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Exponential asymptotic stability for an elliptic equation with memory arising in electrical conduction in biological tissues

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We study an electrical conduction problem in biological tissues in the radiofrequency range, which is governed by an elliptic equation with memory. We prove the time exponential asymptotic stability of the solution. We provide in this way both a theoretical justification to the complex elliptic problem currently used in electrical impedance tomography and additional information on the structure of the complex coefficients appearing in the elliptic equation. Our approach relies on the fact that the elliptic equation with memory is the homogenisation limit of a sequence of problems for which we prove suitable uniform estimates.

1 Introduction

Electric impedance tomography (EIT) is the inverse problem of determining the impedance in the interior of a body, given simultaneous measurements of direct or alternating electric currents and voltages at the boundary [8]. This technique has a wide spectrum of applications, in different fields. Especially important applications of EIT are found in medicine, e.g. detection of pulmonary emboli, monitoring of heart function and blood flow and breast cancer detection [7].

Experimental measurements in clinical applications are currently performed by assigning time-harmonic boundary data and assuming that the resulting electric potential is time harmonic, too. This assumption, which is often referred to as the limiting amplitude principle, leads to the commonly accepted mathematical model based on the complex elliptic problem (1.36)–(1.37) for the electric potential [8, 10].

In this paper we prove that this assumption is correct for sufficiently large times and that the steady-state electric potential does satisfy the well-known equation (1.36). Moreover, we show how the complex admittivity A^{ω_k} appearing in equation (1.36) depends on the frequency ω_k (equation (1.38)). The subject of coefficient reconstruction (i.e. the inverse problem) for the complex elliptic problem (1.36)–(1.37) may significantly benefit of the structure information provided by (1.38).

Our approach relies on a microscopic model of the electrical conduction in the radiofrequency range in biological tissues. The biological tissue is modelled as a composite with two finely mixed phases (intra- and extra-cellular spaces), separated by interfaces (cell membranes). The microscopic structure is periodic, and its period is a small parameter ε, which is assumed to be much smaller than the electromagnetic spatial scale governed by the permittivity of the biological material and the temporal period of the external driving source. The mathematical scheme (problem (1.5)–(1.9)) is derived from the Maxwell equations in the quasi-static approximation, by means of a concentration procedure [4]. The problem involves the unknown electric potential u_ε and consists of partial differential equations of elliptic type prescribed in each phase, coupled with a dynamical boundary condition at the interface.

A macroscopic model has been obtained in [1–3] by letting $\varepsilon \to 0$, via homogenisation theory. The resulting model is governed by an elliptic equation with memory for the homogenised electric potential u_0 (equation (1.1)).

In this paper we are interested in the asymptotic behaviour of the electric potentials u_ε and u_0 for large times. As far as the microscopic problem is concerned, this question is investigated by applying abstract parabolic theory (e.g. [21]). In particular, an asymptotic stability result (Theorem 4) is obtained: roughly speaking, it states that for every $\varepsilon > 0$, u_ε exponentially approaches a time-periodic steady state $u_\varepsilon^\#$ as time increases, provided that a time-periodic Dirichlet boundary condition is assigned. However a non-standard feature of this result is the behaviour with respect to ε of our estimates. Indeed, it appears quite natural to investigate the interplay between the limit with respect to time t and the one with respect to the microstructural parameter ε, i.e. to ask whether the asymptotic behaviour in time of u_ε is uniform with respect to ε. Our main result (Theorem 5) states that this is the case, so that the homogenised electric potential u_0 exponentially approaches a time-periodic steady state $u_0^\#$ as time increases, provided that a time-periodic Dirichlet boundary condition is assigned.

We derive problem (1.21)–(1.22) which uniquely determines the asymptotic limit $u_0^\#$, under time-periodic, not necessarily time-harmonic, boundary data. This result may enhance medical imaging based on EIT by allowing a wider choice of periodic boundary data in experimental measurements. Indeed, under general time-periodic boundary data, the inverse problem of EIT should be based on our problem (1.21)–(1.22) instead of the usual problem (1.36)–(1.37).

From the point of view of mathematical interest, we note that the asymptotic behaviour of evolutive equations with memory is a classical problem (see, e.g., [14, 15, 22]), currently attracting much interest in the literature [6, 16, 18, 20]. It is well known that the exponential decay of the kernel alone, in general, does not imply the existence of bounded solutions [14, 15]. In [12] (see also [11, 13]) an elliptic equation with memory, similar to (1.1), is proved to admit a unique solution in a suitable Sobolev space, under special assumptions of integrability and coercivity of the integral kernel (see (i)–(iii) in [12]), which state its compatibility with thermodynamics.

Our equation (1.1) is the homogenisation limit of problem (1.5)–(1.9), which in turn is derived from the Maxwell equations. Hence, it is natural to expect that it should be compatible with thermodynamics. In fact, the cited coercivity assumptions on the integral kernel are obtained as a by-product of our approach (see Proposition 10, Remark 17 and Remark 18).

Though the required coercivity properties of the integral kernel in (1.1) might be derived directly from the homogenisation limit, our approach has the advantage of yielding the uniformity with respect to ε of the time-asymptotic stability. Such a result is a useful tool both when dealing with the physical case $\varepsilon > 0$ and when refining standard error estimates in homogenisation.

1.1 Detailed exposition of the results

According to the macroscopic model for the electric conduction in biological tissues derived in [1–3], the homogenised electric potential $u_0(x,t)$, obtained as the limit of the solutions u_ε of (1.5)–(1.9), satisfies the equation

$$-\operatorname{div}\left(A\nabla u_0 + \int_0^t B(t-\tau)\nabla u_0(x,\tau)\,\mathrm{d}\tau - \mathscr{F}\right) = 0, \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

where Ω is an open connected bounded subset of $\boldsymbol{R}^N$, $N > 1$, and the matrices A and $B(t)$ and the vector $\mathscr{F}(x,t)$ are given in equations (2.5).

Equation (1.1) is complemented here with a time-periodic Dirichlet boundary condition:

$$u_0(x,t) = \Psi(x)\Phi(t), \quad \text{on } \partial\Omega \times (0,+\infty). \tag{1.2}$$

We assume that

$$\Phi(t) \in H^1_\#(\boldsymbol{R}). \tag{1.3}$$

Here and in the following the subscript # denotes a space of T-periodic functions, for some fixed $T > 0$. Moreover, we assume that Ψ is the trace on $\partial\Omega$ of a function, still denoted by Ψ, such that

$$\Psi(x) \in H^1(\boldsymbol{R}^N), \qquad \Delta\Psi = 0 \text{ in } \Omega. \tag{1.4}$$

Problem (1.1)–(1.2) is the homogenisation limit as $\varepsilon \searrow 0$ of the following microscopic problem for $u_\varepsilon(x,t)$ [2]:

$$-\operatorname{div}(\sigma\nabla u_\varepsilon) = 0, \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (0,+\infty); \tag{1.5}$$

$$[\sigma\nabla u_\varepsilon \cdot \nu] = 0, \qquad \text{on } \Gamma^\varepsilon \times (0,+\infty); \tag{1.6}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] = (\sigma\nabla u_\varepsilon \cdot \nu)^{(\text{out})}, \qquad \text{on } \Gamma^\varepsilon \times (0,+\infty); \tag{1.7}$$

$$u_\varepsilon(x,t) = \Psi(x)\Phi(t), \qquad \text{on } \partial\Omega \times (0,+\infty); \tag{1.8}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x), \qquad \text{on } \Gamma^\varepsilon. \tag{1.9}$$

The operators div and ∇ act with respect to the space variable x; $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$, where Ω_1^ε and Ω_2^ε are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega = \partial\Omega_2^\varepsilon \cap \Omega$, with ν as normal unit vector pointing into Ω_2^ε; the typical geometry we have in mind is depicted in Figure 1. We refer to Section 2 for a precise definition of the structure of Ω_1^ε, Ω_2^ε, Γ^ε.

Moreover, we assume that

$$\sigma = \sigma_1 > 0 \quad \text{in } \Omega_1^\varepsilon, \qquad \sigma = \sigma_2 > 0 \quad \text{in } \Omega_2^\varepsilon, \qquad \alpha > 0, \tag{1.10}$$

where σ_1, σ_2 and α are constant. From a physical point of view, Γ^ε represents the cell membranes, having capacitance α/ε per unit area, whereas Ω_1^ε (respectively Ω_2^ε) is the intra-cellular (respectively extra-cellular) space, whose conductivity is σ_1 (respectively σ_2).

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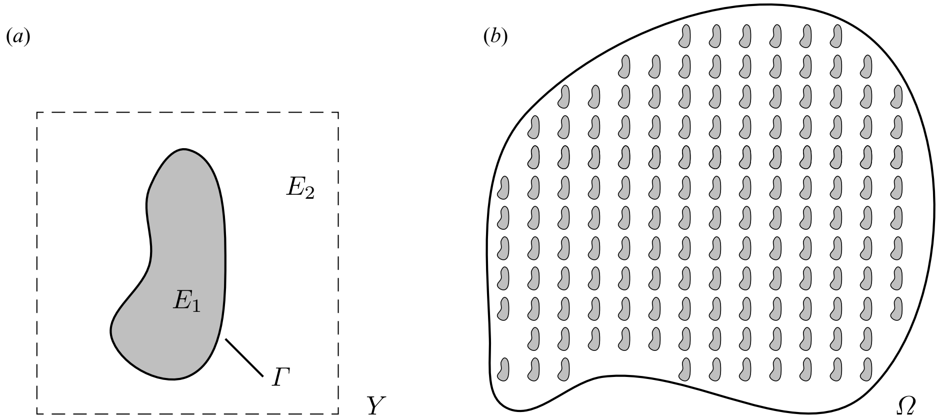

FIGURE 1. (a) An example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\boldsymbol{R}^2$. Here E_1 is the shaded region and Γ is its boundary. The remaining part of Y (the white region) is E_2. (b) The corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the shaded region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε. Note that Ω contains a periodic array of ε-scaled copies of Y.

Since u_ε is not in general continuous across Γ^ε we have set

$$u_\varepsilon^{(\mathrm{out})} := \text{ trace of } u_{\varepsilon|\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon, \quad u_\varepsilon^{(\mathrm{int})} := \text{ trace of } u_{\varepsilon|\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon \text{ and } \quad [u_\varepsilon] := u_\varepsilon^{(\mathrm{out})} - u_\varepsilon^{(\mathrm{int})}.$$

A similar convention is employed for the current flux density across the membrane $\sigma \nabla u_\varepsilon \cdot \nu$.

Remark 1 The physical meaning of problem (1.5)–(1.9) is the following: (1.5) is the standard equation for the Ohmic conduction in intra- and extra-cellular spaces; (1.6) expresses the current density continuity across the cell membranes. Equation (1.7) describes the capacitive behaviour of the cell membranes. Here, $[u_\varepsilon]$ is the potential jump and $(\sigma \nabla u_\varepsilon \cdot \nu)^{(\mathrm{out})}$ is the current flux across the membranes; moreover α/ε is the membrane capacitance per unit area. Hence, (1.7) is the capacitor law, derived in [4] from the Maxwell equations in the quasi-static approximation. The boundary data in (1.8) prescribes a time-periodic boundary value for the electric potential. Finally, the initial datum (1.9) is required by equation (1.7) and prescribes the initial value of the potential jump.

It is known [2] that for every $\overline{T} > 0$, up to a subsequence, u_ε weakly converges in $L^2(\Omega \times (0, \overline{T}))$ and strongly converges in $L^1_{\mathrm{loc}}(0, \overline{T}; L^1(\Omega))$ as $\varepsilon \to 0$, provided that the initial datum $S_\varepsilon(x) \in L^2(\Gamma^\varepsilon)$ satisfies

$$\frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \leqslant \gamma \,, \tag{1.11}$$

for a constant γ independent of ε. If, moreover, $S_\varepsilon(x)$ satisfies (2.3) and (2.4), then any limit $u_0(x,t)$ belongs to $L^2(0, \overline{T}; H^1(\Omega))$ and satisfies problem (1.1)–(1.2). Therefore, by the uniqueness theorem in [1], the limit is uniquely determined, thus implying the convergence of all the sequence $\{u_\varepsilon\}$.

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In this paper we are interested in studying the asymptotic behaviour of $u_\varepsilon(x,t)$ and $u_0(x,t)$ for large times. To this end, after some preliminaries presented in Section 2, we establish in Section 3 the following exponential time-decay estimate for u_ε when homogeneous Dirichlet boundary data prevail on $\partial\Omega \times (0,+\infty)$:

Theorem 2 *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before. Assume that* (1.10) *holds and that the initial datum S_ε satisfies* (1.11). *Let u_ε be the solution of* (1.5)–(1.9), *with homogeneous Dirichlet boundary data on $\partial\Omega \times (0,+\infty)$, i.e. $\Psi \equiv 0$. Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C(\varepsilon + \mathrm{e}^{-\lambda t}) \quad \textit{a.e. in } (1,+\infty), \tag{1.12}$$

where C and λ are independent of ε. Moreover, if S_ε has null mean average over each connected component of Γ^ε, it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda t} \quad \textit{a.e. in } (1,+\infty). \tag{1.13}$$

This result easily yields the following exponential time-decay estimate for u_0 under homogeneous Dirichlet boundary data.

