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THE QUADRATIC FORM IN NINE PRIME VARIABLES

LILU ZHAO

Abstract. Let $f(x_1, \ldots, x_n)$ be a regular indefinite integral quadratic form with $n \geqslant 9$, and let t be an integer. Denote by $\mathbb{U}_p$ the set of p-adic units in $\mathbb{Z}_p$. It is established that $f(x_1, \ldots, x_n) = t$ has solutions in primes if (i) there are positive real solutions, and (ii) there are local solutions in $\mathbb{U}_p$ for all prime p.

§1. Introduction

Let $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ be a symmetric integral matrix with $n \geqslant 4$. In other words,

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \tag{1.1}$$

with $a_{i,j} = a_{j,i} \in \mathbb{Z}$ for all $1 \leqslant i < j \leqslant n$. Let $f(x_1, \ldots, x_n)$ be the quadratic form defined as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j. \tag{1.2}$$

Let t be an integer. We call f regular if A is invertible. For regular indefinite quadratic forms with $n \geqslant 4$, the well-known Hasse principle asserts that $f(x_1, \ldots, x_n) = t$ has integer solutions if and only if $f(x_1, \ldots, x_n) = t$ has local solutions.

In this paper, we consider the equation $f(x_1, \ldots, x_n) = t$, where $x_1, \ldots, x_n$ are prime variables. It is expected that $f(x_1, \ldots, x_n) = t$ has solutions with $x_1, \ldots, x_n$ primes if there are suitable local solutions. The classical theorem of Hua [7] deals with diagonal quadratic forms in five prime variables. In particular, every sufficiently large integer, congruent to 5 modulo 24, can be represented as a sum of five squares of primes. Recently, Liu [9] handled a wide class of quadratic forms f with 10 or more

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prime variables. The general quadratic form in prime variables (or in dense sets) was recently investigated by Cook and Magyar [3], and by Keil [8]. In particular, Cook and Magyar [3] handled all regular quadratic forms in 21 or more prime variables, while the work of Keil [8] can deal with all regular quadratic forms in 17 or more variables. It involves only five prime variables for diagonal quadratic equation due to the effective mean value theorem. This is similar to the problem concerning Diophantine equations for cubic forms. The works of Baker [1], Vaughan [10, 11] and Wooley [13, 14] can deal with the diagonal cubic equation in seven variables. However, more variables are involved for general cubic forms. One can refer to the works of Heath-Brown [4, 5] and Hooley [6] for general cubic forms.

The purpose of this paper is to investigate general regular quadratic forms in nine or more prime variables. We define

$$N_{f,t}(X) = \sum_{\substack{1\leqslant x_1,\ldots,x_n\leqslant X \\ f(x_1,\ldots,x_n)=t}} \prod_{j=1}^{n} \Lambda(x_j),$$

where $\Lambda(\cdot)$ is the von Mangoldt function. Our main result is the following.

THEOREM 1.1. *Suppose that $f(x_1,\ldots,x_n)$ is a regular integral quadratic form with $n\geqslant 9$, and that $t\in\mathbb{Z}$. Let $\mathfrak{S}(f,t)$ and $\mathfrak{I}_{f,t}(X)$ be defined in (3.11) and (3.13), respectively. Suppose that K is an arbitrary large real number. Then we have*

$$N_{f,t}(X) = \mathfrak{S}(f,t)\mathfrak{I}_{f,t}(X) + O(X^{n-2}\log^{-K} X), \tag{1.3}$$

where the implied constant depends on f and K.

Denote by $\mathbb{P}$ the set of all prime numbers. For a prime $p\in\mathbb{P}$, we use $\mathbb{Z}_p$ to denote the ring of p-adic integers. Then we use $\mathbb{U}_p$ to denote the set of p-adic units in $\mathbb{Z}_p$. The general local to global conjecture of Bourgain–Gamburd–Sarnak [2] asserts that $f(x_1,\ldots,x_n)=t$ has prime solutions provided that there are local solutions in $\mathbb{U}_p$ for all $p\in\mathbb{P}$. Liu [9, Theorem 1.1] verified this conjecture for a wide class of regular indefinite integral quadratic forms with ten or more variables. Theorem 1.1 has the following consequence improving upon Liu [9, Theorem 1.1].

THEOREM 1.2. *Let $f(x_1,\ldots,x_n)$ be a regular indefinite integral quadratic form with $n\geqslant 9$, and let $t\in\mathbb{Z}$. Then $f(x_1,\ldots,x_n)=t$ has prime solutions if we have the following two conditions:*

(i) *there are real solutions in* $\mathbb{R}^+$*, and*
(ii) *there are local solutions in* $\mathbb{U}_p$ *for all prime* p*.*

We define $N^*_{f,t}(X)$ to be the number of prime solutions to $f(p_1, \ldots, p_n) = t$ with $1 \leqslant p_1, \ldots, p_n \leqslant X$. Suppose that f is regular with $n \geqslant 9$. Actually, in view of Theorem 1.1, one has $N^*_{f,t}(X) \gg_{f,t} X^{n-2} \log^{-n} X$ for sufficiently large X if the conditions (i) and (ii) in Theorem 1.2 hold.

Theorem 1.2 covers all regular indefinite integral quadratic forms in nine prime variables. The O-constant in the asymptotic formula (1.3) is independent of t. Therefore, Theorem 1.1 can be applied to definite quadratic forms. In particular, if $f(x_1, \ldots, x_n)$ is a positive definite integral quadratic form with $n \geqslant 9$, then there exist $r, q \in \mathbb{N}$ so that all sufficiently large natural numbers N, congruent to r modulo q, can be represented as $N = f(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are prime numbers.

The method in this paper can also be applied to refine Keil [8, Theorem 1.1]. In particular, one may obtain a variant of Keil [8, Theorem 1.1] for a wide class of quadratic forms in nine variables.

§2. Notations

As usual, we write $e(z)$ for $e^{2\pi i z}$. Throughout we assume that X is sufficiently large. Let $L = \log X$. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notations, while the implied constants may depend on the form f. Denote by $\phi(q)$ Euler's totient function.

For a set $\mathcal{S}$ in a field $\mathbb{F}$, we define

$$\mathcal{S}^n = \{(x_1, \ldots, x_n)^T : \ x_1, \ldots, x_n \in \mathcal{S}\}. \tag{2.1}$$

We use $M_{m,n}(\mathcal{S})$ to denote the set of m by n matrixes

$$M_{m,n}(\mathcal{S}) = \{(a_{i,j})_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n} : \ a_{i,j} \in \mathcal{S}\}, \tag{2.2}$$

and $GL_n(\mathcal{S})$ to denote the set of invertible matrixes of order n

$$GL_n(\mathcal{S}) = \{B \in M_{n,n}(\mathcal{S}) : \ B \text{ is invertible}\}, \tag{2.3}$$

respectively. We define the off-diagonal rank of A as

$$\mathrm{rank}_{\mathrm{off}}(A) = \max\{r : \ r \in R\}, \tag{2.4}$$

where

$$R = \{\mathrm{rank}(B) : \ B = (a_{i_k, j_l})_{1 \leqslant k,l \leqslant r} \text{ with } \{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_r\} = \emptyset\}.$$

In other words, $\operatorname{rank}_{\mathrm{off}}(A)$ is the maximal rank of a submatrix in A, which does not contain any diagonal entries. For $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{N}^n$, we write

$$\Lambda(\mathbf{x}) = \Lambda(x_1) \cdots \Lambda(x_n).$$

For $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{Z}^n$, we also use the notation $\mathcal{A}(\mathbf{x})$ to indicate that the argument $\mathcal{A}(x_j)$ holds for all $1 \leqslant j \leqslant s$. The meaning will be clear from the text. For example, we use $1 \leqslant \mathbf{x} \leqslant X$ and $|\mathbf{x}| \leqslant X$ to denote $1 \leqslant x_j \leqslant X$ for $1 \leqslant j \leqslant n$ and $|x_j| \leqslant X$ for $1 \leqslant j \leqslant n$, respectively.

In order to apply the circle method, we introduce the exponential sum

$$S(\alpha) = \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e(\alpha \mathbf{x}^T A \mathbf{x}), \tag{2.5}$$

where A is defined in (1.1). We define

$$\mathcal{M}(Q) = \bigcup_{1 \leqslant q \leqslant Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(q, a; Q), \tag{2.6}$$

where

$$\mathcal{M}(q, a; Q) = \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \leqslant \frac{Q}{qX^2} \right\}.$$

The intervals $\mathcal{M}(q, a; Q)$ are pairwise disjoint for $1 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$ provided that $Q \leqslant X/2$. For $Q \leqslant X/2$, we set

$$\mathfrak{m}(Q) = \mathcal{M}(2Q) \setminus \mathcal{M}(Q). \tag{2.7}$$

Now we introduce the major arcs defined as

$$\mathfrak{M} = \mathcal{M}(P) \quad \text{with } P = L^K, \tag{2.8}$$

where K is a sufficiently large constant throughout this paper. Then we define the minor arcs as

$$\mathfrak{m} = [X^{-1}, 1 + X^{-1}] \setminus \mathfrak{M}. \tag{2.9}$$

§3. The contribution from the major arcs

For $q \in \mathbb{N}$ and $(a, q) = 1$, we define

$$C(q, a) = \sum_{\substack{1 \leqslant \mathbf{h} \leqslant q \\ (\mathbf{h}, q)=1}} e\left(\mathbf{h}^T A \mathbf{h} \frac{a}{q} \right), \tag{3.1}$$

where A is given by (1.1). Throughout, we assume that f is connected to A given by (1.1) and (1.2). Let

$$B_{f,t}(q) = \frac{1}{\phi^n(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C(q,a) e\left(-\frac{at}{q}\right). \tag{3.2}$$

Concerning $B_{f,t}(q)$, we have the following multiplicative property.

LEMMA 3.1. *The arithmetic function $B_{f,t}(q)$ is multiplicative.*

Proof. The desired conclusion can be proved by changing variables. □

LEMMA 3.2. *Suppose that A is invertible. For any prime p, there exists $\gamma_p = \gamma_p(f,t)$ such that $B_{f,t}(p^k) = 0$ for all $k > \gamma_p$. Moreover, if $p \nmid 2\det(A)$, then we have $\gamma_p = 1$.*

Proof. Throughout this proof, we assume that $(a,p) = 1$. We first deal with the case $p \geqslant 3$. We claim that if

$$C(p^k,a) = p^{nj} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-j}\\(\mathbf{h},p)=1\\A\mathbf{h}\equiv\mathbf{0}(\mathrm{mod}\ p^j)}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{p^k}\right) \tag{3.3}$$

for some $j \leqslant (k-2)/2$, then

$$C(p^k,a) = p^{n(j+1)} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-j-1}\\(\mathbf{h},p)=1\\A\mathbf{h}\equiv\mathbf{0}(\mathrm{mod}\ p^{j+1})}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{p^k}\right). \tag{3.4}$$

Indeed, by changing variables, we obtain from (3.3) that

$$C(p^k,a) = p^{nj} \sum_{1\leqslant \mathbf{u}\leqslant p} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-j-1}\\(\mathbf{h},p)=1\\A(\mathbf{u}p^{k-j-1}+\mathbf{h})\equiv\mathbf{0}(\mathrm{mod}\ p^j)}} \times e\left((\mathbf{u}p^{k-j-1}+\mathbf{h})^T A(\mathbf{u}p^{k-j-1}+\mathbf{h})\frac{a}{p^k}\right).$$

It follows from $j \leqslant (k-2)/2$ that $j \leqslant k-j-1$ and $k \leqslant 2(k-j-1)$. Thus we deduce that

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$$C(p^k,a) = p^{nj} \sum_{1\leqslant \mathbf{u}\leqslant p} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-j-1}\\ (\mathbf{h},p)=1\\ A\mathbf{h}\equiv \mathbf{0}(\bmod\ p^j)}} e\left(2p^{k-j-1}\mathbf{u}^T A\mathbf{h}\frac{a}{p^k}\right) e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{p^k}\right)$$
$$= p^{n(j+1)} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-j-1}\\ (\mathbf{h},p)=1\\ A\mathbf{h}\equiv \mathbf{0}(\bmod\ p^{j+1})}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{p^k}\right).$$

This establishes the desired claim, and therefore we arrive at

$$C(p^k,a) = p^{ns} \sum_{\substack{1\leqslant \mathbf{h}\leqslant p^{k-s}\\ (\mathbf{h},p)=1\\ A\mathbf{h}\equiv \mathbf{0}(\bmod\ p^s)}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{p^k}\right), \tag{3.5}$$

where $s=\lfloor k/2\rfloor$. There exists $P\in GL_n(\mathbb{Z}_p)$ with $\det(P)=1$ such that $P^TAP=D=\operatorname{diag}\{d_1,\ldots,d_n\}$ with $d_1,\ldots,d_n\in\mathbb{Z}_p$. Note that A is invertible, one has $d_1\cdots d_n\neq 0$. In particular, we can choose $r\in\mathbb{N}$ such that $p^r\nmid d_j$ for all $1\leqslant j\leqslant n$. The condition $A\mathbf{h}\equiv\mathbf{0}(\bmod\ p^s)$ implies $DP\mathbf{h}\equiv\mathbf{0}(\bmod\ p^s)$. If $s\geqslant r$, then $P\mathbf{h}\equiv\mathbf{0}(\bmod\ p)$. So we obtain $\mathbf{h}\equiv\mathbf{0}(\bmod\ p)$, which is a contradiction to the condition $(\mathbf{h},p)=1$. Therefore, we conclude that

$$C(p^k,a)=0 \quad \text{for all } k\geqslant 2r. \tag{3.6}$$

Moreover, when $p\nmid 2\det(A)$, we can take $r=1$ in (3.6).

For $p=2$, the above argument is still valid with minor modifications. We now claim that if

$$C(2^k,a) = 2^{2nj} \sum_{\substack{1\leqslant \mathbf{h}\leqslant 2^{k-2j}\\ (\mathbf{h},2)=1\\ A\mathbf{h}\equiv \mathbf{0}(\bmod\ 2^j)}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{2^k}\right) \tag{3.7}$$

for some $j\leqslant(k-4)/4$, then

$$C(2^k,a) = 2^{2n(j+1)} \sum_{\substack{1\leqslant \mathbf{h}\leqslant 2^{k-2j-2}\\ (\mathbf{h},2)=1\\ A\mathbf{h}\equiv \mathbf{0}(\bmod\ 2^{j+1})}} e\left(\mathbf{h}^T A\mathbf{h}\frac{a}{2^k}\right). \tag{3.8}$$

This claim can be established by changing variables $\mathbf{h}=\mathbf{u}2^{k-2j-2}+\mathbf{v}$ with $\mathbf{u}(\bmod\ 2^2)$ and $\mathbf{v}(\bmod\ 2^{k-2j-2})$. The argument leading to (3.6) implies that

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there exists k_0 such that

$$C(2^k, a) = 0 \quad \text{for all } k \geqslant k_0. \tag{3.9}$$

The desired conclusion follows from (3.2), (3.6) and (3.9). □

LEMMA 3.3. *Let $B_{f,t}(q)$ be defined as (3.2). If A is invertible and $n \geqslant 5$, then*

$$B_{f,t}(q) \ll_{f,\varepsilon} q^{-3/2+\varepsilon}.$$

Proof. In view of Lemma 3.2, it suffices to prove

$$C(p, a) \ll_f p^{n-5/2} \tag{3.10}$$

for $p \nmid 2\det(A)$ and $(a, p) = 1$. Note that

$$\begin{aligned} C(p, a) = &\sum_{\substack{\mathbf{h} \in \mathbb{N}^n \\ 1 \leqslant \mathbf{h} \leqslant p}} e\left(\mathbf{h}^T A \mathbf{h} \frac{a}{p}\right) - \sum_{j=1}^{n} \sum_{\substack{\mathbf{h} \in \mathbb{N}^{n-1} \\ 1 \leqslant \mathbf{h} \leqslant p}} e\left(\mathbf{h}^T A_j \mathbf{h} \frac{a}{p}\right) \\ &+ \sum_{1 \leqslant i < j \leqslant n} \sum_{\substack{\mathbf{h} \in \mathbb{N}^{n-2} \\ 1 \leqslant \mathbf{h} \leqslant p}} e\left(\mathbf{h}^T A_{ij} \mathbf{h} \frac{a}{p}\right) + O(p^{n-3}), \end{aligned}$$

where A_j denotes the submatrix of A by deleting the jth row and jth column, and A_{ij} denotes the submatrix of A_j by deleting the ith row and ith column. For complete Gauss sums, we have

$$\sum_{\substack{\mathbf{h} \in \mathbb{N}^k \\ 1 \leqslant \mathbf{h} \leqslant p}} e\left(\mathbf{h}^T M \mathbf{h} \frac{a}{p}\right) \ll p^{k-\operatorname{rank}(M)/2},$$

where the implied constant depends on the square matrix M. The estimate (3.10) follows by observing that $\operatorname{rank}(A_j) \geqslant 3$ and $\operatorname{rank}(A_{ij}) \geqslant 1$. We complete the proof. □

Now we introduce the singular series $\mathfrak{S}(f, t)$ defined as

$$\mathfrak{S}(f, t) = \sum_{q=1}^{\infty} B_{f,t}(q), \tag{3.11}$$

where $B_{f,t}(q)$ is given by (3.2). From Lemmas 3.2 and 3.3, we conclude the following result.

