrel{\epsilon}{\Longrightarrow}_F |s$ and Z is 1-active in $C''_{\widetilde{X},Z}$, by Lemmas 5.18 and 4.2, there exists p'' such that $p' \stackrel{\epsilon}{\Longrightarrow} |p''$ and $t \equiv C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p'/Z\} \stackrel{\epsilon}{\Longrightarrow}_F C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p''/Z\} \stackrel{\epsilon}{\Longrightarrow}_F |s$. Moreover $p'' \notin F$ by Lemma 5.5. Hence $p_{i_0} \stackrel{\tau}{\longrightarrow}_F p' \stackrel{\epsilon}{\Longrightarrow}_F |p''$ by Lemma 4.2. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that

$$q_{i_0} \stackrel{\tau}{\longrightarrow}_F q' \stackrel{\epsilon}{\Longrightarrow}_F |q'' \text{ and } p'' \underset{\sim}{\sqsubset}_{RS} q'' \text{ for some } q' \text{ and } q''. \tag{6.6.4}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\tau}{\longrightarrow} C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}$ by (P-τ-4). Therefore $\mathcal{T}$ contains a proper subtree $\mathcal{T}'$ with root $C''_{\widetilde{X},Z}\{\widetilde{q}/\widetilde{X}, q'/Z\}F$. In order to complete the proof, it is sufficient to show that $\mathcal{T}'$ contains a node labelled with $s'F$ for some $s' \in \Omega$. Since Z is 1-active, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, $\widetilde{p} \bowtie \widetilde{q}$, $q' \stackrel{\epsilon}{\Longrightarrow} |q''$, $p'' \underset{\sim}{\sqsubset}_{RS} q''$ and $C''_{\widetilde{X},Z}\{\widetilde{p}/\widetilde{X}, p''/Z\} \notin F$, by Lemma 6.5, there exist $C^*_{\widetilde{X},Z,\widetilde{Y}}$, q'''_Y and p'''_Y for $Y \in \widetilde{Y}$ such that

a.1. $\mathcal{T}'$ has a subtree with root $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, q''/Z, \widetilde{q'''_Y}/\widetilde{Y}\}F$,

a.2. $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, p''/Z, \widetilde{p'''_Y}/\widetilde{Y}\} \notin F$, and

a.3. $\widetilde{p'''_Y} \underset{\sim}{\sqsubset}_{RS} \widetilde{q'''_Y}$.

Clearly, $C^*_{\widetilde{X},Z,\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, q''/Z, \widetilde{q'''_Y}/\widetilde{Y}\} \in \Omega$ due to (a.2), (a.3) and (6.6.4), as desired.

Case 2.2. $\frac{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow} r', \{rF : C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow} r\}}{C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}F} (a \in Act)$.

Since $\widetilde{p} \bowtie \widetilde{q}$, by Lemma 5.6, it follows from $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow}$ that $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ is stable. Further, since $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we get $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\longrightarrow}$ by Lemma 6.1. So, by Theorem 4.1 and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$, we have

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\longrightarrow}_F t \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ for some } t \text{ and } s. \tag{6.6.5}$$

On the one hand, for the a-transition in (6.6.5), by Lemma 5.7, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ that satisfy (CP-a-1) – (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_Y \leqslant |\widetilde{X}|, p'_Y (Y \in \widetilde{Y})$ such that $p_{i_Y} \stackrel{a}{\longrightarrow} p'_Y$ for each $Y \in \widetilde{Y}$ and $t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$. Moreover, by (CP-$a$-1) and (CP-$a$-3-i), $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}$. Hence $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and Lemma 5.12. Further, for each $Y \in \widetilde{Y}$, since Y is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$ (i.e., (CP-a-2)), by Lemma 5.5, $p_{i_Y} \notin F$.

On the other hand, for the transition $t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \stackrel{\epsilon}{\Longrightarrow}_F |s$ in (6.6.5), since Y is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$ for each $Y \in \widetilde{Y}$ (i.e., (CP-a-2)), by Lemma 5.18, there exist p''_Y such that $p'_Y \stackrel{\epsilon}{\Longrightarrow} |p''_Y$ for each $Y \in \widetilde{Y}$ and $t \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \stackrel{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \stackrel{\epsilon}{\Longrightarrow} |s$. Then, by Lemma 4.2, it follows from $s \notin F$ that $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$, which implies that $p''_Y \notin F$ for each $Y \in \widetilde{Y}$ due to Lemma 5.5. Thus $p_{i_Y} \stackrel{a}{\longrightarrow}_F p'_Y \stackrel{\epsilon}{\Longrightarrow}_F |p''_Y$ for each $Y \in \widetilde{Y}$. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that,

$$\text{for each } Y \in \widetilde{Y},\ q_{i_Y} \stackrel{a}{\longrightarrow}_F q'_Y \stackrel{\epsilon}{\Longrightarrow}_F |q''_Y \text{ and } p''_Y \sqsubset_{\sim RS} q''_Y \text{ for some } q'_Y \text{ and } q''_Y. \tag{6.6.6}$$

Then $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{a}{\longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ by (CP-a-3-iii). Hence $\mathcal{T}$ has a proper subtree $\mathcal{T}'$ with root $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}F$. In order to complete the proof, it suffices to show that $\mathcal{T}'$ contains a node labelled with $s'F$ for some $s' \in \Omega$. Since each $Y (\in \widetilde{Y})$ is 1-active in $C''_{\widetilde{X},\widetilde{Y}}$, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$, $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{q'_Y} \stackrel{\epsilon}{\Longrightarrow} |\widetilde{q''_Y}$, $\widetilde{p''_Y} \sqsubset_{\sim RS} \widetilde{q''_Y}$ and $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$, by Lemma 6.5, there exist $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}$, q'''_Z and p'''_Z for $Z \in \widetilde{Z}$ such that

b.1. $\mathcal{T}'$ has a subtree with root $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}, \widetilde{q'''_Z}/\widetilde{Z}\}F$,

b.2. $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}, \widetilde{p'''_Z}/\widetilde{Z}\} \notin F$, and

b.3. $\widetilde{p'''_Z} \sqsubset_{\sim RS} \widetilde{q'''_Z}$.

Obviously, $C^*_{\widetilde{X},\widetilde{Y},\widetilde{Z}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}, \widetilde{q'''_Z}/\widetilde{Z}\} \in \Omega$ due to (b.2), (b.3) and (6.6.6), as desired. □

Before proving that $\sqsubseteq_{RS}$ is indeed precongruent, let us first recall an equivalent formulation of $\sqsubseteq_{RS}$ due to Van Glabbeek (Lüttgen and Vogler 2010).

Definition 6.2. A relation $\mathcal{R} \subseteq T(\Sigma_{\mathrm{CLL}_R}) \times T(\Sigma_{\mathrm{CLL}_R})$ is an alternative ready simulation relation, if for any $(p, q) \in \mathcal{R}$ and $a \in Act$
(RSi) $p \stackrel{\epsilon}{\Longrightarrow}_F |p'$ implies $\exists q'.q \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $(p', q') \in \mathcal{R}$;
(RSiii) $p \stackrel{a}{\Longrightarrow}_F |p'$ and p, q stable implies $\exists q'.q \stackrel{a}{\Longrightarrow}_F |q'$ and $(p', q') \in \mathcal{R}$;
(RSiv) $p \notin F$ and p, q stable implies $\mathcal{I}(p) = \mathcal{I}(q)$.

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We write $p \sqsubseteq_{ALT} q$ if there exists an alternative ready simulation relation $\mathcal{R}$ with $(p,q) \in \mathcal{R}$.

Proposition 6.1 (Lüttgen and Vogler 2010). $\sqsubseteq_{RS} = \sqsubseteq_{ALT}$.

One advantage of Definition 6.2 is that, given p and q, we can prove $p \sqsubseteq_{RS} q$ by means of giving an alternative ready simulation relation relating them. It is well known that up-to technique is a tractable way for such coinduction proof. Here we introduce the notion of an alternative ready relation up to $\underset{\sim RS}{\sqsubset}$ as follows.

Definition 6.3 (ALT up to $\underset{\sim RS}{\sqsubset}$). A relation $\mathcal{R} \subseteq T(\Sigma_{\mathrm{CLL}_R}) \times T(\Sigma_{\mathrm{CLL}_R})$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$, if for any $(p,q) \in \mathcal{R}$ and $a \in Act$

(ALT-upto-1) $p \overset{\epsilon}{\Longrightarrow}_F |p'$ implies $\exists q'.q \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(ALT-upto-2) $p \overset{a}{\Longrightarrow}_F |p'$ and p,q stable implies $\exists q'.q \overset{a}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$;

(ALT-upto-3) $p \notin F$ and p,q stable implies $\mathcal{I}(p) = \mathcal{I}(q)$.

As usual, given a relation $\mathcal{R}$ satisfying the conditions (ALT-upto-1,2,3), in general, $\mathcal{R}$ in itself is not an alternative ready simulation relation. But simple result below ensures that up-to technique based on the above notion is sound.

Lemma 6.7. If $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$ then $\mathcal{R} \subseteq \sqsubseteq_{RS}$.