Corollary 3 *Under the assumptions of Theorem 2, if $u_\varepsilon \to u_0$ weakly in $L^2(\Omega \times (0,\overline{T}))$ for every $\overline{T} > 0$, then*

$$\|u_0(\cdot,t)\|_{L^2(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda t} \quad \textit{a.e. in } (1,+\infty). \tag{1.14}$$

Then, we apply Theorem 2 to the function

$$w_\varepsilon = u_\varepsilon - u_\varepsilon^\#,$$

which satisfies a homogeneous Dirichlet boundary condition on $\partial\Omega \times (0,+\infty)$, where $u_\varepsilon^\#(x,t)$ solves a time-periodic version of the microscopic differential scheme introduced above:

$$-\operatorname{div}(\sigma\nabla u_\varepsilon^\#) = 0, \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times \boldsymbol{R}; \tag{1.15}$$

$$[\sigma\nabla u_\varepsilon^\# \cdot \nu] = 0, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{1.16}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon^\#] = (\sigma\nabla u_\varepsilon^\# \cdot \nu)^{(\mathrm{out})}, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{1.17}$$

$$u_\varepsilon^\#(x,t) = \Psi(x)\Phi(t), \qquad \text{on } \partial\Omega \times \boldsymbol{R}; \tag{1.18}$$

$$u_\varepsilon^\#(x,\cdot) \text{ is } T\text{-periodic}, \qquad \forall x \in \Omega; \tag{1.19}$$

$$[u_\varepsilon^\#(\cdot,t)] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon. \tag{1.20}$$

Indeed, this problem is derived from (1.5)–(1.9), replacing equation (1.9) with (1.19). Equation (1.20) has been added in order to guarantee the uniqueness of the solution and is suggested by the observation that $[u_\varepsilon(\cdot,t)] - S_\varepsilon(\cdot)$ has null average over each connected component of Γ^ε, as a consequence of (1.5)–(1.7) and (1.9).

We show (see Theorem 7) that as $\varepsilon \to 0$, the function $u^{\#}_{\varepsilon}(x,t)$ approaches a time-periodic function $u^{\#}_0 \in H^1_{\#}(\boldsymbol{R}; H^1(\Omega))$ solving

$$-\operatorname{div}\left(A\nabla u^{\#}_0 + \int_0^{+\infty} B(\tau)\nabla u^{\#}_0(x, t-\tau)\,\mathrm{d}\tau\right) = 0, \qquad \text{in } \Omega \times \boldsymbol{R}, \tag{1.21}$$

$$u^{\#}_0 = \Psi(x)\Phi(t), \qquad \text{on } \partial\Omega \times \boldsymbol{R}. \tag{1.22}$$

As a consequence, in Section 7 we get our main results.

Theorem 4 *Let* $\Omega^{\varepsilon}_1, \Omega^{\varepsilon}_2, \Gamma^{\varepsilon}$ *be as before. Assume that* (1.3), (1.4), (1.10) *and* (1.11) *hold. Let* $\{u_\varepsilon\}$ *and* $\{u^{\#}_{\varepsilon}\}$ *be the sequences of the solutions of* (1.5)–(1.9) *and* (1.15)–(1.20), *respectively. Then*

$$\|u_\varepsilon(\cdot,t) - u^{\#}_{\varepsilon}(\cdot,t)\|_{L^2(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda t} \quad \text{a.e. in } (1,+\infty), \tag{1.23}$$

where C *and* λ *are positive constants, independent of* ε.

Theorem 5 *Under the assumption of Theorem 4, if* $u_\varepsilon \to u_0$ *and* $u^{\#}_{\varepsilon} \to u^{\#}_0$ *weakly in* $L^2(\Omega \times (0,\overline{T}))$, *for every* $\overline{T} > 0$, *then the following estimate holds:*

$$\|u_0(\cdot,t) - u^{\#}_0(\cdot,t)\|_{L^2(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda t} \quad \text{a.e. in } (1,+\infty), \tag{1.24}$$

where C *and* λ *are positive constants, independent of* ε.

1.2 Investigation of the behaviour of $u^{\#}_{\varepsilon}$ and $u^{\#}_0$

To solve problem (1.15)–(1.20), we express the function Φ by means of its Fourier series, i.e.

$$\Phi(t) = \sum_{k=-\infty}^{+\infty} c_k\,\mathrm{e}^{i\omega_k t} \tag{1.25}$$

where $\omega_k = 2k\pi/T$ is the kth circular frequency, and we represent the solution $u^{\#}_{\varepsilon}(x,t)$ as follows:

$$u^{\#}_{\varepsilon}(x,t) = \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\,\mathrm{e}^{i\omega_k t}, \tag{1.26}$$

where the complex-valued functions $v_{\varepsilon k}(x) \in L^2(\Omega)$ are such that $v_{\varepsilon k}|_{\Omega^{\varepsilon}_i} \in H^1(\Omega^{\varepsilon}_i)$, $i = 1, 2$, and for $k \neq 0$ satisfy the problem

$$-\operatorname{div}(\sigma\nabla v_{\varepsilon k}) = 0, \qquad \text{in } \Omega^{\varepsilon}_1 \cup \Omega^{\varepsilon}_2; \tag{1.27}$$

$$[\sigma\nabla v_{\varepsilon k}\cdot\nu] = 0, \qquad \text{on } \Gamma^{\varepsilon}; \tag{1.28}$$

$$\frac{i\omega_k\alpha}{\varepsilon}[v_{\varepsilon k}] = (\sigma\nabla v_{\varepsilon k}\cdot\nu)^{(\mathrm{out})}, \qquad \text{on } \Gamma^{\varepsilon}; \tag{1.29}$$

$$v_{\varepsilon k} = c_k\Psi, \qquad \text{on } \partial\Omega; \tag{1.30}$$

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whereas for $k = 0$ they satisfy the problem

$$-\operatorname{div}(\sigma \nabla v_{\varepsilon 0}) = 0, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{1.31}$$

$$[\sigma \nabla v_{\varepsilon 0} \cdot \nu] = 0, \qquad \text{on } \Gamma^\varepsilon; \tag{1.32}$$

$$(\sigma \nabla v_{\varepsilon 0} \cdot \nu)^{(\text{out})} = 0, \qquad \text{on } \Gamma^\varepsilon; \tag{1.33}$$

$$v_{\varepsilon 0} = c_0 \Psi, \qquad \text{on } \partial\Omega; \tag{1.34}$$

$$[v_{\varepsilon 0}] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon. \tag{1.35}$$

Note that any solution $v_{\varepsilon k}$ of problem (1.27)–(1.30) is such that $[v_{\varepsilon k}]$ has null average over each connected component of Γ^ε.

We prove the following homogenisation result.

Theorem 6 *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before and assume that* (1.4) *and* (1.10) *hold. Then, for $k \in \boldsymbol{Z} \setminus \{0\}$* (*respectively $k = 0$, under the further assumption* (1.11)), *the solution $v_{\varepsilon k}$ of problem* (1.27)–(1.30) (*respectively problem* (1.31)–(1.35)) *strongly converges in $L^2(\Omega)$ to a function $v_{0k} \in H^1(\Omega)$ which is the unique solution of the problem*

$$-\operatorname{div}(A^{\omega_k} \nabla v_{0k}) = 0, \qquad \textit{in } \Omega, \tag{1.36}$$

$$v_{0k} = c_k \Psi, \qquad \textit{on } \partial\Omega; \tag{1.37}$$

where

$$A^{\omega_k} = A + \int_0^{+\infty} B(t)\, \mathrm{e}^{-i\omega_k t}\, \mathrm{d}t, \tag{1.38}$$

with A and $B(t)$ defined in (2.5).

The case $k \neq 0$ is dealt with in Section 4, where the subscript k is dropped throughout for the sake of simplicity, and an alternative expression for A^{ω_k} is given (equation (4.31)). The case $k = 0$ is dealt with in Section 5.

In Section 6 we study problem (1.15)–(1.20) and establish the following.

Theorem 7 *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before and assume that* (1.3), (1.4), (1.10) *and* (1.11) *hold. Then,*

(i) the series at the right-hand side of equation (1.26) *strongly converges, uniformly with respect to ε, in $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ and in $H^1_\#(\boldsymbol{R}; H^1(\Omega_i^\varepsilon))$, $i = 1, 2$, to the unique solution $u_\varepsilon^\#(x,t)$ of problem* (1.15)–(1.20)*;*

(ii) the sequence $\{u_\varepsilon^\#(x,t)\}$ strongly converges in $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ as $\varepsilon \to 0$ to a function $u_0^\#(x,t)$, T-periodic in time, which can be represented by means of the Fourier series

$$u_0^\#(x,t) = \sum_{k=-\infty}^{+\infty} v_{0k}(x)\, \mathrm{e}^{i\omega_k t}, \tag{1.39}$$

strongly converging in $H^1_\#(\boldsymbol{R}; H^1(\Omega))$;

(iii) the function $u_0^\#(x,t)$ is the unique solution T-periodic in time of the problem (1.21)–(1.22).

Remark 8 We note that with a change of variables, equation (1.21) can be recast as follows:

$$-\operatorname{div}\left(A\nabla u_0^\# + \int_{-\infty}^{t} B(t-\tau)\nabla u_0^\#(x,\tau)\,\mathrm{d}\tau\right) = 0, \qquad \text{in } \Omega \times \boldsymbol{R}, \tag{1.40}$$

which closely resembles equation (1.1). In fact, equation (1.1) involves a time integration over $(0,t)$ and contains an exponentially time-decaying source $\mathscr{F}$ accounting for the initial data of the original problem (1.5)–(1.9) (see Proposition 10), whereas equation (1.40) involves a time integration over $(-\infty,t)$ and is relevant to periodic functions, i.e. to situations in which any transient phenomenon has elapsed.

2 Notation and preliminary results

Following [2], we introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E+z=E$ for all $z\in\boldsymbol{Z}^N$. For all $\varepsilon>0$ we define $\Omega_1^\varepsilon=\Omega\cap\varepsilon E$, $\Omega_2^\varepsilon=\Omega\setminus\overline{\varepsilon E}$, $\Gamma^\varepsilon=\Omega\cap\partial(\varepsilon E)$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y=(0,1)^N$, and $E_1=E\cap Y$, $E_2=Y\setminus\overline{E}$, $\Gamma=\partial E\cap\overline{Y}$. We stipulate that E_1 is a connected smooth subset of Y such that $\operatorname{dist}(\overline{E_1},\partial Y)>0$. Some generalisations may be possible, but we do not dwell on this point here. Finally, we assume that $\operatorname{dist}(\Gamma^\varepsilon,\partial\Omega)>\gamma\varepsilon$ for some constant $\gamma>0$ independent of ε, by dropping the inclusions contained in the cells $\varepsilon(Y+z)$, $z\in\boldsymbol{Z}^N$ which intersect $\partial\Omega$ (see Figure 1). For later usage, we introduce the set

$$\boldsymbol{Z}_\varepsilon^N := \{z\in\boldsymbol{Z}^N : \varepsilon(Y+z)\subseteq\Omega\}. \tag{2.1}$$

In [3] we prove the existence and uniqueness of a weak solution to (1.5)–(1.9), in the class

$$u_{\varepsilon|\Omega_i^\varepsilon}\in L^2\big(0,\overline{T};H^1\big(\Omega_i^\varepsilon\big)\big), \qquad i=1,2, \quad \overline{T}>0. \tag{2.2}$$

As was recalled in the 'Introduction', if the initial datum $S_\varepsilon(x)$ satisfies (1.11), then for every $\overline{T}>0$, up to a subsequence, as $\varepsilon\to0$, u_ε weakly converges in $L^2(\Omega\times(0,\overline{T}))$ and strongly converges in $L^1_{\mathrm{loc}}(0,\overline{T};L^1(\Omega))$. Under the more stringent assumption

$$S_\varepsilon(x)=\varepsilon S_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon R_\varepsilon(x) \tag{2.3}$$

on S_ε,where $S_1:\Omega\times\partial E\to\boldsymbol{R}$ and

$$\begin{aligned}
&\|S_1\|_{L^\infty(\Omega\times\partial E)}<\infty, \qquad \|R_\varepsilon\|_{L^\infty(\Omega)}\to0, \text{ as } \varepsilon\to0,\\
&S_1(x,y) \text{ is continuous in } x, \text{ uniformly over } y\in\partial E,\\
&\text{and periodic in } y, \text{ for each } x\in\Omega;
\end{aligned} \tag{2.4}$$

then all the sequence $\{u_\varepsilon\}$ converges, and the limit $u_0(x,t)$ belongs to $L^2(0,\overline{T};H^1(\Omega))$ and satisfies problem (1.1)–(1.2) [2]. The two matrices A, B and the vector $\mathscr{F}$ appearing there

