Bull. Aust. Math. Soc. **94** (2016), 110–120
doi:10.1017/S0004972715001549

THE QUENCHING OF SOLUTIONS OF A REACTION–DIFFUSION EQUATION WITH FREE BOUNDARIES

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(Received 6 October 2015; accepted 2 November 2015; first published online 22 January 2016)

Abstract

This paper concerns the quenching phenomena of a reaction–diffusion equation $u_t = u_{xx} + 1/(1-u)$ in a one dimensional varying domain $[g(t), h(t)]$, where $g(t)$ and $h(t)$ are two free boundaries evolving by a Stefan condition. We prove that all solutions will quench regardless of the choice of initial data, and we also show that the quenching set is a compact subset of the initial occupying domain and that the two free boundaries remain bounded.

2010 *Mathematics subject classification*: primary 35K55; secondary 35R35, 35B99.

Keywords and phrases: reaction–diffusion equation, quenching, free-boundary problem.

1. Introduction

The quenching phenomenon has been studied for various types of problems. For example, in [1, 9, 11], the authors considered the initial-boundary problem

$$\begin{cases} v_t = v_{xx} + 1/(1-v) & \text{if } -l < x < l,\ t > 0, \\ v(t,0) = v(t,l) = 0 & \text{if } t > 0, \\ v(0,x) = 0 & \text{if } -l \le x \le l, \end{cases} \tag{1.1}$$

where the first equation in (1.1) arises in the study of electric current transients in polarised ionic conductors (see [9, 11] and references therein for more background). They established the interesting results given in the following theorem.

Theorem 1.1. *There exists a positive constant $l_0 < \sqrt{2}$ such that if $l > l_0$, then the solution of problem (1.1) will quench: that is, there is a finite time $T > 0$ such that the solution v satisfies*

$$0 \le v(t,x) < 1 \quad \textit{for all } (t,x) \in [0,T) \times [-l,l]$$

and

$$\lim_{t\to T} \|v(t,\cdot)\|_{L^\infty([-l,l])} = 1.$$

This research was partly supported by the NSFC (11271285).

However, if $l \le l_0$, then the solution of problem (1.1) cannot quench, even in infinite time.

It is easy to see that the quenching problem is equivalent to the blow-up problem in the sense that, by using the transformation $w = (1 - v)^{-1}$, the first equation in problem (1.1) takes the form

$$w_t = w_{xx} - 2w_x^2/w + w^3.$$

By an argument similar to that in [7], it is easily seen that if the solution v of problem (1.1) quenches, then w will blow up in finite time. Moreover the quenching time of v is equal to the blow-up time of w. We can also easily check that the methods in [6, 14], which are used to study the blow-up problem, will also be useful for the quenching problem. For other forms of quenching problem, please see [2, 8, 10, 12] and the references therein.

Recently, to observe precisely the spreading fronts of an invasive species, Du and Lin [4] used the following free-boundary problem to discuss the spreading of an invasive species in a new environment:

$$\begin{cases} u_t = u_{xx} + u(1-u) & \text{if } g(t) < x < h(t),\ t > 0,\\ u(t, g(t)) = 0,\ g'(t) = -\mu\, u_x(t, g(t)) & \text{if } t > 0,\\ u(t, h(t)) = 0,\ h'(t) = -\mu\, u_x(t, h(t)) & \text{if } t > 0,\\ -g(0) = h(0) = h_0,\ \ u(0, x) = u_0(x) & \text{if } -h_0 \le x \le h_0, \end{cases} \tag{1.2}$$

where $x = g(t)$ and $x = h(t)$ are the free boundaries that represent the spreading fronts of the species whose density is represented by $u(t, x)$, μ is a given positive constant and the initial data $u_0(x)$ and $[-h_0, h_0]$ stand for the initial population and initial occupying interval of the species, respectively (see [4, 13] and the references therein for more background). Du and Lin [4] proved that problem (1.2) admits a unique time-global solution (u, g, h), and obtained a spreading–vanishing dichotomy result for (1.2): as $t \to \infty$, either $u(t, x) \to 1$ and $(g_\infty, h_\infty) = \mathbb{R}$, or $u(t, x) \to 0$ and $h_\infty - g_\infty \le \pi$, where

$$g_\infty := \lim_{t\to\infty} g(t), \quad h_\infty := \lim_{t\to\infty} h(t).$$

Later, Du and Guo [3] extended these conclusions to the problem of higher-space dimensions with radially symmetric parameters. Du and Lou [5] considered problem (1.2) with much more general nonlinear terms, and deduced a complete description of the dynamical behaviour of solutions.