Proof. By Proposition 6.1, it is sufficient to prove that the relation $\sqsubseteq_{RS} \mathcal{R} \sqsubseteq_{RS}$ is an alternative ready simulation. We leave it to the reader. □

Remark 6.1. If we adopt the binary relation $\sqsubseteq_{RS} \mathcal{R} \sqsubseteq_{RS}$ instead of $\underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset}$ in the clauses (ALT-upto-1,2) of Definition 6.3, then it is easy to see that the preceding result is no longer true by considering the counterexample $\mathcal{R} = \{(\tau.a.0, 0)\}$. In fact, this failure is similar to one occurring in Milner's first attempt to define the notion of weak bisimulation up-to (Milner 1989a).

Now we are ready to prove the main result of this section: $\sqsubseteq_{RS}$ is precongruent w.r.t all operations in CLL_R. We shall divide the proof into the next two lemmas.

Lemma 6.8. $C_X\{p/X\} =_{RS} C_X\{\tau.p/X\}$ for any context C_X and stable process p.

Proof. Let p be any stable process. First, we show that $C_X\{p/X\} \sqsubseteq_{RS} C_X\{\tau.p/X\}$. Set $\mathcal{R} \triangleq \{(B_X\{p/X\}, B_X\{\tau.p/X\}) : B_X \text{ is a context}\}$. By Proposition 6.1 and Lemma 6.7, it is sufficient to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$. Let $(C_X\{p/X\}, C_X\{\tau.p/X\}) \in \mathcal{R}$.

(ALT-upto-1) Assume that $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$. For this transition, since p is stable, by Lemma 5.14, there exists a stable context C'_X such that

$$p' \equiv C'_X\{p/X\} \text{ and } C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} C'_X\{\tau.p/X\}. \tag{6.8.1}$$

Moreover, by Lemma 5.15, it follows from $\tau.p \overset{\tau}{\longrightarrow} |p$ that

$$C'_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |r \text{ for some } r. \tag{6.8.2}$$

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For this transition, by Lemma 5.14, there exists a context $C''_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ such that $r \equiv C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$ and

$$p' \equiv C'_X\{p/X\} \Rrightarrow C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}. \quad (6.8.3)$$

Since $p' \notin F$, by Lemma 5.12, we get $C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \notin F$. Further, by Lemma 6.2, $r \equiv C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \notin F$. So, by (6.8.1), (6.8.2) and Lemma 4.2, we have $C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F| C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}$. Moreover, by Lemma 5.13, it follows from (6.8.3) that

$$p' \underset{\sim RS}{\sqsubset} C''_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}\mathcal{R}C''_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}.$$

(ALT-upto-2) Assume that $C_X\{p/X\} \overset{\tau}{\not\longrightarrow}$, $C_X\{\tau.p/X\} \overset{\tau}{\not\longrightarrow}$ and $C_X\{p/X\} \overset{a}{\Longrightarrow}_F |p'$. Then $C_X\{p/X\} \overset{a}{\longrightarrow}_F r \overset{\epsilon}{\Longrightarrow}_F |p'$ for some r. Moreover, by Lemma 6.2 and $C_X\{p/X\} \notin F$, we have

$$C_X\{\tau.p/X\} \notin F. \quad (6.8.4)$$

For the transition $C_X\{p/X\} \overset{a}{\longrightarrow}_F r$, by Lemma 5.7, there exist C'_X, $C'_{X,\widetilde{Y}}$ and $C''_{X,\widetilde{Y}}$ that realize (CP-a-1) – (CP-a-4). By (CP-a-1) and (CP-a-3-i), we have

$$C_X\{\tau.p/X\} \Rrightarrow C'_X\{\tau.p/X\} \equiv C'_{X,\widetilde{Y}}\{\tau.p/X, \tau.p/\widetilde{Y}\}.$$

If $\widetilde{Y} \neq \emptyset$ then, by (CP-a-2) and Lemma 5.4, we have $C'_{X,\widetilde{Y}}\{\tau.p/X, \tau.p/\widetilde{Y}\} \overset{\tau}{\longrightarrow}$, and hence $C_X\{\tau.p/X\} \overset{\tau}{\longrightarrow}$ by Lemma 5.10, which contradicts that $C_X\{\tau.p/X\}$ is stable. Thus $\widetilde{Y} = \emptyset$. So, $r \equiv C''_{X,\widetilde{Y}}\{p/X\}$ by (CP-a-3-ii) and

$$C_X\{\tau.p/X\} \overset{a}{\longrightarrow} C''_{X,\widetilde{Y}}\{\tau.p/X\} \text{ by (CP-}a\text{-3-iii) and } C_X\{\tau.p/X\} \overset{\tau}{\not\longrightarrow}. \quad (6.8.5)$$

Moreover, by (ALT-upto-1), it follows from $(C''_{X,\widetilde{Y}}\{p/X\}, C''_{X,\widetilde{Y}}\{\tau.p/X\}) \in \mathcal{R}$ and $r \equiv C''_{X,\widetilde{Y}}\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ that $C''_{X,\widetilde{Y}}\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$ for some q'. Finally, we also have $C_X\{\tau.p/X\} \overset{a}{\Longrightarrow}_F |q'$ due to (6.8.4) and (6.8.5), as desired.

(ALT-upto-3) Immediately follows from Lemma 6.1.

Next we intend to prove $C_X\{\tau.p/X\} \sqsubseteq_{RS} C_X\{p/X\}$. Set

$$\mathcal{R} \triangleq \{(B_X\{\tau.p/X\}, B_X\{p/X\}) : B_X \text{ is a context}\}.$$

Similarly, it is sufficient to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$. Let $(C_X\{\tau.p/X\}, C_X\{p/X\}) \in \mathcal{R}$. (ALT-upto-3) immediately follows from Lemma 6.1. In the following, we prove the other two conditions.

(ALT-upto-1) Assume that $C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$. For this transition, by Lemma 5.17, there exist r and stable context C^*_X such that $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{p/X\}$ and

$$C_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} C^*_X\{\tau.p/X\} \overset{\epsilon}{\Longrightarrow} |r \Rrightarrow p'. \quad (6.8.6)$$

Moreover, since p is stable, so is $C^*_X\{p/X\}$ by Lemma 5.6. Due to $r \Rrightarrow p'$ and $p' \notin F$, by Lemma 5.12, we get $r \notin F$. Hence $C^*_X\{\tau.p/X\} \notin F$ by (6.8.6) and Lemma 4.2.

Then $C^*_X\{p/X\} \notin F$ by Lemma 6.4. Thus $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |C^*_X\{p/X\}$. It remains to prove that $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} C^*_X\{p/X\}$. For the transition $C^*_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow} |r$ in (6.8.6), by Lemma 5.14, there exists a stable context $C'^*_{X,\widetilde{Y}}$ such that $r \equiv C'^*_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\} \Rightarrow p'$ and $C^*_X\{p/X\} \Rightarrow C'^*_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\}$, which, by Lemma 5.13, implies

$$p' \underset{\sim RS}{\sqsubset} C'^*_{X,\widetilde{Y}}\{\tau.p/X, p/\widetilde{Y}\}\mathcal{R}C'^*_{X,\widetilde{Y}}\{p/X, p/\widetilde{Y}\} \underset{\sim RS}{\sqsubset} C^*_X\{p/X\}.$$

(ALT-upto-2) Assume that $C_X\{\tau.p/X\} \stackrel{\tau}{\not\longrightarrow}$, $C_X\{p/X\} \stackrel{\tau}{\not\longrightarrow}$ and $C_X\{\tau.p/X\} \stackrel{a}{\Longrightarrow}_F \mid p'$. Hence $C_X\{\tau.p/X\} \stackrel{a}{\longrightarrow}_F r \stackrel{\epsilon}{\Longrightarrow}_F |p'$ for some r. Moreover, by Lemma 6.4 and $C_X\{\tau.p/X\} \notin F$, we have $C_X\{p/X\} \notin F$. For the a-transition $C_X\{\tau.p/X\} \stackrel{a}{\longrightarrow}_F r$, by Lemma 5.7 and $\tau.p \stackrel{a}{\not\longrightarrow}$, it is not difficult to see that there exists C'_X such that $C_X\{\tau.p/X\} \stackrel{a}{\longrightarrow} C'_X\{\tau.p/X\} \equiv r$ and $C_X\{p/X\} \stackrel{a}{\longrightarrow} C'_X\{p/X\}$. Moreover, by (ALT-upto-1), it follows from $(C'_X\{\tau.p/X\}, C'_X\{p/X\}) \in \mathcal{R}$ and $r \equiv C'_X\{\tau.p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$ that $C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $p' \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} q'$ for some q'. Clearly, we also have $C_X\{p/X\} \stackrel{a}{\Longrightarrow}_F |q'$, as desired. □

Lemma 6.9. If $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ then $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \sqsubseteq_{RS} C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ for any $C_{\widetilde{X}}$.

Proof. Set $\mathcal{R} \triangleq \{(B_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}, B_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}) : \widetilde{p} \bowtie \widetilde{q}, \widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ and $B_{\widetilde{X}}$ is a context$\}$. Similarly, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim RS}{\sqsubset}$. Suppose $(C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}, C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}) \in \mathcal{R}$. Then, by Lemma 6.1, it is obvious that such pair satisfies the condition (ALT-upto-3). Next we consider the other conditions in turn.

