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ACCESSIBLE SUBRINGS AND KUROSH'S CHAINS OF ASSOCIATIVE RINGS

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Abstract

This article is devoted to the historical study of the ADS-problem with a special emphasis on the use of methods and techniques, emerging with the development of the theory of rings: accessible subrings, iterated maximal essential extensions of rings, completely normal rings. We construct new examples of classes for which Kurosh's chain stabilizes at any given step. We recall the old nontrivial questions, and we pose a new one.

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1. Introduction

All rings in this paper are associative but are not required to have a unity or to be commutative.

Let $\mathcal{M}$ be a homomorphically closed class of associative rings. Put $\mathcal{M}_1 = \mathcal{M}^1 = \mathcal{M}_l^1 = \mathcal{M}$ and for ordinals $\alpha \geq 2$, define $\mathcal{M}_\alpha$, $(\mathcal{M}^\alpha, \mathcal{M}_l^\alpha)$ to be the class of all associative rings R such that every nonzero homomorphic image of R contains a nonzero ideal (one-sided ideal, left ideal) in $\mathcal{M}_\beta$ $(\mathcal{M}^\beta, \mathcal{M}_l^\beta)$ for some $\beta < \alpha$. In this way we obtain a chain $\{\mathcal{M}_\alpha\}$ $(\{\mathcal{M}^\alpha\}, \{\mathcal{M}_l^\alpha\})$, the union of which is equal to the lower radical class $l(\mathcal{M})$ (lower strong radical class $ls(\mathcal{M})$, lower left strong radical class $ls_l(\mathcal{M})$) determined by $\mathcal{M}$. The chain $\{\mathcal{M}_\alpha\}$ is called *Kurosh's chain* of $\mathcal{M}$.

To denote that I is an ideal (one-sided ideal, left ideal) of a ring R, we write $I \lhd R$ $(I < R,\ I <_l R)$. Let n be a positive integer. A subring A of R is said to be *n-accessible (one-sided n-accessible, left n-accessible)* in R, if there are subrings $R = A_0, A_1, \ldots, A_{n-1}, A_n = A$ of R, such that $A_i \lhd A_{i-1}$ $(A_i < A_{i-1},\ A_i <_l A_{i-1})$ for $i = 1, 2, \ldots, n$, and A is said to be *precisely n-accessible* (precisely one-sided n-accessible, precisely left n-accessible), if it is not k-accessible (one-sided k-accessible, left k-accessible) in R for any positive integer $k < n$. A subring A is *accessible* (*one-sided accessible, left accessible*) in R, if it is n-accessible (one-sided n-accessible, left n-accessible) for some $n \in \mathbb{N}$. It is well known [20, Theorem 1], that if A is an idempotent accessible subring of R, then $A \lhd R$.

Recall that a radical class $\mathcal{S}$ is called *stable* (*left stable*), if for every $L < R$ ($L <_l R$), $\mathcal{S}(L) \subseteq \mathcal{S}(R)$. An example of a left stable radical class is the generalized nil radical $\mathcal{N}_g$; this is the upper radical determined by the class of domains. For every homomorphically closed class $\mathcal{N}$, there exists the smallest left stable radical $st(\mathcal{N})$ containing $\mathcal{N}$. Moreover,

$$st(\mathcal{N}) = \left\{ R \;\middle|\; \begin{array}{l} \text{every nonzero homomorphic image of } R \\ \text{contains a nonzero left accessible subring in } \mathcal{N} \end{array} \right\}.$$

For example, $\mathcal{N}_g = st(\{R \mid R^2 = 0\})$. Throughout the paper β denotes the prime radical. For details of radical theory the readers are referred to [14].

Given a ring R, we denote by R^0 the ring with zero-multiplication defined on the additive group of R. A ring with zero-multiplication is called a *zero ring*. By $\mathbb{Z}_p$ we denote the field of p elements, where p is a prime. For a given element a of R, we denote by $[a]$ the subring of R generated by a. By $\{R\}$ we denote the class of all isomorphic images of R.

We call a ring without nontrivial zero divisors a *domain*. A commutative domain with a unity is called an *integral domain*. Let R be an integral domain with the field of fractions K. We say that R is a *completely normal ring* if for any $x \in K$ and $0 \neq a \in R$, we have that $[x]a \subseteq R$ implies $x \in R$. The following are particular examples: noetherian integrally closed domains, unique factorization domains, Krull rings. For details of completely normal ring the readers are referred to [11].

2. The historical outline

The problem with the stabilization of Kurosh's chains appeared in 1966 in [22], where it was shown that the Kurosh's chain determined by a homomorphically closed nonempty class of rings stabilizes on the first infinite ordinal number ω_0. Moreover, the examples of classes for which Kurosh's chain stabilizes precisely at step 2 and 3 were also given. It was pointed out that, if $\mathcal{M}$ is a homomorphically closed class of rings, which contains all zero rings and which is hereditary, then $l(\mathcal{M}) = \mathcal{M}_2$ (Theorem 2). Furthermore, $l(\mathcal{M}) = \mathcal{M}_2$, if $\mathcal{M}$ is the class of all nilpotent rings. However, if $\mathcal{M}$ is the class of all homomorphic images of $\mathbb{Z}^0$, the Kurosh's chain determined by $\mathcal{M}$ stabilizes precisely at step 3.