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are defined by (see [2], equations (3.31), (4.16) and (4.18))

$$\left.\begin{aligned} A &= \sigma_0 I + \int_\Gamma [\sigma]\nu \otimes \chi^0(y)\,\mathrm{d}\sigma, \\ B(t) &= -\alpha \int_\Gamma [\chi^1](y,0) \otimes [\chi^1](y,t)\,\mathrm{d}\sigma = \int_\Gamma \nu \otimes [\sigma\chi^1](y,t)\,\mathrm{d}\sigma, \\ \mathscr{F}(x,t) &:= -\alpha \int_\Gamma S_1(x,y)[\chi^1](y,t)\,\mathrm{d}\sigma = \int_\Gamma [\sigma\mathscr{T}(S_1(x,\cdot))](y,t)\nu\,\mathrm{d}\sigma, \end{aligned}\right\} \tag{2.5}$$

where

$$\sigma_0 = \int_Y \sigma \,\mathrm{d}x = \sigma_1|E_1| + \sigma_2|E_2| \tag{2.6}$$

and two cell functions $\chi^0(y)$ and $\chi^1(y)$ and a transform $\mathscr{T}$ appear. They are defined as follows. The components χ_h^0, $h = 1, \ldots, N$, of $\chi^0 : Y \to \boldsymbol{R}^N$ satisfy

$$-\sigma\,\Delta_y\,\chi_h^0 = 0, \qquad \text{in } E_1,\, E_2; \tag{2.7}$$

$$\left[\sigma\left(\nabla_y \chi_h^0 - \boldsymbol{e}_h\right)\cdot\nu\right] = 0, \qquad \text{on } \Gamma; \tag{2.8}$$

$$\left[\chi_h^0\right] = 0, \qquad \text{on } \Gamma. \tag{2.9}$$

Moreover, χ_h^0 is a periodic function with vanishing integral average over Y. The definition of $\chi^1 : Y \times (0,T) \to \boldsymbol{R}^N$ involves the transform $\mathscr{T}$, defined by

$$\mathscr{T}(s)(y,t) = v(y,t), \qquad y \in Y,\ t > 0, \tag{2.10}$$

where $s : \Gamma \to \boldsymbol{R}$ and v is a periodic null-average function in Y, solving the problem

$$\begin{aligned} -\sigma\,\Delta_y\, v &= 0, && \text{in } (E_1 \cup E_2) \times (0,+\infty); \\ [\sigma\nabla_y v\cdot\nu] &= 0, && \text{on } \Gamma \times (0,+\infty); \\ \alpha\frac{\partial}{\partial t}[v] &= (\sigma\nabla_y v\cdot\nu)^{(\mathrm{out})}, && \text{on } \Gamma \times (0,+\infty); \\ [v](y,0) &= s(y), && \text{on } \Gamma. \end{aligned}$$

Finally, χ_h^1 is defined by

$$\alpha\chi_h^1 = \mathscr{T}\left(\left(\sigma\left(\nabla_y\chi_h^0 - \boldsymbol{e}_h\right)\cdot\nu\right)^{(\mathrm{out})}\right). \tag{2.11}$$

Lemma 9 *For* $s \in L^2(\Gamma)$ *such that* $\int_\Gamma s\,\mathrm{d}\sigma = 0$*, the function* $\mathscr{T}(s)(y,t)$ *defined in equation* (2.10) *satisfies the following estimate, for some constants* $C, \lambda > 0$*:*

$$\|[\mathscr{T}(s)](\cdot,t)\|_{L^2(\Gamma)} \leqslant C\,\mathrm{e}^{-\lambda t}. \tag{2.12}$$

Proof The argument is similar to the one used in Section 3 below, so it is only sketched here. It relies on the application of abstract parabolic theory (e.g. [21], Chapter 7) and leads to the explicit solution

$$[\mathscr{T}(s)](y,t) = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i t}\, w_i(y) \int_\Gamma s w_i\,\mathrm{d}\sigma. \tag{2.13}$$

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Here $\{(\lambda_i, w_i)\}_{i\in N}$ are the eigenvalues and eigenvectors of the spectral problem

$$\text{find } f \in H^{1/2}(\Gamma): \; a(f,g) = \lambda \int_\Gamma \alpha f g \, \mathrm{d}\sigma, \quad \forall g \in H^{1/2}(\Gamma), \tag{2.14}$$

and the bilinear form a is defined as follows:

$$a(f,g) = \int_Y \sigma \nabla z^{(f)} \cdot \nabla z^{(g)} \, \mathrm{d}x, \quad f,g \in H^{1/2}(\Gamma), \tag{2.15}$$

where $z^{(s)}$ is the unique solution with vanishing integral average over Y of the problem

$$-\operatorname{div}(\sigma \nabla z^{(s)}) = 0, \qquad \text{in } E_1 \cup E_2; \tag{2.16}$$

$$[\sigma \nabla z^{(s)} \cdot \nu] = 0, \qquad \text{on } \Gamma; \tag{2.17}$$

$$[z^{(s)}] = s, \qquad \text{on } \Gamma; \tag{2.18}$$

$$z^{(s)} \text{ is } Y\text{-periodic.} \tag{2.19}$$

It is easy to show that a is symmetric and continuous, satisfying the coercivity estimate, for every $\beta > 0$:

$$a(f,f) + \beta \int_\Gamma \alpha f^2 \, \mathrm{d}\sigma \geqslant \gamma(\beta)\big(\|z^{(f)}\|^2_{H^1(E_1)} + \|z^{(f)}\|^2_{H^1(E_2)}\big) \geqslant \gamma(\beta)\|f\|^2_{H^{1/2}(\Gamma)}.$$

Hence, $\{\lambda_i\}$ is an increasing diverging sequence of non-negative eigenvalues and $\{w_i^\varepsilon\}$ constitutes a Hilbert orthonormal basis of $L^2(\Gamma)$. In particular, it is easy to show that $\lambda_1 = 0$, and the corresponding eigenspace is generated by the constant function w_1 on Γ, so that the first term of the sum in (2.13) disappears, since s has null average over Γ. Moreover, $\lambda_2 > 0$ and the assertion follows from (2.13), with $C := \|s\|_{L^2(\Gamma)}$ and $\lambda := \lambda_2$. □

Proposition 10 *The constant matrix A is positive definite and symmetric. The function χ^1 satisfies the estimate*

$$\|[\chi^1_h(\cdot,t)]\|_{L^2(\Gamma)} \leqslant C\,\mathrm{e}^{-\lambda t}, \quad h = 1\ldots N; \tag{2.20}$$

the matrix $B(t)$ belongs to $L^\infty(0,+\infty)$, is symmetric and satisfies the estimate

$$|B_{hj}(t)| \leqslant C\,\mathrm{e}^{-\lambda t}, \quad h,j = 1\ldots N; \tag{2.21}$$

the vector $\mathscr{F}(x,t)$, under the further assumption (2.4)*, belongs to $L^\infty(\Omega \times (0,+\infty))$ and satisfies the estimate*

$$\|\mathscr{F}_h(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda t}, \quad h = 1\ldots N. \tag{2.22}$$

In equations (2.20)–(2.22) *C and λ are positive constants.*

Proof The positive definiteness of A is proved in Proposition 4.1 of [2]. Equation (2.20) follows from (2.11), Lemma 9 and Lemma 7.3 of [2]. Equations (2.21) and (2.22) follow from (2.20) and (2.5), using the Cauchy–Schwarz inequality and, in the proof of (2.22), also the regularity stipulated in (2.4). □

3 Homogeneous Dirichlet boundary data

In this section we prove Theorem 2 and Corollary 3. We introduce the space $\widetilde{H}^{1/2}(\Gamma^{\varepsilon}) \subset H^{1/2}(\Gamma^{\varepsilon})$ of the functions which have null average over each connected component of Γ^{ε}, i.e. on $\varepsilon(\Gamma + z)$, for each z belonging to the set $\boldsymbol{Z}^N_{\varepsilon}$ defined in (2.1).

We decompose the initial datum $S_{\varepsilon}(x)$ in (1.9) as $S_{\varepsilon}(x) = \overline{S}_{\varepsilon}(x) + \widetilde{S}_{\varepsilon}(x)$, where

$$\begin{aligned} &\overline{S}_{\varepsilon}(x) = \fint_{\varepsilon(\Gamma+z)} S_{\varepsilon}\,\mathrm{d}\sigma =: C_{\varepsilon z} \qquad \text{on each } \varepsilon(\Gamma+z),\ z \in \boldsymbol{Z}^N_{\varepsilon}\,; \\ &\widetilde{S}_{\varepsilon}(x) \in \widetilde{H}^{1/2}(\Gamma^{\varepsilon})\,. \end{aligned} \tag{3.1}$$

Accordingly, the solution u_{ε} of problem (1.5)–(1.9) with $\Psi \equiv 0$ is decomposed as $\overline{u}_{\varepsilon} + \widetilde{u}_{\varepsilon}$.

Clearly,

$$\overline{u}_{\varepsilon}(x,t) = \begin{cases} 0 & \text{for } (x,t) \in \Omega^{\varepsilon}_2 \times (0,+\infty)\,, \\ -C_{\varepsilon z} & \text{for } (x,t) \in (\varepsilon(E_1+z)) \times (0,+\infty),\ z \in \boldsymbol{Z}^N_{\varepsilon}\,. \end{cases} \tag{3.2}$$

Using the previous equation, we compute

$$\int_{\Omega} |\overline{u}_{\varepsilon}|^2\,\mathrm{d}x = \sum_{z\in\boldsymbol{Z}^N_{\varepsilon}} \int_{\varepsilon(E_1+z)} |\overline{u}_{\varepsilon}|^2\,\mathrm{d}x = \varepsilon^N |E_1| \sum_{z\in\boldsymbol{Z}^N_{\varepsilon}} \left| \fint_{\varepsilon(\Gamma+z)} S_{\varepsilon}\,\mathrm{d}\sigma \right|^2 .$$

On the other hand, by Hölder's inequality, we estimate

$$\sum_{z\in\boldsymbol{Z}^N_{\varepsilon}} \left| \fint_{\varepsilon(\Gamma+z)} S_{\varepsilon}\,\mathrm{d}\sigma \right|^2 \leqslant \frac{\gamma}{\varepsilon^{N-1}} \int_{\Gamma^{\varepsilon}} S^2_{\varepsilon}\,\mathrm{d}\sigma\,.$$

Hence, as a consequence of (1.11), it follows that

$$\|\overline{u}_{\varepsilon}(\cdot,t)\|_{L^2(\Omega)} \leqslant C\varepsilon\,, \tag{3.3}$$

where C is a constant independent of ε.

An estimate for $\widetilde{u}_{\varepsilon}$ follows from an application of abstract parabolic theory, as summarised for example in Chapter 7 of [21]. We consider the two Hilbert spaces $H^{1/2}(\Gamma^{\varepsilon}) \subset L^2(\Gamma^{\varepsilon})$ and the bilinear form on $H^{1/2}(\Gamma^{\varepsilon})$:

$$a_{\varepsilon}(f,g) = \int_{\Omega} \sigma\nabla z^{(f)}_{\varepsilon} \cdot \nabla z^{(g)}_{\varepsilon}\,\mathrm{d}x\,, \quad f,g \in H^{1/2}(\Gamma^{\varepsilon})\,, \tag{3.4}$$

where $z^{(s)}_{\varepsilon}$ is the unique solution of the problem

$$-\operatorname{div}\left(\sigma\nabla z^{(s)}_{\varepsilon}\right) = 0, \qquad \text{in } \Omega^{\varepsilon}_1 \cup \Omega^{\varepsilon}_2; \tag{3.5}$$

$$\left[\sigma\nabla z^{(s)}_{\varepsilon} \cdot \nu\right] = 0\,, \qquad \text{on } \Gamma^{\varepsilon}; \tag{3.6}$$

$$\left[z^{(s)}_{\varepsilon}\right] = s\,, \qquad \text{on } \Gamma^{\varepsilon}; \tag{3.7}$$

$$z^{(s)}_{\varepsilon} = 0, \qquad \text{on } \partial\Omega. \tag{3.8}$$

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It is easy to show (e.g. [3, Theorem 6]) that a_ε is a symmetric and continuous bilinear form. Moreover, we have the coercivity estimate, for every $\beta > 0$,

$$a_\varepsilon(f,f) + \beta \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} f^2 \, \mathrm{d}\sigma \geqslant \gamma(\beta)\big(\|z_\varepsilon^{(f)}\| H^1(\Omega_1^\varepsilon)^2 + \|z_\varepsilon^{(f)}\|^2_{H^1(\Omega_2^\varepsilon)} \big) \geqslant \gamma(\beta,\varepsilon) \|f\|^2_{H^{1/2}(\Gamma^\varepsilon)},$$

where we have used the Poincaré's inequality in [2, 17] and classical trace inequalities.