(ALT-upto-1) Assume that $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_F |s$. For this transition, by Lemma 5.14, there exist $C'_{\widetilde{X},\widetilde{Y}}$, $i_Y \leqslant |\widetilde{X}|$ and p'_Y for $Y \in \widetilde{Y}$ that satisfy (MS-τ-1) – (MS-τ-7). In particular, by (MS-τ-2,7), we have $p_{i_Y} \stackrel{\tau}{\Longrightarrow} |p'_Y$ for each $Y \in \widetilde{Y}$ and $s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \notin F$. Then, by (MS-$\tau$-1) and Lemma 5.5, $p'_Y \notin F$ and hence $p_{i_Y} \stackrel{\tau}{\Longrightarrow}_F |p'_Y$ by Lemma 4.2 for each $Y \in \widetilde{Y}$. Since $\widetilde{p} \bowtie \widetilde{q}$, it follows from $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that there exist $q'_Y (Y \in \widetilde{Y})$ such that

$$q_{i_Y} \stackrel{\tau}{\Longrightarrow}_F |q'_Y \text{ and } p'_Y \underset{\sim RS}{\sqsubset} q'_Y \text{ for each } Y \in \widetilde{Y}. \tag{6.9.1}$$

So $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow} C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$ by (MS-τ-3-ii). Moreover, by Lemma 5.6, it follows from $s \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \stackrel{\tau}{\not\longrightarrow}$, $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p'_Y} \underset{\sim RS}{\sqsubset} \widetilde{q'_Y}$ that $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \stackrel{\tau}{\not\longrightarrow}$. In addition, by Lemma 6.6 and $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \notin F$, we get $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \notin F$. Hence, by Lemma 4.2, we obtain $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \stackrel{\epsilon}{\Longrightarrow}_F |C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}$. Clearly, by (6.9.1) and the reflexivity of $\underset{\sim RS}{\sqsubset}$, $(C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}, C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}) \in \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset}$.

(ALT-upto-2) Let $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\}$ and $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\}$ be stable and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\Longrightarrow}_F |s$. Then

$$C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \stackrel{a}{\longrightarrow}_F r \stackrel{\epsilon}{\Longrightarrow}_F |s \text{ for some } r. \tag{6.9.2}$$

Moreover, by Lemma 6.6, it follows from $\widetilde{p} \bowtie \widetilde{q}$, $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ and $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ that

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \notin F. \tag{6.9.3}$$

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For the transition $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \xrightarrow{a} r$, by Lemma 5.7, there exist $C'_{\widetilde{X}}$, $C'_{\widetilde{X},\widetilde{Y}}$ and $C''_{\widetilde{X},\widetilde{Y}}$ that satisfy (CP-a-1) – (CP-a-4). In particular, by (CP-a-3-ii), there exist $i_Y \leqslant |\widetilde{X}|, p'_Y$ such that $p_{i_Y} \xrightarrow{a} p'_Y$ for each $Y \in \widetilde{Y}$ and $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\}$. Moreover, by (CP-$a$-1) and (CP-$a$-3-i), we have $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \Rrightarrow C'_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \equiv C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\}$. Hence $C'_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p_{i_Y}}/\widetilde{Y}\} \notin F$ by $C_{\widetilde{X}}\{\widetilde{p}/\widetilde{X}\} \notin F$ and Lemma 5.12. Further, since each $Y (\in \widetilde{Y})$ is 1-active in $C'_{\widetilde{X},\widetilde{Y}}$, by Lemma 5.5, we get

$$p_{i_Y} \notin F \text{ for each } Y \in \widetilde{Y}. \tag{6.9.4}$$

For the transition $r \equiv C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} |s$ in (6.9.2), by Lemma 5.18, it follows that for each $Y \in \widetilde{Y}$, there exists p''_Y such that $p'_Y \overset{\epsilon}{\Longrightarrow} |p''_Y$ and

$$C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} |s.$$

Then $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \notin F$ due to $s \notin F$ and Lemma 4.2, and hence $p''_Y \notin F$ for each $Y \in \widetilde{Y}$ by Lemma 5.5. Therefore, by (6.9.4) and Lemma 4.2, we have $p_{i_Y} \xrightarrow{a}_F p'_Y \overset{\epsilon}{\Longrightarrow}_F | p''_Y$ for each $Y \in \widetilde{Y}$. Then it follows from $\widetilde{p} \bowtie \widetilde{q}$ and $\widetilde{p} \sqsubseteq_{RS} \widetilde{q}$ that for each $Y \in \widetilde{Y}$, there exist q'_Y and q''_Y such that $q_{i_Y} \xrightarrow{a}_F q'_Y \overset{\epsilon}{\Longrightarrow}_F |q''_Y$ and $p''_Y \underset{\sim RS}{\sqsubset} q''_Y$. By (CP-$a$-3-iii),

$$C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \xrightarrow{a} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\}. \tag{6.9.5}$$

Further, by Lemma 5.4 and (CP-a-2), we obtain

$$C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q'_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow} C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\}. \tag{6.9.6}$$

Clearly, $(C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\}, C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\}) \in \mathcal{R}$. So, by $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{p}/\widetilde{X}, \widetilde{p''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow}_F |s$ and (ALT-upto-1), there exists t such that $C''_{\widetilde{X},\widetilde{Y}}\{\widetilde{q}/\widetilde{X}, \widetilde{q''_Y}/\widetilde{Y}\} \overset{\epsilon}{\Longrightarrow}_F |t$ and $s \underset{\sim RS}{\sqsubset} \mathcal{R} \underset{\sim RS}{\sqsubset} t$; moreover $C_{\widetilde{X}}\{\widetilde{q}/\widetilde{X}\} \overset{a}{\Longrightarrow}_F |t$ due to (6.9.3), (6.9.5), (6.9.6) and Lemma 4.2. □

We are now in a position to state the main result of this section.

Theorem 6.1 (precongruence). If $p \sqsubseteq_{RS} q$ then $C_X\{p/X\} \sqsubseteq_{RS} C_X\{q/X\}$.

Proof. Immediately follows from $\tau.p =_{RS} p \sqsubseteq_{RS} q =_{RS} \tau.q$ by Lemmas 6.8 and 6.9. □

7. Unique solution of equations

The main conclusion we reach in this section is that any equation $X =_{RS} t_X$ has at most one consistent solution modulo $=_{RS}$ provided that X is strongly guarded and does not occur in the scope of any conjunction in t_X; moreover the process $\langle X|X = t_X\rangle$ is indeed the unique consistent solution whenever such equation has a consistent solution. The proof of this result (Theorem 7.1) will be divided into two lemmas: Lemmas 7.3 and 7.4. The former considers a particular instance of Theorem 7.1, which asserts the uniqueness of consistent solutions that are uniform w.r.t stability. The latter ensures that $\langle X|X = t_X\rangle$ is consistent whenever a given equation $X =_{RS} t_X$ has some consistent solutions.

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This section is developed under the hypothesis that X does not occur in the scope of any conjunction in t_X. This hypothesis is essential to the proof of Lemma 7.3: first, we cannot generalize its present argument to deal with the case $C_X \equiv B_X \wedge D_X$. Second, Lemma 7.2 is applied to cope with the other cases in the proof of Lemma 7.3, which is no longer valid without this hypothesis. To relax this restriction of Lemma 7.3 (and Theorem 7.1), it is sufficient to verify the proposition that 'for any $p, q \notin F$ with $p \bowtie q$, if $p =_{RS} t_X\{p/X\}$ and $q =_{RS} t_X\{q/X\}$ for some t_X with strongly guarded X, then $C_X\{p/X\} \notin F$ implies $C_X\{q/X\} \notin F$ for any C_X'. Unfortunately, at present, we do not know whether it is true.

Lemma 7.1. For any stable processes $p, q \notin F$ and context C_X such that X does not occur in the scope of any conjunction, if $C_X\{p/X\} \in F$ then $C_X\{q/X\} \in F$.

Proof. We consider only the nontrivial case where C_X is not closed. Assume that $C_X\{p/X\} \in F$ and $\mathcal{T}$ is any proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{p/X\}F$. We proceed by induction on the depth of $\mathcal{T}$. The argument is a routine case analysis on C_X. Moreover, since X does not occur in the scope of any conjunction, the form of C_X is one of the following: X, $\alpha.B_X$, $B_X \odot D_X$ with $\odot \in \{\vee, \Box, \|_A\}$ and $\langle Y|E\rangle$. Here, we give the proof only for the case $C_X \equiv \langle Y|E\rangle$, the other cases are straightforward and omitted.

In case $C_X \equiv \langle Y|E\rangle$, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{p/X\}F}{\langle Y|E\rangle\{p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{p/X\}F}.$$

For the first alternative, we have $\langle t_Y|E\rangle\{q/X\} \in F$ by IH, and hence $C_X\{q/X\} \equiv \langle Y|E\rangle\{q/X\} \in F$. For the second alternative, assume $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$. Since q is stable, by Lemma 5.14, $s \equiv C'_X\{q/X\}$ for some stable C'_X such that X does not occur in the scope of any conjunction in C'_X and $\langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{p/X\}$. Moreover, since p is stable, so is $C'_X\{p/X\}$. Thus there exists a proper subtree of $\mathcal{T}$ with root $C'_X\{p/X\}F$. So, by IH, $s \equiv C'_X\{q/X\} \in F$. Hence $C_X\{q/X\} \in F$ by Theorem 4.1, as desired. □

This result is of independent interest, but its principle use is that it will serve as an important step in demonstrating the next lemma, which reveals that the above result still holds without the hypotheses that q and p are stable. Notice that the next lemma cannot be true for all C_X. In the case where C_X is $a.0 \wedge X$, for example, $a.0 \wedge a.0 \notin F$ but $a.0 \wedge b.0 \in F$.