Motivated by these results, we shall consider the quenching phenomena for the free-boundary problem

$$\begin{cases} u_t = u_{xx} + 1/(1-u) & \text{if } g(t) < x < h(t),\ t > 0,\\ u(t, g(t)) = 0,\ g'(t) = -\mu\, u_x(t, g(t)) & \text{if } t > 0,\\ u(t, h(t)) = 0,\ h'(t) = -\mu\, u_x(t, h(t)) & \text{if } t > 0,\\ -g(0) = h(0) = h_0,\ \ u(0, x) = u_0(x) & \text{if } -h_0 \le x \le h_0, \end{cases} \tag{1.3}$$

where the initial function u_0 satisfies

$$u_0 \in C^2([-h_0, h_0]), \quad u_0(\pm h_0) = 0, \quad u_0'(h_0), -u_0'(-h_0) < 0, \quad 0 < u_0 < 1 \quad \text{in } (-h_0, h_0). \tag{1.4}$$

Before proceeding further, we will introduce the following definitions.

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Definition 1.2. Let $G_\tau := \{(t,x):\ 0 < t \le \tau,\ g(t) \le x \le h(t)\}$. For any given $h_0 > 0$ and u_0 satisfying (1.4), a triple (u, g, h) is defined to be a (classical) solution of (1.3) on the time-interval $[0,\tau]$ for some $\tau > 0$ if it belongs to $C^{1,2}(G_\tau) \times C^1([0,\tau]) \times C^1([0,\tau])$ with $\|u\|_{L^\infty(\overline{G_\tau})} < 1$, and all the identities in (1.3) are satisfied pointwise.

Definition 1.3. A solution (u, g, h) (or simply u) is said to quench if there exists a time T^* called the *quenching time* with $0 < T^* < \infty$ such that

$$0 \le u(t,x) < 1 \quad \text{for } (t,x) \in [0,T^*) \times [g(t), h(t)] \quad \text{and} \quad \lim_{t\to T^*} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 1.$$

It follows, from Lemma 2.4 below, that the quenching time in Definition 1.3 is well defined.

The primary purpose of this paper is to explore the influence of free boundaries on the quenching properties of problem (1.3). Our first main result shows that all solutions of problem (1.3) must quench regardless of the choice of initial data.

Theorem 1.4. *Let $h_0 > 0$ and u_0 satisfy (1.4). Then the solution (u, g, h) of problem (1.3) must quench in a finite time T^*.*

It is easily seen from Theorem 1.4 that problem (1.3) admits no time-global solutions, and thus we only need to consider the properties of the quenching solutions in the rest of this paper. Let us denote by T^* the quenching time of problem (1.3). By the Hopf lemma, we can show that the two free-boundary fronts $g(t)$ and $h(t)$ satisfy

$$g'(t) < 0 < h'(t) \quad \text{for } 0 < t < T^*.$$

Thus $\lim_{t\to T^*} g(t)$ and $\lim_{t\to T^*} h(t)$ always exist, and in the subsequent work we define

$$g^* := \lim_{t\to T^*} g(t) \quad \text{and} \quad h^* := \lim_{t\to T^*} h(t). \tag{1.5}$$

Let us also introduce the following notation. Given a solution (u, g, h) that quenches at $T^* < \infty$, we define its *quenching set* by

$$Q(u_0) := \{x \in [g^*, h^*] : \text{there exist } x_n \to x,\ t_n \to T^* \text{ such that } u(t_n, x_n) \to 1 \text{ as } n \to \infty\},$$

where u_0 denotes the initial function. Any element of $Q(u_0)$ is called a *quenching point* of (u, g, h). Clearly, $Q(u_0)$ is closed.

It is worth studying the quenching set of (1.3) and the boundedness of g^* and h^*. The following theorem shows that the quenching set is a compact subset of the initial domain $[-h_0, h_0]$ and the two free boundaries remain bounded.

Theorem 1.5. *Let $h_0 > 0$, u_0 satisfy (1.4) and T^* be the quenching time. Then:*

(i) *the quenching set $Q(u_0)$ is a compact subset of the initial domain $[-h_0, h_0]$; and*
(ii) *there exists a positive constant $C < \infty$ such that $h^*, -g^* \le C$.*

The plan of the paper is the following. Section 2 covers some basic and known results, which are useful for this research. In Section 3, based on the comparison principle, we prove Theorems 1.4 and 1.5.