The question of a class where the Kurosh's chain will stabilize at precisely the nth step for $n \geq 4$ was also presented. In the following this problem will be named the ADS-problem.

The year 1968 was considerably fruitful in works concerning the Kurosh's chain stabilization. In [16] Hoffman and Leavitt showed that if $\mathcal{M}$ is a homomorphically

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closed and hereditary class of rings, then for every ordinal α, $\mathcal{M}_\alpha$ is hereditary. In particular, $l(\mathcal{M})$ is a hereditary radical class.

In [7] Armendariz and Leavitt proved that if $\mathcal{M}$ is homomorphically closed and hereditary, then $l(\mathcal{M}) = \mathcal{M}_3$.

The first attempt to find the solution to the ADS-problem was presented by Heinicke, who in [15] for any $n \geq 2$ gave an example of a class for which the Kurosh's chain does not stabilize at step $n-1$. In the polynomial ring of two variables $\mathbb{Z}_p[x, t]$, where p is a prime number, Heinicke takes into consideration subrings

$$A_{n,p} = [x] + \mathbb{Z}_p[x, t]x^n, \quad n = 1, 2, \dots. \tag{2.1}$$

He shows that for the class $\mathcal{M}$ of all homomorphic images of $A_{n,p}$, where $n \geq 2$, we have $l(\mathcal{M}) \neq \mathcal{M}_{n-1}$. Making use of the given result, he next forms the class $\mathcal{M}$ consisting of all homomorphic images of $\bigoplus_{n=1}^{\infty} A_{n,p_n}$, where $p_1, p_2, \dots$ is an enumeration of the prime numbers and the A_{n,p_n} are of the form (2.1). For the class $\mathcal{M}$ the Kurosh's chain stabilizes precisely at ω_0. Although Heinicke's deductions concerning the Kurosh's chains are very complicated and technical, the fact that they were the first and general should be appreciated.

Question 2.1. Does Heinicke's example give a positive solution of the ADS-problem to natural numbers as well?

It can be shown (see Theorem 5.10(ii)) that exchanging $\mathbb{Z}_p[x, t]$ for $\mathbb{Z}_p[x]$ and taking into consideration the classes of homomorphic images of $A_n = [x^2] + \mathbb{Z}_p[x]x^{2n}$ for $n \geq 2$ gives the stabilization of the ring at precisely the step $n + 1$, and therefore, a full solution to the ADS-problem. It is, at the same time, the easiest notation of a class for which the Kurosh's chain stabilizes at exactly the step $n + 1$. The examples described by Heinicke overcame a formidable task because in 1968 the attainable subring mathematics was not developed, nor iterated maximal essential extensions of rings (see Definition 4.1). The techniques, used by Heinicke have been further developed, led to proofs of more facts. The idea of searching the solution of the ADS-problem in polynomial rings was pursued later by many other authors (for example, Beidar [9]).

Stewart obtained another quite interesting result in 1974. In [21] he showed that, if $\mathcal{M}$ is a nonempty homomorphically closed class of zero rings, then $l(\mathcal{M}) = \mathcal{M}_3$. In [4] the first author and Puczyłowski generalized Stewart's result and showed that if $\mathcal{M}$ is a class of M-nilpotent rings then $l(\mathcal{M}) = \mathcal{M}_3$ and $l(\mathcal{M})$ is left strong. In [13] Filipowicz and Puczyłowski showed that for a ring A, $l(A)$ is left strong if and only if $l(A) = l(\mathcal{N})$ for a class $\mathcal{N}$ of zero rings.

In [17] there is another example of a class for which the Kurosh's chain stabilizes at step 3. Namely, $l(\mathcal{M}) = \mathcal{M}_3$, if every ring A in $\mathcal{M}$ satisfies one of the following conditions: A is nilpotent or $A = A^2$ or $\mathcal{M}$ contains all the non-nilpotent ideals of A.

For the final solution of the ADS-problem for the chains we had to wait until 1982. Beidar in [8] took into consideration subrings of the ring of Gaussian integers $\mathbb{Z}[i]$. Let

p be a prime number of the form $p = 4k + 3$,

$$A_n = [p] + \mathbb{Z}[i]p^n \tag{2.2}$$

and $\mathcal{M}$ be the class of rings, which consists of all nilpotent rings, all homomorphic images of the ring A_n and all finite commutative rings. Beidar proved that the Kurosh's chain of $\mathcal{M}$ terminates precisely at the step $n + 1$. An adequate magnification of the class of homomorphic images of A_n facilitated the presentation of the stabilization at the step $n + 1$ without disturbing the condition $\mathcal{M}_n \neq \mathcal{M}_{n+1}$ at the same time. Although the article contains ingenious ideas, it is very technical. In [14] you can also find the proof of Beidar's theorem, but it is still complicated. The techniques used in [8] were developed in order to be used for algebras. Beidar has also written many works among which [9], completed in 1988, should be particularly valued. In this paper he considered the subrings $A_{n,f} = f \cdot ([x^r] + K[x]x^{rn})$ of the ring $K[x]$, where K is a field, $r \geq 2$ and $f \in K[x]$ is a polynomial with zero free term. Therefore, for some complicated class $\mathcal{M}$ containing $A_{n,f}$, $l(\mathcal{M})$ is hereditary and the Kurosh's chain stabilizes precisely at the nth step.