LEMMA 3.4. *Suppose that A is invertible and $n \geqslant 5$. Then the singular series $\mathfrak{S}(f,t)$ is absolutely convergent, and*

$$\mathfrak{S}(f,t) = \prod_p \chi_p(f,t),$$

where the local densities $\chi_p(f,t)$ are defined as

$$\chi_p(f,t) = 1 + \sum_{m=1}^{\infty} B_{f,t}(p^m).$$

Moreover, if $f(x_1,\ldots,x_n) = t$ has local solutions in $\mathbb{U}_p$ for all prime p, then one has

$$\mathfrak{S}(f,t) \gg 1.$$

Proof. It suffices to explain $\mathfrak{S}(f,t) \gg 1$ provided that $f(x_1,\ldots,x_n) = t$ has local solutions in $\mathbb{U}_p$ for all prime p. Indeed, in view of Lemma 3.3, one has $\prod_{p \geqslant p_0} \chi_p(f,t) \gg 1$ for some p_0. When $p < p_0$, by Lemma 3.2, for some $\gamma = \gamma_p$ we have

$$\chi_p(f,t) = 1 + \sum_{m=1}^{\gamma} B_{f,t}(p^m) = \sum_{\substack{1 \leqslant \mathbf{h} \leqslant p^\gamma \\ (\mathbf{h},p)=1 \\ F(\mathbf{h}) \equiv t (\mathrm{mod}\ p^\gamma)}} 1.$$

Since $f(x_1,\ldots,x_n) = t$ has local solutions in $\mathbb{U}_p$, one has $\chi_p(f,t) > 0$. This concludes that $\prod_p \chi_p(f,t) \gg 1$. ∎

REMARK 3.5. We point out that in view of the proof of Lemmas 3.2–3.3, one has

$$B_{f,t}(2^k q_1 q_2) \ll_f 2^{-(\mathrm{rank}(A)/4)k} q_1^{-\mathrm{rank}(A)/2} q_2^{-\mathrm{rank}(A)/3},$$

where q_1 is square-free and $(2, q_1 q_2) = (q_1, q_2) = 1$. In particular, the singular series is absolutely convergent if $\mathrm{rank}(A) \geqslant 5$. Therefore, the condition that f is regular with $n \geqslant 9$ in our Theorem 1.1 can be replaced by $\mathrm{rank}(A) \geqslant 9$.

We define

$$I(\beta) = \int_{[0,X]^n} e(\beta \mathbf{x}^T A \mathbf{x}) d\mathbf{x}. \tag{3.12}$$

Since $I(\beta) \ll X^n(1+X^2|\beta|)^{-2}$ for $\operatorname{rank}(A) \geqslant 5$, we introduce the singular integral

$$\mathfrak{I}_{f,t}(X) = \int_{-\infty}^{\infty} I(\beta)e(-t\beta)d\beta, \tag{3.13}$$

where $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Note that $\mathfrak{I}_{f,t}(X) \gg_{f,t} X^{n-2}$ if $f(x_1, \ldots, x_n)$ is indefinite and $f(x_1, \ldots, x_n) = t$ has positive real solutions.

LEMMA 3.6. *Let $t \in \mathbb{Z}$, and let*

$$S(\alpha) = \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x})e(\alpha \mathbf{x}^T A\mathbf{x}),$$

where $A \in M_{n,n}(\mathbb{Z})$ is a symmetric matrix with $\operatorname{rank}(A) \geqslant 5$. *Then one has*

$$\int_{\mathfrak{M}} S(\alpha)e(-t\alpha)\, d\alpha = \mathfrak{S}(f,t)\mathfrak{I}_{f,t}(X) + O(X^{n-2}L^{-K/4}). \tag{3.14}$$

Proof. We write $f(\mathbf{x})$ for $\mathbf{x}^T A\mathbf{x}$. By the definition of $\mathfrak{M}$, one has

$$\int_{\mathfrak{M}} S(\alpha)e(-t\alpha)\, d\alpha = \sum_{q \leqslant P} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \int_{|\beta| \leqslant \frac{P}{qX^2}} \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e\left(f(\mathbf{x})\left(\frac{a}{q}+\beta\right)\right) e\left(-t\left(\frac{a}{q}+\beta\right)\right) d\beta. \tag{3.15}$$

We introduce the congruence condition to deduce that

$$\begin{aligned}
&\sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e\left(f(\mathbf{x})\left(\frac{a}{q}+\beta\right)\right) \\
&\quad = \sum_{1 \leqslant \mathbf{h} \leqslant q} e\left(f(\mathbf{h})\frac{a}{q}\right) \sum_{\substack{1 \leqslant \mathbf{x} \leqslant X \\ \mathbf{x} \equiv \mathbf{h} (\mathrm{mod}\ q)}} \Lambda(\mathbf{x}) e(f(\mathbf{x})\beta) \\
&\quad = \sum_{\substack{1 \leqslant \mathbf{h} \leqslant q \\ (\mathbf{h},q)=1}} e\left(f(\mathbf{h})\frac{a}{q}\right) \sum_{\substack{1 \leqslant \mathbf{x} \leqslant X \\ \mathbf{x} \equiv \mathbf{h} (\mathrm{mod}\ q)}} \Lambda(\mathbf{x}) e(f(\mathbf{x})\beta) + O(X^{n-1}LP).
\end{aligned}$$

Since $q \leqslant P = L^K$, the Siegel–Walfisz theorem together with summation by parts will imply for $(\mathbf{h}, q) = 1$ that

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$$\sum_{\substack{1\leqslant \mathbf{x}\leqslant X\\ \mathbf{x}\equiv \mathbf{h}(\mathrm{mod}\ q)}} \Lambda(\mathbf{x})e(f(\mathbf{x})\beta) = \frac{1}{\phi^n(q)}\int_{[0,X]^n} e(f(\mathbf{x})\beta)d\mathbf{x} + O(X^nL^{-100K})$$

$$= \frac{1}{\phi^n(q)} I(\beta) + O(X^nL^{-100K}).$$

It follows from above

$$\sum_{1\leqslant \mathbf{x}\leqslant X} \Lambda(\mathbf{x})e\left(f(\mathbf{x})\left(\frac{a}{q}+\beta\right)\right) = \frac{C(q,a)}{\phi^n(q)} I(\beta) + O(X^nL^{-10K}). \tag{3.16}$$

By putting (3.16) into (3.15), we obtain

$$\int_{\mathfrak{M}} S(\alpha)e(-t\alpha)\,d\alpha = \sum_{q\leqslant P} B_{f,t}(q)\int_{|\beta|\leqslant \frac{P}{qX^2}} I(\beta)e(-t\beta)d\beta + O(X^nL^{-K}). \tag{3.17}$$

It follows from $I(\beta) \ll X^n(1+X^2|\beta|)^{-2}$ that

$$\mathfrak{I}_{f,t}(X) \ll X^{n-2} \tag{3.18}$$

and

$$\int_{|\beta|\leqslant \frac{P}{qX^2}} I(\beta)e(-t\beta)d\beta = \mathfrak{I}_{f,t}(X) + O(qX^{n-2}P^{-1}). \tag{3.19}$$

Combining (3.17)–(3.19) together with Remark 3.5, we conclude

$$\int_{\mathfrak{M}} S(\alpha)e(-t\alpha)\,d\alpha = \mathfrak{S}(f,t)\mathfrak{I}_{f,t}(X) + O(X^{n-2}L^{-K/4}).$$

The proof of Lemma 3.6 is complete. □

§4. Estimates for exponential sums

LEMMA 4.1. *Let* $\{\xi_z\}$ *be a sequence satisfying* $|\xi_z| \leqslant 1$. *Then one has*

$$\sum_{|y|\ll X}\left|\sum_{|z|\ll X} \xi_z e(\alpha yz)\right|^2 \ll X\sum_{|x|\ll X}\min\{X,\ \|x\alpha\|^{-1}\}.$$

Proof. We expand the square to deduce that

$$\sum_{|y|\ll X}\left|\sum_{|z|\ll X} \xi_z e(\alpha yz)\right|^2 = \sum_{|z_1|\ll X}\sum_{|z_2|\ll X} \xi_{z_1}\overline{\xi_{z_2}}\sum_{|y|\ll X} e(\alpha y(z_1-z_2))$$

$$\leqslant \sum_{|z_1|\ll X}\sum_{|z_2|\ll X}\left|\sum_{|y|\ll X} e(\alpha y(z_1-z_2))\right|.$$

By changing variables, one can obtain

$$\begin{aligned}\sum_{|y|\ll X}\bigg|\sum_{|z|\ll X}\xi_z e(\alpha yz)\bigg|^2 &\ll \sum_{|z|\ll X}\sum_{|x|\ll X}\bigg|\sum_{|y|\ll X} e(\alpha yx)\bigg|\\ &\ll X\sum_{|x|\ll X}\bigg|\sum_{|y|\ll X} e(\alpha yx)\bigg|\\ &\ll X\sum_{|x|\ll X}\min\{X,\ \|x\alpha\|^{-1}\}.\end{aligned}$$

We complete the proof. ∎

LEMMA 4.2. *For $\alpha\in\mathfrak{m}(Q)$, one has*

$$\sum_{|x|\ll X}\min\{X,\ \|x\alpha\|^{-1}\}\ll LQ^{-1}X^2.$$

Proof. For $\alpha\in\mathfrak{m}(Q)$, there exist a and q such that $1\leqslant a\leqslant q\leqslant 2Q$, $(a,q)=1$ and $|\alpha-a/q|\leqslant 2Q(qX^2)^{-1}$. By a variant of Vaughan [12, Lemma 2.2] (see also Exercise 2 in Chapter 2 [12]), one has

$$\sum_{|x|\ll X}\min\{X,\ \|x\alpha\|^{-1}\}\ll LX^2\bigg(\frac{1}{q(1+X^2|\beta|)}+\frac{1}{X}+\frac{q(1+X^2|\beta|)}{X^2}\bigg).$$

Since $\alpha\in\mathfrak{m}(Q)$, one has either $q>Q$ or $|\alpha-a/q|>Q(qX^2)^{-1}$. Then the desired estimate follows immediately. ∎

LEMMA 4.3. *Let $\alpha\in\mathfrak{m}$ and $\beta\in\mathbb{R}$. For $d\in\mathbb{Q}$, we define*

$$f(\alpha,\beta)=\sum_{1\leqslant x\leqslant X}\Lambda(x)e(\alpha dx^2+x\beta).$$

If $d\neq 0$, then one has

$$f(\alpha,\beta)\ll XL^{-K/5}, \tag{4.1}$$

where the implied constant depends only on d and K.

Proof. The result is essentially classical. In particular, the method used to handle $\sum_{1\leqslant x\leqslant X}\Lambda(x)e(\alpha x^2)$ can be modified to establish the desired conclusion. We only explain that the implied constant is independent of β. By Vaughan's identity, we essentially consider two types of exponential sums

$$\sum_y \eta_y \sum_x e(\alpha d x^2 y^2 + xy\beta) \tag{4.2}$$

and

$$\sum_x \sum_y \xi_x \eta_y e(\alpha d x^2 y^2 + xy\beta). \tag{4.3}$$

By Cauchy's inequality, to handle the summation (4.3), it suffices to deal with

$$\sum_{y_1} \sum_{y_2} \eta_{y_1} \overline{\eta_{y_2}} \sum_x e(\alpha d x^2 (y_1^2 - y_2^2) + x(y_1 - y_2)\beta).$$

One can apply the differencing argument to the summation of the type $\sum_x e(\alpha' x^2 + x\beta')$ as follows

$$\begin{aligned}\left|\sum_x e(\alpha' x^2 + x\beta')\right|^2 &= \sum_{x_1}\sum_{x_2} e(\alpha'(x_1^2 - x_2^2) + (x_1 - x_2)\beta') \\ &= \sum_h \sum_x e(2\alpha' h x + h\beta') \leqslant \sum_h \left|\sum_x e(2\alpha' h x)\right|.\end{aligned}$$

This leads to the fact that the estimate (4.1) is uniformly for β. □

Lemma 4.4. *Let $\alpha \in \mathfrak{m}(Q)$. Suppose that A is in the form*

$$A = \begin{pmatrix} A_1 & B & 0 \\ B^T & A_2 & C \\ 0 & C^T & A_3 \end{pmatrix}, \tag{4.4}$$

where $\operatorname{rank}(B) \geqslant 3$ *and* $\operatorname{rank}(C) \geqslant 2$. *Then we have*

$$S(\alpha) \ll X^n Q^{-5/2} L^{n+5/2}. \tag{4.5}$$

Remark 4.5. In view of the proof, the estimate (4.5) still holds provided that $\operatorname{rank}(B) + \operatorname{rank}(C) \geqslant 5$.

Proof. By (4.4), we can write $S(\alpha)$ in the form

$$\begin{aligned}S(\alpha) = &\sum_{\substack{1\leqslant \mathbf{x}\leqslant X \\ 1\leqslant \mathbf{y}\leqslant X \\ 1\leqslant \mathbf{z}\leqslant X}} \Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z}) \\ &\times e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T B\mathbf{y} + \mathbf{y}^T A_2 \mathbf{y} + 2\mathbf{y}^T C\mathbf{z} + \mathbf{z}^T A_3 \mathbf{z})),\end{aligned}$$

where $\mathbf{x} \in \mathbb{N}^r$, $\mathbf{y} \in \mathbb{N}^s$ and $\mathbf{z} \in \mathbb{N}^t$. Then we have

$$S(\alpha) \leqslant L^s \sum_{1 \leqslant \mathbf{y} \leqslant X} \bigg| \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T B \mathbf{y}))\bigg| \times \bigg| \sum_{1 \leqslant \mathbf{z} \leqslant X} \Lambda(\mathbf{z}) e(\alpha(2\mathbf{y}^T C \mathbf{z} + \mathbf{z}^T A_3 \mathbf{z}))\bigg|.$$

By Cauchy's inequality, we obtain

$$S(\alpha) \leqslant L^s \bigg(\sum_{1 \leqslant \mathbf{y} \leqslant X} \bigg| \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T B \mathbf{y}))\bigg|^2 \bigg)^{1/2} \times \bigg(\sum_{1 \leqslant \mathbf{y} \leqslant X} \bigg| \sum_{1 \leqslant \mathbf{z} \leqslant X} \Lambda(\mathbf{z}) e(\alpha(2\mathbf{y}^T C \mathbf{z} + \mathbf{z}^T A_3 \mathbf{z}))\bigg|^2 \bigg)^{1/2}. \tag{4.6}$$

We deduce by expanding the square that

$$\begin{aligned} &\sum_{1 \leqslant \mathbf{y} \leqslant X} \bigg| \sum_{1 \leqslant \mathbf{x} \leqslant X} \Lambda(\mathbf{x}) e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T B \mathbf{y}))\bigg|^2 \\ &\quad = \sum_{1 \leqslant \mathbf{x}_1 \leqslant X} \sum_{1 \leqslant \mathbf{x}_2 \leqslant X} \xi(\mathbf{x}_1, \mathbf{x}_2) \sum_{1 \leqslant \mathbf{y} \leqslant X} e(2\alpha(\mathbf{x}_1 - \mathbf{x}_2)^T B \mathbf{y}) \\ &\quad = \sum_{|\mathbf{h}| \leqslant X} \sum_{\substack{1 \leqslant \mathbf{x} \leqslant X \\ 1 \leqslant \mathbf{x}+\mathbf{h} \leqslant X}} \xi(\mathbf{x}+\mathbf{h}, \mathbf{x}) \sum_{1 \leqslant \mathbf{y} \leqslant X} e(2\alpha(\mathbf{h}^T B \mathbf{y})) \\ &\quad \leqslant X^r L^{2r} \sum_{|\mathbf{h}| \leqslant X} \bigg| \sum_{1 \leqslant \mathbf{y} \leqslant X} e(2\alpha(\mathbf{h}^T B \mathbf{y}))\bigg|, \end{aligned}$$

where $\xi(\mathbf{x}_1, \mathbf{x}_2)$ is defined as

$$\xi(\mathbf{x}_1, \mathbf{x}_2) = \Lambda(\mathbf{x}_1)\Lambda(\mathbf{x}_2) e(\alpha(\mathbf{x}_1^T A_1 \mathbf{x}_1 - \mathbf{x}_2^T A_1 \mathbf{x}_2)).$$

We write

$$B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,s} \\ \vdots & \cdots & \vdots \\ b_{r,1} & \cdots & b_{r,s} \end{pmatrix}.$$