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2. Some basic results

In this section we recall some basic and known results which will play a fundamental role in this paper. In what follows, we set $f(u) := 1/(1-u)$ for convenience. Let $T_{\max}$ be the maximal existence time of (u, g, h):

$$T_{\max} := \sup\{\tau > 0 : \|u\|_{L^\infty([0,t]\times[g(t),h(t)])} < 1 \text{ and } g(t) > -\infty,\ h(t) < \infty \text{ for all } t \in [0, \tau)\}.$$

It is easy to check that $f \in C^2$ when $t \in (0, T_{\max})$, and thus some of the proofs here are omitted or only sketched, since they are similar to those of [4, 5]. Firstly, the comparison principle suitable for our needs is stated in the following lemma.

Lemma 2.1. *Suppose for some $\overline{T} \in (0, \infty)$, $\overline{g}$, $\overline{h} \in C^1([0, \overline{T}])$, $\overline{u} \in C(\overline{D_{\overline{T}}}) \cap C^{1,2}(D_{\overline{T}})$ satisfying $\|\overline{u}\|_{L^\infty(\overline{D_{\overline{T}}})} < 1$, with $D_{\overline{T}} = \{(t, x) \in \mathbb{R}^2 : 0 < t \le \overline{T},\ \overline{g}(t) < x < \overline{h}(t)\}$, and*

$$\begin{cases} \overline{u}_t \ge \overline{u}_{xx} + f(\overline{u}) & \text{if } 0 < t \le \overline{T},\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{g}'(t) \le -\mu \overline{u}_x & \text{if } 0 < t \le \overline{T},\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu \overline{u}_x & \text{if } 0 < t \le \overline{T},\ x = \overline{h}(t). \end{cases}$$

If $[-h_0, h_0] \subseteq [\overline{g}(0), \overline{h}(0)]$ and $u_0(x) \le \overline{u}(0, x)$ in $[-h_0, h_0]$, then the solution (u, g, h) of problem (1.3) satisfies

$$[g(t), h(t)] \subseteq [\overline{g}(t), \overline{h}(t)] \quad \text{for } t \in (0, \overline{T}],$$
$$u(t, x) \le \overline{u}(t, x) \quad \text{for } (t, x) \in (0, \overline{T}] \times [g(t), h(t)].$$

Remark 2.2. Note that $f(0) > 0$ in our problem which is very different from the case in [4]. However, Lemma 2.1 can be proved by an argument similar to that in [4]. The function $\overline{u}$, or the triple $(\overline{u}, \overline{g}, \overline{h})$, in Lemma 2.1, is often called a supersolution of problem (1.3). A subsolution can be defined analogously by reversing all the inequalities. We also have corresponding comparison results for subsolutions.

Next, we present the local existence and uniqueness result for problem (1.3).

Theorem 2.3. *For any given $h_0 > 0$, any u_0 satisfying (1.4) and any $\alpha \in (0, 1)$, there is a $\tau > 0$ such that problem (1.3) admits a unique solution*

$$(u, g, h) \in C^{(1+\alpha)/2, 1+\alpha}(\overline{G_\tau}) \times C^{1+\alpha/2}([0, \tau]) \times C^{1+\alpha/2}([0, \tau]),$$

with

$$\|u\|_{L^\infty(\overline{G_\tau})} < 1,$$

where $G_\tau := \{(t, x) : t \in (0, \tau],\ x \in [g(t), h(t)]\}$, and τ only depends on h_0, α and $\|u_0\|_{C^2([-h_0, h_0])}$.

Proof. The main idea of the proof is to change problem (1.3) into the equivalent fixed-boundary problem, define a complete metric space and construct a contraction mapping on this space. Then we can prove the theorem by using the contraction mapping theorem. Since $f \in C^2$ when $t \in (0, T_{\max})$, the proof of the theorem follows the same argument as in [4] and we omit the details here. As in [4], by applying the Schauder estimates to the equivalent fixed-boundary problem used in the proof, additional regularity for u can be obtained: namely, $u \in C^{1+\alpha/2, 2+\alpha}(G_\tau)$. $\square$

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The following lemma gives a general boundedness result for $g'(t)$ and $h'(t)$.

Lemma 2.4. *Assume that (u, g, h) is a solution of problem (1.3) defined for $t \in [0, \hat{T})$ for some $\hat{T} \in (0, \infty)$, and there exists $0 < C_1 < 1$ such that*

$$u(t, x) \le C_1 \quad \text{for all } (t, x) \in [0, \hat{T}) \times [g(t), h(t)].$$

Then there exists C_2, depending on C_1 but independent of $\hat{T}$, such that

$$-g'(t),\ h'(t) \in (0, C_2] \quad \text{for } t \in (0, \hat{T}).$$

Moreover, the solution can be extended to some interval $(0, \tau)$ with $\tau > \hat{T}$.