Lemma 7.2. For any processes $p, q \notin F$ and context C_X such that X does not occur in the scope of any conjunction, if $C_X\{p/X\} \in F$ then $C_X\{q/X\} \in F$.

Proof. Suppose that $C_X\{p/X\} \in F$. We proceed by induction on the depth of the proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash C_X\{p/X\}F$. Similar to the preceding lemma, we handle only the case $C_X \equiv \langle Y|E\rangle$. In this situation, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Y|E\rangle\{p/X\}F}{\langle Y|E\rangle\{p/X\}F} \text{ with } Y = t_Y \in E \text{ or } \frac{\{rF : \langle Y|E\rangle\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |r\}}{\langle Y|E\rangle\{p/X\}F}.$$

The argument for the former is the same as the one in Lemma 7.1 and omitted. In the following, we consider the latter and suppose $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$. By Theorem 4.1, to

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complete the proof, it suffices to prove that $s \in F$. By Lemma 5.17, there exist t and stable C^*_X such that $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |t \Rrightarrow s$ and

$$\langle Y|E\rangle\{r/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{r/X\} \text{ for any } r. \tag{7.2.1}$$

In particular, we have $\langle Y|E\rangle\{a_X.0/X\} \stackrel{\epsilon}{\Longrightarrow} C^*_X\{a_X.0/X\}$ where a_X is a fresh visible action. For this transition, applying Lemma 5.6 finitely many times (notice that, in this procedure, since $a_X.0$ is stable, the clause (2) in Lemma 5.6 is always false), and by the clause (1) in Lemma 5.6, we get the sequence

$$\langle Y|E\rangle\{a_X.0/X\} \equiv C^0_X\{a_X.0/X\} \stackrel{\tau}{\longrightarrow} C^1_X\{a_X.0/X\} \stackrel{\tau}{\longrightarrow} \cdots \stackrel{\tau}{\longrightarrow} C^n_X\{a_X.0/X\} \equiv C^*_X\{a_X.0/X\}.$$

Here $n \geqslant 0$ and for each $1 \leqslant i \leqslant n$, C^i_X satisfies (C-τ-1,2,3) in Lemma 5.6. Since X does not occur in the scope of any conjunction in $\langle Y|E\rangle$, by (C-τ-3-iv), neither does X in C^n_X. Moreover we have $C^n_X \equiv C^*_X$ by Lemma 5.16. Hence X does not occur in the scope of any conjunction in C^*_X.

If p is stable then so is $C^*_X\{p/X\}$ by Lemma 5.6. Thus, by (7.2.1), $C^*_X\{p/X\}F$ is one of premises in the last inferring step in $\mathcal{T}$. Hence $C^*_X\{q/X\} \in F$ by applying IH. Then $t \in F$ by Lemma 4.2. Further, by Lemma 5.12, it follows from $t \Rrightarrow s$ that $s \in F$.

Next we consider the other case where $p \stackrel{\tau}{\longrightarrow}$. In this situation, due to $p \notin F$, we have

$$p \stackrel{\tau}{\Longrightarrow}_F |p^* \text{ for some } p^*. \tag{7.2.2}$$

In the following, we distinguish two cases based on whether q is stable.

Case 1. q is stable.

Then, for the transition $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$, by Lemma 5.14, we have $s \equiv C'_X\{q/X\}$ for some stable C'_X such that X does not occur in the scope of any conjunction and $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_X\{p/X\}$. Moreover, by Lemma 5.15, it follows from (7.2.2) that

$$C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} |p' \text{ for some } p'.$$

For this transition, by Lemma 5.14, there exist a stable context $C'^*_{X,\widetilde{Y}}$ and stable processes p'_Y for $Y \in \widetilde{Y}$ that realize (MS-τ-1) – (MS-τ-7). In particular, by (MS-τ-3-ii) it follows from (7.2.2) that $C'_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}$. Then, since $C'^*_{X,\widetilde{Y}}$, p and p^* are stable, by Lemma 5.6, so is $C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}$. Thus, $C'^*_{X,\widetilde{Y}}\{p/X, p^*/\widetilde{Y}\}F$ is one of premises of the last inferring step in $\mathcal{T}$. Moreover $p' \equiv C'^*_{X,\widetilde{Y}}\{p/X, \widetilde{p'_Y}/\widetilde{Y}\}$ by (MS-τ-2). Then, by (MS-τ-6) and IH, we obtain $C'^*_{X,\widetilde{Y}}\{q/X, p^*/\widetilde{Y}\} \in F$. Further, by (MS-$\tau$-6) and Lemma 7.1, we get $C'^*_{X,\widetilde{Y}}\{q/X, q/\widetilde{Y}\} \in F$. Finally, due to the stableness of C'_X, by (MS-τ-4), we also have $C'_X\{q/X\} \Rrightarrow C'^*_{X,\widetilde{Y}}\{q/X, q/\widetilde{Y}\}$. Hence $s \equiv C'_X\{q/X\} \in F$ by Lemma 5.12, as desired.

Case 2. q is not stable.

By Lemma 5.14, for the transition $\langle Y|E\rangle\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |s$, there exist a stable context $C'_{X,\widetilde{Z}}$ and q'_Z for $Z \in \widetilde{Z}$ that satisfy (MS-τ-1) – (MS-τ-7). Amongst them, by (MS-τ-2,7),

$$q \stackrel{\tau}{\Longrightarrow} |q'_Z \text{ for each } Z \in \widetilde{Z} \text{ and } s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\}. \tag{7.2.3}$$

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If $q'_Z \in F$ for some $Z \in \widetilde{Z}$ then by Lemma 5.5, we get $s \in F$ (notice that each Z in $\widetilde{Z}$ is 1-active), as desired. In the following, we handle the other case where

$$q'_Z \notin F \text{ for each } Z \in \widetilde{Z}. \tag{7.2.4}$$

By (MS-τ-3-ii), it follows from (7.2.2) that $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}$. Since $p \stackrel{\tau}{\longrightarrow}$, $q \stackrel{\tau}{\longrightarrow}$, $p^* \stackrel{\tau}{\not\longrightarrow}$, $q'_Z \stackrel{\tau}{\not\longrightarrow}$ for each $Z \in \widetilde{Z}$ and $s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\} \stackrel{\tau}{\not\longrightarrow}$, by Lemma 5.6, $C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}$ is stable. Hence $\mathcal{T}$ has a proper subtree with root $C'_{X,\widetilde{Z}}\{p/X, p^*/\widetilde{Z}\}F$. Then $C'_{X,\widetilde{Z}}\{q/X, p^*/\widetilde{Z}\} \in F$ by (MS-τ-6) and IH. Further, by Lemma 7.1, it follows from (7.2.3) and (7.2.4) that $s \equiv C'_{X,\widetilde{Z}}\{q/X, \widetilde{q'_Z}/\widetilde{Z}\} \in F$, as desired. □

We shall use $Dep(\mathcal{T})$ to denote the depth of a given proof tree $\mathcal{T}$. Given p, q and $\alpha \in Act_\tau$, for any proof tree $\mathcal{T}$ of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p \stackrel{\alpha}{\longrightarrow} q$, it is evident that $\mathcal{T}$ involves only rules in Table 1. Moreover, since each rule in Table 1 has only finitely many premises, it is not difficult to show that $Dep(\mathcal{T}) < \omega$ by induction on the depth of $\mathcal{T}$. This makes it legitimate to use arithmetical expressions with the form like $\sum_{\mathcal{T}\in\Omega} Dep(\mathcal{T})$ where Ω is a finite set and each $\mathcal{T} \in \Omega$ is a proof tree for some transition $p \stackrel{\alpha}{\longrightarrow} r$.

Definition 7.1. Given $p \stackrel{\epsilon}{\Longrightarrow} q$ and a finite set Ω of proof trees, we say that Ω is a *proof forest* for $p \stackrel{\epsilon}{\Longrightarrow} q$ if there exist $p_i (0 \leqslant i \leqslant n)$ such that

1. $p \equiv p_0 \stackrel{\tau}{\longrightarrow} p_1 \stackrel{\tau}{\longrightarrow} \cdots \stackrel{\tau}{\longrightarrow} p_n \equiv q$,
2. for each $i < n$, Ω contains exactly one proof tree for $p_i \stackrel{\tau}{\longrightarrow} p_{i+1}$ and
3. for each $\mathcal{T} \in \Omega$, $\mathcal{T}$ is a proof tree for $p_i \stackrel{\tau}{\longrightarrow} p_{i+1}$ for some $i < n$.

In this case, we also say that Ω is a proof forest for the sequence $p_0 \stackrel{\tau}{\longrightarrow} p_1 \ldots p_{n-1} \stackrel{\tau}{\longrightarrow} p_n$. The depth of Ω is defined as $Dep(\Omega) \triangleq \sum_{\mathcal{T}\in\Omega} Dep(\mathcal{T})$. Similarly, we may define the notion of a proof forest for $p \stackrel{a}{\Longrightarrow} q$.

The following lemma will prove extremely useful in establishing the main result in this section and its proof involves induction on the depths of proof forests.