Since $\mathrm{rank}(B) \geqslant 3$, without loss of generality, we assume that $\mathrm{rank}(B_0) = 3$, where $B_0 = (b_{i,j})_{1 \leqslant i,j \leqslant 3}$. Let $B' = (b_{i,j})_{4 \leqslant i \leqslant r, 1 \leqslant j \leqslant 3}$. Then one has

$$\sum_{|\mathbf{h}|\leqslant X}\left|\sum_{1\leqslant\mathbf{y}\leqslant X}e(2\mathbf{h}^TB\mathbf{y}\alpha)\right|$$
$$\leqslant X^{s-3}\sum_{|h_4|,\ldots,|h_r|\leqslant X}\sum_{|\mathbf{u}|\leqslant X}\left|\sum_{1\leqslant\mathbf{v}\leqslant X}e(2\alpha(\mathbf{u}^TB_0+\mathbf{k}^T)\mathbf{v})\right|,$$

where $\mathbf{u}^T=(h_1,h_2,h_3)$, $\mathbf{v}^T=(y_1,y_2,y_3)$ and $\mathbf{k}^T=(h_4,\ldots,h_r)B'$. By changing variables $\mathbf{x}^T=2(\mathbf{u}^TB_0+\mathbf{k}^T)$, we obtain

$$\sum_{|\mathbf{h}|\leqslant X}\left|\sum_{1\leqslant\mathbf{y}\leqslant X}e(2\mathbf{h}^TB\mathbf{y}\alpha)\right|\leqslant X^{s-3}\sum_{|h_4|,\ldots,|h_r|\leqslant X}\sum_{|\mathbf{x}|\ll X}\left|\sum_{1\leqslant\mathbf{v}\leqslant X}e(\alpha(\mathbf{x}^T\mathbf{v}))\right|$$
$$\ll X^{r+s-6}\sum_{|\mathbf{x}|\ll X}\left|\sum_{1\leqslant\mathbf{v}\leqslant X}e(\alpha(\mathbf{x}^T\mathbf{v}))\right|.$$

We apply Lemma 4.2 to conclude that

$$\sum_{|\mathbf{h}|\leqslant X}\left|\sum_{1\leqslant\mathbf{y}\leqslant X}e(2\mathbf{h}^TB\mathbf{y}\alpha)\right|\ll X^{r+s}Q^{-3}L^3,$$

and therefore,

$$\sum_{1\leqslant\mathbf{y}\leqslant X}\left|\sum_{1\leqslant\mathbf{x}\leqslant X}\Lambda(\mathbf{x})e(\alpha(\mathbf{x}^TA_1\mathbf{x}+2\mathbf{x}^TB\mathbf{y}))\right|^2\ll X^{2r+s}Q^{-3}L^{2r+3}. \tag{4.7}$$

Similar to (4.7), we can prove

$$\sum_{1\leqslant\mathbf{y}\leqslant X}\left|\sum_{1\leqslant\mathbf{z}\leqslant X}\Lambda(\mathbf{z})e(\alpha(2\mathbf{y}^TC\mathbf{z}+\mathbf{z}^TA_3\mathbf{z}))\right|^2\ll X^{2t+s}Q^{-2}L^{2t+2}. \tag{4.8}$$

The proof is completed by invoking (4.6)–(4.8). ∎

LEMMA 4.6. *Suppose that A is in the form (4.4) with* $\operatorname{rank}(B)\geqslant 3$ *and* $\operatorname{rank}(C)\geqslant 2$. *Then we have*

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha\ll X^{n-2}L^{-K/3}.$$

Proof. By Dirichlet's approximation theorem, for any $\alpha\in[X^{-1},1+X^{-1}]$, there exist a and q with $1\leqslant a\leqslant q\leqslant X$ and $(a,q)=1$ such that $|\alpha-a/q|\leqslant(qX)^{-1}$. Thus the desired conclusion follows from Lemma 4.4 by the dyadic argument. ∎

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§5. Quadratic forms with off-diagonal rank $\leqslant 3$

PROPOSITION 5.1. *Let A be given by (1.1), and let $S(\alpha)$ be defined in (2.5). Suppose that* $\mathrm{rank}(A) \geqslant 9$ *and* $\mathrm{rank}_{\mathrm{off}}(A) \leqslant 3$*. Then we have*

$$\int_{\mathfrak{m}} |S(\alpha)|\ d\alpha \ll X^{n-2}L^{-K/6},$$

where the implied constant depends on A and K.

From now on, we assume throughout Section 5 that $\mathrm{rank}(A) \geqslant 9$ and

$$\mathrm{rank}_{\mathrm{off}}(A) = \mathrm{rank}(B) = 3, \tag{5.1}$$

where

$$B = \begin{pmatrix} a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,4} & a_{3,5} & a_{3,6} \end{pmatrix}. \tag{5.2}$$

Then we introduce $B_1, B_2, B_3 \in M_{3,n-4}(\mathbb{Z})$ defined as

$$B_1 = \begin{pmatrix} a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots & a_{1,n} \\ a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots & a_{2,n} \\ a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots & a_{3,n} \end{pmatrix}, \tag{5.3}$$

$$B_2 = \begin{pmatrix} a_{1,4} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots & a_{1,n} \\ a_{2,4} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots & a_{2,n} \\ a_{3,4} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots & a_{3,n} \end{pmatrix}, \tag{5.4}$$

and

$$B_3 = \begin{pmatrix} a_{1,4} & a_{1,5} & a_{1,7} & a_{1,8} & \cdots & a_{1,n} \\ a_{2,4} & a_{2,5} & a_{2,7} & a_{2,8} & \cdots & a_{2,n} \\ a_{3,4} & a_{3,5} & a_{3,7} & a_{3,8} & \cdots & a_{3,n} \end{pmatrix}. \tag{5.5}$$

Subject to the assumption (5.1), we have the following.

LEMMA 5.2. *If* $\mathrm{rank}(B_1) = \mathrm{rank}(B_2) = \mathrm{rank}(B_3) = 2$*, then one has*

$$\int_{\mathfrak{m}} |S(\alpha)|\ d\alpha \ll X^{n-2}L^{-K/6}.$$

LEMMA 5.3. *If* $\mathrm{rank}(B_1) = \mathrm{rank}(B_2) = 2$ *and* $\mathrm{rank}(B_3) = 3$*, then one has*

$$\int_{\mathfrak{m}} |S(\alpha)|\ d\alpha \ll X^{n-2}L^{-K/6}.$$

LEMMA 5.4. *If* $\mathrm{rank}(B_1) = 2$ *and* $\mathrm{rank}(B_2) = \mathrm{rank}(B_3) = 3$*, then one has*

$$\int_{\mathfrak{m}} |S(\alpha)| \; d\alpha \ll X^{n-2} L^{-K/6}.$$

LEMMA 5.5. *If* $\mathrm{rank}(B_1) = \mathrm{rank}(B_2) = \mathrm{rank}(B_3) = 3$*, then one has*

$$\int_{\mathfrak{m}} |S(\alpha)| \; d\alpha \ll X^{n-2} L^{-K/6}.$$

Remark for the Proof of Proposition 5.1. If $\mathrm{rank}_{\mathrm{off}}(A) = 0$, then A is a diagonal matrix and the conclusion is classical. When $\mathrm{rank}_{\mathrm{off}}(A) = 3$, our conclusion follows from Lemmas 5.2–5.5 immediately. The method applied to establish Lemmas 5.2–5.5 can also be used to deal with the case $1 \leqslant \mathrm{rank}_{\mathrm{off}}(A) \leqslant 2$. Indeed, the proof of Proposition 5.1 under the condition $1 \leqslant \mathrm{rank}_{\mathrm{off}}(A) \leqslant 2$ is easier, and we omit the details. Therefore, our main task is to establish Lemmas 5.2–5.5.

LEMMA 5.6. *Let* $C \in M_{n,n}(\mathbb{Q})$ *be a symmetric matrix, and let* $H \in M_{n,k}(\mathbb{Q})$*. For* $\alpha \in \mathbb{R}$ *and* $\boldsymbol{\beta} \in \mathbb{R}^k$*, we define*

$$\mathcal{F}(\alpha, \; \boldsymbol{\beta}) = \sum_{\mathbf{x} \in \mathcal{X}} w(\mathbf{x}) e(\alpha \mathbf{x}^T C \mathbf{x} + \mathbf{x}^T H \boldsymbol{\beta}),$$

where $\mathcal{X} \subset \mathbb{Z}^n$ *is a finite subset of* $\mathbb{Z}^n$*. Let*

$$\mathcal{N}(\mathcal{F}) = \sum_{\substack{\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{X} \\ \mathbf{x}^T C \mathbf{x} = \mathbf{y}^T C \mathbf{y} \\ \mathbf{x}^T H = \mathbf{y}^T H}} w(\mathbf{x}) w(\mathbf{y}).$$

Then we have

$$\int_{[0,1]^{k+1}} |\mathcal{F}(\alpha, \; \boldsymbol{\beta})|^2 \; d\alpha \; d\boldsymbol{\beta} \ll \mathcal{N}(\mathcal{F}),$$

where the implied constant may depend on C *and* H*.*

Proof. We can choose a natural number $h \in \mathbb{N}$ such that $hC \in M_{n,n}(\mathbb{Z})$ and $hH \in M_{n,k}(\mathbb{Z})$. Then we deduce that

$$\begin{aligned}&\int_{[0,1]^{k+1}} |\mathcal{F}(\alpha,\ \boldsymbol{\beta})|^2\, d\alpha\, d\boldsymbol{\beta}\\ &\quad\leqslant \int_{[0,h]^{k+1}} \left| \sum_{\mathbf{x}\in\,\mathcal{X}} w(\mathbf{x}) e(h^{-1}\alpha \mathbf{x}^T(hC)\mathbf{x} + \mathbf{x}^T(hH)(h^{-1}\boldsymbol{\beta}))\right|^2 d\alpha\, d\boldsymbol{\beta}\\ &\quad= h^{k+1} \int_{[0,1]^{k+1}} \left| \sum_{\mathbf{x}\in\,\mathcal{X}} w(\mathbf{x}) e(\alpha \mathbf{x}^T(hC)\mathbf{x} + \mathbf{x}^T(hH)\boldsymbol{\beta})\right|^2 d\alpha\, d\boldsymbol{\beta}.\end{aligned}$$

By orthogonality, we have

$$\begin{aligned}&\int_{[0,1]^{k+1}} \left| \sum_{\mathbf{x}\in\,\mathcal{X}} w(\mathbf{x}) e(\alpha \mathbf{x}^T(hC)\mathbf{x} + \mathbf{x}^T(hH)\boldsymbol{\beta})\right|^2 d\alpha\, d\boldsymbol{\beta}\\ &\quad= \sum_{\substack{\mathbf{x}\in\,\mathcal{X},\ \mathbf{y}\in\,\mathcal{X}\\ \mathbf{x}^T(hC)\mathbf{x}=\mathbf{y}^T(hC)\mathbf{y}\\ \mathbf{x}^T(hH)=\mathbf{y}^T(hH)}} w(\mathbf{x})w(\mathbf{y}) = \mathcal{N}(\mathcal{F}).\end{aligned}$$

Therefore, one obtains

$$\int_{[0,1]^{k+1}} |\mathcal{F}(\alpha,\ \boldsymbol{\beta})|^2\, d\alpha\, d\boldsymbol{\beta} \leqslant h^{k+1}\mathcal{N}(\mathcal{F}),$$

and this completes the proof. ∎

Lemma 5.7. *Let $C \in M_{n,n}(\mathbb{Q})$ be a symmetric matrix, and let $H \in M_{n,k}(\mathbb{Q})$. We have*

$$\mathcal{N}_1 \ll \mathcal{N}_2,$$

where

$$\mathcal{N}_1 = \sum_{\substack{|\mathbf{x}|\ll X,\ |\mathbf{y}|\ll X\\ \mathbf{x}^T C\mathbf{x}=\mathbf{y}^T C\mathbf{y}\\ \mathbf{x}^T H=\mathbf{y}^T H}} 1 \quad \text{and} \quad \mathcal{N}_2 = \sum_{\substack{|\mathbf{x}|\ll X,\ |\mathbf{y}|\ll X\\ \mathbf{x}^T C\mathbf{y}=0\\ \mathbf{x}^T H=0}} 1.$$

Proof. By changing variables $\mathbf{x} - \mathbf{y} = \mathbf{h}$ and $\mathbf{x} + \mathbf{y} = \mathbf{z}$, the desired conclusion follows immediately. ∎

The following result is well known.

Lemma 5.8. *Let $C \in M_{k,m}(\mathbb{Q})$. If* $\operatorname{rank}(C) \geqslant 2$, *then one has*

$$\sum_{\substack{|\mathbf{x}|\ll X, |\mathbf{y}|\ll X\\ \mathbf{x}^T C\mathbf{y}=0}} 1 \ll X^{k+m-2}L,$$

where the implied constant depends on the matrix C.

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5.1 Proof of Lemma 5.2

LEMMA 5.9. *If* $\mathrm{rank}(B_1) = \mathrm{rank}(B_2) = \mathrm{rank}(B_3) = 2$, *then we can write* A *in the form*

$$A = \begin{pmatrix} A_1 & B & 0 \\ B^T & A_2 & C \\ 0 & C^T & D \end{pmatrix}, \tag{5.6}$$

where $B \in GL_3(\mathbb{Z})$, $C \in M_{3,n-6}(\mathbb{Z})$ *and* $D = \mathrm{diag}\{d_1, \ldots, d_{n-6}\}$ *is a diagonal matrix.*

Proof. We write for $1 \leqslant j \leqslant n-3$ that

$$\gamma_j = \begin{pmatrix} a_{1,\,3+j} \\ a_{2,\,3+j} \\ a_{3,\,3+j} \end{pmatrix}. \tag{5.7}$$

Since $B = (\gamma_1, \gamma_2, \gamma_3) \in GL_3(\mathbb{Z})$, γ_1, γ_2 and γ_3 are linearly independent. For any $4 \leqslant j \leqslant n-3$, one has $\mathrm{rank}(\gamma_2, \gamma_3, \gamma_j) \leqslant \mathrm{rank}(B_1) = 2$. Therefore, we obtain $\gamma_j \in <\gamma_2, \gamma_3>$. Similarly, one has $\gamma_j \in <\gamma_1, \gamma_3>$ and $\gamma_j \in <\gamma_1, \gamma_2>$. Then we can conclude that $\gamma_j = 0$ for $4 \leqslant j \leqslant n-3$.