Proof. Construct the auxiliary function

$$w(t, x) := C_1[2L(h(t) - x) - L^2(h(t) - x)^2]$$

over the domain

$$\Omega_L =: \{(t, x) : 0 < t < \hat{T},\ h(t) - L^{-1} < x < h(t)\},$$

where $L := \max\{h_0^{-1},\ (2C_1(1 - C_1))^{-1/2},\ 4\|u_0\|_{C^1([-h_0,h_0])}/(3C_1)\}$. Following the proof of [4, Lemma 2.2],

$$h'(t) = -\mu u_x(t, h(t)) \le -\mu w_x(t, h(t)) = 2LC_1\mu := C_2 \quad \text{for } t \in [0, \hat{T}).$$

Similarly, one can deduce that $g'(t) \ge -C_2$ for $t \in [0, \hat{T})$.

Now fix $\delta_0 \in (0, \hat{T})$. By a standard L^p estimate, the Sobolev embedding theorem and the Hölder estimates for the parabolic equation, there exists a constant $C_3 > 0$ depending only on δ_0, $\hat{T}$, L, C_1 and C_2 such that

$$\|u(t, \cdot)\|_{C^{2+\alpha}([g(t),h(t)])} \le C_3 \quad \text{for } t \in [\delta_0, \hat{T}).$$

Since $0 \le u(t, x) \le C_1 < 1$ for $(t, x) \in [0, \hat{T}) \times [g(t), h(t)]$, it then follows from the proof of Theorem 2.3 that there exists a $t_0 > 0$ depending only on C_1, C_2 and C_3 such that the solution of problem (1.3) with initial time $\hat{T} - t_0$ can be extended uniquely to the time $\hat{T} + t_0$. This completes the proof of the lemma. □

The above lemma implies that the solution of problem (1.3) can be extended as long as u remains smaller than one. In particular, the free boundaries remain bounded as long as u is less than one. Therefore we have the following result.

Corollary 2.5. *Suppose that the solution (u, g, h) of problem (1.3) is defined on some maximal interval $(0, T_{\max})$ and $T_{\max} < \infty$. Then (u, g, h) must quench and the quenching time satisfies $T^* = T_{\max}$.*

Proof. We will use an indirect argument and assume that $T_{\max} < \infty$ and (u, g, h) does not quench. Then there exists a constant $0 < \tilde{C} < 1$ such that

$$u(t, x) \le \tilde{C} \quad \text{for } (t, x) \in (0, T_{\max}) \times [g(t), h(t)].$$

By Lemma 2.4, u can be extended beyond $T_{\max}$, which is in contradiction to the definition of $T_{\max}$. □

3. Quenching phenomenon

In this section, we study the profiles of the solution of problem (1.3) and prove Theorems 1.4 and 1.5.

3.1. Quenching of solutions. This subsection covers the proof of Theorem 1.4, which is based on the comparison principle and some well-known results. Let $T_{\max}$ be the maximal existence time of (u, g, h).

Proof of Theorem 1.4. Define the auxiliary function $\underline{f}(u) := u^m(1-u)$ with $m = 1/2$. There exists a small constant $0 < \epsilon_0 \leq 1/12$ such that, for any $0 < \epsilon \leq \epsilon_0$, the equation $\underline{f}(u) = \underline{f}(\epsilon)$ has roots ϵ and $1 - a(\epsilon)$ with $0 < a(\epsilon) < 1 - \epsilon$ and $a(\epsilon) \to 0$ as $\epsilon \to 0$. Define $f_\epsilon(u) := \underline{f}(u+\epsilon) - \underline{f}(\epsilon)$. It is easily checked that

$$\begin{cases} f_\epsilon \in C^2([0,\infty)), \quad f_\epsilon'(0) = \underline{f}'(\epsilon) > 0, \quad f_\epsilon(u) \nearrow \underline{f}(u) \quad \text{as } \epsilon \to 0, \\ f_\epsilon(0) = f_\epsilon(1 - a(\epsilon) - \epsilon) = 0, \quad f_\epsilon(u) > 0 \quad \text{for } u \in (0, 1 - a(\epsilon) - \epsilon). \end{cases}$$