Then we consider the spectral problem

$$\text{find } f \in H^{1/2}(\Gamma^\varepsilon): \; a_\varepsilon(f,g) = \lambda^\varepsilon \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} f g \, \mathrm{d}\sigma, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon) \tag{3.9}$$

and the associate evolution problem, for an arbitrary $\overline{T} > 0$,

$$\begin{gathered} \text{given } f_0 \in L^2(\Gamma^\varepsilon), \text{ find } F \in L^2(0,\overline{T}; H^{1/2}(\Gamma^\varepsilon)) \cap C([0,\overline{T}]; L^2(\Gamma^\varepsilon)): \\ F(0) = f_0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} F(t) g \, \mathrm{d}\sigma + a_\varepsilon(F(t),g) = 0, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon). \end{gathered} \tag{3.10}$$

Problem (3.9) admits an increasing diverging sequence $\{\lambda_i^\varepsilon\}$ of non-negative eigenvalues, and there exists a Hilbert orthonormal basis of $L^2(\Gamma^\varepsilon)$ composed by eigenvectors w_i^ε such that [21, Theorem 6.2-1]

$$a_\varepsilon(w_i^\varepsilon, g) = \lambda_i^\varepsilon \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} w_i^\varepsilon g \, \mathrm{d}\sigma, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon), \; i \in \boldsymbol{N}.$$

Moreover, for every $f_0 \in L^2(\Gamma^\varepsilon)$, problem (3.10) admits a unique solution [21, Theorem 7.2-1], which can be represented as follows [21, Lemma 7.2-1]:

$$F(x,t) = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i^\varepsilon t} w_i^\varepsilon(x) \int_{\Gamma^\varepsilon} f_0 w_i^\varepsilon \, \mathrm{d}\sigma. \tag{3.11}$$

Since problem (3.10) is a weak formulation of problem (1.5)–(1.9) with homogeneous Dirichlet boundary conditions, i.e. $\Psi \equiv 0$, and initial data f_0, we conclude that

$$[\widetilde{u}_\varepsilon(x,t)] = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i^\varepsilon t} w_i^\varepsilon(x) \int_{\Gamma^\varepsilon} \widetilde{S}_\varepsilon w_i^\varepsilon \, \mathrm{d}\sigma. \tag{3.12}$$

Let N_ε be the number of connected components of Γ^ε. It is easy to show that

$$\lambda_i^\varepsilon = 0, \qquad i \in \{1, \dots, N_\varepsilon\},$$

and the corresponding eigenspace is generated by the characteristic functions of $\varepsilon(\Gamma + z)$, $z \in \boldsymbol{Z}_\varepsilon^N$: indeed, by (3.4)–(3.8), $a_\varepsilon(f,g) = 0$ for all $g \in H^{1/2}(\Gamma^\varepsilon)$ when f is piecewise constant on Γ^ε. However we can neglect those eigenvalues, since $\widetilde{S}_\varepsilon \in \widetilde{H}^{1/2}(\Gamma^\varepsilon)$, and hence they disappear from equation (3.12).

Our aim is to prove that the next eigenvalue, i.e. $\lambda^\varepsilon_{N_\varepsilon+1}$, here denoted by $\widetilde{\lambda}^\varepsilon$, is bounded below by a positive constant independent of ε. To this purpose, we introduce the space

$$\widetilde{H}^1(\Omega) := \{ v \in L^2(\Omega) \; : \; v_{|\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon), \; i = 1,2, \; [v] \in \widetilde{H}^{1/2}(\Gamma^\varepsilon) \}, \tag{3.13}$$

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and using Lemma 11 and Remark 12, we estimate, for any $v \in \widetilde{H}^1(\Omega)$,

$$\int_\Omega \sigma|\nabla v|^2\,\mathrm{d}x \geqslant \sum_{z\in \mathbf{Z}_\varepsilon^N} \int_{\varepsilon(Y+z)} \sigma|\nabla v|^2\,\mathrm{d}x \geqslant \frac{\alpha\widetilde{\lambda}}{\varepsilon} \sum_{z\in \mathbf{Z}_\varepsilon^N} \int_{\varepsilon(\Gamma+z)} [v]^2\,\mathrm{d}\sigma = \frac{\alpha\widetilde{\lambda}}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2\,\mathrm{d}\sigma\,, \tag{3.14}$$

where $\widetilde{\lambda}$ is defined in (3.18). Hence ([21], equation (6.2-20)),

$$\widetilde{\lambda}^\varepsilon := \min_{s\in \widetilde{H}^{1/2}(\Gamma^\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sigma|\nabla z_\varepsilon^{(s)}|^2\,\mathrm{d}x}{\displaystyle\frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} s^2\,\mathrm{d}\sigma} \geqslant \widetilde{\lambda}, \tag{3.15}$$

for $\widetilde{\lambda} > 0$ and independent of ε.

Estimating (3.15), together with (3.12), gives

$$\|\widetilde{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C\,\mathrm{e}^{-\widetilde{\lambda}t/2} \quad \text{a.e. in } (1,+\infty). \tag{3.16}$$

In order to prove (3.16), we reason in the following manner. For every $t > 0$ fixed, using Poincar's inequality in [2,17], Lemma 13 and equation (3.23), equations (3.12) and (3.15), the Parseval identity and equation (1.11), we have

$$\begin{aligned}
\int_t^{t+h}\int_\Omega |\widetilde{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant C\left(\int_t^{t+h}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_t^{t+h}\int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\\
&\leqslant \frac{Cf(h,t)}{\varepsilon}\int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon(x,t/2)]^2\,\mathrm{d}\sigma \leqslant \frac{Cf(h,t)}{\varepsilon}\sum_{i=1}^{+\infty} \mathrm{e}^{-2\lambda_i^\varepsilon t/2}\left(\int_{\Gamma^\varepsilon} \widetilde{S}_\varepsilon w_i^\varepsilon\,\mathrm{d}\sigma\right)^2\\
&\leqslant \frac{Cf(h,t)}{\varepsilon}\,\mathrm{e}^{-\widetilde{\lambda}t}\|S_\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \leqslant Cf(h,t)\,\mathrm{e}^{-\widetilde{\lambda}t}\,,
\end{aligned} \tag{3.17}$$

where $f(h,t) = \log(1+h/t)+h$. Dividing by h and letting $h\to 0$, equation (3.16) follows. Setting $\lambda = \widetilde{\lambda}/2$, this equation gives (1.13), if S_ε has null mean average over each connected component of Γ^ε (since in this case $\overline{u}_\varepsilon = 0$ and $u_\varepsilon = \tilde{u}_\varepsilon$). Moreover, together with (3.3) it gives (1.12) and completes the proof of Theorem 2.

In order to prove Corollary 3, we use the L^2-weak convergence of u_ε to u_0 in $\Omega\times(t,t+h)$, for every fixed $t > 1$ and $h > 0$, and estimate (1.12) as follows:

$$\int_t^{t+h}\int_\Omega u_0^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \liminf_{\varepsilon\to 0}\int_t^{t+h}\int_\Omega u_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant h(C\,\mathrm{e}^{-\lambda t})^2\,.$$

Dividing by h and letting $h\to 0$, equation (1.14) follows.

Lemma 11 *Set $\widetilde{H}^1(Y) := \{v \in L^2(Y) \ :\ v_{|E_i} \in H^1(E_i),\ i = 1,2,\ [v] \in \widetilde{H}^{1/2}(\Gamma)\}$, where $\widetilde{H}^{1/2}(\Gamma)$ is comprised by the functions of $H^{1/2}(\Gamma)$ with null integral average. Then, it results that*

$$\widetilde{\lambda} := \min_{v\in\widetilde{H}^1(Y),\ [v]\not\equiv 0} \frac{\displaystyle\int_Y \sigma|\nabla v|^2\,\mathrm{d}y}{\displaystyle\alpha\int_\Gamma [v]^2\,\mathrm{d}\sigma} > 0. \tag{3.18}$$

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Proof We consider the bilinear form in (2.15), where now $z^{(s)}$ is the unique solution with vanishing integral average over Y of the problem (2.16)–(2.18) complemented with a homogeneous Neumann boundary condition on ∂Y.

Reasoning as in the proof of Lemma 9, it can be shown that the spectral problem (2.14) admits an increasing diverging sequence of non-negative eigenvalues $\{\lambda_i\}$, with $\lambda_1 = 0$ and the corresponding eigenspace composed by the constant functions on Γ. The space orthogonal to the first eigenspace is $\widetilde{H}^{1/2}(\Gamma)$ and hence the second eigenvalue, denoted by $\overline{\lambda}$, satisfies ([21], equation (6.2-20))

$$\overline{\lambda} = \min_{s\in\widetilde{H}^{1/2}(\Gamma)\setminus\{0\}} \frac{\displaystyle\int_Y \sigma|\nabla z^{(s)}|^2\,\mathrm{d}y}{\displaystyle\alpha\int_\Gamma s^2\,\mathrm{d}\sigma}; \tag{3.19}$$

thus we have that $\overline{\lambda} > 0$, since otherwise the corresponding eigenvector would be constant and hence zero.

Clearly, the infimum at the right-hand side of (3.18) is less than or equal to $\overline{\lambda}$, since for $s \in \widetilde{H}^{1/2}(\Gamma)\setminus\{0\}$, it results that $z^{(s)} \in \widetilde{H}^1(Y)$, and $[z^{(s)}] = s$.

On the other hand, for every $v \in \widetilde{H}^1(Y)\setminus\{0\}$, the function $z^{([v])} \in \widetilde{H}^1(Y)$ is such that

$$\int_Y \sigma|\nabla v|^2\,\mathrm{d}y = \int_Y \sigma|\nabla z^{([v])} + \nabla(v - z^{([v])})|^2\,\mathrm{d}y \geqslant \int_Y \sigma|\nabla z^{([v])}|^2\,\mathrm{d}y, \tag{3.20}$$

since by (2.16)–(2.18) and the homogeneous Neumann boundary condition on ∂Y we have

$$\begin{aligned}\int_Y \sigma\nabla z^{([v])}\cdot\nabla(v - z^{([v])})\,\mathrm{d}y &= -\int_Y (v - z^{([v])})\,\mathrm{div}(\sigma\nabla z^{([v])})\,\mathrm{d}y\\ &\quad - \int_\Gamma [v - z^{([v])}](\sigma\nabla z^{([v])}\cdot\nu)^{(\mathrm{out})}\,\mathrm{d}\sigma + \int_{\partial Y}(v - z^{([v])})(\sigma_2\nabla z^{([v])}\cdot n)\,\mathrm{d}\sigma = 0,\end{aligned}$$

where n is the outward unit normal to ∂Y. As a consequence of (3.20), we conclude that the infimum at the right-hand side of (3.18) is attained and is equal to $\overline{\lambda}$. □

Remark 12 The change of variables $y = x/\varepsilon$ applied to equation (3.18) yields

$$\min_{v\in\widetilde{H}^1(\varepsilon Y),\ [v]\not\equiv 0} \frac{\displaystyle\int_{\varepsilon Y} \sigma|\nabla v|^2\,\mathrm{d}x}{\displaystyle\frac{\alpha}{\varepsilon}\int_{\varepsilon\Gamma} [v]^2\,\mathrm{d}\sigma} = \widetilde{\lambda} > 0, \tag{3.21}$$

where $\widetilde{H}^1(\varepsilon Y) := \{v \in L^2(\varepsilon Y) \ : \ v_{|\varepsilon E_i} \in H^1(\varepsilon E_i),\ i = 1, 2,\ [v] \in \widetilde{H}^{1/2}(\varepsilon\Gamma)\}$, and $\widetilde{H}^{1/2}(\varepsilon\Gamma)$ comprises the functions of $H^{1/2}(\varepsilon\Gamma)$ with null integral average. In particular, we emphasise that $\widetilde{\lambda}$ is a positive constant independent of ε.

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Lemma 13 *Under the assumptions of Theorem 2, there exists a constant* $\gamma > 0$ *independent of* ε*, such that the following estimate holds for* $t > 0$*:*

$$\sup_{\tau \geqslant t} \int_{\Omega} \sigma |\nabla u_\varepsilon(x,\tau)|^2 \,\mathrm{d}x \leqslant \frac{\gamma}{\varepsilon t} \int_{\Gamma^\varepsilon} [u_\varepsilon(x,t/2)]^2 \,\mathrm{d}\sigma \,. \tag{3.22}$$

Proof For $0 \leqslant t_1 \leqslant t_2$, we multiply equation (1.5) by u_ε, integrate by parts over $(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (t_1, t_2)$, use equations (1.6), (1.7) and the homogeneous Dirichlet boundary data on $\partial\Omega$ and obtain

$$\int_{t_1}^{t_2} \int_{\Omega} \sigma |\nabla u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(x,t_2)]^2 \,\mathrm{d}\sigma = \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(x,t_1)]^2 \,\mathrm{d}\sigma \,. \tag{3.23}$$

Then we fix $t > 0$ and choose a cutoff function $\zeta(\tau) \in C^1(0, +\infty)$ such that

$$\zeta(\tau) = \begin{cases} 0, & \tau \leqslant t/2\,; \\ 1\,, & \tau \geqslant t\,; \end{cases} \qquad 0 \leqslant \zeta' \leqslant \frac{\widetilde{\gamma}}{t}\,. \tag{3.24}$$

We multiply equation (1.5) by $u_{\varepsilon t}\zeta$ and integrate by parts over $(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (t/2, t)$. These computations can be made rigorous using a Steklov averaging procedure. Using equations (1.6), (1.7) and (3.24) and the homogeneous Dirichlet boundary data on $\partial\Omega$, we obtain

$$\int_{\Omega} \frac{\sigma}{2} |\nabla u_\varepsilon(x,t)|^2 \,\mathrm{d}x + \frac{\alpha}{\varepsilon} \int_{t/2}^{t} \int_{\Gamma^\varepsilon} \zeta\, [u_{\varepsilon t}]^2 \,\mathrm{d}\sigma \,\mathrm{d}\tau = \int_{t/2}^{t} \int_{\Omega} \frac{\sigma}{2} |\nabla u_\varepsilon|^2 \zeta' \,\mathrm{d}x \,\mathrm{d}\tau \,.$$