Lemma 7.3. Let C_X be any context where X is strongly guarded and does not occur in the scope of any conjunction. For any processes $p, q \notin F$ with $p \bowtie q$, if $p =_{RS} C_X\{p/X\}$ and $q =_{RS} C_X\{q/X\}$ then $p =_{RS} q$.

Proof. Suppose $p, q \notin F$ with $p \bowtie q$, $p =_{RS} C_X\{p/X\}$ and $q =_{RS} C_X\{q/X\}$. It is sufficient to prove that $p \sqsubseteq_{RS} q$. Put

$$\mathcal{R} \triangleq \{(B_X\{p/X\}, B_X\{q/X\}) : X \text{ does not occur in the scope of any conjunction in } B_X\}.$$

By Proposition 6.1 and Lemma 6.7, it suffices to prove that $\mathcal{R}$ is an alternative ready simulation relation up to $\underset{\sim}{\sqsubset}_{RS}$. Let $(B_X\{p/X\}, B_X\{q/X\}) \in \mathcal{R}$.

(**ALT-upto-1**) Assume that $B_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$. Hence $B_X\{p/X\} \equiv p_0 \stackrel{\tau}{\longrightarrow}_F p_1 \stackrel{\tau}{\longrightarrow} \ldots p_{n-1} \stackrel{\tau}{\longrightarrow}_F |p_n \equiv p'$ for some $p_i (0 \leqslant i \leqslant n)$. Then, for each $0 \leqslant i < n$, there exists a proof

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tree $\mathcal{T}_i$ for $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash p_i \xrightarrow{\tau} p_{i+1}$. Let Ω be the set of all these proof trees $\mathcal{T}_i$. Clearly, it is a proof forest of $p_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} p_n$. We intend to prove that there exists q' such that $B_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{RS} \mathcal{R} \sqsubset_{RS} q'$ by induction on $Dep(\Omega)$. It is a routine case analysis on B_X. We treat only three cases as examples.

Case 1. $B_X \equiv X$.

Then $B_X\{p/X\} \equiv p \overset{\epsilon}{\Longrightarrow}_F |p'$. Since $p =_{RS} C_X\{p/X\}$, it follows that $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F | s$ and $p' \sqsubset_{RS} s$ for some s. Since X is strongly guarded and does not occur in the scope of any conjunction in C_X, by Lemma 5.14, there exists a stable context C'_X such that (a.1) $s \equiv C'_X\{p/X\}$, (a.2) X is strongly guarded and does not occur in the scope of any conjunction in C'_X, and (a.3) $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow} C'_X\{q/X\}$. Since $s \equiv C'_X\{p/X\} \not\xrightarrow{\tau}$, we have $C'_X\{q/X\} \not\xrightarrow{\tau}$ by (a.2) and Lemma 5.8. Moreover, by Lemma 7.2, $C'_X\{q/X\} \notin F$ follows from $C'_X\{p/X\} \notin F$ and $p, q \notin F$. Hence $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |C'_X\{q/X\}$ by (a.3) and Lemma 4.2. Further, it follows from $q =_{RS} C_X\{q/X\}$ that $q \overset{\epsilon}{\Longrightarrow}_F |q'$ and $C'_X\{q/X\} \sqsubset_{RS} q'$ for some q'. Therefore $B_X\{q/X\} \equiv q \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{RS} s \equiv C'_X\{p/X\}\mathcal{R}C'_X\{q/X\} \sqsubset_{RS} q'$.

Case 2. $B_X \equiv \langle Y|E\rangle$.

If $\langle Y|E\rangle\{p/X\}$ is stable then so is $\langle Y|E\rangle\{q/X\}$ by $p \bowtie q$ and Lemma 5.6. By Lemma 7.2, $\langle Y|E\rangle\{q/X\} \notin F$ because of $\langle Y|E\rangle\{p/X\} \notin F$. Hence $\langle Y|E\rangle\{q/X\} \overset{\epsilon}{\Longrightarrow}_F | \langle Y|E\rangle\{q/X\}$ and $(\langle Y|E\rangle\{p/X\}, \langle Y|E\rangle\{q/X\}) \in \sqsubset_{RS} \mathcal{R} \sqsubset_{RS}$ due to the reflexivity of $\sqsubset_{RS}$. Next we handle the other case where $\langle Y|E\rangle\{p/X\}$ is not stable. Clearly, the last rule applied in $\mathcal{T}_0$ is $\frac{\langle t_Y|E\rangle\{p/X\} \xrightarrow{\tau} p_1}{\langle Y|E\rangle\{p/X\} \xrightarrow{\tau} p_1}$ with $Y = t_Y \in E$. Thus, $\mathcal{T}_0$ contains a proper subtree, say $\mathcal{T}'_0$, which is a proof tree of $Strip(\mathcal{P}_{\mathrm{CLL}_R}, M_{\mathrm{CLL}_R}) \vdash \langle t_Y|E\rangle\{p/X\} \xrightarrow{\tau} p_1$ and $Dep(\mathcal{T}'_0) < Dep(\mathcal{T}_0)$. Thus $\Omega' \triangleq \{\mathcal{T}'_0, \mathcal{T}_i : 1 \leqslant i \leqslant n-1\}$ is a proof forest for $\langle t_Y|E\rangle\{p/X\} \xrightarrow{\tau} p_1 \xrightarrow{\tau} p_2 \xrightarrow{\tau} \cdots \xrightarrow{\tau} p_n \equiv p'$; moreover $Dep(\Omega') < Dep(\Omega)$. Then, by Lemma 5.2(5) and IH, we have $\langle t_Y|E\rangle\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{RS} \mathcal{R} \sqsubset_{RS} q'$ for some q'. Moreover we also have $B_X\{q/X\} \equiv \langle Y|E\rangle\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |q'$.

Case 3. $B_X \equiv D_X \Box D'_X$.

If $B_X\{p/X\}$ is stable then we can proceed analogously to Case 2 with $\langle Y|E\rangle\{p/X\} \not\xrightarrow{\tau}$. In the following, we consider the case $B_X\{p/X\} \xrightarrow{\tau}$.

For the transitions $D_X\{p/X\}\Box D'_X\{p/X\} \equiv p_0 \xrightarrow{\tau}_F \cdots \xrightarrow{\tau}_F |p_n \equiv p'(n \geqslant 1)$, there exist two sequences of processes $t_0(\equiv D_X\{p/X\}), \ldots, t_n$ and $s_0(\equiv D'_X\{p/X\}), \ldots, s_n$ such that t_n, s_n are consistent and stable, $p_n \equiv t_n \Box s_n$, and for each $0 \leqslant i < n$, $p_i \equiv t_i \Box s_i$ and the last rule applied in $\mathcal{T}_i$ is

$$\text{either } \frac{t_i \xrightarrow{\tau} t_{i+1}}{t_i \Box s_i \xrightarrow{\tau} t_{i+1} \Box s_{i+1}} \text{ or } \frac{s_i \xrightarrow{\tau} s_{i+1}}{t_i \Box s_i \xrightarrow{\tau} t_{i+1} \Box s_{i+1}}.$$

For the former, $s_{i+1} \equiv s_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}'_i$ which is a proof tree for $t_i \xrightarrow{\tau} t_{i+1}$. We use Ω_1 to denote the (finite) set of all these proof trees $\mathcal{T}'_i$. Similarly, for the latter, $t_{i+1} \equiv t_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}''_i$ which is a proof tree for $s_i \xrightarrow{\tau} s_{i+1}$. We use Ω_2 to denote the (finite) set of all these proof trees $\mathcal{T}''_i$. Clearly, Ω_1

is a proof forest for $D_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow} t_n$; moreover $Dep(\Omega_1) < Dep(\Omega)$. Thus, by IH, we have $D_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q_1'$ and $t_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q_1'$ for some q_1'. Similarly, for the transition $D_X'\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s_n$, we also have $D_X'\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q_2'$ and $s_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q_2'$ for some q_2'. Then, by Theorem 4.2, it is easy to check that $p' \equiv t_n \Box s_n \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q_1' \Box q_2'$. Moreover we also have $B_X\{q/X\} \equiv D_X\{q/X\} \Box D_X'\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q_1' \Box q_2'$.

(**ALT-upto-2**) Suppose that $B_X\{p/X\}$ and $B_X\{q/X\}$ are stable. Let $B_X\{p/X\} \stackrel{a}{\Longrightarrow}_F |p'$. So, there exist $p_0, \ldots, p_n (n \geqslant 1)$ such that $B_X\{p/X\} \equiv p_0 \stackrel{a}{\longrightarrow}_F p_1 \stackrel{\tau}{\longrightarrow}_F \cdots \stackrel{\tau}{\longrightarrow}_F | p_n \equiv p'$. Then there exists a proof tree $\mathcal{T}_i$ for $p_i \stackrel{\alpha_i}{\longrightarrow} p_{i+1}$ for $i < n$, where $\alpha_0 = a$ and $\alpha_j = \tau (1 \leqslant j < n)$. Let Ω be the set of all these proof trees $\mathcal{T}_i$. Clearly, it is a proof forest for $B_X\{p/X\} \equiv p_0 \stackrel{a}{\longrightarrow} p_1 \stackrel{\tau}{\longrightarrow} \cdots \stackrel{\tau}{\longrightarrow} p_n \equiv p'$. In the following, we want to prove that there exists q' such that $B_X\{q/X\} \stackrel{a}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'$ by induction on $Dep(\Omega)$. Since $B_X\{p/X\}$ is stable and X does not occur in the scope of any conjunction in B_X, the topmost operator of B_X is neither disjunction nor conjunction. Thus, we distinguish five cases based on the form of B_X.