For $7 \leqslant i < j \leqslant n$, we write

$$B_{i,j} = \begin{pmatrix} a_{1,4} & a_{1,5} & a_{1,6} & a_{1,j} \\ a_{2,4} & a_{2,5} & a_{2,6} & a_{2,j} \\ a_{3,4} & a_{3,5} & a_{3,6} & a_{3,j} \\ a_{i,4} & a_{i,5} & a_{i,6} & a_{i,j} \end{pmatrix} = \begin{pmatrix} \eta_1^T \\ \eta_2^T \\ \eta_3^T \\ \eta_4^T \end{pmatrix}.$$

Since $3 \leqslant \mathrm{rank}(B_{i,j}) \leqslant \mathrm{rank}_{\mathrm{off}}(A) = 3$, we conclude that η_4^T can be linearly represented by η_1^T, η_2^T and η_3^T. Then we obtain $a_{i,j} = 0$ due to $a_{1,j} = a_{2,j} = a_{3,j} = 0$. Therefore, the matrix A is in the form (5.6). We complete the proof. ∎

Proof of Lemma 5.2. By Lemma 5.9, we have

$$\begin{aligned} S(\alpha) = & \sum_{\substack{\mathbf{x} \in \mathbb{N}^3 \\ 1 \leqslant \mathbf{x} \leqslant X}} \sum_{\substack{\mathbf{y} \in \mathbb{N}^3 \\ 1 \leqslant \mathbf{y} \leqslant X}} \sum_{\substack{\mathbf{z} \in \mathbb{N}^{n-6} \\ 1 \leqslant \mathbf{z} \leqslant X}} \\ & \times e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T B \mathbf{y} + \mathbf{y}^T A_2 \mathbf{y} + 2\mathbf{z}^T C^T \mathbf{y} + \mathbf{z}^T D \mathbf{z})) \\ & \times \Lambda(\mathbf{x}) \Lambda(\mathbf{y}) \Lambda(\mathbf{z}). \end{aligned}$$

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By orthogonality, we have

$$S(\alpha)=\int_{[0,1]^3}\sum_{\substack{\mathbf{w}\in\mathbb{Z}^3\\|\mathbf{w}|\ll X}}\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\1\leqslant\mathbf{x}\leqslant X}}\sum_{\substack{\mathbf{y}\in\mathbb{N}^3\\1\leqslant\mathbf{y}\leqslant X}}\sum_{\substack{\mathbf{z}\in\mathbb{N}^{n-6}\\1\leqslant\mathbf{z}\leqslant X}} e(\alpha(\mathbf{x}^TA_1\mathbf{x}+\mathbf{w}^T\mathbf{y}+\mathbf{z}^TD\mathbf{z}))$$
$$\times\, e((2\mathbf{x}^TB+\mathbf{y}^TA_2+2\mathbf{z}^TC^T-\mathbf{w}^T)\boldsymbol{\beta})\Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z})\,d\boldsymbol{\beta},$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)^T$ and we use $d\boldsymbol{\beta}$ to denote $d\beta_1\,d\beta_2\,d\beta_3$. We define

$$\mathcal{F}(\alpha,\boldsymbol{\beta})=\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\1\leqslant\mathbf{x}\leqslant X}} e(\alpha\mathbf{x}^TA_1\mathbf{x}+2\mathbf{x}^TB\boldsymbol{\beta})\Lambda(\mathbf{x}),$$

and

$$f_j(\alpha,\boldsymbol{\beta})=\sum_{1\leqslant z\leqslant X} e(\alpha d_jz^2+2z\xi_j^T\boldsymbol{\beta})\Lambda(z),$$

where $\xi_j=(a_{4,6+j},a_{5,6+j},a_{6,6+j})^T$ for $1\leqslant j\leqslant n-6$. On writing $I_3=(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$, we introduce

$$\mathcal{H}_j(\alpha,\boldsymbol{\beta})=\sum_{|w|\ll X}\sum_{1\leqslant y\leqslant X} e(\alpha wy+y\gamma_j^T\boldsymbol{\beta}-w\mathbf{e}_j^T\boldsymbol{\beta})\Lambda(y),$$

where $\gamma_j^T=(a_{3+j,4},a_{3+j,5},a_{3+j,6})$ for $1\leqslant j\leqslant 3$. With above notations, we have

$$\int_{\mathfrak{m}} S(\alpha)\,d\alpha=\int_{\mathfrak{m}}\int_{[0,1]^3}\mathcal{F}(\alpha,\boldsymbol{\beta})\mathcal{H}_1(\alpha,\boldsymbol{\beta})\mathcal{H}_2(\alpha,\boldsymbol{\beta})\mathcal{H}_3(\alpha,\boldsymbol{\beta})$$
$$\times\prod_{j=1}^{n-6} f_j(\alpha,\boldsymbol{\beta})\,d\boldsymbol{\beta}\,d\alpha.$$

Therefore, one has the following inequality

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha\leqslant\int_{\mathfrak{m}}\int_{[0,1]^3}\bigg|\mathcal{F}(\alpha,\boldsymbol{\beta})\mathcal{H}_1(\alpha,\boldsymbol{\beta})\mathcal{H}_2(\alpha,\boldsymbol{\beta})\mathcal{H}_3(\alpha,\boldsymbol{\beta})$$
$$\times\prod_{j=1}^{n-6} f_j(\alpha,\boldsymbol{\beta})\bigg|\,d\boldsymbol{\beta}\,d\alpha. \tag{5.8}$$

We first consider the case $\operatorname{rank}(D)\geqslant 3$. Without loss of generality, we assume $d_1d_2d_3\neq 0$. By (5.8) and the Cauchy–Schwarz inequality, one has

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha\leqslant\mathcal{I}_1^{1/2}\mathcal{I}_2^{1/2}\sup_{\substack{\alpha\in\mathfrak{m}\\\boldsymbol{\beta}\in[0,1]^3}}\bigg|\prod_{j=3}^{n-6} f_j(\alpha,\boldsymbol{\beta})\bigg|, \tag{5.9}$$

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where

$$\mathcal{I}_1 = \int_{[0,1]^4} |\mathcal{F}(\alpha, \boldsymbol{\beta}) f_1(\alpha, \boldsymbol{\beta}) f_2(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha \tag{5.10}$$

and

$$\mathcal{I}_2 = \int_{[0,1]^4} |\mathcal{H}_1(\alpha, \boldsymbol{\beta}) \mathcal{H}_2(\alpha, \boldsymbol{\beta}) \mathcal{H}_3(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha. \tag{5.11}$$

By Lemmas 5.6 and 5.7, one has

$$\begin{aligned} \mathcal{I}_1 &\ll L^{10} \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|z_1|,|z_1'|,|z_2|,|z_2'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x} + d_1 z_1^2 + d_2 z_2^2 = \mathbf{x}'^T A_1 \mathbf{x}' + d_1 z_1'^2 + d_2 z_2'^2 \\ \mathbf{x}^T B + z_1 \gamma_1^T + z_2 \gamma_2^T = \mathbf{x}'^T B + z_1' \gamma_1^T + z_2' \gamma_2^T}} 1 \\ &\ll L^{10} \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|z_1|,|z_1'|,|z_2|,|z_2'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + d_1 z_1 z_1' + d_2 z_2 z_2' = 0 \\ \mathbf{x}^T B + z_1 \gamma_1^T + z_2 \gamma_2^T = 0}} 1. \end{aligned}$$

Since B is invertible, we obtain

$$\mathcal{I}_1 \ll L^{10} \sum_{\substack{|\mathbf{x}'|,|z_1|,|z_1'|,|z_2|,|z_2'| \ll X \\ -(z_1 \gamma_1^T + z_2 \gamma_2^T) B^{-1} A_1 \mathbf{x}' + d_1 z_1 z_1' + d_2 z_2 z_2' = 0}} 1.$$

Then we conclude from Lemma 5.8 that

$$\mathcal{I}_1 \ll X^5 L^{11}. \tag{5.12}$$

It follows from Lemmas 5.6–5.7 that

$$\begin{aligned} \mathcal{I}_2 &\ll L^6 \sum_{\substack{|w_1|,|w_1'|,|w_2|,|w_2'|,|w_3|,|w_3'|,|y_1|,|y_1'|,|y_2|,|y_2'|,|y_3|,|y_3'| \ll X \\ w_1 y_1 + w_2 y_2 + w_3 y_3 = w_1' y_1' + w_2' y_2' + w_3' y_3' \\ y_1 \gamma_1^T - w_1 \mathbf{e}_1^T + y_2 \gamma_2^T - w_2 \mathbf{e}_2^T + y_3 \gamma_3^T - w_3 \mathbf{e}_3^T = y_1' \gamma_1^T - w_1' \mathbf{e}_1^T + y_2' \gamma_2^T - w_2' \mathbf{e}_2^T + y_3' \gamma_3^T - w_3' \mathbf{e}_3^T}} 1 \\ &\ll L^6 \sum_{\substack{|w_1|,|w_1'|,|w_2|,|w_2'|,|w_3|,|w_3'|,|y_1|,|y_1'|,|y_2|,|y_2'|,|y_3|,|y_3'| \ll X \\ w_1 y_1' + w_1' y_1 + w_2 y_2' + w_2' y_2 + w_3 y_3' + w_3' y_3 = 0 \\ y_1 \gamma_1^T - w_1 \mathbf{e}_1^T + y_2 \gamma_2^T - w_2 \mathbf{e}_2^T + y_3 \gamma_3^T - w_3 \mathbf{e}_3^T = 0}} 1 \\ &\ll L^6 \sum_{\substack{|w_1'|,|w_2'|,|w_3'|,|y_1|,|y_1'|,|y_2|,|y_2'|,|y_3|,|y_3'| \ll X \\ \mathbf{y}^T (\gamma_1, \gamma_2, \gamma_3)^T \mathbf{y}' + \mathbf{y}^T \mathbf{w}' = \mathbf{0}}} 1, \end{aligned}$$

where $\mathbf{y} = (y_1, y_2, y_3)^T$, $\mathbf{y}' = (y_1', y_2', y_3')^T$ and $\mathbf{w}' = (w_1', w_2', w_3')^T$. Then by Lemma 5.8, we have

$$\mathcal{I}_2 \ll X^7 L^7. \tag{5.13}$$

Since $d_3 \neq 0$, we obtain by Lemma 4.3

$$\sup_{\substack{\alpha \in \mathfrak{m} \\ \boldsymbol{\beta} \in [0,1]^3}} |f_3(\alpha, \boldsymbol{\beta})| \ll X L^{-K/5},$$

and thereby

$$\sup_{\substack{\alpha \in \mathfrak{m} \\ \boldsymbol{\beta} \in [0,1]^3}} \left| \prod_{j=3}^{n-6} f_j(\alpha, \boldsymbol{\beta}) \right| \ll X^{n-8} L^{-K/5}. \tag{5.14}$$

Now we conclude from (5.9), (5.12)–(5.14) that

$$\int_{\mathfrak{m}} |S(\alpha)| \, d\alpha \ll X^{n-2} L^{-K/6}.$$

Next we consider the case $1 \leqslant \operatorname{rank}(D) \leqslant 2$. Without loss of generality, we suppose that $d_1 \neq 0$ and $d_k = 0$ for $3 \leqslant k \leqslant n$. Since $\operatorname{rank}(A) \geqslant 9$, there exists k with $3 \leqslant k \leqslant n-6$ such that $\xi_k \neq 0 \in \mathbb{Z}^3$. Then we can find i, j with $1 \leqslant i < j \leqslant 3$ so that $\operatorname{rank}(\mathbf{e}_i, \mathbf{e}_j, \xi_k) = 3$. Without loss of generality, we can assume that $i = 1$, $j = 2$ and $k = 3$. One has

$$\int_{\mathfrak{m}} |S(\alpha)| \, d\alpha \leqslant \sup_{\substack{\alpha \in \mathfrak{m} \\ \boldsymbol{\beta} \in [0,1]^3}} \left| \prod_{j \neq 3} f_j(\alpha, \boldsymbol{\beta}) \right| \left(\int_{[0,1]^4} |\mathcal{F}(\alpha, \boldsymbol{\beta}) \mathcal{H}_3(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha \right)^{1/2}$$
$$\times \left(\int_{[0,1]^4} |\mathcal{H}_1(\alpha, \boldsymbol{\beta}) \mathcal{H}_2(\alpha, \boldsymbol{\beta}) f_3(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha \right)^{1/2}.$$

We deduce from Lemmas 5.6–5.7 that

$$\int_{[0,1]^4} |\mathcal{F}(\alpha, \boldsymbol{\beta}) \mathcal{H}_3(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha \ll L^8 \sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |w|, |w'|, |y|, |y'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x} + wy = \mathbf{x}'^T A_1 \mathbf{x}' + w'y' \\ 2\mathbf{x}^T B + y\gamma_3^T - w\mathbf{e}_3^T = 2\mathbf{x}'^T B + y'\gamma_3^T - w'\mathbf{e}_3^T}} 1$$
$$\ll L^8 \sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |w|, |w'|, |y|, |y'| \ll X \\ 2\mathbf{x}^T A_1 \mathbf{x}' + wy' + w'y = 0 \\ 2\mathbf{x}^T B + y\gamma_3^T - w\mathbf{e}_3^T = 0}} 1$$

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$$\ll L^8 \sum_{\substack{|\mathbf{x}'|,|w|,|w'|,|y|,|y'|\ll X \\ -(y\gamma_3^T - w\mathbf{e}_3^T)B^{-1}A_1\mathbf{x}' + wy' + w'y = 0}} 1.$$

Then by Lemma 5.8, one has

$$\int_{[0,1]^4} |\mathcal{F}(\alpha,\boldsymbol{\beta})\mathcal{H}_3(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha \ll X^5 L^9. \tag{5.15}$$

We deduce from Lemmas 5.6–5.7 again that

$$\int_{[0,1]^4} |\mathcal{H}_1(\alpha,\boldsymbol{\beta})\mathcal{H}_2(\alpha,\boldsymbol{\beta})f_3(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha$$

$$\ll L^6 \sum_{\substack{|w_1|,|w_1'|,|w_2|,|w_2'|,|y_1|,|y_1'|,|y_2|,|y_2'|,|z|,|z'|\ll X \\ w_1y_1 + w_2y_2 + d_3z^2 = w_1'y_1' + w_2'y_2' + d_3z'^2 \\ y_1\gamma_1^T + y_2\gamma_2^T - w_1\mathbf{e}_1^T - w_2\mathbf{e}_2^T + 2z\xi_3^T = y_1'\gamma_1^T + y_2'\gamma_2^T - w_1'\mathbf{e}_1^T - w_2'\mathbf{e}_2^T + 2z'\xi_3^T}} 1$$

$$\ll L^6 \sum_{\substack{|w_1|,|w_1'|,|w_2|,|w_2'|,|y_1|,|y_1'|,|y_2|,|y_2'|,|z|,|z'|\ll X \\ w_1y_1' + w_1'y_1 + w_2y_2' + w_2'y_2 + 2d_3zz' = 0 \\ y_1\gamma_1^T + y_2\gamma_2^T - w_1\mathbf{e}_1^T - w_2\mathbf{e}_2^T + 2z\xi_3^T = 0}} 1.$$

On applying $\operatorname{rank}(\mathbf{e}_1, \mathbf{e}_2, \xi_3) = 3$ and Lemma 5.8, we obtain

$$\int_{[0,1]^4} |\mathcal{H}_1(\alpha,\boldsymbol{\beta})\mathcal{H}_2(\alpha,\boldsymbol{\beta})f_3(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha \ll X^5 L^7. \tag{5.16}$$

It follows from Lemma 4.3 that

$$\sup_{\substack{\alpha\in\mathfrak{m} \\ \boldsymbol{\beta}\in[0,1]^3}} \left| \prod_{j\neq 3} f_j(\alpha,\boldsymbol{\beta}) \right| \ll X^{n-7} L^{-K/5}. \tag{5.17}$$

Then we conclude from (5.15)–(5.17) that

$$\int_{\mathfrak{m}} |S(\alpha)| \, d\alpha \ll X^{n-2} L^{-K/6}.$$

Now it suffices to assume $D = 0$. Then the matrix A is in the form

$$A = \begin{pmatrix} A_1 & B & 0 \\ B^T & A_2 & C \\ 0 & C^T & 0 \end{pmatrix}. \tag{5.18}$$

It follows from $\operatorname{rank}(A) \geqslant 9$ that $\operatorname{rank}(C) \geqslant 3$. By Lemma 4.6,

$$\int_{\mathfrak{m}} |S(\alpha)| \, d\alpha \ll X^{n-2} L^{-K/3}.$$

This completes the proof of Lemma 5.2.

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5.2 Proof of Lemma 5.3

LEMMA 5.10. *If* $\operatorname{rank}(B_1) = \operatorname{rank}(B_2) = 2$ *and* $\operatorname{rank}(B_3) = 3$*, then the symmetric integral matrix* A *can be written in the form*

$$A = \begin{pmatrix} A_1 & C & \gamma_3\xi^T \\ C^T & A_2 & V \\ \xi\gamma_3^T & V^T & D + h\xi\xi^T \end{pmatrix},$$

where $C = (\gamma_1, \gamma_2) \in M_{3,2}(\mathbb{Z})$*,* $\gamma_3 \in \mathbb{Q}^3$*,* $\xi \in \mathbb{Z}^{n-5}$*,* $V \in M_{2,n-5}(\mathbb{Z})$*,* $h \in \mathbb{Q}$ *and* $D = \operatorname{diag}\{d_1, \ldots, d_{n-5}\} \in M_{n-5,n-5}(\mathbb{Q})$ *is a diagonal matrix. Moreover, one has* $(\gamma_1, \gamma_2, \gamma_3) \in GL_3(\mathbb{Q})$*.*

Proof. Let us write

$$\gamma'_j = \begin{pmatrix} a_{1,\,3+j} \\ a_{2,\,3+j} \\ a_{3,\,3+j} \end{pmatrix} \quad \text{for } 1 \leqslant j \leqslant n-3.$$

Since $\operatorname{rank}(\gamma'_1, \gamma'_2, \gamma'_3) = \operatorname{rank}(B) = 3$, we conclude that γ'_1, γ'_2 and γ'_3 are linearly independent. For any $4 \leqslant j \leqslant n-3$, we deduce from $\operatorname{rank}(B_1) = \operatorname{rank}(B_2) = 2$ that $\gamma'_j \in <\gamma'_2, \gamma'_3> \cap <\gamma'_1, \gamma'_3> = <\gamma'_3>$. Therefore, we can write A in the form

$$A = \begin{pmatrix} A_1 & C & \gamma_3\xi^T \\ C^T & A_2 & V \\ \xi\gamma_3^T & V^T & A_3 \end{pmatrix},$$

where $C = (\gamma_1, \gamma_2) \in M_{3,2}(\mathbb{Z})$, $\gamma_3 \in \mathbb{Q}^3$, $\xi \in \mathbb{Z}^{n-5}$, $V \in M_{2,n-5}(\mathbb{Z})$ and $A_3 \in M_{n-5,n-5}(\mathbb{Q})$.