Hence,

$$\sup_{\tau \geqslant t} \int_{\Omega} \sigma |\nabla u_\varepsilon(x,\tau)|^2 \,\mathrm{d}x \leqslant \int_{t/2}^{+\infty} \int_{\Omega} \sigma |\nabla u_\varepsilon|^2 \zeta' \,\mathrm{d}x \,\mathrm{d}\tau \,, \tag{3.25}$$

and the assertion follows from equations (3.23), with $t_1 = t/2$ and $t_2 = t$, and (3.24). □

4 Homogenisation limit of time-harmonic solutions: Case $k \neq 0$

In this section we prove Theorem 6 in the case $k \neq 0$. For the sake of simplicity, we omit here the subscript k and set

$$\psi(x) := c_k \Psi(x)\,. \tag{4.1}$$

4.1 Energy estimate

We establish the following energy estimate:

$$\int_{\Omega} \sigma |\nabla v_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega}{\varepsilon} \int_{\Gamma^\varepsilon} |[v_\varepsilon]|^2 \,\mathrm{d}\sigma \leqslant \gamma \int_{\Omega} \sigma |\nabla \psi|^2 \,\mathrm{d}x \,, \tag{4.2}$$

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where γ is independent of ε and ω. This estimate, together with Poincaré's inequality in [2,17], implies the following L^2 estimate:

$$\int_\Omega v_\varepsilon^2 \,\mathrm{d}x \leqslant \gamma(1+\omega^{-1}) \int_\Omega \sigma |\nabla \psi|^2 \,\mathrm{d}x\,. \tag{4.3}$$

In order to carry out the proof, we set

$$z_\varepsilon = v_\varepsilon - \psi\,. \tag{4.4}$$

The complex-valued function $z_\varepsilon(x,t)$ satisfies the equations

$$-\operatorname{div}(\sigma \nabla z_\varepsilon) = 0, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{4.5}$$

$$[\sigma \nabla z_\varepsilon \cdot \nu] = -[\sigma] \nabla \psi \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.6}$$

$$\frac{i\omega\alpha}{\varepsilon}[z_\varepsilon] = (\sigma \nabla z_\varepsilon \cdot \nu)^{(\mathrm{out})} + \sigma_2 \nabla \psi \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.7}$$

$$z_\varepsilon = 0, \qquad \text{on } \partial\Omega. \tag{4.8}$$

We multiply (4.5) by $\bar{z}_\varepsilon$, integrate over $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$, use the Gauss–Green identity and equation (4.8) and arrive at

$$\int_\Omega \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} [\bar{z}_\varepsilon \,\sigma \nabla z_\varepsilon \cdot \nu] \,\mathrm{d}\sigma = 0\,. \tag{4.9}$$

Using equations (4.6)–(4.7) and then the Gauss–Green identity and equations (1.4) and (4.8), we obtain

$$\int_\Omega \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma = \int_{\Gamma^\varepsilon} [\bar{z}_\varepsilon \,\sigma \nabla \psi \cdot \nu] \,\mathrm{d}\sigma = -\int_\Omega \sigma \nabla \bar{z}_\varepsilon \cdot \nabla \psi \,\mathrm{d}x\,. \tag{4.10}$$

Taking the real and imaginary parts of equation (4.10) and adding them, we get

$$\int_\Omega \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma = -\int_\Omega \sigma (\Re \nabla z_\varepsilon - \Im \nabla z_\varepsilon) \cdot \nabla \psi \,\mathrm{d}x\,. \tag{4.11}$$

Then, we estimate, using Young's inequality,

$$\int_\Omega \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma \leqslant \frac{1}{2} \int_\Omega \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + 2 \int_\Omega \sigma |\nabla \psi|^2 \,\mathrm{d}x\,, \tag{4.12}$$

from which equation (4.2) follows.

4.2 Existence

We prove existence of solution of problem (4.5)–(4.8), for the unknown z_ε defined in equation (4.4), in the class

$$\mathscr{H} = \{z_\varepsilon \in L^2(\Omega)\,, \quad z_{\varepsilon|\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon)\,,\ i = 1\,,2\,, \quad z_{\varepsilon|\partial\Omega} = 0\}\,, \tag{4.13}$$

which is identified with the Hilbert space $H^1(\Omega_1^\varepsilon) \times H_0^1(\Omega_2^\varepsilon)$. The weak formulation of problem (4.5)–(4.8) is

$$a(z_\varepsilon, \phi) := \int_\Omega \sigma \nabla z_\varepsilon \cdot \nabla \overline{\phi} \, \mathrm{d}x + \frac{i\alpha\omega}{\varepsilon} \int_{\Gamma^\varepsilon} [z_\varepsilon][\overline{\phi}] \, \mathrm{d}\sigma = \int_{\Gamma^\varepsilon} [\overline{\phi} \, \sigma \nabla \psi \cdot \nu] \, \mathrm{d}\sigma , \quad \forall \phi \in \mathscr{H} . \quad (4.14)$$

Existence of $z_\varepsilon \in \mathscr{H}$ satisfying (4.14) follows from the Lax–Milgram theorem [23, Chapter 6, Theorem 1.4]: indeed, the continuity of the bilinear form $a(\cdot, \cdot)$ and of the linear functional at the right-hand side of (4.14) follows from standard trace inequalities, and the coercivity estimate $|a(\phi, \phi)| \geqslant m \|\phi\|^2_{\mathscr{H}}$, for some $m > 0$, follows from Poincar's inequality in [2, 17].

4.3 Formal homogenisation asymptotics

Here we aim to identify the homogenised equation of problem (1.27)–(1.30), via the two-scale method. The argument is standard, so we only sketch it.

Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$v_\varepsilon = v_\varepsilon(x, y) = v_0(x, y) + \varepsilon v_1(x, y) + \varepsilon^2 v_2(x, y) + \dots . \quad (4.15)$$

Note that v_0, v_1, v_2 are periodic in y, and v_1, v_2 are assumed to have zero integral average over Y.

Applying (4.15) to (1.27)–(1.29) we find, at the leading-order term,

$$-\sigma \, \Delta_y \, v_0 = 0, \qquad \text{in } E_1, E_2; \quad (4.16)$$

$$[\sigma \nabla_y v_0 \cdot \nu] = 0 , \qquad \text{on } \Gamma ; \quad (4.17)$$

$$i\omega\alpha[v_0] = (\sigma \nabla_y v_0 \cdot \nu)^{(\mathrm{out})} , \qquad \text{on } \Gamma . \quad (4.18)$$

Multiplying (4.16) by $\overline{v}_0$, integrating by parts over $E_1 \cup E_2$ and taking into account (4.17)–(4.18), it easily follows that

$$v_0 = v_0(x) . \quad (4.19)$$

Proceeding as above, but taking into consideration the next-order terms in the ε-expansion, we obtain

$$-\sigma \, \Delta_y \, v_1 = 0, \qquad \text{in } E_1, E_2; \quad (4.20)$$

$$[\sigma \nabla_y v_1 \cdot \nu] = -[\sigma \nabla_x v_0 \cdot \nu] , \qquad \text{on } \Gamma ; \quad (4.21)$$

$$i\omega\alpha[v_1] = (\sigma \nabla_y v_1 \cdot \nu)^{(\mathrm{out})} + \sigma_2 \nabla_x v_0 \cdot \nu , \qquad \text{on } \Gamma . \quad (4.22)$$

In (4.20) and in (4.22) we have made use of (4.19) and of its consequence $[v_0] = 0$.

We represent v_1 in the form

$$v_1(x, y) = -\chi^\omega(y) \cdot \nabla_x v_0(x) , \quad (4.23)$$

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where the cell function $\chi^\omega : Y \to \boldsymbol{C}^N$ is such that its components χ_h^ω, $h = 1, \dots, N$, satisfy

$$-\sigma\,\Delta_y\,\chi_h^\omega = 0, \qquad \text{in } E_1,\, E_2, \tag{4.24}$$

$$\left[\sigma\left(\nabla_y\chi_h^\omega - \boldsymbol{e}_h\right)\cdot\nu\right] = 0, \qquad \text{on } \Gamma, \tag{4.25}$$

$$i\omega\alpha\left[\chi_h^\omega\right] = \left(\sigma\left(\nabla_y\chi_h^\omega - \boldsymbol{e}_h\right)\cdot\nu\right)^{(\mathrm{out})}, \qquad \text{on } \Gamma, \tag{4.26}$$

and are periodic functions with vanishing integral average over Y. Existence and uniqueness of the solution of problem (4.24)–(4.26) is proved in Lemma 14.

Finally, the next-order terms in the ε-expansion give

$$-\sigma\,\Delta_y\,v_2 = \sigma\,\Delta_x\,v_0 + 2\sigma\frac{\partial^2 v_1}{\partial x_j \partial y_j}, \qquad \text{in } E_1, E_2 \tag{4.27}$$

$$[\sigma\nabla_y v_2\cdot\nu] = -[\sigma\nabla_x v_1\cdot\nu], \qquad \text{on } \Gamma; \tag{4.28}$$

$$i\omega\alpha[v_2] = (\sigma\nabla_y v_2\cdot\nu)^{(\mathrm{out})} + (\sigma\nabla_x v_1\cdot\nu)^{(\mathrm{out})}, \qquad \text{on } \Gamma. \tag{4.29}$$

Integrating equation (4.27) by parts both in E_1 and in E_2, using equation (4.28) and adding the two contributions, we get

$$-\sigma_0\,\Delta_x\,v_0 = -2\int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma + \int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma = -\int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma,$$

where σ_0 is defined in equation (2.6).

Then, we use representation (4.23) and infer from the equality above the partial differential equation for v_0 as

$$-\operatorname{div}(A^\omega\,\nabla v_0) = 0, \qquad \text{in } \Omega, \tag{4.30}$$

where the matrix A^ω is given by (here the superscript t denotes transposition)

$$A^\omega = \sigma_0 I + \int_\Gamma \nu\otimes[\sigma\chi^\omega]\,\mathrm{d}\sigma = \sigma_0 I - \int_Y \sigma\nabla^t\chi^\omega\,\mathrm{d}y. \tag{4.31}$$

Lemma 14 *Under the assumptions on E_1, E_2, Γ reported in Section 2, problem* (4.24)–(4.26) *admits a unique solution in the class*

$$\widehat{H}^1(Y) := \{f\in L^2(\boldsymbol{R}^N) : f|_{E_i}\in H^1(E_i),\ i = 1,2,\ f \text{ is } Y\text{-periodic with vanishing integral average over } Y\}. \tag{4.32}$$

Proof First, we prove the uniqueness result. Assuming, by contradiction, that two different solutions $\chi_{h,1}^\omega$ and $\chi_{h,2}^\omega$ to problem (4.24)–(4.26) exist, the function $z_h^\omega := \chi_{h,2}^\omega - \chi_{h,1}^\omega$ satisfies

$$-\sigma\,\Delta_y\,z_h^\omega = 0, \qquad \text{in } E_1,\, E_2; \tag{4.33}$$

$$\left[\sigma\nabla_y z_h^\omega\cdot\nu\right] = 0, \qquad \text{on } \Gamma; \tag{4.34}$$

$$i\omega\alpha\left[z_h^\omega\right] = \left(\sigma\nabla_y z_h^\omega\cdot\nu\right)^{(\mathrm{out})}, \qquad \text{on } \Gamma. \tag{4.35}$$

Multiplying (4.33) by $\overline{z}_h^{\omega}$, integrating by parts and using (4.34) and (4.35), we obtain

$$\int_Y \sigma |\nabla z_h^{\omega}|^2 \,\mathrm{d}y + i\omega\alpha \int_{\Gamma} \left| \left[z_h^{\omega} \right] \right|^2 \,\mathrm{d}\sigma = 0. \tag{4.36}$$

This estimate, recalling that $z_h^{\omega} \in \widehat{H}^1(Y)$, implies that $z_h^{\omega} \equiv 0$.