In [10] Beidar defines the term iterated maximal essential extension of a ring and points out its relation with the solution of the ADS-problem. The study of rings which have an iterated maximal extension was undertaken in 2002 by the first author in [2]. The findings were quite interesting (see Propositions 4.2–4.4) thanks to them, the solution to the ADS-problem has been considerably shortened and the proofs have become more clear (see Theorem 5.1).

Another solution to the ADS-problem was given by Guo Jinyun *et al.* in 1987 in [17]. He took into consideration subrings $A_n = [x^n] + \mathbb{Z}_p[x]x^{n^2}$ of $\mathbb{Z}_p[x]$. Let $\mathcal{K} = \{A_n, \mathbb{Z}^0\}$ and $\mathcal{P}$ be a class of rings which can homomorphically map to a subring of the ideal $x\mathbb{Z}_p[x]$ which contains A_n. Therefore, for the class $\mathcal{M} = \mathcal{K} \cup \{l(\mathcal{K}) \setminus \mathcal{P}\}$ we have $l(\mathcal{M}) = \mathcal{M}_{n+1} \neq \mathcal{M}_n$.

The first full solution to the ADS-problem for algebras was presented in 1984 in [18] by Lvov and Sidorov. They considered a completely normal F-algebra R such that the transcendental degree is 1. For $a \in R$ such that $\dim_F(R/Ra) \geq 2$ let the ring $A_n = F[a] + Ra^n$ be given. If $\mathcal{M}$ is a class of rings consisting of homomorphic images of the ring A_n, algebraic commutative F-algebras and nilpotent F-algebras, then $l(\mathcal{M}) = \mathcal{M}_n \neq \mathcal{M}_{n-1}$. Lvov and Sidorov discovered the meaning of completely normal rings (see Proposition 4.4).

Watter's preprint [23] should also be taken into consideration for the class that he constructed is relatively simple. It consists of nilpotent F-algebras and algebras isomorphic with $A_n = xF[[x]] + K[[x]]x^n$, where K is a proper field extension of a field F. The Kurosh's chain of such class of algebras stabilizes precisely at the step $n + 1$.

The research of the ADS-problem for the one-sided Kurosh's chains was initiated by Divinsky *et al.* in [12]. They proved, among other things, that if $\mathcal{M}$ is a nonempty homomorphically closed hereditary class of rings and contains all zero rings then $ls(\mathcal{M}) = \mathcal{M}^{\omega_0}$. It was also shown that for every limit ordinal α and for every class $\mathcal{N}$, $\mathcal{N}^\alpha$ is a radical class. Puczyłowski in [19] has concentrated on the left Kurosh's chains and showed that for every $\alpha \geq \omega_0$ and every class $\mathcal{M}$, $\mathcal{M}_l^\alpha$ is a radical. Moreover, if $\mathcal{M}$ is a hereditary radical then $ls_l(\mathcal{M}) = \mathcal{M}_l^2$.

Recently, we addressed the subject of the stabilization of the one-sided Kurosh's chains in [6]. The level of our research can be compared to Heinicke's results. We have constructed a class for which the one-sided Kurosh's chain does not stabilize at any finite step [6, Theorem 3.6]. We have constructed a hereditary radical class for which the one-sided Kurosh's chain does not stabilize at step 2. In [6] we also raise questions.

Question 2.2. Is it true that for every hereditary radical class $\mathcal{S}$, $ls(\mathcal{S}) = \mathcal{S}^3$?

Question 2.3. Let R be a simple domain with unity which is not a division ring. Then there exists $0 \neq a \in R$ such that $Ra \neq R$. Let n be a positive integer and let $A = Ra^n + [a]$. Let $\mathcal{S} = l(\{A\} \cup \beta)$. Does the Kurosh's chain $\{\mathcal{S}^\alpha\}$ stabilize at some finite step?

The research on the one-sided Kurosh's chains has been hindered by the fact that there is no equivalent to an iterated maximal essential extension of a ring for one-sided accessible subrings. There is a lack of methods and techniques which would facilitate a classic research of the Kurosh's chains.

In 1990 the first author and Puczyłowski in [4] took into consideration the one-sided Kurosh's chains and proved, among other things, the following theorem.

Theorem 2.4 [4, 6]. *If $\mathcal{S}$ is a stable radical class containing $\mathcal{N}_g$ then for $\mathcal{N} = \mathcal{S} \cup \mathcal{P}$, where $\mathcal{P}$ is a homomorphically closed class of commutative rings, $\mathcal{N}_\alpha = \mathcal{N}_l^\alpha = \mathcal{N}^\alpha$ for every ordinal α, and $l(\mathcal{N})$ is stable. Moreover, if $\mathcal{N}$ is hereditary then $l(\mathcal{N})$ is hereditary.*

The ADS-problem was solved by means of Theorem 2.4 and Beidar's example.

Theorem 2.5 [4, Theorem 2]. *If $\mathcal{N} = \mathcal{N}_g \cup \mathcal{T} \cup \{A_n\}$, where $\mathcal{T}$ is the class of rings with torsion additive groups and $\{A_n\}$ is the class of all isomorphic images of A_n for any $n \geq 1$, where A_n is of the form (2.2), then $\mathcal{N}_1 = \mathcal{N}_l^1 \subsetneq \mathcal{N}_2 = \mathcal{N}_l^2 \subsetneq \cdots \subsetneq \mathcal{N}_{n+1} = \mathcal{N}_l^{n+1} = \mathcal{N}_{n+2} = \mathcal{N}_l^{n+2}$.*

It is, at the same time, an answer to Question 6 from [19].