For any given $h_0 > 0$, since $\lim_{\epsilon\to 0} f_\epsilon'(0) = \infty$, no matter how small h_0 is, we can find $\epsilon^* \in (0, \epsilon_0]$ sufficiently small such that $h_0 \geq \pi/(2\sqrt{f_{\epsilon^*}'(0)})$. Then the solution (u_*, g_*, h_*) of problem (1.3) with $f(\cdot) = f_{\epsilon^*}(\cdot)$ satisfies

$$-\lim_{t\to\infty} g_*(t) = \lim_{t\to\infty} h_*(t) = \infty.$$

Thus there exists a constant $T_1 < \infty$ such that

$$h_*(T_1) - g_*(T_1) \geq 2l_0,$$

where l_0 is given in Theorem 1.1.

Noting that $f_{\epsilon^*}(s) < f(s)$ for $s \in [0, 1)$, then applying the comparison principle,

$$[g_*(t), h_*(t)] \subseteq (g(t), h(t)) \quad \text{for } 0 < t < T_{\max}.$$

It follows, from Corollary 2.5, that if $T_{\max} \leq T_1$, then the solution (u, g, h) must quench. If $T_{\max} > T_1$, there exists $\sigma > 0$ sufficiently small such that $T_1 < \tilde{T} < T_{\max}$ and $h(\tilde{T}) - g(\tilde{T}) > 2l_0$ with $\tilde{T} := T_1 + \sigma$. Theorem 1.1 implies that the solution $\underline{u}$ of problem (1.1) with $l = (h(\tilde{T}) - g(\tilde{T}))/2$ quenches. By the comparison principle, (u, g, h) must quench in this case and $T_{\max} < \infty$ since $\underline{u}$ is a subsolution of problem (1.3). Therefore all solutions of problem (1.3) will quench. The proof of the theorem is complete. □

Remark 3.1. From the proof, we see that all solutions of problem (1.3) will quench for any initial function u_0 satisfying (1.4), and thus there exist no time-global solutions.

In the rest of this subsection, we show some properties of solutions of (1.3) which will be used in the next subsection. We use T^* to denote the quenching time of (u, g, h).

Lemma 3.2. *Let (u, g, h) be a solution of problem (1.3) with u_0 satisfying (1.4). Then*

$$-2h_0 < g(t) + h(t) < 2h_0 \quad \text{for } t \in [0, T^*),$$
$$u_x(t, x) > 0 > u_x(t, y) \, \text{for } t \in [0, T^*), x \in [g(t), -h_0] \text{ and } y \in [h_0, h(t)].$$

Proof. The proof is similar to that of [5, Lemma 2.8] and we omit the details here. □

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3.2. Quenching set. In this subsection we study the quenching set and the boundedness of g^* and h^* (where g^* and h^* are defined by (1.5)) and prove Theorem 1.5.

PROOF OF THEOREM 1.5. We prove this theorem in two steps.

Step 1. $Q(u_0)$ is a compact subset of $[-h_0, h_0]$.

Let us first use an indirect argument to prove that

$$(h_0, h^*) \not\subseteq Q(u_0).$$

Suppose, by way of contradiction, that there exists $h_1 \in (h_0, h^*)$ such that $h_1 \in Q(u_0)$. As $u_x < 0$ in $(0, T^*) \times [h_0, h(t)]$, then $[h_0, h_1] \subseteq Q(u_0)$. Recalling that $h'(t) > 0$ for $0 < t < T^*$, we can find a unique $t_1 \in (0, T^*)$ such that

$$h(t_1) = h_1.$$

By an argument similar to that in [7, 14], construct the auxiliary function

$$V(t, x) = u_x(t, x) + p(x) \quad \text{for } (t, x) \in [t_1, T^*) \times [h_0, h_1],$$

where $p(x) := \varepsilon \sin(\pi(x - h_0)/(h_1 - h_0))$ for sufficiently small $\varepsilon > 0$. By direct computation,

$$V_t - V_{xx} - f'(u)V = -p(x)[(1-u)^{-2} - \pi^2/(h_1 - h_0)^2] \quad \text{for } (t, x) \in [t_1, T^*) \times [h_0, h_1].$$