As far as existence is concerned, we refer to equation (4.53), where a solution to problem (4.24)–(4.26) is explicitly exhibited. Alternatively, one could appeal to the Lax–Milgram theorem as in Section 4.2. □

4.4 Homogenisation limit

Introduce for $i = 1, \dots, N$ the functions

$$q_i^{\varepsilon}(x,t) = x_i - \varepsilon \chi_i^{\omega}\left(\frac{x}{\varepsilon}\right), \tag{4.37}$$

so that explicit calculations reveal

$$-\sigma\,\Delta\, q_i^{\varepsilon} = 0, \qquad \text{in } \Omega_1^{\varepsilon},\ \Omega_2^{\varepsilon}; \tag{4.38}$$

$$\left[\sigma \nabla q_i^{\varepsilon} \cdot \nu\right] = 0, \qquad \text{on } \Gamma^{\varepsilon}; \tag{4.39}$$

$$\frac{i\omega\alpha}{\varepsilon}\left[q_i^{\varepsilon}\right] = (\sigma \nabla q_i^{\varepsilon} \cdot \nu)^{(\mathrm{out})}, \qquad \text{on } \Gamma^{\varepsilon}. \tag{4.40}$$

Let $\varphi \in C_o^{\infty}(\Omega)$, and select $q_i^{\varepsilon}\varphi$ as a test function in the weak formulation of (1.27)–(1.30) and use equations (1.28) and (1.29). We obtain

$$\int_{\Omega} \sigma \nabla v_{\varepsilon} \cdot \nabla q_i^{\varepsilon}\, \varphi \,\mathrm{d}x + \int_{\Omega} \sigma \nabla v_{\varepsilon} \cdot \nabla \varphi\, q_i^{\varepsilon} \,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon} \int_{\Gamma^{\varepsilon}} [v_{\varepsilon}] \left[q_i^{\varepsilon}\right] \varphi \,\mathrm{d}\sigma = 0\,. \tag{4.41}$$

Next select $v_{\varepsilon}\varphi$ as a test function in the weak formulation of (4.38)–(4.40). We get

$$\int_{\Omega} \sigma \nabla q_i^{\varepsilon} \cdot \nabla v_{\varepsilon}\, \varphi \,\mathrm{d}x + \int_{\Omega} \sigma \nabla q_i^{\varepsilon} \cdot \nabla \varphi\, v_{\varepsilon} \,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon} \int_{\Gamma^{\varepsilon}} \left[q_i^{\varepsilon}\right] \left[v_{\varepsilon}\right] \varphi \,\mathrm{d}\sigma = 0\,. \tag{4.42}$$

Subtracting (4.42) from (4.41), we find

$$\int_{\Omega} \sigma \nabla v_{\varepsilon} \cdot \nabla \varphi\, q_i^{\varepsilon} \,\mathrm{d}x = \int_{\Omega} \sigma \nabla q_i^{\varepsilon} \cdot \nabla \varphi\, v_{\varepsilon} \,\mathrm{d}x\,. \tag{4.43}$$

The energy inequality (4.2), the L^2 estimate (4.3) and Corollary 3.5 in [17] imply that extracting subsequences if needed, we may assume

$$-\sigma \nabla v_{\varepsilon} \to \xi^{\omega}\,, \qquad \text{weakly in } L^2(\Omega), \tag{4.44}$$

$$v_{\varepsilon} \to v_0\,, \qquad \text{strongly in } L^2(\Omega), \tag{4.45}$$

for some $\xi^{\omega} \in L^2(\Omega)^N$, $v_0 \in L^2(\Omega)$. On the other hand, recalling (4.31) and (4.37), it is easy to show that

$$q_i^{\varepsilon} \to x_i\,, \qquad \text{strongly in } L^2(\Omega), \tag{4.46}$$

$$\sigma \nabla q_i^{\varepsilon} \to A^{\omega} \boldsymbol{e}_i\,, \qquad \text{weakly in } L^2(\Omega). \tag{4.47}$$

Thus, using [2, Lemma 7.5], it follows that

$$-\int_\Omega \xi^\omega \cdot \nabla\varphi\, x_i\, \mathrm{d}x = \int_\Omega A^\omega \boldsymbol{e}_i \cdot \nabla\varphi\, v_0\, \mathrm{d}x\,. \tag{4.48}$$

As usual, next we take φx_i as a test function in the weak formulation of (1.27)–(1.30). On letting $\varepsilon \to 0$, we get

$$-\int_\Omega \xi^\omega \cdot \nabla\varphi\, x_i\, \mathrm{d}x - \int_\Omega \xi^\omega \cdot \boldsymbol{e}_i\, \varphi\, \mathrm{d}x = 0\,. \tag{4.49}$$

We substitute (4.49) in (4.48), and recalling that A^ω is symmetric (see Section 4.6), we obtain

$$\int_\Omega v_0 A^\omega \nabla\varphi\, \mathrm{d}x = \int_\Omega \xi^\omega\, \varphi\, \mathrm{d}x\,.$$

By the arbitrariness of $\varphi \in C_o^\infty(\Omega)$, recalling also equation (4.49), it follows that

$$\xi^\omega = -A^\omega \nabla v_0 \quad \text{and} \quad \operatorname{div} \xi^\omega = 0 \text{ in the sense of distributions,}$$

and hence equation (1.36) holds.

4.5 Dirichlet boundary condition for v_0

Here we prove equation (1.37) using an argument similar to [2, Section 5.1]. We define

$$V_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{in } \Omega, \\ \psi & \text{in } \boldsymbol{R}^N \setminus \overline{\Omega}. \end{cases}$$

Since the jump of V_ε across $\partial\Omega$ is zero, we infer that for each bounded open set $G \subset \boldsymbol{R}^N$, the variation $|DV_\varepsilon|(G)$ is given by

$$|DV_\varepsilon|(G) = \int_G |\nabla V_\varepsilon|\, \mathrm{d}x + \int_{\Gamma^\varepsilon \cap G} |[V_\varepsilon]|\, \mathrm{d}\sigma \leqslant \gamma\big(|G|^{1/2} + (\varepsilon|\Gamma^\varepsilon \cap G|_{N-1})^{1/2}\big)\,, \tag{4.50}$$

where we have made use of Hölder's inequality and of equations (1.4), (4.1) and (4.2). As a first consequence of this estimate, we may invoke classical compactness and semi-continuity results to show that (extracting subsequences if needed)

$$V_\varepsilon \to V_0\,, \qquad \text{in } L^1(\boldsymbol{R}^N), \qquad |DV_0|(G) \leqslant \liminf_{\varepsilon\to 0} |DV_\varepsilon|(G)\,, \tag{4.51}$$

for every set $G \subset \boldsymbol{R}^N$ as above. On the other hand, according to [5, Theorem 3.77],

$$|DV_0|(\partial\Omega) = \int_{\partial\Omega} |V_0^+ - V_0^-|\, \mathrm{d}\sigma = \int_{\partial\Omega} |V_0^+ - \psi|\, \mathrm{d}\sigma\,, \tag{4.52}$$

where the symbol V_0^+ (respectively V_0^-) denotes the trace on $\partial\Omega$ of $V_{0|\Omega}$ (respectively of $V_{0|\boldsymbol{R}^N\setminus\overline{\Omega}} \equiv \psi$).

Define for $0 < h < 1$ the open set

$$G_h = \{x \in \boldsymbol{R}^N \mid \operatorname{dist}(x, \partial\Omega) < h\}\,.$$

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Combining (4.50)–(4.52), we obtain, as $\partial\Omega \subset G_h$ for all h,

$$\int_{\partial\Omega} |V_0^+ - \psi| \, \mathrm{d}\sigma \leqslant |DV_0(G_h)| \leqslant \gamma \liminf_{\varepsilon\to 0} \left(|G_h|^{1/2} + (\varepsilon|\Gamma^\varepsilon \cap G_h|_{N-1})^{1/2} \right) \leqslant \gamma h^{1/2} .$$

Indeed, it is readily seen that $|G_h| \leqslant \gamma h$ and that $|\Gamma^\varepsilon \cap G_h|_{N-1} \leqslant \gamma h/\varepsilon$ for all sufficiently small h. Therefore, letting $h \to 0$ above we obtain that $V_0^+ = \psi$ a.e. on $\partial\Omega$. As a consequence, $v_0 = \psi$ a.e. on $\partial\Omega$.

4.6 Structure of the limit equation

First, we show that equations (1.38) and (4.31) yield the same matrix A^ω. To this end, we set

$$\theta^\omega = \chi^0 + \int_0^{+\infty} \chi^1(\cdot, t) \, \mathrm{e}^{-i\omega t} \, \mathrm{d}t . \tag{4.53}$$

Recalling (2.7), (2.8) and (2.11), it follows that θ^ω satisfies equations (4.24) and (4.25). Indeed, it satisfies also equation (4.26):

$$\begin{aligned}\left(\sigma\left(\nabla_y \theta_h^\omega - \boldsymbol{e}_h\right) \cdot \nu\right)^{(\mathrm{out})} &= \left(\sigma\left(\nabla_y \chi_h^0 - \boldsymbol{e}_h\right) \cdot \nu\right)^{(\mathrm{out})} + \int_0^{+\infty} \left(\sigma(\nabla_y \chi^1(\cdot, t)) \cdot \nu\right)^{(\mathrm{out})} \mathrm{e}^{-i\omega t} \, \mathrm{d}t \\ &= \alpha\left[\chi_h^1(\cdot, 0)\right] + \int_0^{+\infty} \alpha \frac{\partial}{\partial t}\left[\chi_h^1(\cdot, t)\right] \mathrm{e}^{-i\omega t} \, \mathrm{d}t = i\omega\alpha\left[\theta_h^\omega\right],\end{aligned} \tag{4.54}$$

where we have used (2.9), (2.11) and Proposition 10. Thus $\theta_h^\omega = \chi_h^\omega$, since both of them satisfy problem (4.24)–(4.26), which admits a unique solution in the class $\widehat{H}^1(Y)$ (see Lemma 14). In turn, recalling (2.5), this implies the equivalence between equation (1.38) and equation (4.31).

Then we prove the following result, which, in particular, implies the well-posedness of problem (1.36)–(1.37) for $k > 0$.

Proposition 15 *A^ω is symmetric; its real part and its imaginary part are positive definite; $|A_{hj}^\omega|$, $h, j = 1, \dots, N$, is uniformly bounded with respect to ω. Moreover, $\Re(A^\omega\zeta, \zeta) \geqslant \gamma|\zeta|^2$, for all $\zeta \in \boldsymbol{C}^N$, where $(\cdot, \cdot)$ is the scalar product in $\boldsymbol{C}^N$ and γ is a positive constant, independent of ω.*

Proof The symmetry of A^ω follows from equation (1.38) and the fact that the matrices A and $B(t)$ therein are symmetric (see Proposition 10). The uniform upper bound on $\|A^\omega\|$ follows from equation (1.38) and Proposition 10.

In order to prove the strict positivity of $\Re(A^\omega)$ and $\Im(A^\omega)$, we compute

$$\begin{aligned}\int_Y \sigma\left(\nabla\chi_j^\omega - \boldsymbol{e}_j\right) \cdot \left(\nabla\overline{\chi}_h^\omega - \boldsymbol{e}_h\right) \mathrm{d}y &= -\int_\Gamma \left(\sigma\left(\nabla\chi_j^\omega - \boldsymbol{e}_j\right) \cdot \nu\right)^{(\mathrm{out})} \left[\overline{\chi}_h^\omega\right] \mathrm{d}\sigma \\ &\quad - \int_Y \sigma\left(\nabla\chi_j^\omega - \boldsymbol{e}_j\right) \cdot \boldsymbol{e}_h \, \mathrm{d}y = -i\omega\alpha \int_\Gamma \left[\chi_j^\omega\right]\left[\overline{\chi}_h^\omega\right] \mathrm{d}\sigma + A_{hj}^\omega ,\end{aligned} \tag{4.55}$$

where we have used the Gauss–Green theorem, equations (4.24)–(4.26) and (4.31) and the fact that χ^ω is Y-periodic. As a consequence,

$$\Re(A^\omega) = S^\omega + W^\omega \quad \text{and} \quad \Im(A^\omega) = T^\omega + Z^\omega , \tag{4.56}$$

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where, setting $\alpha^\omega = \Re(\chi^\omega)$ and $\beta^\omega = \Im(\chi^\omega)$,

$$\begin{aligned} S^\omega_{hj} &= \int_Y \sigma\big(\nabla\alpha^\omega_j - \boldsymbol{e}_j\big)\cdot\big(\nabla\alpha^\omega_h - \boldsymbol{e}_h\big)\,\mathrm{d}y + \int_Y \sigma\nabla\beta^\omega_j\cdot\nabla\beta^\omega_h\,\mathrm{d}y,\\ W^\omega_{hj} &= -\omega\alpha\int_\Gamma \big([\alpha^\omega_j]\,[-\beta^\omega_h] + [\beta^\omega_j]\,[\alpha^\omega_h]\big)\,\mathrm{d}\sigma\,,\\ T^\omega_{hj} &= \omega\alpha\int_\Gamma \big([\alpha^\omega_j]\,[\alpha^\omega_h] + [\beta^\omega_j]\,[\beta^\omega_h]\big)\,\mathrm{d}\sigma \quad \text{and}\\ Z^\omega_{hj} &= \int_Y \sigma\nabla\beta^\omega_j\cdot\big(\nabla\alpha^\omega_h - \boldsymbol{e}_h\big)\,\mathrm{d}y - \int_Y \sigma\big(\nabla\alpha^\omega_j - \boldsymbol{e}_j\big)\cdot\nabla\beta^\omega_h\,\mathrm{d}y\,. \end{aligned} \tag{4.57}$$

Clearly, the matrices S^ω and T^ω are symmetric, whereas the matrices W^ω and Z^ω are skew-symmetric: hence, $W^\omega = Z^\omega = 0$, due to the symmetry of A^ω. Exploiting the Y-periodicity of α^ω, we have, for all $\eta \in \boldsymbol{R}^N$,

$$\begin{aligned} &(\Re(A^\omega)\eta,\eta) = \textstyle\sum_{j,h} S^\omega_{jh}\eta_j\eta_h\\ &\geqslant\ \sigma_m\int_Y |\nabla\textstyle\sum_j\big(\alpha^\omega_j\eta_j - y_j\eta_j\big)|^2\,\mathrm{d}y + \sigma_m\int_Y|\nabla\textstyle\sum_j\big(\beta^\omega_j\eta_j\big)|^2\,\mathrm{d}y \geqslant \gamma(\omega)|\eta|^2\,, \end{aligned} \tag{4.58}$$

where $\sigma_m = \min(\sigma_1,\sigma_2)$ and $\gamma(\omega)$ is a positive constant depending on ω. In order to prove the last inequality, first we fix η such that $|\eta| = 1$ and observe that

$$\sigma_m\int_Y|\nabla\textstyle\sum_j\big(\alpha^\omega_j\eta_j - y_j\eta_j\big)|^2\,\mathrm{d}y = 0$$

implies that $\sum_j(\alpha^\omega_j\eta_j - y_j\eta_j)$ is constant in E_2, which is a contradiction, since the functions α^ω_j are Y-periodic, while y_j are not. Then, the result follows by compactness and homogeneity with respect to η.