In 1997 in [5] the first author and Puczyłowski constructed new general examples showing that the Kurosh's construction of the lower radical can terminate at any finite or the first infinite ordinal.

Theorem 2.6 [5, Theorem 4.1]. *Suppose that P is a commutative noetherian integrally closed domain with unity, $\mathcal{S}$ is a radical class such that $\beta \subseteq \mathcal{S} \subseteq \mathcal{N}_g$ and $\mathcal{M} = \mathcal{S} \cup \mathcal{P} \cup \{\mathcal{A}\}$, where $\mathcal{P}$ is the class of all proper homomorphic images of all accessible subrings of P and $\mathcal{A}$ is a nonempty set of precisely n-accessible subrings of P. Then $\mathcal{M}_n \neq \mathcal{M}_{n+1} = l(\mathcal{M})$.*

The generality facilitates clearer and more visible arguments which avoid particular calculations, as well as the construction of radicals which satisfy some extra properties.

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3. The construction of precisely n-accessible subrings

The following proposition collects some well-known properties of the classes $\mathcal{N}_\alpha$.

PROPOSITION 3.1.

(i) *The classes $\mathcal{N}_\alpha$ are homomorphically closed for all α.*
(ii) *$R \in l(\mathcal{N})$ if and only if every nonzero homomorphic image of R contains a nonzero accessible subring in $\mathcal{N}$.*
(iii) *$R \in \mathcal{N}_{n+1}$, where n is an integer greater than 1, if and only if every nonzero homomorphic image of R contains a nonzero n-accessible subring in $\mathcal{N}$.*

Let n be a positive integer and let $\mathcal{M}$ be a homomorphically closed class of rings such that $\mathcal{M}_{n+1} \neq \mathcal{M}_n$. Then by Proposition 3.1(iii), there is a nonzero ring $R \in \mathcal{M}_{n+1}$ which does not contain any nonzero $(n-1)$-accessible subring in $\mathcal{M}$. But R contains any nonzero n-accessible subring $A \in \mathcal{M}$. Therefore, A is a precisely n-accessible subring of R. We can see then the first technical difficulties which arise in connection with solving the ADS-problem.

D1. How should n-accessible subrings in a ring R be constructed?
D2. When does a precisely n-accessible subring for fixed $n \in \mathbb{N}$ exist in a ring R?
D3. Which rings does contain R for every natural number n a nonzero precisely n-accessible subring?

In order to answer the above questions, it is relevant to obtain the most general classes of rings and subrings.

Heinicke should be given recognition for the first relevant ideas concerning the questions above for he had proved that subrings of the form (2.1) are precisely n-accessible in the ring $\mathbb{Z}_p[x, t]$. Similar ideas then have been continued and developed in other works. Beidar in [8] considered precisely n-accessible subrings of the ring of Gaussian integers $\mathbb{Z}[i]$ of the form (2.2). It should be emphasized that the search for the solution of the problems **D1**–**D3** has been conducted mainly in a class of commutative rings. It is not difficult to notice that the following proposition holds.

PROPOSITION 3.2. *Let A be a subring of a commutative ring R. Then:*

(i) *A is n-accessible in R if and only if $RA^n \subseteq A$;*
(ii) *$A + RA^n$ is the smallest n-accessible subring in R containing A;*
(iii) *A is precisely n-accessible in R if and only if $RA^n \subseteq A$ and $RA^{n-1} \nsubseteq A$.*

Proposition 3.2 gives an answer to the question **D1** in a class of commutative rings. In addition, for $A = [a]$ and $n \in \mathbb{N}$ we have $A + RA^n = [a] + Ra^n$.

Let us recall that a ring R is filial if every accessible subring of R is an ideal of R. Therefore, if in a ring R for each positive integer n there is a precisely n-accessible subring, then R is not filial. The next proposition shows that in the class of integral domains the reverse implication holds. This unexpected result has been proven by the first author and Puczyłowski in [5]; it provides satisfactory answers to questions **D2–D3**.

Proposition 3.3 [5]. *Suppose that R is an integral domain, $0 \neq a \in R$ and $R \neq [1] + Ra$. Then:*

(i) *if $Ra \cap [1] = 0$, then $[a] + Ra^n$ is a precisely n-accessible subring of R for every $n \in \mathbb{N}$;*

(ii) *if $Ra \cap [1] \neq 0$, then $Ra \cap [1] = b[1]$ for some $b \in [1]$ and $[b] + Rb^n$ is a precisely n-accessible subring of R for every $n \in \mathbb{N}$.*

By [3, Proposition 2.2] and Proposition 3.3 we can easily obtain the following proposition.