For $6 \leqslant j \leqslant n$. we define $\eta_j^T = (a_{j,4}, \ldots, a_{j,j-1}, a_{j,j+1}, \ldots, a_{j,n})^T \in \mathbb{Z}^{n-4}$. Then we set $\theta_{i,j}^T = (a_{i,4}, \ldots, a_{i,j-1}, a_{i,j+1}, \ldots, a_{i,n})^T \in \mathbb{Z}^{n-4}$ for $1 \leqslant i \leqslant 3$. Since $\operatorname{rank}_{\mathrm{off}}(A) = \operatorname{rank}(B) = \operatorname{rank}(B_3) = 3$, η_j can be linearly represented by $\theta_{1,j}$, $\theta_{2,j}$ and $\theta_{3,j}$. Let

$$\theta_i^T = (a_{i,4}, \ldots, a_{i,n})^T \in \mathbb{Z}^{n-3} \quad \text{for } 1 \leqslant i \leqslant 3.$$

Then one can choose $a'_{j,j} \in \mathbb{Q}$ such that $(a_{j,4}, \ldots, a_{j,j-1}, a'_{j,j}, a_{j,j+1}, \ldots, a_{j,n})$ is linearly represented by θ_1, θ_2 and θ_3. We consider A_3 and A'_3 defined as

$$A_3 = \begin{pmatrix} a_{6,6} & \cdots & a_{6,n} \\ \vdots & \cdots & \vdots \\ a_{n,6} & \cdots & a_{n,n} \end{pmatrix} \quad \text{and} \quad A'_3 = \begin{pmatrix} a'_{6,6} & \cdots & a'_{6,n} \\ \vdots & \cdots & \vdots \\ a'_{n,6} & \cdots & a'_{n,n} \end{pmatrix},$$

where $a'_{i,j} = a_{i,j}$ for $6 \leqslant i \neq j \leqslant n$. Since A'_3 is symmetric, we conclude from above that $A'_3 = h\xi\xi^T$ for some $h \in \mathbb{Q}$. The proof is completed by noting that $D = A_3 - A'_3$ is a diagonal matrix. □

Proof of Lemma 5.3. One can deduce from Lemma 5.10 that

$$S(\alpha) = \sum_{\substack{\mathbf{x}\in\mathbb{N}^3 \\ 1\leqslant\mathbf{x}\leqslant X}} \sum_{\substack{\mathbf{y}\in\mathbb{N}^2 \\ 1\leqslant\mathbf{y}\leqslant X}} \sum_{\substack{\mathbf{z}\in\mathbb{N}^{n-5} \\ 1\leqslant\mathbf{z}\leqslant X}} e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T \gamma_3 \xi^T \mathbf{z} + \mathbf{z}^T D\mathbf{z} + h\mathbf{z}^T \xi\xi^T \mathbf{z}))$$
$$\times e(\alpha(2\mathbf{x}^T C\mathbf{y} + \mathbf{y}^T A_2 \mathbf{y} + 2\mathbf{z}^T V^T \mathbf{y}))\Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z}).$$

We introduce new variables $\mathbf{w} \in \mathbb{Z}^2$ and $s \in \mathbb{Z}$ to replace $2\mathbf{x}^T C + \mathbf{y}^T A_2 + 2\mathbf{z}^T V^T$ and $\xi^T \mathbf{z}$, respectively. Therefore, we have

$$S(\alpha) = \int_{[0,1]^3} \sum_{|s|\ll X} \sum_{\substack{\mathbf{w}\in\mathbb{Z}^2 \\ |\mathbf{w}|\ll X}} \sum_{\substack{\mathbf{x}\in\mathbb{N}^3 \\ 1\leqslant\mathbf{x}\leqslant X}} \sum_{\substack{\mathbf{y}\in\mathbb{N}^2 \\ 1\leqslant\mathbf{y}\leqslant X}} \sum_{\substack{\mathbf{z}\in\mathbb{N}^{n-5} \\ 1\leqslant\mathbf{z}\leqslant X}} \Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z})$$
$$\times e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + \mathbf{w}^T \mathbf{y} + \mathbf{z}^T D\mathbf{z} + 2\mathbf{x}^T \gamma_3 s + hs^2))$$
$$\times e((2\mathbf{x}^T C + \mathbf{y}^T A_2 + 2\mathbf{z}^T V^T - \mathbf{w}^T)\boldsymbol{\beta}')$$
$$\times e((\xi^T \mathbf{z} - s)\beta_3)\, d\boldsymbol{\beta},$$

where $\boldsymbol{\beta}' = (\beta_1, \beta_2)^T$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$ and $d\boldsymbol{\beta} = d\beta_1\, d\beta_2\, d\beta_3$. We define

$$\mathcal{F}(\alpha, \boldsymbol{\beta}) = \sum_{|s|\ll X} \sum_{\substack{\mathbf{x}\in\mathbb{N}^3 \\ 1\leqslant\mathbf{x}\leqslant X}} e(\alpha(\mathbf{x}^T A_1 \mathbf{x} + 2\mathbf{x}^T \gamma_3 s + hs^2) + 2\mathbf{x}^T C\boldsymbol{\beta}' - s\beta_3)\Lambda(\mathbf{x}).$$

On writing $I_2 = (\mathbf{e}_1, \mathbf{e}_2)$, we introduce

$$\mathcal{H}_j(\alpha, \boldsymbol{\beta}) = \sum_{|w|\ll X} \sum_{1\leqslant y\leqslant X} e(\alpha wy + y\rho_j^T \boldsymbol{\beta}' - w\mathbf{e}_j^T \boldsymbol{\beta}')\Lambda(y),$$

where $\rho_j = (a_{3+j,4}, a_{3+j,5})^T$ for $1 \leqslant j \leqslant 2$. Let $\xi = (\epsilon_1, \ldots, \epsilon_{n-5})^T$. Then we define

$$f_j(\alpha, \boldsymbol{\beta}) = \sum_{1\leqslant z\leqslant X} e(\alpha d_j z^2 + 2z\upsilon_j^T \boldsymbol{\beta}' + \epsilon_j z\beta_3)\Lambda(z),$$

where $V = (\upsilon_1, \ldots, \upsilon_{n-5})$ with $\upsilon_j = (a_{4,5+j}, a_{5,5+j})^T$ for $1 \leqslant j \leqslant n-5$. With above notations, we obtain

(5.19)
$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \leqslant \int_{\mathfrak{m}} \int_{[0,1]^3} \left| \mathcal{F}(\alpha, \boldsymbol{\beta})\mathcal{H}_1(\alpha, \boldsymbol{\beta})\mathcal{H}_2(\alpha, \boldsymbol{\beta}) \prod_{j=1}^{n-5} f_j(\alpha, \boldsymbol{\beta}) \right| d\boldsymbol{\beta}\, d\alpha.$$

Let

$$\mathcal{J}_1 = \int_{[0,1]^4} |\mathcal{F}(\alpha, \boldsymbol{\beta}) f_i(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha$$

and

$$\mathcal{J}_2 = \int_{[0,1]^4} |\mathcal{H}_1(\alpha, \boldsymbol{\beta}) \mathcal{H}_2(\alpha, \boldsymbol{\beta}) f_j(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta} \, d\alpha.$$

By (5.19) and the Cauchy–Schwarz inequality, one has for $i \neq j$ that

$$\int_{\mathfrak{m}} |S(\alpha)| \, d\alpha \leqslant \mathcal{J}_1^{1/2} \mathcal{J}_2^{1/2} \sup_{\substack{\alpha \in \mathfrak{m} \\ \boldsymbol{\beta} \in [0,1]^3}} \bigg| \prod_{k \neq i,j} f_k(\alpha, \boldsymbol{\beta}) \bigg|. \tag{5.20}$$

One can deduce by Lemmas 5.6 and 5.7 that

$$\mathcal{J}_1 \ll L^8 \sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |s|, |s'|, |z|, |z'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{x}^T \gamma_3 s' + s \gamma_3^T \mathbf{x}' + hss' + d_i zz' = 0 \\ \mathbf{x}^T C + z v_i^T = 0 \\ s = \epsilon_i z}} 1.$$

Note that

$$\sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |s|, |s'|, |z|, |z'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{x}^T \gamma_3 s' + s \gamma_3^T \mathbf{x}' + hss' + d_i zz' = 0 \\ \mathbf{x}^T C + z v_i^T = 0 \\ s = \epsilon_i z}} 1 = \sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |s'|, |z|, |z'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{x}^T \gamma_3 s' + \epsilon_i z \gamma_3^T \mathbf{x}' + h \epsilon_i z s' + d_i zz' = 0 \\ \mathbf{x}^T C + z v_i^T = 0}} 1$$

$$= \sum_{\substack{|\mathbf{x}|, |\mathbf{x}'|, |s|, |s'|, |z|, |z'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + ss' + \epsilon_i z \gamma_3^T \mathbf{x}' + h \epsilon_i z s' + d_i zz' = 0 \\ \mathbf{x}^T (C, \gamma_3) + (z v_i^T, -s) = 0}} 1.$$

Recalling $\operatorname{rank}(C, \gamma_3) = 3$, one can replace $\mathbf{x}$ by $-(z v_i^T, -s)(C, \gamma_3)^{-1}$. Therefore, by Lemma 5.8, one has

$$\mathcal{J}_1 \ll X^5 L^9 \quad \text{if } d_i \neq 0. \tag{5.21}$$

The argument leading to (5.16) also implies

$$\mathcal{J}_2 \ll X^5 L^5 \quad \text{if } \epsilon_j \neq 0. \tag{5.22}$$

Now we are able to handle the case $\operatorname{rank}(D) \geqslant 2$. Since $\operatorname{rank}(B_3) = 3$, one has $\epsilon_l \neq 0$ for some l satisfying $2 \leqslant l \leqslant n - 5$. We may assume $\epsilon_2 \neq 0$. We also

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have $\epsilon_1 \neq 0$ due to $\operatorname{rank}(B) = 3$. If $d_l \neq 0$ for some $l \geqslant 3$, then we can find i, j, k pairwise distinct so that $\epsilon_j \neq 0$ and $d_i d_k \neq 0$. If $d_1 d_2 \neq 0$ and $\epsilon_j \neq 0$ for some $j \geqslant 3$, then we can also find i, j, k pairwise distinct so that $\epsilon_j \neq 0$ and $d_i d_k \neq 0$. In these cases, we can conclude from (5.20)–(5.22) that

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll X^{n-2} L^{-K/6}.$$

Next we assume $d_l = \epsilon_l = 0$ for all $l \geqslant 3$. Then we can represent A in the form

$$A = \begin{pmatrix} A_1 & H & 0 \\ H^T & Y & W \\ 0 & W^T & 0 \end{pmatrix},$$

where $H \in M_{3,4}(\mathbb{Z})$, $Y \in M_{4,4}(\mathbb{Z})$ and $W \in M_{4,n-7}(\mathbb{Z})$. It follows from $\operatorname{rank}(B) = 3$ and $\operatorname{rank}(A) \geqslant 9$ that $\operatorname{rank}(H) \geqslant 3$ and $\operatorname{rank}(W) \geqslant 2$. We apply Lemma 4.6 to conclude

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll X^{n-2} L^{-K/3}.$$

We are left to handle the case $\operatorname{rank}(D) \leqslant 1$. Since $\operatorname{rank}(D) + \operatorname{rank}(V) + 1 + 5 \geqslant \operatorname{rank}(A) \geqslant 9$, we obtain $\operatorname{rank}(D) \geqslant 1$. Therefore, $\operatorname{rank}(D) = 1$. We have

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \leqslant \mathcal{J}_3^{1/2} \mathcal{J}_4^{1/2} \sup_{\substack{\alpha \in \mathfrak{m} \\ \boldsymbol{\beta} \in [0,1]^3}} \left| \prod_{u \neq i,j,k} f_u(\alpha, \boldsymbol{\beta}) \right|, \tag{5.23}$$

where

$$\mathcal{J}_3 = \int_{[0,1]^4} |\mathcal{F}(\alpha, \boldsymbol{\beta}) H_1(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha$$

and

$$\mathcal{J}_4 = \int_{[0,1]^4} |\mathcal{H}_2(\alpha, \boldsymbol{\beta}) f_i(\alpha, \boldsymbol{\beta}) f_j(\alpha, \boldsymbol{\beta}) f_k(\alpha, \boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha.$$

By Lemmas 5.6–5.7, we have

$$\mathcal{J}_3 \ll L^8 \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|s|,|s'|,|y|,|y'|,|w|,|w'| \ll X \\ 2\mathbf{x}^T A_1 \mathbf{x}' + 2\mathbf{x}^T \gamma_3 s' + 2hss' + 2s\gamma_3^T \mathbf{x}' + wy' + yw' = 0 \\ 2\mathbf{x}^T C + y\rho_1^T - w\mathbf{e}_1^T = 0 \\ s = 0}} 1.$$

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Since $\mathrm{rank}(C) \geqslant 3$, we can represent two of x_1, x_2, x_3 (say x_1 and x_2) in terms of x_3, y and w. Then by Lemma 5.8, one has

$$\mathcal{J}_3 \ll X^7 L^9. \tag{5.24}$$

We deduce from Lemma 5.10 that $\mathrm{rank}\begin{pmatrix}\xi^T\\ V\end{pmatrix} \geqslant \mathrm{rank}(A) - 5 - \mathrm{rank}(D) \geqslant 3$. Therefore, there exist distinct i, j, k, s such that $\mathrm{rank}\begin{pmatrix}\upsilon_i^T & \upsilon_j^T & \upsilon_k^T\\ \epsilon_i & \epsilon_j & \epsilon_k\end{pmatrix} = 3$ and $d_s \neq 0$. By Lemmas 5.6–5.7, we also have

$$\mathcal{J}_4 \ll L^8 \sum_{\substack{|y|,|y'|,|w|,|w'|,|z_1|,|z_1'|,|z_2|,|z_2'|,|z_3|,|z_3'| \ll X\\ wy'+yw'+2d_iz_1z_1'+2d_jz_2z_2'+2d_kz_3z_3'=0\\ y\rho_2^T - w\mathbf{e}_2^T + 2z_1\upsilon_i^T + 2z_2\upsilon_j^T + 2z_3\upsilon_k^T = 0\\ \epsilon_i z_1 + \epsilon_j z_2 + \epsilon_k z_3 = 0}} 1.$$

Hence we can replace z_1, z_2 and z_3 by linear functions of y and w, and it follows that

$$\mathcal{J}_4 \ll X^5 L^9. \tag{5.25}$$

Hence we can obtain again that

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll X^{n-2} L^{-K/3}.$$

The proof of Lemma 5.3 is finished.

5.3 Proof of Lemma 5.4

The proof of Lemma 5.10 can be modified to establish the following result. The detail of the proof is omitted.

LEMMA 5.11. *If* $\mathrm{rank}(B_1) = 2$ *and* $\mathrm{rank}(B_2) = \mathrm{rank}(B_3) = 3$*, then we can write* A *in the form*

$$A = \begin{pmatrix} A_1 & \gamma_1 & (\gamma_2, \gamma_3)C\\ \gamma_1^T & a & \upsilon^T\\ C^T(\gamma_2, \gamma_3)^T & \upsilon & D + C^T H C \end{pmatrix}, \tag{5.26}$$

where $\gamma_1 \in \mathbb{Z}^3$*,* $\gamma_2, \gamma_3 \in \mathbb{Q}^3$*,* $C \in M_{2,n-4}(\mathbb{Z})$*,* $a \in \mathbb{Z}$*,* $\upsilon \in \mathbb{Z}^{n-4}$*,* $H \in M_{2,2}(\mathbb{Q})$ *and* $D = \mathrm{diag}\{d_1, \ldots, d_{n-4}\} \in M_{n-4,n-4}(\mathbb{Q})$ *is a diagonal matrix. Moreover, one has* $(\gamma_1, \gamma_2, \gamma_3) \in GL_3(\mathbb{Q})$.

LEMMA 5.12. *Let A be given by (5.26). We write*

$$C=(\xi_1,\ldots,\xi_{n-4}) \quad \text{and} \quad \upsilon^T=(\upsilon_1,\ldots,\upsilon_{n-4}). \tag{5.27}$$

Let

$$R_{i,j,k}=\begin{pmatrix}\xi_i & \xi_j & \xi_k\\ \upsilon_i & \upsilon_j & \upsilon_k\end{pmatrix}. \tag{5.28}$$

Under the conditions in Lemma 5.11, one can find pairwise distinct i,j,k,u with $1\leqslant i,j,k,u\leqslant n-4$ such that at least one of the following two statements holds: (i) $\mathrm{rank}(R_{i,j,k})=3$ and $d_u\neq 0$; (ii) $\mathrm{rank}(\xi_i,\xi_j)=2$ and $d_kd_u\neq 0$.