Case 1. $B_X \equiv X$.

Due to $B_X\{p/X\} \equiv p \stackrel{a}{\Longrightarrow}_F |p'$, we have $p \notin F$. Moreover, since $p (\equiv B_X\{p/X\})$ is stable, we get $p \stackrel{\epsilon}{\Longrightarrow}_F |p$. Hence it follows from $p =_{RS} C_X\{p/X\}$ that $C_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |s$ and $p \sqsubset_{\sim RS} s$ for some s. Further, since X is strongly guarded and does not occur in the scope of any conjunction in C_X, by Lemma 5.14, there exists a stable C_X' such that (b.1) X is strongly guarded and does not occur in the scope of any conjunction in C_X', (b.2) $s \equiv C_X'\{p/X\}$ and (b.3) $C_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow} C_X'\{q/X\}$. Then it follows from $p \sqsubset_{\sim RS} s \equiv C_X'\{p/X\}$ and $p \stackrel{a}{\Longrightarrow}_F |p'$ that $C_X'\{p/X\} \stackrel{a}{\Longrightarrow}_F |s'$ and $p' \sqsubset_{\sim RS} s'$ for some s'. Since $p \stackrel{\tau}{\not\longrightarrow}$, by (b.1), Lemmas 5.8 and 5.14, there exists a stable context C_X'' such that (c.1) $s' \equiv C_X''\{p/X\}$, (c.2) X does not occur in the scope of any conjunction in C_X'' and (c.3) $C_X'\{q/X\} \stackrel{a}{\longrightarrow} \stackrel{\epsilon}{\Longrightarrow} C_X''\{q/X\}$. Moreover, since $q (\equiv B_X\{q/X\})$ is stable, so is $C_X''\{q/X\}$. Then, by (b.3) and (c.3), we have $C_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow} |C_X'\{q/X\} \stackrel{a}{\Longrightarrow} |C_X''\{q/X\}$. By Lemmas 7.2 and 4.2, it follows from $p, q, C_X\{p/X\}, C_X'\{p/X\}, C_X''\{p/X\} \notin F$ that

$$C_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |C_X'\{q/X\} \stackrel{a}{\Longrightarrow}_F |C_X''\{q/X\}. \tag{7.3.1}$$

Then, since $C_X\{q/X\} =_{RS} q$ and $q \stackrel{\tau}{\not\longrightarrow}$, we get $C_X'\{q/X\} \sqsubset_{\sim RS} q$. Further, due to (7.3.1), it follows that $B_X\{q/X\} (\equiv q) \stackrel{a}{\Longrightarrow}_F |q'$ and $C_X''\{q/X\} \sqsubset_{\sim RS} q'$ for some q'. Moreover $p' \sqsubset_{\sim RS} s' \equiv C_X''\{p/X\} \mathcal{R} C_X''\{q/X\} \sqsubset_{\sim RS} q'$, as desired.

Case 2. $B_X \equiv \alpha.D_X$.

So $\alpha = a$ and $D_X\{p/X\} \stackrel{\epsilon}{\Longrightarrow}_F |p'$. Clearly, $(D_X\{p/X\}, D_X\{q/X\}) \in R$. By (ALT-upto-1), there exists q' such that $D_X\{q/X\} \stackrel{\epsilon}{\Longrightarrow}_F |q'$ and $p' \sqsubset_{\sim RS} \mathcal{R} \sqsubset_{\sim RS} q'$. Moreover it is evident that $\alpha.D_X\{q/X\} \stackrel{a}{\Longrightarrow}_F |q'$.

Case 3. $B_X \equiv D_X \Box D_X'$.

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W.l.o.g, we assume that the last rule applied in $\mathcal{T}_0$ is $\frac{D_X\{p/X\}\xrightarrow{a}p_1}{D_X\{p/X\}\Box D'_X\{p/X\}\xrightarrow{a}p_1}$ with $D'_X\{p/X\}\stackrel{\tau}{\not\longrightarrow}$. Then $\mathcal{T}_0$ has a proper subtree, say $\mathcal{T}'_0$, which is a proof tree for the transition $D_X\{p/X\}\xrightarrow{a}p_1$.Clearly, $\Omega'\triangleq\{\mathcal{T}'_0,\mathcal{T}_i : 1\leqslant i\leqslant n-1\}$ is a proof forest for $D_X\{p/X\}\stackrel{a}{\Longrightarrow}p'$ and $Dep(\Omega')<Dep(\Omega)$. Moreover, since $B_X\{q/X\}$ is stable, so are $D_X\{q/X\}$ and $D'_X\{q/X\}$. Then, by IH, we have $D_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'$ and $p'\sqsubseteq_{RS}\mathcal{R}\sqsubseteq_{RS}q'$ for some q'. Moreover, $D'_X\{p/X\}\notin F$ because of $B_X\{p/X\}\notin F$, which, by Lemma 7.2, implies $D'_X\{q/X\}\notin F$. Hence $B_X\{q/X\}\equiv D_X\{q/X\}\Box D'_X\{q/X\}\notin F$, and $B_X\{q/X\}\equiv D_X\{q/X\}\Box D'_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'$, as desired.

Case 4. $B_X\equiv D_X\parallel_A D'_X$.

Then the last rule applied in $\mathcal{T}_0$ is one of the following three formats:

1. $\frac{D_X\{p/X\}\xrightarrow{a}t_1,D'_X\{p/X\}\xrightarrow{a}s_1}{D_X\{p/X\}\parallel_A D'_X\{p/X\}\xrightarrow{a}t_1\parallel_A s_1}$ with $a\in A$ and $p_1\equiv t_1\parallel_A s_1$;
2. $\frac{D_X\{p/X\}\xrightarrow{a}t_1}{D_X\{p/X\}\parallel_A D'_X\{p/X\}\xrightarrow{a}t_1\parallel_A D'_X\{p/X\}}$ with $D'_X\{p/X\}\stackrel{\tau}{\not\longrightarrow}$, $p_1\equiv t_1\parallel_A D'_X\{p/X\}$ and $a\notin A$;
3. $\frac{D'_X\{p/X\}\xrightarrow{a}s_1}{D_X\{p/X\}\parallel_A D'_X\{p/X\}\xrightarrow{a}D_X\{p/X\}\parallel_A s_1}$ with $D_X\{p/X\}\stackrel{\tau}{\not\longrightarrow}$, $p_1\equiv D_X\{p/X\}\parallel_A s_1$ and $a\notin A$.

We treat the first one, and the proof of the later two runs as in Case 3. Clearly, $\mathcal{T}_0$ has two proper subtrees $\mathcal{T}'_0$ and $\mathcal{T}''_0$, which are proof trees for $D_X\{p/X\}\xrightarrow{a}t_1$ and $D'_X\{p/X\}\xrightarrow{a}s_1$ respectively. Moreover, for the transitions $p_1\xrightarrow{\tau}\cdots\xrightarrow{\tau}|p_n$, there exist two processes sequences $t_1,\ldots,t_n$ and $s_1,\ldots,s_n$ such that t_n,s_n are stable, $p_n\equiv t_n\parallel_A s_n$, and for each $1\leqslant i<n$, $p_i\equiv t_i\parallel_A s_i$ and the last rule applied in $\mathcal{T}_i$ is

$$\text{either }\frac{t_i\xrightarrow{\tau}t_{i+1}}{t_i\parallel_A s_i\xrightarrow{\tau}t_{i+1}\parallel_A s_{i+1}}\text{ or }\frac{s_i\xrightarrow{\tau}s_{i+1}}{t_i\parallel_A s_i\xrightarrow{\tau}t_{i+1}\parallel_A s_{i+1}}.$$

For the former, $s_{i+1}\equiv s_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}'_i$ which is a proof tree for $t_i\xrightarrow{\tau}t_{i+1}$. We use Ω_1 to denote the (finite) set of all these proof tree $\mathcal{T}'_i$. Similarly, for the latter, $t_{i+1}\equiv t_i$ and $\mathcal{T}_i$ contains a proper subtree $\mathcal{T}''_i$ which is a proof tree for $s_i\xrightarrow{\tau}s_{i+1}$. We use Ω_2 to denote the (finite) set of all these proof tree $\mathcal{T}''_i$. Clearly, $\Omega'\triangleq\{\mathcal{T}'_0\}\cup\Omega_1$ is a proof forest for $D_X\{p/X\}\stackrel{a}{\Longrightarrow}t_n$ and $Dep(\Omega')<Dep(\Omega)$. Thus, by IH, we have $D_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'_1$ and $t_n\sqsubseteq_{RS}\mathcal{R}\sqsubseteq_{RS}q'_1$ for some q'_1. Similarly, for the transition $D'_X\{p/X\}\stackrel{a}{\Longrightarrow}_F|s_n$, we also have $D'_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'_2$ and $s_n\sqsubseteq_{RS}\mathcal{R}\sqsubseteq_{RS}q'_2$ for some q'_2. Therefore we obtain $p'\equiv t_n\parallel_A s_n\sqsubseteq_{RS}\mathcal{R}\sqsubseteq_{RS}q'_1\parallel_A q'_2$ by Theorem 4.2. Moreover it is not difficult to see that $B_X\{q/X\}\equiv D_X\{q/X\}\parallel_A D'_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'_1\parallel_A q'_2$ because of $B_X\{q/X\}\stackrel{\tau}{\not\longrightarrow}$, $D_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'_1$ and $D'_X\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'_2$.