Proposition 3.4. *Suppose that R is an integral domain. The following conditions are equivalent:*

(i) *R is not filial;*

(ii) *there is a nonzero element $a \in R$ such that $R \neq [1] + Ra$;*

(iii) *for every $n \in \mathbb{N}$ there is a precisely n-accessible subring of R;*

(iv) *in every nonzero ideal of R there is a $b \in R$ such that $[b] + Rb^n$ is a precisely n-accessible subring of R for every $n \in \mathbb{N}$.*

Filial integral domains were completely classified in [3]. According to this classification, filial integral domains are Euclidean domains, so they are principal ideal domains. Moreover, if they are of positive characteristic, they are fields. Furthermore, if R is a filial integral domain of characteristic zero which is not a field, then there exists an embedding of R into the ring of p-adic integers. Applying Proposition 3.3, one can easily find an integral domain R and $a \in R$ such that $[a] + Ra^n$ is a precisely n-accessible subring of R for every $n \in \mathbb{N}$.

Example 3.5. Let R be an integral domain of positive characteristic p and let $a \in R$ be a reducible element. Then there are nonzero nonirrevertible elements $x, y \in R$ such that $a = xy$. By assumption on the characteristic of R, $Rx \cap [1] = 0$, and hence, in particular, $Ra \cap [1] = 0$. If $x \in [1] + Ra$, then there are $k \in [1]$ and $r \in R$ such that $x = k + rxy$. Hence $k \in Rx \cap [1]$, so $k = 0$. Then $1 = ry$, which is a contradiction. Therefore, $R \neq [1] + Ra$. By Proposition 3.3, $[a] + Ra^n$ is a precisely n-accessible subring of R for every $n \in \mathbb{N}$. In particular, $[x^2] + \mathbb{Z}_p[x]x^{2n}$ is a precisely n-accessible subring of $\mathbb{Z}_p[x]$ for every $n \in \mathbb{N}$.

In [1] the first author proved the following surprising result.

Theorem 3.6 [1, Theorem 2]. *If A is a ring such that the group $(A/A^2)^+$ is not divisible or is not torsion, then for every natural number n there is a ring R, in which A is a precisely n-accessible subring.*

4. Iterated maximal essential extensions of rings

We will discuss now another technical difficulty connected with the verification of the stabilization of the Kurosh's chains of $\mathcal{M}$ at precisely the step $n + 1$. Let us notice that then there is a ring $R \in \mathcal{M}_{n+1}$ with a nonzero precisely n-accessible subring A such that R has no nonzero $(n-1)$-accessible subring in M. In particular, if B is a nonzero subring accessible in R, and B is a homomorphic image of A, then B is not $(n-1)$-accessible in R. The first who managed to succeed in this laborious task was Heinicke in [15]. Later, Beidar succeeded in the same task, which helped him to find a final solution to the ADS-problem. During his deductions Beidar discovered a useful tool: the iterated maximal essential extension of a ring.

We say that a subring A of a ring R is *essential* in R if, for every nonzero ideal I of R, we have $A \cap I \neq 0$.

Definition 4.1 [10, Beidar]. We say that a ring R is an *iterated maximal essential extension* of a ring A and we write $R = IME(A)$ if A is an essential accessible subring of R and, for every ring S in which A is accessible, there exists a homomorphism of S into R which is the identity map on A.

The next propositions give some properties of IME.

Proposition 4.2 [2, Corollary 2.6]. *Assume A is a semiprime ring for which there exists $R = IME(A)$. If A is n-accessible in R, then for every semiprime ring S in which A is accessible, A is n-accessible in S.*

Proposition 4.3 [2, Proposition 3.4]. *Let R be a ring such that $R = IME(I)$ for every nonzero ideal I of R. Let A and B be nonzero accessible isomorphic subrings of R. If A is precisely n-accessible in R, then so is B.*

Proposition 4.4 [2, Theorem 3.2, Proposition 3.3]. *If R is a nonzero commutative ring then the following conditions are equivalent:*

(i) *$R = IME(I)$ for every nonzero ideal I of R;*
(ii) *$R = IME(A)$ for every nonzero accessible subring A of R;*
(iii) *R is a completely normal ring.*

Now we prove the following proposition.

Proposition 4.5. *Let R be a ring such that $R = IME(I)$ for every nonzero ideal I of R. Then the following conditions are equivalent:*

(i) *if $f\colon A \to B$ is a surjective homomorphism of nonzero accessible subrings of R, then f is an isomorphism;*
(ii) *if $f\colon A \to B$ is a surjective homomorphism of nonzero ideals of R, then f is an isomorphism;*
(iii) *if $0 \neq I \lhd R$ and $g\colon R \to R$ is a homomorphism such that* $\operatorname{Ker} g \neq 0$, *then $I \not\subseteq g(R)$.*

Proof. By assumption, $R = IME(I)$ for every nonzero ideal I of R. By [2, Theorem 3.1], R is prime and every nonzero accessible subring of R is an essential subring of R. Moreover $R = IME(A)$ for every nonzero accessible subring A of R.

(i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). Let $g\colon R \to R$ be a homomorphism of rings such that $\operatorname{Ker} g \neq 0$. Assume that there exists a nonzero ideal I of R such that $I \subseteq g(R)$. Then $I \lhd g(R)$, and thus there

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exists $J \triangleleft R$ such that $g(J) = I$. But $I \neq 0$, so $J \nsubseteq \mathrm{Ker}(g)$. Moreover, $J \cap \mathrm{Ker}(g) \neq 0$, then taking $A = J$, $B = I$ and $f = g|_J$, we obtain a contradiction.