Proof. It follows from $9\leqslant \mathrm{rank}(A)\leqslant \mathrm{rank}(D)+\mathrm{rank}(\upsilon)+\mathrm{rank}(C)+4$ that $\mathrm{rank}(D)\geqslant 2$. If $\mathrm{rank}(D)=2$, say $d_1d_2\neq 0$, then $\mathrm{rank}(R)\geqslant 3$, where

$$R=\begin{pmatrix}\xi_3 & \cdots & \xi_{n-4}\\ \upsilon_3 & \cdots & \upsilon_{n-4}\end{pmatrix}.$$

Then statement (i) holds. Next we assume $\mathrm{rank}(D)\geqslant 3$. Note that $\mathrm{rank}(\xi_1,\xi_2)=2$ due to $\mathrm{rank}(B)=3$. If $d_rd_s\neq 0$ for some $r>s\geqslant 3$, then statement (ii) follows by choosing $i=1,j=2,k=r$ and $u=s$. Therefore, we now assume that $\mathrm{rank}(D)=3$ and $d_1d_2\neq 0$. Without loss of generality, we suppose that $d_3\neq 0$ and $d_s=0(4\leqslant s\leqslant n-4)$. We consider $\mathrm{rank}(\xi_1,\xi_s)$ and $\mathrm{rank}(\xi_2,\xi_s)$ for $4\leqslant s\leqslant n-4$. If $\mathrm{rank}(\xi_1,\xi_s)=2$ for some s with $4\leqslant s\leqslant n-4$, then one can choose $i=1$, $j=s$ $k=2$ and $u=3$ to establish statement (ii). Similarly, statement (ii) follows if $\mathrm{rank}(\xi_2,\xi_s)=2$ for some s with $4\leqslant s\leqslant n-4$. Thus it remains to consider the case $\mathrm{rank}(\xi_1,\xi_s)=\mathrm{rank}(\xi_2,\xi_s)=1$ for $4\leqslant s\leqslant n-4$. However, it follows from $\mathrm{rank}(\xi_1,\xi_2)=\mathrm{rank}(\xi_1,\xi_2,\xi_s)=2$ that $\xi_s=0$, and this is contradictory to the condition $\mathrm{rank}(A)\geqslant 9$. We complete the proof of Lemma 5.12. ∎

Proof of Lemma 5.4. We deduce from Lemma 5.11 that

$$\begin{aligned}S(\alpha)=&\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\1\leqslant\mathbf{x}\leqslant X}}\sum_{1\leqslant y\leqslant X}\sum_{\substack{\mathbf{z}\in\mathbb{N}^{n-4}\\1\leqslant\mathbf{z}\leqslant X}}\\&\times e(\alpha(\mathbf{x}^TA_1\mathbf{x}+2\mathbf{x}^T(\gamma_2,\gamma_3)C\mathbf{z}+\mathbf{z}^TD\mathbf{z}+\mathbf{z}^TC^THC\mathbf{z}))\\&\times e(\alpha(2\mathbf{x}^T\gamma_1y+ay^2+2\mathbf{z}^T\upsilon y))\Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z}).\end{aligned}$$

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We introduce new variables $w \in \mathbb{Z}$ and $\mathbf{h} \in \mathbb{Z}^2$ to replace $2\mathbf{x}^T\gamma_1 + ay + 2\mathbf{z}^T\upsilon^T$ and $C\mathbf{z}$, respectively. Therefore, we have

$$\begin{aligned} S(\alpha) = \int_{[0,1]^3} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^2 \\ |\mathbf{h}|\ll X}} \sum_{|w|\ll X} \sum_{\substack{\mathbf{x}\in\mathbb{N}^3 \\ 1\leqslant\mathbf{x}\leqslant X}} \sum_{1\leqslant y\leqslant X} \sum_{\substack{\mathbf{z}\in\mathbb{N}^{n-4} \\ 1\leqslant\mathbf{z}\leqslant X}} \Lambda(\mathbf{x})\Lambda(\mathbf{y})\Lambda(\mathbf{z}) \\ \times e(\alpha(\mathbf{x}^T A_1\mathbf{x} + 2\mathbf{x}^T(\gamma_2, \gamma_3)\mathbf{h} + \mathbf{z}^T D\mathbf{z} + \mathbf{h}^T H\mathbf{h} + wy)) \\ \times e((2\mathbf{x}^T\gamma_1 + ay + 2\mathbf{z}^T\upsilon - w)\beta_1) \\ \times e((C\mathbf{z} - \mathbf{h})^T\boldsymbol{\beta}')\, d\boldsymbol{\beta}, \end{aligned}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$, $\boldsymbol{\beta}' = (\beta_2, \beta_3)^T$ and $d\boldsymbol{\beta} = d\beta_1\, d\beta_2\, d\beta_3$. Now we introduce

$$\begin{aligned} \mathcal{F}(\alpha, \boldsymbol{\beta}) = \sum_{\substack{\mathbf{h}\in\mathbb{Z}^2 \\ |\mathbf{h}|\ll X}} \sum_{\substack{\mathbf{x}\in\mathbb{N}^3 \\ 1\leqslant\mathbf{x}\leqslant X}} e(\alpha(\mathbf{x}^T A_1\mathbf{x} + 2\mathbf{x}^T(\gamma_2, \gamma_3)\mathbf{h} + \mathbf{h}^T H\mathbf{h})) \\ \times e(2\mathbf{x}^T\gamma_1\beta_1 - \mathbf{h}^T\boldsymbol{\beta}')\Lambda(\mathbf{x}), \end{aligned}$$

and

$$\mathcal{H}(\alpha, \boldsymbol{\beta}) = \sum_{|w|\ll X} \sum_{1\leqslant y\leqslant X} e(\alpha wy + (ay - w)\beta_1)\Lambda(y).$$

On recalling notations in (5.27), we define

$$f_j(\alpha, \boldsymbol{\beta}) = \sum_{1\leqslant z\leqslant X} e(\alpha d_j z^2 + 2zv_j\beta_1 + z\xi_j^T\boldsymbol{\beta}')\Lambda(z).$$

Then we obtain from above

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \leqslant \int_{\mathfrak{m}} \int_{[0,1]^3} \left| \mathcal{F}(\alpha, \boldsymbol{\beta})\mathcal{H}(\alpha, \boldsymbol{\beta}) \prod_{j=1}^{n-4} f_j(\alpha, \boldsymbol{\beta}) \right| d\boldsymbol{\beta}\, d\alpha. \tag{5.29}$$

One can deduce from Lemmas 5.6 and 5.7 that

$$\begin{aligned} \int_{[0,1]^4} |\mathcal{H}(\alpha, \boldsymbol{\beta}) f_i(\alpha, \boldsymbol{\beta}) f_j(\alpha, \boldsymbol{\beta}) f_k(\alpha, \boldsymbol{\beta})|^2\, d\boldsymbol{\beta}\, d\alpha \\ \ll L^8 \sum_{\substack{|w|,|w'|,|y|,|y'|,|z_i|,|z_i'|,|z_j|,|z_j'|,|z_k|,|z_k'|\ll X \\ wy' + yw' + 2(d_i z_i z_i' + d_j z_j z_j' + d_k z_k z_k') = 0 \\ ay - w + 2(v_i z_i + v_j z_j + v_k z_k) = 0 \\ z_i\xi_i + z_j\xi_j + z_k\xi_k = 0}} 1. \end{aligned}$$

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If $\mathrm{rank}(R_{i,j,k}) = 3$, then we can represent z_i, z_j and z_k by linear functions of y and w. Then by Lemma 5.8,

$$\int_{[0,1]^4} |\mathcal{H}(\alpha,\boldsymbol{\beta}) f_i(\alpha,\boldsymbol{\beta}) f_j(\alpha,\boldsymbol{\beta}) f_k(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha \ll X^5 L^9.$$

If $\mathrm{rank}(\xi_i, \xi_j) = 2$, then we can represent z_i, z_j and w by linear functions of y and z_k. Then we obtain by Lemma 5.8 again

$$\int_{[0,1]^4} |\mathcal{H}(\alpha,\boldsymbol{\beta}) f_i(\alpha,\boldsymbol{\beta}) f_j(\alpha,\boldsymbol{\beta}) f_k(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha \ll X^5 L^9$$

provided that $d_k \neq 0$. By Lemmas 5.6–5.7, we can obtain

$$\begin{aligned}\int_{[0,1]^4} |\mathcal{F}(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha &\ll L^6 \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|\mathbf{h}|,|\mathbf{h}'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{x}^T(\gamma_2,\gamma_3)\mathbf{h}' + \mathbf{x}'^T(\gamma_2,\gamma_3)\mathbf{h} + \mathbf{h}^T H \mathbf{h}' = 0 \\ \mathbf{x}^T \gamma_1 = 0 \\ \mathbf{h}^T = 0}} 1 \\ &= L^6 \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|\mathbf{h}'| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{x}^T(\gamma_2,\gamma_3)\mathbf{h}' = 0 \\ \mathbf{x}^T \gamma_1 = 0}} 1.\end{aligned}$$

Then we deduce that

$$\begin{aligned}\int_{[0,1]^4} |\mathcal{F}(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha &\ll L^6 \sum_{\substack{|\mathbf{x}|,|\mathbf{x}'|,|\mathbf{h}'|,|\mathbf{h}| \ll X \\ \mathbf{x}^T A_1 \mathbf{x}' + \mathbf{h}^T \mathbf{h}' = 0 \\ \mathbf{x}^T(\gamma_1,\gamma_2,\gamma_3) = (0,\mathbf{h}^T)}} 1 \\ &\ll L^6 \sum_{\substack{|\mathbf{x}'|,|\mathbf{h}'|,|\mathbf{h}| \ll X \\ (0,\mathbf{h}^T)(\gamma_1,\gamma_2,\gamma_3)^{-1} A_1 \mathbf{x}' + \mathbf{h}^T \mathbf{h}' = 0}} 1.\end{aligned}$$

On invoking Lemma 5.8, we arrive at

$$\int_{[0,1]^4} |\mathcal{F}(\alpha,\boldsymbol{\beta})|^2 \, d\boldsymbol{\beta}\, d\alpha \ll X^5 L^7.$$

If $1 \leqslant i, j, k \leqslant n-4$ are pairwise distinct, then one has by (5.29) and the Cauchy–Schwarz inequality

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \leqslant \sup_{\substack{\alpha\in\mathfrak{m}\\ \boldsymbol{\beta}\in[0,1]^3}} \left| \prod_{u\neq i,j,k} f_u(\alpha,\boldsymbol{\beta})\right| \left(\int_{[0,1]^4} |\mathcal{F}(\alpha,\boldsymbol{\beta})|^2\, d\boldsymbol{\beta}\, d\alpha\right)^{1/2}$$
$$\times \left(\int_{[0,1]^4} |\mathcal{H}(\alpha,\boldsymbol{\beta}) f_i(\alpha,\boldsymbol{\beta}) f_j(\alpha,\boldsymbol{\beta}) f_k(\alpha,\boldsymbol{\beta})|^2\, d\boldsymbol{\beta}\, d\alpha\right)^{1/2}.$$

Now it follows from above together with Lemmas 4.3 and 5.12 that

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll X^{n-2} L^{-K/6}.$$

We complete the proof of Lemma 5.4.

5.4 Proof of Lemma 5.5

Similar to Lemmas 5.9–5.11, we also have the following result.

LEMMA 5.13. *If* $\operatorname{rank}(B_1) = \operatorname{rank}(B_2) = \operatorname{rank}(B_3) = 3$, *then we can write* A *in the form*

$$A = \begin{pmatrix} A_1 & (\gamma_1,\gamma_2,\gamma_3)C \\ C^T(\gamma_1,\gamma_2,\gamma_3)^T & D + C^T H C \end{pmatrix}, \tag{5.30}$$

where $C \in M_{3,n-3}(\mathbb{Z})$, $\gamma_1,\gamma_2,\gamma_3 \in \mathbb{Q}^3$, $H \in M_{3,3}(\mathbb{Q})$ *and* $D = \operatorname{diag}\{d_1,\ldots, d_{n-3}\} \in M_{n-3,n-3}(\mathbb{Q})$ *is a diagonal matrix. Furthermore, we have* $(\gamma_1, \gamma_2,\gamma_3) \in GL_3(\mathbb{Q})$.

LEMMA 5.14. *Let* A *be given by (5.30) satisfying the conditions in Lemma 5.13. We write*

$$C = (\xi_1,\ldots,\xi_{n-3}). \tag{5.31}$$

Then we can find pairwise distinct $u_j (1 \leqslant j \leqslant 6)$ *with* $1 \leqslant u_1,u_2,u_3,u_4, u_5,u_6 \leqslant n-3$ *so that* $\operatorname{rank}(\xi_{u_1},\xi_{u_2},\xi_{u_3}) = 3$ *and* $d_{u_4}d_{u_5}d_{u_6} \neq 0$.

Proof. It follows from $\operatorname{rank}(A) \geqslant 9$ that $\operatorname{rank}(D) \geqslant 3$. If $\operatorname{rank}(D) = 3$, then we may assume that $d_1 d_2 d_3 \neq 0$ and $d_j = 0$ for $j \geqslant 4$. Thus $\operatorname{rank}(\xi_4,\ldots,\xi_{n-3}) = 3$, and the desired conclusion follows. Next we assume $\operatorname{rank}(D) \geqslant 4$. Since $\operatorname{rank}(\xi_1,\xi_2,\xi_3) = 3$, the desired conclusion follows again if there are distinct k_1, k_2 and k_3 such that $d_{k_1}d_{k_2}d_{k_3} \neq 0$ and $k_1,k_2,k_3 \geqslant 4$. Thus we now assume that for any distinct $k_1,k_2,k_3 \geqslant 4$, one has $d_{k_1}d_{k_2}d_{k_3} = 0$. This yields $\operatorname{rank}(D) \leqslant 5$. We first consider the case $\operatorname{rank}(D) = 4$. There are at least two distinct $j_1, j_2 \leqslant 3$ such that $d_{j_1}d_{j_2} \neq 0$. Suppose that

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$d_{s_i}=0$ for $1\leqslant i\leqslant n-7$. Then the rank of $\{\xi_{s_i}\}_{1\leqslant i\leqslant n-7}$ is at least 2, say $\mathrm{rank}(\xi_{s_1},\xi_{s_2})=2$. Since $\mathrm{rank}(\xi_1,\xi_2,\xi_3)=3$, we can find j with $1\leqslant j\leqslant 3$ such that $\mathrm{rank}(\xi_j,\xi_{s_1},\xi_{s_2})=3$. The desired conclusion follows easily by choosing $u_1=j$, $u_2=s_1$ and $u_3=s_2$. Now we consider the case $\mathrm{rank}(D)=5$, and we may assume that $d_1d_2d_3d_4d_5\neq 0$ and $d_r=0$ for $r\geqslant 6$. Since $\mathrm{rank}(A)\geqslant 9$, there exist $r\geqslant 6$ (say $r=6$) such that $\xi_r\neq 0$. Then one can choose $j_1,j_2\leqslant 3$ so that $\mathrm{rank}(\xi_{j_1},\xi_{j_2},\xi_6)=3$. The desired conclusion follows by choosing $u_1=j_1$, $u_2=j_2$ and $u_3=6$. The proof of Lemma 5.14 is completed. ∎

Proof of Lemma 5.5. We apply Lemma 5.13 to conclude that

$$S(\alpha)=\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\ 1\leqslant\mathbf{x}\leqslant X}}\sum_{\substack{\mathbf{y}\in\mathbb{N}^{n-3}\\ 1\leqslant\mathbf{y}\leqslant X}}\Lambda(\mathbf{x})\Lambda(\mathbf{y})e(\alpha(\mathbf{x}^TA_1\mathbf{x}+\mathbf{y}^TD\mathbf{y}))\\ \times e(\alpha(2\mathbf{x}^T(\gamma_1,\gamma_2,\gamma_3)C\mathbf{y}+\mathbf{y}^TC^THC\mathbf{y})).$$

By orthogonality, one has

$$S(\alpha)=\int_{[0,1]^3}\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\ 1\leqslant\mathbf{x}\leqslant X}}\sum_{\substack{\mathbf{y}\in\mathbb{N}^{n-3}\\ 1\leqslant\mathbf{y}\leqslant X}}\sum_{\substack{\mathbf{z}\in\mathbb{Z}^3\\ |\mathbf{z}|\ll X}}\Lambda(\mathbf{x})\Lambda(\mathbf{y})e(\alpha(\mathbf{x}^TA_1\mathbf{x}+\mathbf{y}^TD\mathbf{y}))\\ \times e(\alpha(2\mathbf{x}^T(\gamma_1,\gamma_2,\gamma_3)\mathbf{z}+\mathbf{z}^TH\mathbf{z}))\\ \times e((\mathbf{y}^TC^T-\mathbf{z}^T)\boldsymbol{\beta})\,d\boldsymbol{\beta},$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)^T$ and $d\boldsymbol{\beta}=d\beta_1\,d\beta_2\,d\beta_3$. Now we introduce

$$\mathcal{F}(\alpha,\boldsymbol{\beta})=\sum_{\substack{\mathbf{x}\in\mathbb{N}^3\\ 1\leqslant\mathbf{x}\leqslant X}}\sum_{\substack{\mathbf{z}\in\mathbb{Z}^3\\ 1\leqslant\mathbf{z}\leqslant X}}\Lambda(\mathbf{x})e(\alpha(\mathbf{x}^TA_1\mathbf{x}+2\mathbf{x}^T(\gamma_1,\gamma_2,\gamma_3)\mathbf{z}+\mathbf{z}^TH\mathbf{z})-\mathbf{z}^T\boldsymbol{\beta}),$$

and

$$f_j(\alpha,\boldsymbol{\beta})=\sum_{1\leqslant y\leqslant X}e(d_j\alpha y^2+2y\xi_j^T\boldsymbol{\beta})\Lambda(y),$$

where $\xi_1,\ldots,\xi_{n-3}$ is given by (5.31). We conclude from above

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha\leqslant\int_{\mathfrak{m}}\int_{[0,1]^3}\left|\mathcal{F}(\alpha,\boldsymbol{\beta})\prod_{j=1}^{n-3}f_j(\alpha,\boldsymbol{\beta})\right|d\boldsymbol{\beta}\,d\alpha. \tag{5.32}$$