Analogously, for all $\eta \in \boldsymbol{R}^N$ we compute

$$(\Im(A^\omega)\eta,\eta) = \textstyle\sum_{j,h} T^\omega_{jh}\eta_j\eta_h = \omega\alpha\displaystyle\int_\Gamma\Big[\Big(\textstyle\sum_j[\alpha^\omega_j\eta_j]\Big)^2 + \Big(\textstyle\sum_j[\beta^\omega_j\eta_j]\Big)^2\Big]\,\mathrm{d}\sigma \geqslant \gamma(\omega)|\eta|^2\,.$$

Indeed, reasoning as above, if a vector $\eta\in\boldsymbol{R}^N$ exists, such that $|\eta| = 1$ and that $\sum_j[\chi^\omega_j\eta_j] = 0$, by (4.24)–(4.26) it results that $\sum_j(\chi^\omega_j - y_j)\eta_j$ is constant, and this contradicts the Y-periodicity of χ^ω.

Finally, for $\zeta\in\boldsymbol{C}^N$ we set $\eta = \Re(\zeta)$, $\upsilon = \Im(\zeta)$ and compute, by exploiting (4.58) and the symmetry of $\Im(A^\omega)$,

$$\Re(A^\omega\zeta,\zeta) = (\Re(A^\omega)\eta,\eta) + (\Re(A^\omega)\upsilon,\upsilon) \geqslant \gamma(\omega)|\zeta|^2\,.$$

This estimate is uniform with respect to ω. Indeed, using (1.38) we have

$$(\Re(A^\omega)\eta,\eta) = (A\eta,\eta) + \int_0^{+\infty}(B(t)\eta,\eta)\cos(\omega t)\,\mathrm{d}t\,. \tag{4.59}$$

The constant matrix A is positive definite and the matrix $B(t)$ belongs to $L^1(0,+\infty)$ (see Proposition 10). Hence, by the Riemann–Lebesgue lemma, a sufficiently large ω_0 can be found such that for $\omega > \omega_0$ the right-hand side of the previous equation is minorised by

$\gamma|\eta|^2$, for a constant γ independent of ω. The assertion follows noting that finitely many values of ω are less than ω_0, since $\omega = 2\pi k/T$, and using (4.58). □

Remark 16 Due to Proposition 15 and the Lax–Milgram theorem, the problem

$$-\operatorname{div}(A^{\omega}\,\nabla v) = 0, \qquad \text{in } \Omega, \tag{4.60}$$

$$v = \psi\,, \qquad \text{on } \partial\Omega, \tag{4.61}$$

is uniformly elliptic with respect to k and admits a unique solution $v \in H^1(\Omega)$. As a consequence, the function $v_0 = \lim_{\varepsilon\to 0} v_\varepsilon$, which was proved to satisfy the problem above, coincides with v. Hence, $v_0 \in H^1(\Omega)$ and the following estimate holds:

$$\int_{\Omega} (|v_0|^2 + |\nabla v_0|^2)\,\mathrm{d}x \leqslant \gamma \int_{\Omega} \sigma|\nabla\psi|^2\,\mathrm{d}x\,, \tag{4.62}$$

for a constant γ independent of k. We note that the uniqueness of v_0 also implies that actually the whole sequence $\{v_\varepsilon\}$ converges to v_0.

Remark 17 We emphasise that the condition of strict positivity of $\Im(A^{\omega})$ implies assumption (iii) in [12]. This assumption was stipulated there as a consequence of the second law of thermodynamics. In this paper, the same condition is proved to be a direct consequence of the homogenisation of equations (1.5)–(1.9), which are derived from Maxwell equations.

5 Homogenisation limit of time-harmonic solutions: The case $k = 0$

Here we prove Theorem 6 in the case $k = 0$, so that we study problem (1.31)–(1.35). It amounts to solving independent Neumann problems on Ω_2^{ε} and on each connected component $\varepsilon(E_1 + z)$, $z \in \boldsymbol{Z}_{\varepsilon}^N$, of Ω_1^{ε}. The first one was considered in [9, Chapter 1] in the context of homogenisation in perforated media, where the authors obtained that there exists a positive constant γ, independent of ε such that

$$\int_{\Omega_2^{\varepsilon}} |\nabla v_{\varepsilon 0}|^2\,\mathrm{d}x \leqslant \gamma\,. \tag{5.1}$$

Moreover, they proved that

$$P_\varepsilon v_{\varepsilon 0} \to v_{00} \qquad \text{weakly in } H^1(\Omega), \text{ as } \varepsilon \to 0\,, \tag{5.2}$$

where we have used the following notation. Setting $V_\varepsilon = \{v \in H^1(\Omega_2^{\varepsilon}) : v = c_0\Psi \text{ on } \partial\Omega\}$, P_ε is any extension operator from $L^2(\Omega_2^{\varepsilon})$ to $L^2(\Omega)$ and from V_ε to $H^1(\Omega)$ such that for any $v \in V_\varepsilon$,

$$\|P_\varepsilon v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega_2^{\varepsilon})} \qquad \text{and} \qquad \|\nabla P_\varepsilon v\|_{[L^2(\Omega)]^N} \leqslant C\|\nabla v\|_{[L^2(\Omega_2^{\varepsilon})]^N} \tag{5.3}$$

for a constant C independent of ε. Moreover, v_{00} is the solution of (1.36) and (1.37) and

$$A^0 = \sigma_2|E_2|I + \int_{\Gamma} \sigma_2 \nu \otimes \chi^{00}(y)\,\mathrm{d}\sigma\,. \tag{5.4}$$

The components χ_h^{00}, $h = 1, \ldots, N$, of $\chi^{00} : E_2 \to \boldsymbol{R}^N$ satisfy

$$-\sigma_2 \Delta_y \chi_h^{00} = 0, \qquad \text{in } E_2; \tag{5.5}$$

$$\sigma_2(\nabla_y \chi_h^{00} - \boldsymbol{e}_h) \cdot \nu = 0, \qquad \text{on } \Gamma. \tag{5.6}$$

In addition, χ_h^{00} is a Y-periodic function with vanishing integral average over E_2.

For every $z \in \boldsymbol{Z}_\varepsilon^N$, the Neumann problem in $\varepsilon(E_1 + z)$ can be explicitly solved, giving

$$v_{\varepsilon 0}(x) = \fint_{\varepsilon(\Gamma+z)} v_{\varepsilon 0}^{(\text{out})} \, \mathrm{d}\sigma - \fint_{\varepsilon(\Gamma+z)} S_\varepsilon(x) \, \mathrm{d}\sigma =: v_{\varepsilon 0}^{(a)}(x) + v_{\varepsilon 0}^{(b)}(x), \quad x \in \varepsilon(E_1 + z). \tag{5.7}$$

By (5.2), it follows that $v_{\varepsilon 0} \to v_{00}$ strongly in $L^2(\Omega)$, since

$$\|v_{\varepsilon 0} - v_{00}\|_{L^2(\Omega)} \leqslant \|v_{\varepsilon 0} - P_\varepsilon v_{\varepsilon 0}\|_{L^2(\Omega)} + \|P_\varepsilon v_{\varepsilon 0} - v_{00}\|_{L^2(\Omega)}, \tag{5.8}$$

and the first term at the right-hand side of the previous inequality is estimated as follows:

$$\|v_{\varepsilon 0} - P_\varepsilon v_{\varepsilon 0}\|^2_{L^2(\Omega)} \leqslant \gamma \varepsilon \int_{\Gamma^\varepsilon} [v_{\varepsilon 0}]^2 \, \mathrm{d}\sigma + \gamma \varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla v_{\varepsilon 0}|^2 \, \mathrm{d}x \leqslant \gamma \varepsilon^2, \tag{5.9}$$

where we have used [19, Lemma 6], the fact that $P_\varepsilon v_{\varepsilon 0} = v_{\varepsilon 0}$ on Ω_2^ε, estimate (5.1) and the estimate

$$\int_{\Gamma^\varepsilon} [v_{\varepsilon 0}]^2 \, \mathrm{d}\sigma \leqslant 2 \int_{\Gamma^\varepsilon} \left(v_{\varepsilon 0}^{(\text{out})} - v_{\varepsilon 0}^{(a)}\right)^2 \mathrm{d}\sigma + 2 \int_{\Gamma^\varepsilon} \left(v_{\varepsilon 0}^{(b)}\right)^2 \mathrm{d}\sigma =: I_1 + I_2 \leqslant \gamma \varepsilon. \tag{5.10}$$

Indeed, I_1 is estimated as follows: the inequality

$$I_1 \leqslant \gamma \varepsilon \int_{\Omega_2^\varepsilon} |\nabla v_{\varepsilon 0}|^2 \, \mathrm{d}x,$$

is obtained reasoning as in (3.14), and using Lemma 11 and Remark 12 applied to the function

$$w_\varepsilon(x) = \begin{cases} v_{\varepsilon 0}^{(a)}(x) & \text{for } x \in \Omega_1^\varepsilon, \\ v_{\varepsilon 0}^{(\text{out})}(x) & \text{for } x \in \Omega_2^\varepsilon, \end{cases}$$

whose jump across Γ^ε, $[w_\varepsilon] = v_{\varepsilon 0}^{(\text{out})} - v_{\varepsilon 0}^{(a)}$, has null average over each connected component of Γ^ε by (5.7). On the other hand, using (1.11), we compute

$$I_2 \leqslant 2\varepsilon^{N-1} |\Gamma| \sum_{z \in \boldsymbol{Z}_\varepsilon^N} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon(x) \, \mathrm{d}\sigma \right|^2 \leqslant 2 \sum_{z \in \boldsymbol{Z}_\varepsilon^N} \int_{\varepsilon(\Gamma+z)} S_\varepsilon^2(x) \, \mathrm{d}\sigma = 2 \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \leqslant \gamma \varepsilon.$$

In particular, by (5.9), the classical Poincar's inequality, (5.3) and (5.1) we obtain

$$\|v_{\varepsilon 0}\|^2_{L^2(\Omega)} \leqslant \gamma. \tag{5.11}$$

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It remains to prove equation (1.38) in the present case $k = 0$. To this end, we set

$$\theta^0 = \chi^0 + \int_0^{+\infty} \chi^1(\cdot, t)\, \mathrm{d}t\,. \tag{5.12}$$

We remark that θ^0 coincides with θ^ω defined in (4.53) after setting $\omega = 0$. Using equations (2.7), (2.8) and (2.11) and Proposition 10, we note that the components θ_h^0, $h = 1, \ldots, N$, of $\theta^0 : Y \to \boldsymbol{R}^N$ satisfy

$$-\sigma\, \Delta_y\, \theta_h^0 = 0, \qquad \text{in } E_1,\, E_2; \tag{5.13}$$

$$[\sigma(\nabla_y \theta_h^0 - \boldsymbol{e}_h) \cdot \nu] = 0\,, \qquad \text{on } \Gamma\,; \tag{5.14}$$

$$(\sigma_2(\nabla_y \theta_h^0 - \boldsymbol{e}_h) \cdot \nu)^{(\text{out})} = 0, \qquad \text{on } \Gamma\,. \tag{5.15}$$

In addition, θ_h^0 is a Y-periodic function with vanishing integral average over Y. The above problem is comprised of two independent Neumann problems in E_1 and E_2. Comparing with problem (5.5)–(5.6), we obtain that

$$\theta_h^0(y) = \begin{cases} y_h + d_1 & \text{for } y \in E_1\,, \\ \chi_h^{00}(y) + d_2 & \text{for } y \in E_2\,, \end{cases}$$

for some constants d_1, d_2. Hence, recalling (2.5) and (2.6), we get

$$A + \int_0^{+\infty} B(t)\, \mathrm{d}t = \sigma_0 I + \int_\Gamma \nu \otimes [\sigma\theta^0](y)\, \mathrm{d}\sigma = \sigma_0 I - \sigma_1 |E_1| I + \int_\Gamma \nu \otimes \sigma_2 \chi^{00}(y)\, \mathrm{d}\sigma = A^0$$

Remark 18 We note that our hypotheses on the geometry of Ω_2^ε imply that A^0 is a positive definite real symmetric matrix [9, Chapter 1] and $v_{00} \in H^1(\Omega)$.