(iii) $\Rightarrow$ (i). Suppose there are nonzero accessible subrings A and B of R and there is a surjective homomorphism of rings $f: A \to B$, which is not an isomorphism. Then $\mathrm{Ker}(f) \neq 0$. Moreover, each nonzero accessible subring of R is a prime ring. There exists $m \in \mathbb{N}$ such that $A_R^m \subseteq A$ and $A_R^m \nsubseteq \mathrm{Ker}(f)$. Otherwise, $(A/\mathrm{Ker}(f))^m = 0$, which contradicts the primeness of $A/\mathrm{Ker}(f) \cong B$. Next, $A_R^m \triangleleft R$, so $A_R^m \triangleleft A$, whence $f(A_R^m) \triangleleft B$. Therefore, $f(A_R^m)$ is a nonzero accessible subring of R and $A_R^m \cap \mathrm{Ker}(f) \neq 0$. We can therefore assume $A \triangleleft R$. Then $\mathrm{Ker}(f) \triangleleft A \triangleleft R$ and $A/\mathrm{Ker}(f)$ is semiprime, so $\mathrm{Ker}(f) \triangleleft R$. Hence $B \cong A/\mathrm{Ker}(f) \triangleleft R/\mathrm{Ker}(f)$. Thus there exists an isomorphism of rings $h: A/\mathrm{Ker}(f) \to B$. But $R = IME(B)$, so there exists a homomorphism $\varphi: R/\mathrm{Ker}(f) \to R$ such that $\varphi(a + \mathrm{Ker}(f)) = h(a + \mathrm{Ker}(f))$ for $a \in A$. Let $\pi: R \to R/\mathrm{Ker}(f)$ be the natural surjective homomorphism of rings and let $g = \varphi \circ \pi$. Then g is an endomorphism of R and $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(g)$, and therefore $\mathrm{Ker}(g) \neq 0$ and $B \subseteq g(R)$. But $B_R^n \subseteq B$ for some $n \in \mathbb{N}$, so $g(R)$ contains a nonzero ideal of R, which is a contradiction. □

Question 4.6. If the ring R from Proposition 4.4 is a completely normal ring, can the condition (iii) be simplified ?

Question 4.7. If the ring R from Proposition 4.4 is completely normal, is (ii) equivalent to the condition

(iv) Every surjective endomorphism $f: R \to R$ is an automorphism of R?

Examples of rings R that satisfy the condition (i) of Proposition 4.5:

- (i) integral domains R such that R^+ is torsion-free and $(R/I)^+$ is torsion for every $0 \neq I \triangleleft R$ (see [8]);
- (ii) commutative domains R such that R/I is nilpotent for every $0 \neq I \triangleleft R$ (see [23]);
- (iii) noetherian integrally closed domains (see [5]);
- (iv) integral domains R with $K \neq R$ the field of fractions of positive characteristic p such that the transcendence degree is 1 ($F = \mathbb{Z}/(p)$) (see [18]).

5. Main results

Theorem 5.1. *Suppose that P is a nonfilial completely normal ring such that if $f: A \to B$ is a surjective homomorphism of nonzero accessible subrings of P, then f is an isomorphism; $\mathcal{P}$ is the class of all proper homomorphic images of all accessible subrings of P and $\mathcal{S} = st(\mathcal{P} \cup \{R | R^2 = 0\})$. Then, for every $n \in \mathbb{N}$ the set $\mathcal{P}(n)$ of all precisely n-accessible subrings of P is nonempty. Moreover, if $\emptyset \neq \mathcal{A}(n) \subseteq \mathcal{P}(n)$ and $\mathcal{N}(n) = \mathcal{S} \cup \{\mathcal{A}(n)\}$, then $\mathcal{N}(n)_n = \mathcal{N}(n)^n \neq \mathcal{N}(n)_{n+1} = \mathcal{N}(n)^{n+1} = l(\mathcal{N}(n)) = ls(\mathcal{N}(n)) = st(\mathcal{N}(n))$. If $\mathcal{A}(n) = \mathcal{P}(n)$, then the radical class $l(\mathcal{N}(n))$ is hereditary.*

Proof. Applying Proposition 3.3, we get that $\mathcal{P}(n) \neq \emptyset$ for every $n \in \mathbb{N}$. By Proposition 4.4, $P = IME(A)$ for every nonzero accessible subring A of P. Moreover, by Theorem 2.4, $\mathcal{N}(n)_\alpha = \mathcal{N}(n)^\alpha$ for every ordinal α, and $l(\mathcal{N}(n))$ is left stable.

Clearly, every nonzero homomorphic image of P contains a nonzero accessible subring in $\mathcal{N}(n)$. Hence, $P \in l(\mathcal{N}(n))$. By the assumptions and Proposition 4.5, $\mathcal{S}(P) = 0$ and by Proposition 4.3, no nonzero $(n-1)$-accessible subrings of P is in $\{\mathcal{A}(n)\}$. Consequently, $P \in l(\mathcal{N}(n)) \setminus \mathcal{N}(n)_n$.

We show that $l(\mathcal{N}(n)) = \mathcal{N}(n)_{n+1}$ or, equivalently, that every nonzero ring $R \in l(\mathcal{N}(n))$ contains a nonzero n-accessible subring in $\mathcal{N}(n)$. This is obvious if $\mathcal{S}(R) \neq 0$. Assume that $\mathcal{S}(R) = 0$. Then R is semiprime. Since $R \in l(\mathcal{N}(n))$, R contains a nonzero accessible subring $D \in \mathcal{N}(n)$. Obviously, $D \notin \mathcal{S}$. Therefore, $D \in \{\mathcal{A}(n)\}$, and by Proposition 4.2, D is a n-accessible subring of R.