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One applying Lemmas 5.6–5.8, we can easily establish

$$\int_{[0,1]^4}\left|\prod_{i=1}^{5} f_{u_i}(\alpha,\boldsymbol{\beta})\right|^2 d\boldsymbol{\beta}\,d\alpha \ll X^5L^{11} \tag{5.33}$$

provided that $\operatorname{rank}(\xi_{u_1},\xi_{u_2},\xi_{u_3})=3$ and $d_{u_4}d_{u_5}\neq 0$. Similarly, we also have

$$\int_{[0,1]^4}|\mathcal{F}(\alpha,\boldsymbol{\beta})|^2\,d\boldsymbol{\beta}\,d\alpha \ll X^7L^7. \tag{5.34}$$

By (5.32) and the Cauchy–Schwarz inequality, one has for distinct u_1,u_2,u_3,u_4 and u_5 that

$$\begin{aligned}\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha \leqslant{}& \sup_{\substack{\alpha\in\mathfrak{m}\\ \boldsymbol{\beta}\in[0,1]^3}}\left|\prod_{k\neq u_1,u_2,u_3,u_4,u_5} f_k(\alpha,\boldsymbol{\beta})\right|\\ &\times\left(\int_{[0,1]^4}|\mathcal{F}(\alpha,\boldsymbol{\beta})|^2\,d\boldsymbol{\beta}\,d\alpha\right)^{1/2}\\ &\times\left(\int_{[0,1]^4}\left|\prod_{i=1}^{5} f_{u_i}(\alpha,\boldsymbol{\beta})\right|^2 d\boldsymbol{\beta}\,d\alpha\right)^{1/2}.\end{aligned} \tag{5.35}$$

Combining (5.33)–(5.35), Lemma 4.3 and Lemma 5.14, one has

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha \ll X^{n-2}L^{-K/6}.$$

The proof of Lemma 5.5 is finished.

§6. Quadratic forms with off-diagonal rank $\geqslant 4$

PROPOSITION 6.1. *Let A be defined in (1.1), and let $S(\alpha)$ be defined in (2.5). We write*

$$G=\begin{pmatrix} a_{1,5} & \cdots & a_{1,9}\\ \vdots & \cdots & \vdots\\ a_{5,5} & \cdots & a_{5,9}\end{pmatrix}. \tag{6.1}$$

Suppose that $\det(G)\neq 0$. Then we have

$$\int_{\mathfrak{m}}|S(\alpha)|\,d\alpha \ll X^{n-2}L^{-K/20},$$

where the implied constant depends on A and K.

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Throughout this section, we shall assume that the matrix G given by (6.1) is invertible.

LEMMA 6.2. *Let $\tau \neq 0$ be a real number. Then we have*

$$\int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} \sum_{|x| \ll X} \min\{X, \|x\tau(\alpha-\beta)\|^{-1}\}\, d\alpha\, d\beta \ll LQ^{7/2}X^{-2},$$

where the implied constant depends on τ.

Proof. Without loss of generality, we assume that $0 < |\tau| \leqslant 1$. Thus $|\tau(\alpha-\beta)| \leqslant 1$. We introduce

$$\mathcal{M} = \bigcup_{1 \leqslant q \leqslant Q^{1/2}} \bigcup_{\substack{-q \leqslant a \leqslant q \\ (a,q)=1}} \left\{ |\alpha - \frac{a}{q}| \leqslant \frac{Q^{1/2}}{qX^2} \right\}.$$

By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \leqslant q \leqslant X^2 Q^{-1/2}$ and $|\tau(\alpha-\beta) - a/q| \leqslant Q^{1/2}(qX^2)^{-1}$. Since $|\tau(\alpha-\beta)| \leqslant 1$, one has $-q \leqslant a \leqslant q$. If $\tau(\alpha-\beta) \notin \mathcal{M}$, then $q > Q^{1/2}$. By Vaughan [12, Lemma 2.2],

$$\sum_{|x| \ll X} \min\{X,\ \|x\tau(\alpha-\beta)\|^{-1}\} \ll LQ^{-1/2}X^2.$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathfrak{m}(Q)} \int_{\substack{\mathfrak{m}(Q) \\ \tau(\alpha-\beta) \notin \mathcal{M}}} \sum_{|x| \ll X} \min\{X,\ \|x\tau(\alpha-\beta)\|^{-1}\}\, d\alpha d\beta \\ \ll LQ^{-1/2}X^2 \int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} d\alpha\, d\beta \ll LQ^{7/2}X^{-2}.\end{aligned}$$

When $\tau(\alpha-\beta) \in \mathcal{M}$, we apply the trivial bound to the summation over x to deduce that

$$\begin{aligned}\int_{\mathfrak{m}(Q)} \int_{\substack{\mathfrak{m}(Q) \\ \tau(\alpha-\beta) \in \mathcal{M}}} \sum_{|x| \ll X} \min\{X,\ \|x\tau(\alpha-\beta)\|^{-1}\}\, d\alpha\, d\beta \\ \ll X^2 \int_{\mathfrak{m}(Q)} \int_{\substack{\mathfrak{m}(Q) \\ \tau(\alpha-\beta) \in \mathcal{M}}} d\alpha\, d\beta \ll X^2(Q^2X^{-2}QX^{-2}) = Q^3X^{-2}.\end{aligned}$$

The desired conclusion follows from above immediately. $\square$

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To introduce the next lemma, we define

$$\Phi(\alpha) = \min\{X, \ \|\alpha\|^{-1}\}.$$

For $\mathbf{v} = (v_1, \ldots, v_5) \in \mathbb{Z}^5$ and G given by (6.1), we write

$$2G\mathbf{v} = \begin{pmatrix} g_1(\mathbf{v}) \\ \vdots \\ g_5(\mathbf{v}) \end{pmatrix}. \tag{6.2}$$

LEMMA 6.3. *One has*

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|^2 \, d\alpha \ll X^{2n-10} L^{2n-6} \int_0^1 \left(\int_{\mathfrak{m}(Q)} J_\gamma(\alpha) \, d\alpha \right) \Phi(\gamma) d\gamma, \tag{6.3}$$

where

$$J_\gamma(\alpha) = \sum_{|\mathbf{v}| \leqslant X} \left| \sum_{|z| \leqslant X} \Lambda(z)\Lambda(z+v_1) e(\alpha z g_5(\mathbf{v})) e(\gamma z) \right| \prod_{j=1}^{4} \Phi(g_j(\mathbf{v})\alpha). \tag{6.4}$$

Proof. Let

$$r(\mathbf{y}) = \sum_{i=1}^{4} \sum_{j=1}^{4} a_{i,j} y_i y_j, \quad q(\mathbf{z}) = \sum_{i=5}^{9} \sum_{j=5}^{9} a_{i,j} z_i z_j \quad \text{and}$$

$$p(\mathbf{w}) = \sum_{i=10}^{n} \sum_{j=10}^{n} a_{i,j} w_i w_j.$$

We set

$$B = (2a_{i,j})_{1 \leqslant i \leqslant 4, 10 \leqslant j \leqslant n} \quad \text{and} \quad C = (2a_{i,j})_{5 \leqslant i \leqslant 9, 10 \leqslant j \leqslant n}.$$

Then f can be written in the form

$$f(\mathbf{x}) = r(\mathbf{y}) + y_1 g_1(\mathbf{z}) + \cdots + y_4 g_4(\mathbf{z}) + q(\mathbf{z}) + \mathbf{y}^T B \mathbf{w} + \mathbf{z}^T C \mathbf{w} + p(\mathbf{w}),$$

where $\mathbf{z} = (z_1, \ldots, z_5)$, $\mathbf{y} = (y_1, \ldots, y_4)$, $\mathbf{w} = (w_1, \ldots, w_{n-9})$. Note that $\mathbf{y}^T B \mathbf{w} + \mathbf{z}^T C \mathbf{w} + p(\mathbf{w})$ vanishes if $n = 9$. Therefore, one has

$$\begin{aligned} S(\alpha) = & \sum_{\substack{1 \leqslant \mathbf{y} \leqslant X \\ 1 \leqslant \mathbf{w} \leqslant X}} \sum_{1 \leqslant \mathbf{z} \leqslant X} \Lambda(\mathbf{z}) e(\alpha(y_1 g_1(\mathbf{z}) + \cdots + y_4 g_4(\mathbf{z}) + q(\mathbf{z}) + \mathbf{z}^T C \mathbf{w})) \\ & \times \Lambda(\mathbf{y}) \Lambda(\mathbf{w}) e(\alpha(r(\mathbf{y}) + \mathbf{y}^T B \mathbf{w} + p(\mathbf{w}))). \end{aligned}$$

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By Cauchy's inequality,

$$|S(\alpha)|^2 \leqslant X^{n-5}L^{2n-10}T(\alpha), \tag{6.5}$$

where

$$T(\alpha) = \sum_{\substack{1\leqslant \mathbf{y}\leqslant X\\ 1\leqslant \mathbf{w}\leqslant X}} \left| \sum_{1\leqslant \mathbf{z}\leqslant X} \Lambda(\mathbf{z}) e\left(\alpha\left(\sum_{j=1}^{4} y_j g_j(\mathbf{z}) + q(\mathbf{z}) + \mathbf{z}^T C\mathbf{w}\right)\right)\right|^2.$$

Then we deduce that

$$\begin{aligned} T(\alpha) &= \sum_{\substack{1\leqslant \mathbf{y}\leqslant X\\ 1\leqslant \mathbf{w}\leqslant X}} \sum_{1\leqslant \mathbf{z}_1\leqslant X} \sum_{1\leqslant \mathbf{z}_2\leqslant X} \Lambda(\mathbf{z}_1)\Lambda(\mathbf{z}_2) \\ &\quad \times e\left(\alpha\left(\sum_{j=1}^{4} y_j g_j(\mathbf{z}_1 - \mathbf{z}_2) + q(\mathbf{z}_1) - q(\mathbf{z}_2)\right)\right) e(\alpha(\mathbf{z}_1 - \mathbf{z}_2)^T C\mathbf{w}) \\ &= \sum_{1\leqslant \mathbf{z}_1\leqslant X} \sum_{1\leqslant \mathbf{z}_2\leqslant X} \Lambda(\mathbf{z}_1)\Lambda(\mathbf{z}_2) \sum_{\substack{1\leqslant \mathbf{y}\leqslant X\\ 1\leqslant \mathbf{w}\leqslant X}} \\ &\quad \times e\left(\alpha\left(\sum_{j=1}^{4} y_j g_j(\mathbf{z}_1 - \mathbf{z}_2) + q(\mathbf{z}_1) - q(\mathbf{z}_2)\right)\right) e(\alpha(\mathbf{z}_1 - \mathbf{z}_2)^T C\mathbf{w}). \end{aligned}$$

By changing variables $\mathbf{z}_1 = \mathbf{z}_2 + \mathbf{v}$, we have

$$\begin{aligned} T(\alpha) &= \sum_{1\leqslant \mathbf{z}\leqslant X} \sum_{\substack{|\mathbf{v}|\leqslant X\\ 1\leqslant \mathbf{v}+\mathbf{z}\leqslant X}} \Lambda(\mathbf{z})\Lambda(\mathbf{z}+\mathbf{v}) \sum_{\substack{1\leqslant \mathbf{y}\leqslant X\\ 1\leqslant \mathbf{w}\leqslant X}} \\ &\quad \times e\left(\alpha\left(\sum_{j=1}^{4} y_j g_j(\mathbf{v}) + q(\mathbf{z}+\mathbf{v}) - q(\mathbf{z})\right)\right) e(\alpha \mathbf{v}^T C\mathbf{w}). \end{aligned}$$

We exchange the summation over $\mathbf{z}$ and the summation over $\mathbf{v}$ to obtain

$$T(\alpha) = \sum_{|\mathbf{v}|\leqslant X} \left(\sum_{1\leqslant \mathbf{y}\leqslant X} e\left(\alpha \sum_{j=1}^{4} y_j g_j(\mathbf{v})\right)\right) R(\mathbf{v}) \prod_{j=1}^{5} \mathcal{K}_{j,\mathbf{v}}(\alpha), \tag{6.6}$$

where

$$R(\mathbf{v}) = e(\alpha q(\mathbf{v})) \sum_{1\leqslant \mathbf{w}\leqslant X} e(\alpha(\mathbf{v}^T C\mathbf{w}))$$

and

$$\mathcal{K}_{j,\mathbf{v}}(\alpha) = \sum_{\substack{1\leqslant z_j\leqslant X\\ 1-v_j\leqslant z_j\leqslant X-v_j}} \Lambda(z_j)\Lambda(z_j+v_j)e\bigg(2\alpha z_j \sum_{k=1}^{5} a_{j+4,k+4}v_k\bigg). \tag{6.7}$$

The range of z_j in summation (6.7) depends on v_j. We first follow the standard argument (see for example the argument around (15) in [15]) to remove the dependence on v_j. We write

$$\mathcal{G}_{v_1}(\gamma) = \sum_{\substack{1\leqslant z\leqslant X\\ 1-v_1\leqslant z\leqslant X-v_1}} e(-z\gamma) \tag{6.8}$$

and

$$\mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma) = \sum_{|z|\leqslant X} \Lambda(z)\Lambda(z+v_1)e(\alpha z g_5(\mathbf{v}))e(\gamma z). \tag{6.9}$$

Then we deduce from (6.7)–(6.9) that

$$\mathcal{K}_{1,\mathbf{v}}(\alpha) = \int_0^1 \mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma)\mathcal{G}_{v_1}(\gamma)\,d\gamma. \tag{6.10}$$

On substituting (6.10) into (6.6), we obtain

$$\begin{aligned} T(\alpha) &= \sum_{|\mathbf{v}|\leqslant X} R(\mathbf{v}) \prod_{j=2}^{5} \mathcal{K}_{j,\mathbf{v}}(\alpha)\bigg(\sum_{1\leqslant \mathbf{y}\leqslant X} e\bigg(\alpha \sum_{j=1}^{4} y_j g_j(\mathbf{v})\bigg)\bigg)\\ &\quad\times \int_0^1 \mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma)\mathcal{G}_{v_1}(\gamma)\,d\gamma\\ &= \int_0^1 \sum_{|\mathbf{v}|\leqslant X} R(\mathbf{v}) \prod_{j=2}^{5} \mathcal{K}_{j,\mathbf{v}}(\alpha)\bigg(\sum_{1\leqslant \mathbf{y}\leqslant X} e\bigg(\alpha \sum_{j=1}^{4} y_j g_j(\mathbf{v})\bigg)\bigg)\\ &\quad\times \mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma)\mathcal{G}_{v_1}(\gamma)d\gamma. \end{aligned}$$

Then we conclude that

$$|T(\alpha)| \ll X^{n-5}L^4 \int_0^1 \sum_{|\mathbf{v}|\leqslant X} |\mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma)| \prod_{j=1}^{4} \Phi(g_j(\mathbf{v})\alpha)\Phi(\gamma)\,d\gamma. \tag{6.11}$$

By putting (6.11) into (6.5), one has

$$|S(\alpha)|^2 \ll X^{2n-10}L^{2n-6}\int_0^1 \sum_{|\mathbf{v}|\leqslant X} |\mathcal{K}_{0,\mathbf{v}}(\alpha,\gamma)| \prod_{j=1}^{4} \Phi(g_j(\mathbf{v})\alpha)\Phi(\gamma)\,d\gamma.$$

Therefore,

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|^2\,d\alpha \ll X^{2n-10}L^{2n-6}\int_0^1 \left(\int_{\mathfrak{m}(Q)} J_\gamma(\alpha)\,d\alpha\right)\Phi(\gamma)\,d\gamma.$$