Since $[h_0, h_1] \subseteq Q(u_0)$, we can find a $T_1 \in (t_1, T^*)$ such that, if $t \geq T_1$, then

$$(1-u)^{-2} - \pi^2/(h_1 - h_0)^2 > 0 \quad \text{in } [h_0, h_1],$$

which implies that, for $(t, x) \in [T_1, T^*) \times [h_0, h_1]$,

$$V_t - V_{xx} - f'(u)V < 0.$$

Moreover, for this T_1, we can find $\varepsilon > 0$ small enough such that $V(T_1, x) \leq 0$ in $[h_0, h_1]$, since $u_x < 0$ in $[h_0, h_1]$. It is clear that

$$V(t, h_0) = u_x(t, h_0) < 0 \quad \text{and} \quad V(t, h_1) = u_x(t, h_1) < 0 \quad \text{for } T_1 < t < T^*.$$

Therefore, by the comparison principle, for $(t, x) \in [T_1, T^*) \times [h_0, h_1]$,

$$-u_x \geq \varepsilon \sin(\pi(x - h_0)/(h_1 - h_0)).$$

Integrating with respect to x from h_0 to y for any $h_0 < y \leq h_1$,

$$u(t, h_0) - u(t, y) \geq \int_{h_0}^{y} \varepsilon \sin(\pi(x - h_0)/(h_1 - h_0))\, dx. \tag{3.1}$$

As $t \nearrow T^*$, the left-hand side of (3.1) tends to zero, but the right-hand side of (3.1) remains positive, which is a contraction. Thus $(h_0, h^*) \not\subseteq Q(u_0)$.

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By a similar argument, $(g^*, -h_0) \nsubseteq Q(u_0)$. Clearly $(-\infty, g^*] \cup [h^*, \infty) \nsubseteq Q(u_0)$, and hence $Q(u_0) \subseteq [-h_0, h_0]$. Since $Q(u_0)$ is a closed set, $Q(u_0)$ is a compact subset of $[-h_0, h_0]$ and the proof of Step 1 is complete.

Step 2. We claim that g^* and h^* are bounded.

In fact, it follows from Step 1 that $Q(u_0) \subseteq [-h_0, h_0]$, and so

$$\tilde{M} := \limsup_{t \uparrow T^*} u(t, h(\tilde{t})) < 1,$$

where we set $\tilde{t} := T^*/4$. Thus, there exists $0 < \epsilon < \tilde{t}$ sufficiently small that

$$u(t, h(\tilde{t})) \le \tilde{M} + (1 - \tilde{M})/2 := M_1 \quad \text{for } T^* - \epsilon \le t < T^*.$$

Choose

$$M := \max\Big\{ \sup_{t \in [\tilde{t}, T^* - \epsilon]} u(t, h(\tilde{t})), M_1 \Big\}.$$

Then $M < 1$ and, thanks to Lemma 3.2,

$$u(t, x) \le M \quad \text{for } (t, x) \in [\tilde{t}, T^*) \times [h(\tilde{t}), h(t)].$$

Let us construct the auxiliary function

$$\bar{u}(t, x) = M[2K(h(t) - x) - K^2(h(t) - x)^2]$$

over the region $D := \{(t, x) : 2\tilde{t} < t < T^*, h(t) - K^{-1} < x < h(t)\}$, where

$$K := \max\{(h(2\tilde{t}) - h(\tilde{t}))^{-1},\ (2M(1 - M))^{-1/2},\ 4\|u(2\tilde{t}, \cdot)\|_{C^1([h_0, h(2\tilde{t})])}/(3M)\}.$$

It is easy to check that

$$u(t, x) \le M \quad \text{and} \quad \bar{u}(t, x) \le M \quad \text{for } (t, x) \in D \subseteq [\tilde{t}, T^*) \times [h(\tilde{t}), h(t)].$$

Moreover, by the definition of $\bar{u}$ we see that for $2\tilde{t} < t < T^*$,

$$\bar{u}(t, h(t) - K^{-1}) = M \ge u(t, h(t) - K^{-1}), \quad \bar{u}(t, h(t)) = 0 = u(t, h(t)).$$

It is easy to see from the definition of K that, for $(t, x) \in D$,

$$\bar{u}_t - \bar{u}_{xx} \ge 2MK^2 \ge f(\bar{u}).$$

Then, following the same argument as in [4], we obtain

$$u(2\tilde{t}, x) \le \bar{u}(2\tilde{t}, x) \quad \text{for } x \in [h(2\tilde{t}) - K^{-1}, h(2\tilde{t})].$$

Hence the comparison principle yields

$$u(t, x) \le \bar{u}(t, x) \quad \text{for } (t, x) \in D,$$

which implies that

$$h'(t) = -\mu u_x(t, h(t)) \le 2\mu KM := M_2 \quad \text{for } t \in [2\tilde{t}, T^*),$$

and then

$$h^* = \lim_{t \to T^*} h(t) \le \max\{-g(T^*/2), h(T^*/2)\} + M_2 T^*/2 := C.$$

In a similar way, we can prove $-g^* \le C$. Thus Step 2 is finished and the proof of the theorem is complete. □

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From Theorem 1.5, one can easily deduce the following result.