Finally, suppose $\mathcal{A}(n) = \mathcal{P}(n)$. Let $\mathcal{A}$ be the class of all homomorphic images of all accessible subrings of P. Then, by Theorem 2.4 the radical class $\mathcal{T} = st(\mathcal{A} \cup \{R | R^2 = 0\})$ is hereditary. We show that $l(\mathcal{N}(n)) = \mathcal{T}$ for every $n \in \mathbb{N}$. It suffices to show that every nonzero accessible subring D of P is in $l(\mathcal{N}(n))$. By Proposition 3.2, there is a positive integer m such that $PD^m \subseteq D$. Since P is nonfilial, there is $0 \neq a \in P$ such that $P \neq [1] + Pa$. Let $0 \neq s \in D$. Then $0 \neq b = as^m \in D \cap Pa$. Since $P \neq [1] + Pa$, $P \neq [1] + Pb$. Hence, by Proposition 3.3, there is $c \in Pb$, such that $C = [c] + Pc^n \in \mathcal{A}(n)$. Obviously, $Pc \subseteq D$, so $C \subseteq D$. Hence, $0 \neq C \subseteq (l(\mathcal{N}(n)))(D)$. Every proper homomorphic image of D is in $\mathcal{P}$, so $D/(l(\mathcal{N}(n)))(D) \in \mathcal{P}$. Consequently, $D \in l(\mathcal{N}(n))$. The result follows. □

Theorem 5.2. *Let R be a ring such that $R = IME(J)$ for every nonzero $J \lhd R$. Moreover, let for all nonzero ideals B and C of R, every surjective homomorphism $f\colon B \to C$ be an isomorphism. Let $\mathcal{S}$ be the lower radical determined by the class of all zero-rings and the class of all proper homomorphic images of all nonzero accessible subrings of R. Let $\mathcal{A}$ be a nonempty family of precisely n-accessible subrings of R for some $n \in \mathbb{N}$ and $\mathcal{M} = \{\mathcal{A}\} \cup \mathcal{S}$. Then $l(\mathcal{M}) = \mathcal{M}_{n+1}$ and $R \in l(\mathcal{M}) \setminus \mathcal{M}_n$.*

Proof. By assumption, every proper homomorphic image of R is in $\mathcal{M}$ and by [2, Theorem 3.1], R is prime. Moreover, the family $\mathcal{A}$ is nonempty, so in R there is a nonzero n-accessible subring, which is in $\mathcal{M}$. Thus $R \in \mathcal{M}_{n+1}$, and hence $R \in l(\mathcal{M})$.

Suppose $R \in \mathcal{M}_n$. By assumption, $\mathcal{S}(R) = 0$. Therefore, there is a nonzero subring D, which is $(n-1)$-accessible in R and $D \simeq A$ for some $A \in \mathcal{A}$. But by Proposition 4.4, $R = IME(T)$ for every nonzero accessible subring T of R, so by Proposition 4.3, A is an $(n-1)$-accessible subring of R, which is a contradiction. Therefore, $R \notin \mathcal{M}_n$.

It remains to show that every nonzero ring $P \in l(\mathcal{M})$ has a nonzero n-accessible subring in $\mathcal{M}$. This is clear if $\mathcal{S}(P) \neq 0$. Therefore, let $\mathcal{S}(P) = 0$. Then $\beta(P) = 0$. Since $P \in l(\mathcal{M})$, there is a nonzero accessible subring $D \in \mathcal{M}$ of P. But $\mathcal{S}(P) = 0$, so $D \simeq A$ for some $A \in \mathcal{A}$. Then by Proposition 4.2, D is n-accessible in P. □

Question 5.3. Does a noncommutative ring satisfying the assumptions of Theorem 5.2 exist?

Theorem 5.4. *Let A be a semiprime ring such that $pA = 0$ for some prime number p and let A be a precisely n-accessible subring of a ring R for some positive integer $n \geq 2$, and $R = IME(A)$. If A has no semiprime proper homomorphic image, or every*

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proper semiprime homomorphic image of A is idempotent, and if $\mathcal{M}$ is the class of all homomorphic images of A, then $l(\mathcal{M}) = \mathcal{M}_{n+1}$.

Proof. Let $A = A^2$. The subring A is accessible in R, whence $A \triangleleft R$, which contradicts $n \geq 2$. Therefore, $A \neq A^2$ and $A/A^2 \neq 0$. But $pA = 0$, and thus A/A^2 is a nonzero $\mathbb{Z}_p$-algebra with zero-multiplication. So A/A^2 can be homomorphically mapped on $\mathbb{Z}_p^0$. Hence $\mathbb{Z}_p^0 \in \mathcal{M}$. Let $\mathcal{T}_p$ be the class of all rings such that their additive group is a p-group. Then $\mathcal{T}_p$ is a radical class and $A \in \mathcal{T}_p$, so $l(\mathcal{M}) \subseteq \mathcal{T}_p$. It remains to show that every nonzero ring $S \in l(\mathcal{M})$ has a nonzero n-accessible subring in $\mathcal{M}$. If $\beta(S) \neq 0$, then there is a nonzero ideal I of S such that $I^2 = 0$. But I^+ is a p-group, because $S \in \mathcal{T}_p$, therefore, there is a nonzero $c \in I$ such that $pc = 0$. Then $[c] \simeq \mathbb{Z}_p^0$, whence $[c] \in \mathcal{M}$ and $[c] \triangleleft I \triangleleft S$. But $n \geq 2$, so $[c]$ is an n-accessible subring of S. Now let $\beta(S) = 0$. Since $S \in l(\mathcal{M})$, there is a nonzero subring $D \in \mathcal{M}$ accessible in S. If $D \simeq A$, then by Proposition 4.2, D is n-accessible in S. Suppose that D is a proper homomorphic image of A. Since D is an accessible subring of the semiprime ring S, it follows that D is a semiprime ring. Thus, by assumptions $D = D^2$, and hence $D \triangleleft S$, so D is an n-accessible subring of S. □