The proof is completed. □

LEMMA 6.4. *Let $J_\gamma(\alpha)$ be defined in (6.4). Then one has uniformly for $\gamma\in[0,1]$ that*

$$\int_{\mathfrak{m}(Q)} J_\gamma(\alpha)\,d\alpha \ll L^{25/4}Q^{-17/8}X^8.$$

Proof. We deduce by changing variables $\mathbf{h}=2G\mathbf{v}$ that

$$J_\gamma(\alpha)=\sum_{\substack{|\mathbf{h}|\leqslant cX\\ (2G)^{-1}\mathbf{h}\in\mathbb{Z}^5\\ |(2G)^{-1}\mathbf{h}|\leqslant X}} \left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda(z+\sum_{j=1}^{5} b_jh_j)e(\alpha z h_5)e(\gamma z)\right| \prod_{j=1}^{4}\Phi(h_j\alpha)$$

for some constants $c, b_1, \ldots, b_5$ depending only on G. We point out that $b_1, \ldots, b_5$ are rational numbers, and we extend the domain of function $\Lambda(x)$ by taking $\Lambda(x)=0$ if $x\in\mathbb{Q}\setminus\mathbb{N}$. Then we have

$$J_\gamma(\alpha)\leqslant \sum_{|\mathbf{u}|\leqslant cX}\sum_{|h|\leqslant cX}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\left(z+\sum_{j=1}^{4} b_ju_j+b_5h\right)e(\alpha z h)e(\gamma z)\right| \times\prod_{j=1}^{4}\Phi(u_j\alpha).$$

We first handle the easier case $b_5=0$. In this case, we can easily obtain a nontrivial estimate for the summation over h. By Cauchy's inequality and Lemma 4.1, one has

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$$\Big(\sum_{|h|\leqslant cX}\Big|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\Big(z+\sum_{j=1}^{4}b_ju_j\Big)e(\alpha zh)e(\gamma z)\Big|\Big)^2$$
$$\leqslant (2cX+1)\sum_{|h|\leqslant cX}\Big|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\Big(z+\sum_{j=1}^{4}b_ju_j\Big)e(\alpha zh)e(\gamma z)\Big|^2$$
$$\ll X^2L^4\sum_{|x|\ll X}\min\{X,\|x\alpha\|^{-1}\}.$$

For $\alpha\in\mathfrak{m}(Q)$, we apply Lemma 4.2 to deduce from above

$$\sum_{|h|\leqslant cX}\Big|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\Big(z+\sum_{j=1}^{4}b_ju_j\Big)e(\alpha zh)e(\gamma z)\Big|\ll L^{5/2}Q^{-1/2}X^2.$$

Then for $\alpha\in\mathfrak{m}(Q)$, we obtain

$$J_\gamma(\alpha)\ll L^{1/2}Q^{-1/2}X^2\sum_{|\mathbf{u}|\leqslant cX}\prod_{j=1}^{4}\Phi(u_j\alpha)\ll L^{5/2}Q^{-9/2}X^{10},$$

and thereby

$$\int_{\mathfrak{m}(Q)}J_\gamma(\alpha)\,d\alpha\ll L^{13/2}Q^{-5/2}X^8 \tag{6.12}$$

provided that $b_5=0$. From now on, we assume $b_5\neq 0$. Then we have

$$\sum_{|h|\leqslant cX}\Big|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\Big(z+\sum_{j=1}^{4}b_ju_j+b_5h\Big)e(\alpha zh)e(\gamma z)\Big|$$
$$=\sum_{\substack{|k|\leqslant c'X\\ \frac{1}{b_5}(k-\sum_{j=1}^{4}b_ju_j)\in\mathbb{Z}\\ |\frac{1}{b_5}(k-\sum_{j=1}^{4}b_ju_j)|\leqslant cX}}\Big|\sum_{|z|\leqslant X}\Lambda(z)\Lambda(z+k)e\Big(\frac{\alpha}{b_5}z\Big(k-\sum_{j=1}^{4}b_ju_j\Big)\Big)e(\gamma z)\Big|$$

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for some constant c' depending only on $b_1, \ldots, b_5$ and c. Therefore, one has

$$\sum_{|h|\leqslant cX}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\left(z+\sum_{j=1}^{4}b_ju_j+b_5h\right)e(\alpha zh)e(\gamma z)\right|$$
$$\leqslant \sum_{|k|\leqslant c'X}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda(z+k)e\left(\frac{\alpha}{b_5}z\left(k-\sum_{j=1}^{4}b_ju_j\right)\right)e(\gamma z)\right|.$$

We apply Cauchy's inequality to deduce that

$$\sum_{|h|\leqslant cX}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda\left(z+\sum_{j=1}^{4}b_ju_j+b_5h\right)e(\alpha zh)e(\gamma z)\right|\leqslant(2c'X+1)^{1/2}$$
$$\times\left(\sum_{|k|\leqslant c'X}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda(z+k)e\left(\frac{\alpha}{b_5}z\left(k-\sum_{j=1}^{4}b_ju_j\right)\right)e(\gamma z)\right|^2\right)^{1/2}.$$

We apply Cauchy's inequality again to obtain

$$J_\gamma(\alpha)\leqslant(2c'X+1)^{1/2}\Xi_\gamma(\alpha)^{1/2}\left(\sum_{|\mathbf{u}|\leqslant cX}\prod_{j=1}^{4}\Phi(u_j\alpha)\right)^{1/2},$$

where $\Xi_\gamma(\alpha)$ is defined as

$$\Xi_\gamma(\alpha)=\sum_{|\mathbf{u}|\leqslant cX}\sum_{|k|\leqslant c'X}\left|\sum_{|z|\leqslant X}\Lambda(z)\Lambda(z+k)e\left(\frac{\alpha}{b_5}z\left(k-\sum_{j=1}^{4}b_ju_j\right)\right)e(\gamma z)\right|^2$$
$$\times\prod_{j=1}^{4}\Phi(u_j\alpha).$$

By Lemma 4.2,

$$J_\gamma(\alpha)\ll L^2Q^{-2}X^{9/2}\Xi_\gamma(\alpha)^{1/2}.$$

Therefore, we have

$$\int_{\mathfrak{m}(Q)}J_\gamma(\alpha)\,d\alpha\ll L^2Q^{-2}X^{9/2}\left(\int_{\mathfrak{m}(Q)}d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}(Q)}\Xi_\gamma(\alpha)\,d\alpha\right)^{1/2}$$
$$\ll L^2Q^{-1}X^{7/2}\left(\int_{\mathfrak{m}(Q)}\Xi_\gamma(\alpha)\,d\alpha\right)^{1/2}. \tag{6.13}$$

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Now it suffices to estimate $\int_{\mathfrak{m}(Q)} \Xi_\gamma(\alpha)\, d\alpha$. We observe

$$\begin{aligned}
&\int_{\mathfrak{m}(Q)} \Xi_\gamma(\alpha)\, d\alpha \\
&\quad = \int_{\mathfrak{m}(Q)} \sum_{|\mathbf{u}|\leqslant cX} \sum_{|k|\leqslant c'X} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \varpi(z_1, z_2, k) e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)k\biggr) \\
&\qquad \times \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \\
&\quad = \int_{\mathfrak{m}(Q)} \sum_{|k|\leqslant c'X} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \varpi(z_1, z_2, k) e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)k\biggr) \sum_{|\mathbf{u}|\leqslant cX} \\
&\qquad \times \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha,
\end{aligned}$$

where

$$\varpi(z_1, z_2, k) = \Lambda(z_1)\Lambda(z_1 + k)\Lambda(z_2)\Lambda(z_2 + k)e(\gamma(z_1 - z_2))$$

and

$$\Pi(\alpha, \mathbf{u}, z_1, z_2) = e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)\sum_{j=1}^{4} b_j u_j\biggr) \prod_{j=1}^{4} \Phi(u_j\alpha).$$

We exchange the order of summation and integration to conclude that

$$\begin{aligned}
&\int_{\mathfrak{m}(Q)} \Xi_\gamma(\alpha)\, d\alpha \\
&\quad = \sum_{|k|\leqslant c'X} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \varpi(z_1, z_2, k) \int_{\mathfrak{m}(Q)} e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)k\biggr) \\
&\qquad \times \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \\
&\quad \ll L^4 \sum_{|k|\leqslant c'X} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \biggl| \int_{\mathfrak{m}(Q)} e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)k\biggr) \\
&\qquad \times \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \biggr| \\
&\quad = L^4 \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \sum_{|k|\leqslant c'X} \biggl| \int_{\mathfrak{m}(Q)} e\biggl(\frac{\alpha}{b_5}(z_1 - z_2)k\biggr) \\
&\qquad \times \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \biggr|.
\end{aligned}$$

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Then the Cauchy–Schwarz inequality implies

$$\begin{aligned}
&\left(\int_{\mathfrak{m}(Q)} \Xi_\gamma(\alpha)\, d\alpha\right)^2 \\
&\ll L^8 X^3 \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \sum_{|k|\leqslant c'X} \left| \int_{\mathfrak{m}(Q)} e\left(\frac{\alpha}{b_5}(z_1 - z_2)k\right) \right. \\
&\qquad \left. \times \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \right|^2. \qquad (6.14)
\end{aligned}$$

Now we apply the method developed by the author [16] to deduce that

$$\begin{aligned}
&\sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \sum_{|k|\leqslant c'X} \left| \int_{\mathfrak{m}(Q)} e\left(\frac{\alpha}{b_5}(z_1 - z_2)k\right) \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \right|^2 \\
&= \int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \sum_{|k|\leqslant c'X} e\left(\frac{\alpha - \beta}{b_5}(z_1 - z_2)k\right) \\
&\quad \times \sum_{|\mathbf{u}_1|\leqslant cX} \Pi(\alpha, \mathbf{u}_1, z_1, z_2) \sum_{|\mathbf{u}_2|\leqslant cX} \Pi(-\beta, \mathbf{u}_2, z_1, z_2)\, d\alpha\, d\beta \\
&\leqslant \int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \left| \sum_{|k|\leqslant c'X} e\left(\frac{\alpha - \beta}{b_5}(z_1 - z_2)k\right) \right| \\
&\quad \times \sum_{|\mathbf{u}_1|\leqslant cX} \prod_{j=1}^{4} \Phi(u_j \alpha) \sum_{|\mathbf{u}_2|\leqslant cX} \prod_{j=1}^{4} \Phi(u_j' \beta)\, d\alpha\, d\beta,
\end{aligned}$$

where $\mathbf{u}_1 = (u_1, \ldots, u_4)^T \in \mathbb{Z}^4$ and $\mathbf{u}_2 = (u_1', \ldots, u_4')^T \in \mathbb{Z}^4$. Therefore, we obtain by Lemma 4.2

$$\begin{aligned}
&\sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \sum_{|k|\leqslant c'X} \left| \int_{\mathfrak{m}(Q)} e\left(\frac{\alpha}{b_5}(z_1 - z_2)k\right) \sum_{|\mathbf{u}|\leqslant cX} \Pi(\alpha, \mathbf{u}, z_1, z_2)\, d\alpha \right|^2 \\
&\ll \int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} \sum_{|z_1|\leqslant X} \sum_{|z_2|\leqslant X} \min\left\{X, \left\| \frac{\alpha - \beta}{b_5}(z_1 - z_2) \right\|^{-1}\right\} \\
&\quad \times (L^4 Q^{-4} X^8)^2\, d\alpha\, d\beta \\
&\ll L^8 Q^{-8} X^{17} \int_{\mathfrak{m}(Q)} \int_{\mathfrak{m}(Q)} \sum_{|x|\leqslant X} \min\left\{X, \left\| \frac{\alpha - \beta}{b_5} x \right\|^{-1}\right\} d\alpha\, d\beta.
\end{aligned}$$

Then we conclude from Lemma 6.2 that

$$\sum_{|z_1|\leqslant X}\sum_{|z_2|\leqslant X}\sum_{|k|\leqslant c'X}\left|\int_{\mathfrak{m}(Q)} e\left(\frac{\alpha}{b_5}(z_1-z_2)k\right)\sum_{|\mathbf{u}|\leqslant cX}\Pi(\alpha,\mathbf{u},z_1,z_2)\,d\alpha\right|^2$$

$$\ll L^9Q^{-9/2}X^{15}. \tag{6.15}$$

By (6.14) and (6.15),

$$\int_{\mathfrak{m}(Q)} \Xi_\gamma(\alpha)\,d\alpha \ll L^{17/2}Q^{-9/4}X^9. \tag{6.16}$$

By substituting (6.16) into (6.13), we obtain

$$\int_{\mathfrak{m}(Q)} J_\gamma(\alpha)\,d\alpha \ll L^{25/4}Q^{-17/8}X^8 \tag{6.17}$$

provided that $b_5 \neq 0$.

We complete the proof in view of the argument around (6.12) and (6.17). ∎

Lemma 6.5. *One has*

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|\,d\alpha \ll L^{n+1}Q^{-1/16}X^{n-2}.$$

Proof. By Cauchy's inequality,

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|\,d\alpha \leqslant \left(\int_{\mathfrak{m}(Q)} d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}(Q)} |S(\alpha)|^2\,d\alpha\right)^{1/2}$$

$$\ll QX^{-1}\left(\int_{\mathfrak{m}(Q)} |S(\alpha)|^2\,d\alpha\right)^{1/2}. \tag{6.18}$$

It follows from Lemmas 6.3–6.4 that

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|^2\,d\alpha \ll L^{2n+1}Q^{-17/8}X^{2n-2}\int_0^1 \Phi(\gamma)\,d\gamma \ll L^{2n+2}Q^{-17/8}X^{2n-2}. \tag{6.19}$$

We complete the proof by putting (6.19) into (6.18). ∎

We finish Section 6 by pointing out that Proposition 6.1 follows from Lemma 6.5 by the dyadic argument.

§7. The Proof of Theorem 1.1

By orthogonality, we have

$$N_{f,t}(X) = \int_{X^{-1}}^{1+X^{-1}} S(\alpha)e(-t\alpha)\,d\alpha.$$

Recalling the definitions of $\mathfrak{M}$ and $\mathfrak{m}$ in (2.8) and (2.9), we have

$$N_{f,t}(X) = \int_{\mathfrak{M}} S(\alpha)e(-t\alpha)\,d\alpha + \int_{\mathfrak{m}} S(\alpha)e(-t\alpha)\,d\alpha. \tag{7.1}$$

In light of Lemma 3.6, to establish the asymptotic formula (1.3), it suffices to prove

$$\int_{\mathfrak{m}} |S(\alpha)|\,d\alpha \ll X^{n-2}L^{-K/20}. \tag{7.2}$$

In view of Proposition 6.1 and the work of Liu [9] (see also Remark of Lemma 4.4), the estimate (7.2) holds if there exists an invertible matrix

$$B = \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_5,j_5} \\ \vdots & \cdots & \vdots \\ a_{i_5,j_1} & \cdots & a_{i_5,j_5} \end{pmatrix}$$

with

$$|\{i_1,\ldots,i_5\} \cap \{j_1,\ldots,j_5\}| \leqslant 1.$$

Next we assume $\operatorname{rank}(B) \leqslant 4$ for all $B = (a_{i_k,j_l})_{1\leqslant k,l\leqslant 5}$ satisfying $|\{i_1,\ldots,i_5\} \cap \{j_1,\ldots,j_5\}| \leqslant 1$. This yields $\operatorname{rank}_{\text{off}}(A) \leqslant 4$. By Proposition 5.1, we can establish (7.2) again if $\operatorname{rank}_{\text{off}}(A) \leqslant 3$. It remains to consider the case $\operatorname{rank}_{\text{off}}(A) = 4$. Without loss of generality, we assume that $\operatorname{rank}(C) = 4$, where

$$C = \begin{pmatrix} a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} \\ a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} \\ a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} \\ a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} \end{pmatrix}.$$

Let $\gamma_j = (a_{j,5},\ldots,a_{j,n})^T \in \mathbb{Z}^{n-4}$ for $1 \leqslant j \leqslant n$. Then γ_1, γ_2, γ_3 and γ_4 are linear independent due to $\operatorname{rank}(C) = 4$. For $5 \leqslant k \leqslant n$, we consider

$$B = \begin{pmatrix} a_{1,5} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{4,5} & \cdots & a_{4,n} \\ a_{k,5} & \cdots & a_{k,n} \end{pmatrix} \in M_{5,n-4}(\mathbb{Z}).$$

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According to our assumption, one has $\mathrm{rank}(B) \leqslant 4$. Then we conclude from above that γ_k can be linear represented by γ_1, γ_2, γ_3 and γ_4. Therefore, one has $\mathrm{rank}(H) = 4$, where

$$H = \begin{pmatrix} a_{1,5} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{n,5} & \cdots & a_{n,n} \end{pmatrix} \in M_{n,n-4}(\mathbb{Z}).$$

We obtain $\mathrm{rank}(A) \leqslant \mathrm{rank}(H) + 4 \leqslant 8$. This is contradictory to the condition that $\mathrm{rank}(A) \geqslant 9$. Therefore, we complete the proof of Theorem 1.1.

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School of Mathematics
Hefei University of Technology
Hefei 230009
People's Republic of China
zhaolilu@gmail.com

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