Case 5. $B_X\equiv\langle Y|E\rangle$.

Clearly, the last rule applied in $\mathcal{T}_0$ is $\frac{\langle t_Y|E\rangle\{p/X\}\xrightarrow{a}p_1}{\langle Y|E\rangle\{p/X\}\xrightarrow{a}p_1}$. Hence $\mathcal{T}_0$ contains a proper subtree, say $\mathcal{T}'_0$, which is a proof tree for $Strip(\mathcal{P}_{\mathrm{CLL}_R},M_{\mathrm{CLL}_R})\vdash\langle t_Y|E\rangle\{p/X\}\xrightarrow{a}p_1$ and $Dep(\mathcal{T}'_0)<Dep(\mathcal{T}_0)$. So, $\Omega'\triangleq\{\mathcal{T}'_0,\mathcal{T}_i : 1\leqslant i<n\}$ is a proof forest for $\langle t_Y|E\rangle\{p/X\}\stackrel{a}{\Longrightarrow}p'$ and $Dep(\Omega')<Dep(\Omega)$. Then, by IH, we have $\langle t_Y|E\rangle\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'$ and $p'\sqsubseteq_{RS}\mathcal{R}\sqsubseteq_{RS}q'$ for some q'; moreover $B_X\{q/X\}\equiv\langle Y|E\rangle\{q/X\}\stackrel{a}{\Longrightarrow}_F|q'$, as desired.

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(**ALT-upto-3**) Let $B_X\{p/X\}$ and $B_X\{q/X\}$ be stable and $B_X\{p/X\} \notin F$. We shall prove $\mathcal{I}(B_X\{p/X\}) \supseteq \mathcal{I}(B_X\{q/X\})$, the converse inclusion may be proved in a similar manner and is omitted. Assume that $B_X\{q/X\} \xrightarrow{a} q'$. Then, for this transition, by Lemma 5.7, there exist B'_X, $B'_{X,\widetilde{Y}}$ and $B''_{X,\widetilde{Y}}$ with $X \notin \widetilde{Y}$ that satisfy (CP-*a*-1) – (CP-*a*-4). In case $\widetilde{Y} = \varnothing$, it immediately follows from (CP-*a*-3-iii) that $B_X\{p/X\} \xrightarrow{a} B''_{X,\widetilde{Y}}\{p/X\}$. Next we handle the other case $\widetilde{Y} \neq \varnothing$. In this situation, by (CP-*a*-3-iii), to complete the proof, it suffices to prove that $\mathcal{I}(p) = \mathcal{I}(q)$. By (CP-*a*-1) and (CP-*a*-3-i), we have

$$B_X\{r/X\} \Rrightarrow B'_{X,\widetilde{Y}}\{r/X, r/\widetilde{Y}\} \text{ for any } r.$$

Then, since $B_X\{p/X\}$ and $B_X\{q/X\}$ are stable, by $\widetilde{Y} \neq \varnothing$, (CP-*a*-2) and Lemmas 5.10 and 5.4, it follows that both p and q are stable. Hence $p \overset{\epsilon}{\Longrightarrow}_F |p$ by $p \notin F$. Then, due to $p =_{RS} C_X\{p/X\}$, we have $C_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |s$ and $p \sqsubseteq_{RS} s$ for some s. For this transition, since X is strongly guarded in C_X, by Lemma 5.14, there exists a stable context D_X such that (d.1) $s \equiv D_X\{p/X\} \overset{\tau}{\not\longrightarrow}$, (d.2) X is strongly guarded and does not occur in the scope of any conjunction in D_X, and (d.3) $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow} D_X\{q/X\}$. Hence $\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\})$ by (d.1), $p \sqsubseteq_{RS} s$ and $p \notin F$. Moreover, by (d.1), (d.2) and Lemma 5.8, we have $D_X\{q/X\} \overset{\tau}{\not\longrightarrow}$ and $\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\}) = \mathcal{I}(D_X\{q/X\})$. By Lemma 7.2, we also get $D_X\{q/X\} \notin F$ due to $p \notin F$, $q \notin F$ and $s \equiv D_X\{p/X\} \notin F$. So, $C_X\{q/X\} \overset{\epsilon}{\Longrightarrow}_F |D_X\{q/X\}$ by Lemma 4.2. Further, it follows from $q =_{RS} C_X\{q/X\}$ and $q \overset{\tau}{\not\longrightarrow}$ that $D_X\{q/X\} \sqsubseteq_{RS} q$. Hence $\mathcal{I}(D_X\{q/X\}) = \mathcal{I}(q)$ because of $D_X\{q/X\} \notin F$. Therefore $\mathcal{I}(p) = \mathcal{I}(D_X\{p/X\}) = \mathcal{I}(D_X\{q/X\}) = \mathcal{I}(q)$, as desired. □

The next lemma is the crucial step in the demonstrating the assertion that $\langle X|X = t_X\rangle$ is a consistent solution of a given equation $X =_{RS} t_X$ whenever consistent solutions exist.

Lemma 7.4. For any term t_X where X is strongly guarded and does not occur in the scope of any conjunction, if $q =_{RS} t_X\{q/X\}$ for some $q \notin F$ then $\langle X|X = t_X\rangle \notin F$.

Proof. Assume $p =_{RS} t_X\{p/X\}$ for some $p \notin F$. Then $t_X\{p/X\} \notin F$. Set

$$\Omega = \left\{ B_Y\{\langle X|X = t_X\rangle/Y\} : \begin{array}{l} B_Y\{p/Y\} \notin F \text{ and } Y \text{ does not occur in the scope of} \\ \text{any conjunction in } B_Y \end{array} \right\}.$$

It is obvious that $\langle X|X = t_X\rangle \in \Omega$ by taking $B_Y \triangleq Y$. Thus we intend to show that $\Omega \cap F = \varnothing$. Assume $C_Y\{\langle X|X = t_X\rangle/Y\} \in \Omega$. Let $\mathcal{T}$ be any proof tree for $Strip(\mathcal{P}_{\text{CLL}_R}, M_{\text{CLL}_R}) \vdash C_Y\{\langle X|X = t_X\rangle/Y\}F$. Similar to Lemma 6.2, it is sufficient to prove that $\mathcal{T}$ has a proper subtree with root sF for some $s \in \Omega$, which is a routine case analysis based on the last rule applied in $\mathcal{T}$. Here we treat only two cases.

Case 1. $C_Y \equiv Y$.

Then $C_Y\{\langle X|X = t_X\rangle/Y\} \equiv \langle X|X = t_X\rangle$. Clearly, the last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_X|X = t_X\rangle F}{\langle X|X = t_X\rangle F} \text{ or } \frac{\{rF : \langle X|X = t_X\rangle \overset{\epsilon}{\Longrightarrow} |r\}}{\langle X|X = t_X\rangle F}.$$

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For the former, $\mathcal{T}$ has a proper subtree with root $\langle t_X|X = t_X\rangle F$; moreover $\langle t_X|X = t_X\rangle \equiv t_X\{\langle X|X = t_X\rangle/X\} \in \Omega$ due to $t_X\{p/X\} \notin F$, as desired. For the latter, if $\langle X|X = t_X\rangle \not\xrightarrow{\tau}$, then, in $\mathcal{T}$, the unique node directly above the root is labelled with $\langle X|X = t_X\rangle F$; moreover $\langle X|X = t_X\rangle \in \Omega$, as desired. In the following, we consider the nontrivial case $\langle X|X = t_X\rangle \xrightarrow{\tau}$. Since $t_X\{p/X\} \notin F$, by Theorem 4.1, we get $t_X\{p/X\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. For this transition, since X is strongly guarded and does not occur in the scope of any conjunction, by Lemma 5.14, there exists a stable context B_X such that (a.1) X is strongly guarded and does not occur in the scope of any conjunction, (a.2) $p' \equiv B_X\{p/X\}$ and (a.3) $t_X\{\langle X|X = t_X\rangle/X\} \overset{\epsilon}{\Longrightarrow} B_X\{\langle X|X = t_X\rangle/X\}$. Since $p' \equiv B_X\{p/X\} \not\xrightarrow{\tau}$, by (a.1) and Lemma 5.8, $B_X\{\langle X|X = t_X\rangle/X\} \not\xrightarrow{\tau}$. Then it follows from (a.3) and $\langle X|X = t_X\rangle \xrightarrow{\tau}$ that $\langle X|X = t_X\rangle \overset{\epsilon}{\Longrightarrow} |B_X\{\langle X|X = t_X\rangle/X\}$. Hence $\mathcal{T}$ has a proper subtree with root $B_X\{\langle X|X = t_X\rangle/X\}F$; moreover $B_X\{\langle X|X = t_X\rangle/X\} \in \Omega$ because of $p' \notin F$, (a.1) and (a.2).

Case 2. $C_Y \equiv \langle Z|E\rangle$.