6 Time-periodic solutions

In this section we prove Theorem 7.

6.1 Fourier representation of the time-periodic solution $\{u_\varepsilon^\#\}$

Here we prove Theorem 7, part (i). In order to show the convergence in $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ of the series at the right-hand side of equation (1.26), we use the Parseval identity and equations (1.3), (4.1), (4.3) and (5.11), and we get

$$\begin{aligned} \|u_\varepsilon^\#\|_{H^1_\#(\boldsymbol{R};L^2(\Omega))} &= \int_0^T \int_\Omega \left[\left| \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\, \mathrm{e}^{i\omega_k t} \right|^2 + \left| \sum_{k=-\infty}^{+\infty} i\omega_k v_{\varepsilon k}(x)\, \mathrm{e}^{i\omega_k t} \right|^2 \right] \mathrm{d}x\, \mathrm{d}t \\ &= T \int_\Omega \sum_{k=-\infty}^{+\infty} \left(1 + \omega_k^2\right) |v_{\varepsilon k}(x)|^2\, \mathrm{d}x \leqslant \gamma \sum_{k=-\infty}^{+\infty} \left(1 + \omega_k^2\right) |c_k|^2 < +\infty. \end{aligned}$$

The convergence in $H^1_\#(\boldsymbol{R}; H^1(\Omega_i^\varepsilon))$, $i = 1, 2$, can be shown analogously.

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It remains to show that the function $u^{\#}_{\varepsilon}(x,t)$ defined in (1.26) solves problem (1.15)–(1.20). Weak solutions to this problem are defined to be in the class

$$u_{\varepsilon}(x,\cdot) \text{ is } T\text{-periodic in time;} \qquad u_{\varepsilon|\Omega^{\varepsilon}_i} \in L^2_{\#}\big(\boldsymbol{R};H^1\big(\Omega^{\varepsilon}_i\big)\big), \quad i=1,2; \tag{6.1}$$

and $u_{\varepsilon|\partial\Omega} = \Psi\Phi$ in the sense of traces. The weak formulation is

$$\int_0^T \int_{\Omega} \sigma \nabla u_{\varepsilon} \cdot \nabla \overline{\psi} \, \mathrm{d}x \, \mathrm{d}t - \frac{\alpha}{\varepsilon} \int_0^T \int_{\Gamma^{\varepsilon}} [u_{\varepsilon}] \frac{\partial}{\partial t} [\overline{\psi}] \, \mathrm{d}\sigma \, \mathrm{d}t = 0, \tag{6.2}$$

for each $\psi \in L^2_{\#}(\boldsymbol{R};L^2(\Omega))$ such that ψ is in class (6.1), $[\psi] \in H^1_{\#}(\boldsymbol{R};L^2(\Gamma^{\varepsilon}))$ and ψ vanishes on $\partial\Omega \times (0,T)$.

The left-hand side in equation (6.2), after substituting u_{ε} from the series at the right-hand side of (1.26), becomes

$$\sum_{k=-\infty}^{+\infty} \int_0^T \left[\int_{\Omega} \sigma \nabla v_{\varepsilon k} \cdot \nabla \overline{\psi} \, \mathrm{d}x + \frac{i\alpha\omega_k}{\varepsilon} \int_{\Gamma^{\varepsilon}} [v_{\varepsilon k}][\overline{\psi}] \, \mathrm{d}\sigma \right] \mathrm{e}^{i\omega_k t} \, \mathrm{d}t,$$

which vanishes, since $v_{\varepsilon k}$ satisfies problem (1.27)–(1.29) for $k \neq 0$ and problem (1.31)–(1.33) for $k=0$. The series over k can be exchanged with the integrals, since using Hölder's inequality and equations (1.3), (4.1), (4.2) and (5.1) we obtain

$$\sum_{k=-\infty}^{+\infty} \int_0^T \left[\int_{\Omega} |\sigma \nabla v_{\varepsilon k} \cdot \nabla \overline{\psi}| \, \mathrm{d}x + \frac{\alpha\omega_k}{\varepsilon} \int_{\Gamma^{\varepsilon}} |[v_{\varepsilon k}][\overline{\psi}]| \, \mathrm{d}\sigma \right] \mathrm{d}t \leqslant \gamma(\varepsilon) \sum_{k=-\infty}^{+\infty} |c_k|^2 < +\infty .$$

On the other hand, the boundary condition (1.8) is satisfied, as it is easily verified by exchanging the trace operator on $\partial\Omega$ with the series and recalling equations (1.25), (1.30) and (1.34), taking into account the linearity and continuity of the trace operator in the space $H^1_{\#}(\boldsymbol{R};H^1(\Omega^{\varepsilon}_2))$.

The uniqueness of T-periodic solutions to problem (1.15)–(1.20) is easily proved. Indeed, by linearity, the difference $w^{\#}_{\varepsilon}(x,t)$ of two such solutions satisfies

$$\int_0^T \int_{\Omega} \sigma |\nabla w^{\#}_{\varepsilon}|^2 \, \mathrm{d}x = 0 \, ;$$

hence it is piecewise constant. This relation follows integrating (1.15) over $\Omega \times (0,T)$, using the Gauss–Green identity, the homogeneous Dirichlet boundary data for $w^{\#}_{\varepsilon}$ and equations (1.16), (1.17) and (1.19). By equation (1.20), it follows that $w^{\#}_{\varepsilon}$ has null average over each connected component of Γ^{ε}; hence it is constant over $\Omega \times \boldsymbol{R}$, and so it vanishes, due to the homogeneous Dirichlet boundary data.

6.2 Convergence of $\{u^{\#}_{\varepsilon}\}$ to $u^{\#}_0$ as $\varepsilon \to 0$

Here we prove Theorem 7, part (ii). The strong convergence in $H^1_{\#}(\boldsymbol{R};H^1(\Omega))$ of the series at the right-hand side of (1.39) easily follows from the Parseval identity, equations (4.1) and (4.62), Remark 18 and assumptions (1.3) and (1.4).

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In order to show that $\{u_\varepsilon^\#\}$ strongly converges in $H^1_\#(\boldsymbol{R};L^2(\Omega))$ as $\varepsilon \to 0$ to $u_0^\#$, we compute, for $k_0 \in \boldsymbol{N}$ fixed,

$$\begin{aligned}
&\int_0^T \int_\Omega \left[|u_\varepsilon^\#(x,t) - u_0^\#(x,t)|^2 + |u_{\varepsilon t}^\#(x,t) - u_{0t}^\#(x,t)|^2\right] \mathrm{d}x\,\mathrm{d}t \\
&\quad = T \int_\Omega \sum_{k=-\infty}^{+\infty} \left(1+\omega_k^2\right)|v_{\varepsilon k}(x) - v_{0k}(x)|^2\,\mathrm{d}x \\
&\quad = T \sum_{|k|\leqslant k_0} \left(1+\omega_k^2\right)\|v_{\varepsilon k} - v_{0k}\|^2_{L^2(\Omega)} + T \sum_{|k|>k_0} \left(1+\omega_k^2\right)\|v_{\varepsilon k} - v_{0k}\|^2_{L^2(\Omega)} =: I_1 + I_2,
\end{aligned}$$

where we have used the monotone convergence theorem. Using equations (4.1), (4.3) and (4.62) we compute

$$|I_2| \leqslant \gamma \sum_{|k|>k_0} \left(1+\omega_k^2\right)\left(\|v_{\varepsilon k}\|^2_{L^2(\Omega)} + \|v_{0k}\|^2_{L^2(\Omega)}\right) \leqslant \gamma \sum_{|k|>k_0} \left(1+\omega_k^2\right)|c_k^2|.$$

By hypothesis (1.3), the right-hand term of the above inequality can be made arbitrarily small by choosing k_0 sufficiently large. For such fixed k_0, I_1 can be made arbitrarily small letting $\varepsilon \to 0$, by virtue of the strong L^2 convergence of $v_{\varepsilon k}$ to v_{0k} as $\varepsilon \to 0$, and the assertion follows.

6.3 Equation for the time-periodic asymptotic solution $u_0^\#$

Here we prove Theorem 7, part (iii). Equation (1.22) follows from equations (1.25) and (1.37) and the $H^1_\#(\boldsymbol{R};H^1(\Omega))$-convergence of series (1.39). In order to prove equation (1.21), we consider its weak formulation:

$$\int_0^T \int_\Omega \nabla\phi \cdot A\nabla u_0^\#\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nabla\phi \cdot \int_0^{+\infty} B(\tau)\nabla u_0^\#(x,t-\tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t = 0,$$
$$\forall \phi \in L^2_\#(\boldsymbol{R};H^1(\Omega)). \quad (6.3)$$

A direct computation shows that any partial sum of the series (1.39), i.e.

$$\hat{u}_0^N(x,t) = \sum_{k=-N}^{N} v_{0k}(x)\,\mathrm{e}^{i\omega_k t}, \quad N \in \boldsymbol{N}, \quad (6.4)$$

satisfies equation (6.3), by virtue of equations (1.36) and (1.38). Then we let $N \to +\infty$: to this regard, as far as the second integral in equation (6.3) is concerned, we proceed as follows. We exchange the integration order and use Hölder's inequality, the Parseval identity, Beppo–Levi theorem, Proposition 10 and equations (4.1) and (4.62), thus obtaining the following estimate:

$$\left| \int_0^{+\infty} B(\tau)\,\mathrm{e}^{-i\omega_k\tau}\,\mathrm{d}\tau \int_0^T \int_\Omega \nabla\phi \cdot \sum_{|k|>N} \nabla v_{0k}(x)\,\mathrm{e}^{i\omega_k t}\,\mathrm{d}x\,\mathrm{d}t \right| \leqslant \gamma\|\phi\|_{L^2_\#(\boldsymbol{R};H^1(\Omega))} \sum_{|k|>N} |c_k|^2, \quad (6.5)$$

which tends to zero by (1.25) and (1.3).

Remark 19 Theorem 7, part (iii) is related to the results in [12], where, however, the setting is slightly different.

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7 Stability result

In this section we prove Theorems 4 and 5. Let u_ε and $u_\varepsilon^\#$ be the solutions of problem (1.5)–(1.9) and problem (1.15)–(1.20), respectively. We set

$$w_\varepsilon = u_\varepsilon - u_\varepsilon^\# . \tag{7.1}$$

Since w_ε satisfies problem (1.5)–(1.9) with homogeneous Dirichlet boundary data on $\partial\Omega \times (0, +\infty)$, i.e. $\Psi \equiv 0$, and with S_ε replaced by $S_\varepsilon - u_\varepsilon^\#(\cdot, 0)$, which has null mean average over each connected component of Γ^ε, the assertion of Theorems 4 and 5 respectively follows from Theorem 2 and Corollary 3, after proving that $u_\varepsilon^\#(\cdot, 0)$ satisfies (1.11).

To this end, we first observe that a classical trace inequality implies that

$$\frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} \left[u_\varepsilon^\#(x,0)\right]^2 \mathrm{d}\sigma \leqslant \frac{\gamma}{\varepsilon}\int_0^T \int_{\Gamma^\varepsilon} \left(|[u_\varepsilon^\#]|^2 + |[u_{\varepsilon t}^\#]|^2\right) \mathrm{d}\sigma\, \mathrm{d}t. \tag{7.2}$$

Then we use equation (1.26), the Parseval identity, (4.1), (4.2) and (5.10) and estimate

$$\begin{aligned} \frac{\gamma}{\varepsilon}\int_0^T \int_{\Gamma^\varepsilon} \left(|[u_\varepsilon^\#]|^2 + |[u_{\varepsilon t}^\#]|^2\right) \mathrm{d}\sigma\, \mathrm{d}t &= \frac{\gamma}{\varepsilon}\sum_{k=-\infty}^{+\infty} \int_{\Gamma^\varepsilon} |[v_{\varepsilon k}]|^2 \left(1+\omega_k^2\right) \mathrm{d}\sigma \\ &\leqslant \gamma \sum_{k=-\infty}^{+\infty} |c_k|^2 (1+k^2). \end{aligned} \tag{7.3}$$

The assertion follows, since the right-hand term of (7.3) is estimated by a constant independent of ε, by (1.25) and (1.3).

8 Conclusions

In this work we studied the electric conduction in biological tissues in the radiofrequency range. Both a microscopic model and a homogenised macroscopic model have been considered, for time-periodic Dirichlet boundary data. We proved that the solution to the microscopic problem approaches a time-periodic steady state for large times. Further, this limit is uniform with respect to the microstructural parameter ε.

As a consequence, the solution to the homogenised problem also approaches a time-periodic steady state for large times. We derived the equation determining this steady state and connected its coefficients to geometrical and material properties of the biological structure.

These results are relevant from the point of view of applications, since we gave here a theoretical justification for the complex elliptic problem (1.36)–(1.37) presently used in electrical impedance tomography. Even more significantly, we provided an explicit connection between the coefficients in equation (1.36) and the properties of the biological tissue.

We think that the applications to electrical impedance tomography may significantly benefit from further investigations on the subject of coefficient reconstruction (i.e. the inverse problem) for the homogenised model introduced above ((1.21)–(1.22)).

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