Question 5.5. Does a noncommutative ring satisfying the assumptions of Theorem 5.4 exist?

In [6, 19] some questions, which were recalled during the workshop Radicals of Rings and Related Topics [24], are still open for discussion. They can be formulated as the following questions.

Question 5.6. Is for every class $\mathcal{N}$, $ls\mathcal{N} = \mathcal{N}^{\omega_0}$?

Question 5.7. Does, for every natural number n, a radical class $\mathcal{N}$ exist such that $ls\mathcal{N} = \mathcal{N}^{n+1} \neq \mathcal{N}^n$?

Question 5.8. Does a radical class $\mathcal{N}$ exist such that $ls\mathcal{N} = \mathcal{N}^{\omega_0} \neq \mathcal{N}^n$ for every natural number n?

Question 5.9. Does a radical class $\mathcal{N}$ exist such that $ls\mathcal{N} \neq \mathcal{N}^\alpha$ for every ordinal α?

Theorem 5.10. *For every positive integer n there is a ring A such that the class $\mathcal{M}$ of all homomorphic images of A satisfies the condition $l(\mathcal{M}) = \mathcal{M}_{n+1} \neq \mathcal{M}_n$. Namely:*

(i) *for $n = 1$, A is a nonzero idempotent ring;*

(ii) *for $n \geq 2$, $A = [x^2] + \mathbb{Z}_p[x]x^{2n}$, where p is a prime number.*

Proof. (i) By assumption, every ring in $\mathcal{M}$ is idempotent. Let us take any $R \in l(\mathcal{M})$ and let R' be a nonzero homomorphic image of R. Then by Proposition 3.1(ii), there is a nonzero $B \in \mathcal{M}$ such that B is an accessible subring of R'. But $B = B^2$, so $B \triangleleft R'$. Hence $R \in \mathcal{M}_2$. Moreover, for any cardinal number $\alpha > |A|$, a ring P, which is a direct sum of α copies of A, has cardinality greater than that of A, so P is not in $\mathcal{M}$ but $P \in l(\mathcal{M})$.

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(ii) Let $n \geq 2$ and let p be a prime number. Since $\mathbb{Z}_p[x]$ is a completely normal ring, by [2, Proposition 2.18], $\mathbb{Z}_p[x] = IME(T)$ for every nonzero accessible subring T of $\mathbb{Z}_p[x]$. By Example 3.5, the subring $A = [x^2] + \mathbb{Z}_p[x]x^{2n}$ is precisely n-accessible in $\mathbb{Z}_p[x]$. Let $\phi\colon \mathbb{Z}_p[x] \to \mathbb{Z}_p$ be given by the formula $\phi(w) = w(1)$. Obviously, ϕ is a surjective homomorphism of rings. But $\phi(x^2) = 1$, so $\phi(A) = \mathbb{Z}_p$. Hence $\mathbb{Z}_p \in \mathcal{M}$. Moreover, $J = x^2[x^2] + \mathbb{Z}_p[x]x^{2n} \triangleleft A$ and $A/J \simeq \mathbb{Z}_p^0$, so consequently $\mathbb{Z}_p^0 \in \mathcal{M}$. Further, $x^2\mathbb{Z}_p[x]$ is an ideal of $\mathbb{Z}_p[x]$ generated by A, so by [22, Lemma 2], $x^2\mathbb{Z}_p[x] \in l(\mathcal{M})$. But $x\mathbb{Z}_p[x]/x^2\mathbb{Z}_p[x] \simeq \mathbb{Z}_p^0$, and therefore, $x\mathbb{Z}_p[x] \in l(\mathcal{M})$. Moreover, $\mathbb{Z}_p[x]/x\mathbb{Z}_p[x] \simeq \mathbb{Z}_p$, from where $\mathbb{Z}_p[x] \in l(\mathcal{M})$. But $\mathbb{Z}_p[x]$ is a noetherian ring, so by [5, Corollary 3.7] for all nonzero accessible subrings B and C of $\mathbb{Z}_p[x]$ every homomorphism of B onto C is an isomorphism. Therefore, by Theorem 5.2, $\mathbb{Z}_p[x] \notin \mathcal{M}_n$. It is easy to see that every proper homomorphic image of A is a finite ring, because it is a homomorphic image of a ring of the form A/I for some nonzero ideal I of $\mathbb{Z}_p[x]$. But every finite semiprime ring is idempotent, so by Theorem 5.4, $l(\mathcal{M}) = \mathcal{M}_{n+1}$. □

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