Then $C_Y\{\langle X|X = t_X\rangle/Y\} \equiv \langle Z|E\{\langle X|X = t_X\rangle/Y\}\rangle$. The last rule applied in $\mathcal{T}$ is

$$\text{either } \frac{\langle t_Z|E\rangle\{\langle X|X = t_X\rangle/Y\}F}{\langle Z|E\rangle\{\langle X|X = t_X\rangle/Y\}F}(Z = t_Z \in E) \text{ or } \frac{\{rF : \langle Z|E\rangle\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Longrightarrow} |r\}}{\langle Z|E\rangle\{\langle X|X = t_X\rangle/Y\}F}.$$

For the first alternative, by Lemma 4.1(8), it follows from $\langle Z|E\rangle\{p/Y\} \notin F$ that $\langle t_Z|E\rangle\{p/Y\} \notin F$. Since Y does not occur in the scope of any conjunction in $\langle Z|E\rangle$, by Lemma 5.2(5), neither does it in $\langle t_Z|E\rangle$. Therefore $\langle t_Z|E\rangle\{\langle X|X = t_X\rangle/Y\} \in \Omega$.

For the second alternative, since $\langle Z|E\rangle\{p/Y\} \notin F$ and $p =_{RS} t_X\{p/X\}$, we get $\langle Z|E\rangle\{t_X\{p/X\}/Y\} \notin F$ by Theorem 6.1. So $\langle Z|E\rangle\{t_X\{p/X\}/Y\} \overset{\epsilon}{\Longrightarrow}_F |p'$ for some p'. Then, for this transition, by Lemma 5.14, there exist processes q_W for $W \in \widetilde{W}$ and a context $D_{Y,\widetilde{W}}$ with $Y \notin \widetilde{W}$ such that

b.1. $t_X\{p/X\} \overset{\tau}{\Longrightarrow} |q_W$ for each $W \in \widetilde{W}$ and $p' \equiv D_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{q_W}/\widetilde{W}\}$,

b.2. none of Y and $\widetilde{W}$ occur in the scope of any conjunction in $D_{Y,\widetilde{W}}$ and

b.3. $\langle Z|E\rangle\{r/Y\} \overset{\epsilon}{\Longrightarrow} D_{Y,\widetilde{W}}\{r/Y, \widetilde{r_W}/\widetilde{W}\}$ for any r and $r_W (W \in \widetilde{W})$ such that $r \overset{\tau}{\Longrightarrow} r_W$ for each $W \in \widetilde{W}$.

Then, since X is strongly guarded and does not occur in the scope of any conjunction in t_X, by Lemmas 5.14 and 5.8, for each transition $t_X\{p/X\} \overset{\tau}{\Longrightarrow} |q_W$, there exists a stable context t_X^W such that (c.1) X is strongly guarded and does not occur in the scope of any conjunction in t_X^W, (c.2) $q_W \equiv t_X^W\{p/X\}$ and (c.3) $t_X\{\langle X|X = t_X\rangle/X\} \overset{\tau}{\Longrightarrow} |t_X^W\{\langle X|X = t_X\rangle/X\}$. For the simplicity of notation, we let Q_W stand for $t_X^W\{\langle X|X = t_X\rangle/X\}$ for each $W \in \widetilde{W}$. So, by (c.3), $\langle X|X = t_X\rangle \overset{\tau}{\Longrightarrow} |Q_W$ for each $W \in \widetilde{W}$. Hence, by (b.3),

$$\langle Z|E\rangle\{\langle X|X = t_X\rangle/Y\} \overset{\epsilon}{\Longrightarrow} D_{Y,\widetilde{W}}\{\langle X|X = t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\}. \tag{7.4.1}$$

By (b.2) and (c.1), it is not difficult to see that X is strongly guarded and does not occur in the scope of any conjunction in $D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}$. So, by Lemma 5.8 and $p' \equiv D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}\{p/X\} \not\xrightarrow{\tau}$, we get $D_{Y,\widetilde{W}}\{t_X/Y, \widetilde{t_X^W}/\widetilde{W}\}\{\langle X|X = t_X\rangle/X\} \not\xrightarrow{\tau}$. Hence $D_{Y,\widetilde{W}}\{\langle X|X = t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\} \not\xrightarrow{\tau}$ by Lemma 5.6 and $\mathcal{I}(\langle X|X = t_X\rangle) = \mathcal{I}(t_X\{\langle X|X = t_X\rangle/X\})$. So, by (7.4.1), $\mathcal{T}$ has a proper subtree with root $D_{Y,\widetilde{W}}\{\langle X|X = t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\}F$.

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Moreover, by Theorem 6.1 and $p =_{RS} t_X\{p/X\}$, it follows from $p' \equiv D_{Y,\widetilde{W}}\{t_X\{p/X\}/Y, \widetilde{t_X^W\{p/X\}}/\widetilde{W}\} \notin F$ that $D_{Y,\widetilde{W}}\{p/Y, \widetilde{t_X^W\{p/X\}}/\widetilde{W}\} \notin F$. Therefore $D_{Y,\widetilde{W}}\{\langle X|X = t_X\rangle/Y, \widetilde{Q_W}/\widetilde{W}\} \equiv D'_Y\{\langle X|X = t_X\rangle/Y\} \in \Omega$ by setting $D'_Y \triangleq D_{Y,\widetilde{W}}\{\widetilde{t_X^W\{Y/X\}}/\widetilde{W}\}$. □

Theorem 7.1 (unique solution). For any $p, q \notin F$ and t_X where X is strongly guarded and does not occur in the scope of any conjunction, if $p =_{RS} t_X\{p/X\}$ and $q =_{RS} t_X\{q/X\}$ then $p =_{RS} q$. Moreover $\langle X|X = t_X\rangle$ is the unique consistent solution (modulo $=_{RS}$) of the equation $X =_{RS} t_X$ whenever consistent solutions exist.

Proof. If $p \bowtie q$ then $p =_{RS} q$ follows from Lemma 7.3, otherwise, w.l.o.g, we assume that p is stable and q is not. By Theorem 6.1, $\tau.p =_{RS} p =_{RS} t_X\{p/X\} =_{RS} t_X\{\tau.p/X\}$. Then, by Lemma 7.3, it follows from $\tau.p, q \notin F$, $\tau.p \bowtie q$, $\tau.p =_{RS} t_X\{\tau.p/X\}$ and $q =_{RS} t_X\{q/X\}$ that $\tau.p =_{RS} q$. Hence $p =_{RS} q$.

Suppose that $X =_{RS} t_X$ has consistent solutions. It is obvious that $\langle X|X = t_X\rangle =_{RS} t_X\{\langle X|X = t_X\rangle/X\}$ due to $\langle X|X = t_X\rangle \Rrightarrow_1 \langle t_X|X = t_X\rangle \equiv t_X\{\langle X|X = t_X\rangle/X\}$ and Lemma 5.13. Further, by Lemma 7.4, $\langle X|X = t_X\rangle$ is the unique consistent solution of the equation $X =_{RS} t_X$. □

Corollary 7.1. For any term t_X where X is strongly guarded and does not occur in the scope of any conjunction, then the equation $X =_{RS} t_X$ has consistent solutions iff $\langle X|X = t_X\rangle \notin F$.

We provide a brief discussion to end this section. For Theorem 7.1, the condition that X is strongly guarded cannot be relaxed to that X is weakly guarded. For instance, consider the equation $X =_{RS} \tau.X$, it has infinitely many consistent solutions. In fact, for any p, it always holds that $p =_{RS} \tau.p$. Moreover the condition that $p, q \notin F$ is also necessary. For example, both $\langle X|X = a.X\rangle$ and $\perp$ are solutions of the equation $X =_{RS} a.X$, but they are not equivalent modulo $=_{RS}$.

8. Conclusions and future work

This paper considers recursive operations over LLTSs in a pure process-algebraic style. We show that the behavioural relation $\sqsubseteq_{RS}$ is precongruent w.r.t all operations in CLL$_R$, which reveals that this calculus supports compositional reasoning. Moreover, we also provide a result on the uniqueness of consistent solution of a given equation $X =_{RS} t_X$ where X is required to be strongly guarded and does not occur in the scope of any conjunction in t_X.

We conclude this paper by giving several possible avenues for further work. First, it would be desirable to relax the restriction of Theorem 7.1, that is, establish the uniqueness of consistent solution without the hypothesis that X does not occur in the scope of any conjunction. Second, it is well known that the operator hiding is useful in specifying systems. In Lüttgen and Vogler (2010), such operator has been considered. To preserve τ-purity, Lüttgen and Vogler give a complicated setting to introduce it in the framework of LLTS. As a future work, we plan to enrich CLL$_R$ by adding this operator. Although it is relatively easy to capture Lüttgen and Vogler's setting in terms of SOS rules, it seems nontrivial to reason about F-predicate in the presence of both recursion and hiding as the operator hiding may lead to divergence and hence introduce inconsistency because of Condition (LTS2), for instance, considering the process $\langle X|X = a.X\rangle/a$. Third, it would be interesting to find a (ground) complete proof system for regular processes in CLL_R along lines adopted in Baeten and Bravetti (2008) and Milner (1989b). Here a process is regular if its LTS has only finitely many states and transitions. To this end, it is necessary to adopt the restriction that recursive variables do not occur in the scope of any conjunction in recursive specifications. Otherwise, non-regular expressions would occur, for instance, consider the process $\langle X|X = a.X \wedge \tau.a.X\rangle$. Thus we think that Theorem 7.1 may be enough for this aim.

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