Corollary 3.3. *Let $h_0 > 0$, u_0 satisfy (1.4) and g^* and h^* be defined by (1.5). Assume that $w(t,x)$ is the solution of*

$$\begin{cases} w_t = w_{xx} + 1/(1-w) & \text{if } g^* < x < h^*,\ t > 0,\\ w(t,g^*) = w(t,h^*) = 0 & \text{if } t > 0,\\ w(0,x) = w_0(x) & \text{if } g^* \le x \le h^*, \end{cases}$$

with $w_0(x) \in C^2([g^*,h^*])$ satisfying $w_0(x) \ge 0$ in $[g^*,h^*]$ and $w_0(x) \ge u_0(x)$ for x in $[-h_0,h_0]$. Then $w(t,x)$ must quench.

Proof. It is easy to check that $w(t,x)$ is a supersolution of problem (1.3), and hence

$$w(t,x) \ge u(t,x) \quad \text{in } (0,T_w^*) \times [g(t),h(t)] \text{ and } T_w^* \le T^* < \infty,$$

where $(0,T_w^*)$ is the maximal interval of w. Since 1 is a singular point of $1/(1-w)$, it is easy to see that w must quench in time T_w^*, as desired. The proof is complete. □

The following lemma shows that if the initial function u_0 is even and decreasing, then the quenching set is $\{0\}$ and the two free boundaries remain bounded.

Lemma 3.4. *Suppose $h_0 > 0$, u_0 satisfies (1.4) and*

$$u_0 \text{ is even} \quad \text{and} \quad u_0'(x) < 0 \text{ in } (0,h_0]. \tag{3.2}$$

Then $Q(u_0) = \{0\}$ and $h^* = -g^*$ are bounded.

Proof. Thanks to the condition (3.2), by Lemma 3.2 and a simple moving plane consideration, we can firstly deduce that

$$-g(t) = h(t), \quad u_x(t,0) = 0 \quad \text{for } 0 < t < T^*$$

and

$$u_x(t,x) < 0 < u_x(t,y) \quad \text{for } 0 < t < T^*, \quad 0 < x \le h(t),\ g(t) \le y < 0.$$

Next, we claim that $(0,h_0) \not\subseteq Q(u_0)$. If not, then we can assume that there exists $x_0 \in (0,h_0)$ such that $x_0 \in Q(u_0)$. It is then clear that, for any $\theta \in (0,x_0)$, we have $[\theta,x_0] \subseteq Q(u_0)$, since $u_x < 0$ in $(0,T^*) \times (0,h(t)]$.

Construct the auxiliary function

$$G(t,x) = u_x(t,x) + b(x) \quad \text{for } (t,x) \in (0,T^*) \times [\theta,x_0],$$

where $b(x) := \varepsilon \sin(\pi(x-\theta)/(x_0-\theta))$ for small $\varepsilon > 0$. Arguing as in the proof of Theorem 1.5 we reach a contradiction, which implies that $(0,h_0) \not\subseteq Q(u_0)$, since $\theta \in (0,x_0)$ is arbitrary. Recalling that, for $(t,x) \in (0,T^*) \times (0,h(t)]$,

$$u_x(t,x) < 0 \quad \text{and} \quad u(t,x) = 0,$$

we see that $(0,\infty) \not\subseteq Q(u_0)$. Similarly, $(-\infty,0) \not\subseteq Q(u_0)$, therefore

$$Q(u_0) = \{0\}.$$

This, together with the proof of Theorem 1.5, gives the conclusion that $h^* = -g^*$ are bounded, which completes the proof of the lemma. □

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In the rest of this subsection, under some additional assumptions on the initial datum u_0, we show that u_t will blow up in time T^*.

LEMMA 3.5. *Suppose $h_0 > 0$, u_0 satisfies (1.4) and*

$$u_0'' + f(u_0) \geq 0 \quad in\ [-h_0, h_0]. \tag{3.3}$$

Then

$$\lim_{t \to T^*} \sup_{x \in [g(t), h(t)]} u_t(t, x) = \infty.$$

PROOF. The proof is based on the ideas of [6, 10]. The condition (3.3) enables us to apply the strong maximum principle to deduce that

$$u_t > 0 \quad \text{in } (0, T^*) \times (g(t), h(t)).$$

For any $\eta \in (0, T^*)$, there exists a constant $k_1 > 0$ such that $u_t > k_1$ on the parabolic boundary of $(\eta, T^*) \times [g(\eta), h(\eta)]$. Since we have proved that $Q(u_0) \subseteq [-h_0, h_0]$, there exists a constant $0 < k_2 < 1$ such that $u < k_2$ on the parabolic boundary of $(\eta, T^*) \times [g(\eta), h(\eta)]$.

Construct the auxiliary function

$$W(t, x) := u_t - \delta f(u) \quad \text{in } [\eta, T^*) \times [g(\eta), h(\eta)],$$

where δ is a constant to be determined later. Direct calculations yield

$$W_t - W_{xx} - f'(u)W = \delta f''(u) u_x^2 \geq 0 \quad \text{for } (t, x) \in [\eta, T^*) \times [g(\eta), h(\eta)].$$

Choose $0 < \delta < k_1(1 - k_2)$. Then, on the parabolic boundary of $(\eta, T^*) \times [g(\eta), h(\eta)]$,

$$W(t, x) = u_t - \delta f(u) > 0.$$

It then follows, from the comparison principle, that

$$W(t, x) \geq 0 \quad \text{in } [\eta, T^*) \times [g(\eta), h(\eta)],$$

which implies that

$$\lim_{t \to T^*} \sup_{x \in [g(t), h(t)]} u_t(t, x) \geq \delta \lim_{u \to 1} f(u) = \infty.$$

Hence the desired result follows. □

REMARK 3.6. From Lemmas 3.4 and 3.5, we see that if the initial function u_0 satisfies (1.4), (3.2) and (3.3), then

$$0 \leq u(t, x) < 1 \quad \text{for } t \in [0, T^*],\ x \in [g(t), 0) \cup (0, h(t)]$$

and

$$\lim_{t \to T^*} u(t, 0) = 1 \quad \text{and} \quad \lim_{t \to T^*} u_t(t, 0) = \infty.$$

References

[1] A. Acker and W. Walter, 'On the global existence of solutions of parabolic differential equations with a singular nonlinear term', *Nonlinear Anal.* **2** (1978), 499–505.

[2] C. Y. Chan and Hans G. Kaper, 'Quenching for semilinear singular parabolic problems', *SIAM J. Math. Anal.* **20** (1989), 558–566.

[3] Y. Du and Z. M. Guo, 'Spreading-vanishing dichotomy in a diffusion logistic model with a free boundary, II', *J. Differential Equations* **250** (2011), 4336–4366.

https://doi.org/10.1017/S0004972715001549 Published online by Cambridge University Press

[4] Y. Du and Z. G. Lin, ‘Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary’, *SIAM J. Math. Anal.* **42** (2010), 377–405.
[5] Y. Du and B. D. Lou, ‘Spreading and vanishing in nonlinear diffusion problems with free boundaries’, *J. Eur. Math. Soc.* **17** (2015), 2673–2724.
[6] A. Friedman and B. Mcleod, ‘Blow-up of positive solutions of semilinear heat equations’, *Indiana Univ. Math. J.* **34** (1985), 425–447.
[7] J.-S. Guo, ‘On the quenching behavior of the solution of a semilinear parabolic equation’, *J. Math. Anal. Appl.* **151** (1990), 58–79.
[8] J.-S. Guo and B. Hu, ‘Quenching problem for a singular semilinear heat equation’, *Nonlinear Anal.* **30** (1997), 905–910.
[9] H. Kawarada, ‘On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$’, *Publ. RIMS, Kyoto Univ.* **10** (1975), 729–736.
[10] L. Ke and S. Ning, ‘Quenching for degenerate parabolic equations’, *Nonlinear Anal.* **34** (1998), 1123–1135.
[11] H. A. Levine and J. T. Montgomery, ‘The quenching of solutions of some nonlinear parabolic equations’, *SIAM J. Math. Anal.* **11** (1980), 842–847.
[12] M. Montenegro, ‘Complete quenching for singular parabolic equations’, *J. Math. Anal. Appl.* **384** (2011), 591–596.
[13] L. I. Rubinstein, *The Stefan Problem* (American Mathematical Society, Providence, RI, 1971).
[14] N. K. Sun, ‘Blow-up and asymptotic behavior of solutions for reaction-diffusion equations with free boundaries’, *J. Math. Anal. Appl.* **428** (2015), 838–854.

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