Dissecting attached squares

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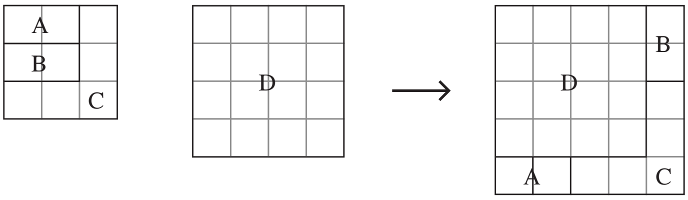
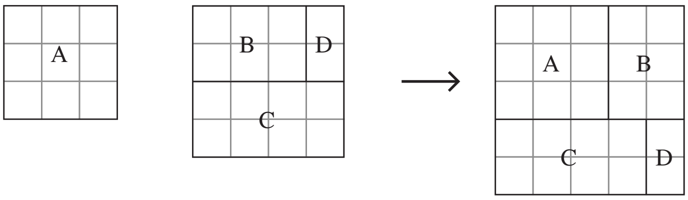
Many papers and indeed books have been written about the problem of dissecting a number of squares of different integer side length and reassembling the pieces to form a single square (see [1], [2] and [3]). For example, in the case of $3^2 + 4^2 = 5^2$, we can achieve a four-piece dissection as follows:

Diagram 1



This can be achieved in many ways, and can obey further constraints, for example, if we insist that all of the pieces are rectangular, we can still achieve a four-piece dissection as follows:

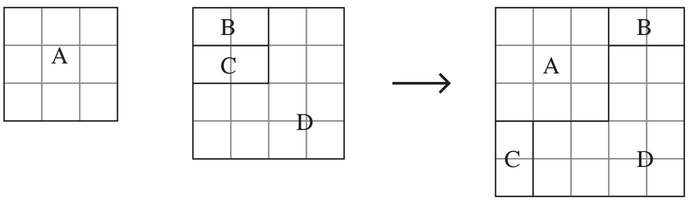
Diagram 2



The reader is challenged to produce a four-piece dissection where all of the pieces can be moved into place by translation only, without rotation.

In this case, four would appear to be the minimum number of pieces possible, but a rigorous proof of this claim could be quite demanding. Here is yet another four-piece dissection.

Diagram 3



Of course, the problem can also be tackled by dissecting the largest square into a number of pieces and reassembling them to form the smaller squares individually.

In this Article we look at the situation where the smaller squares are

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already attached to each other in some way and one often finds that the total number of pieces needed can be reduced, sometimes considerably.

We start with the case where the four-square and the three-square are already attached to each other; now we can in several ways achieve a three-piece dissection where the pieces can be reassembled to form a five-square.

Diagram 4

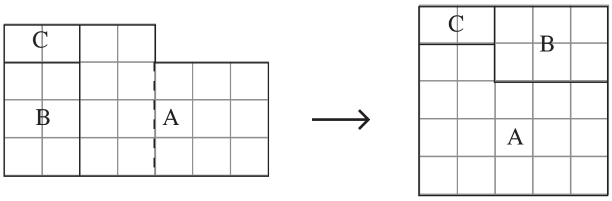
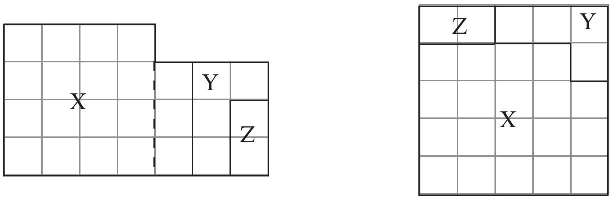
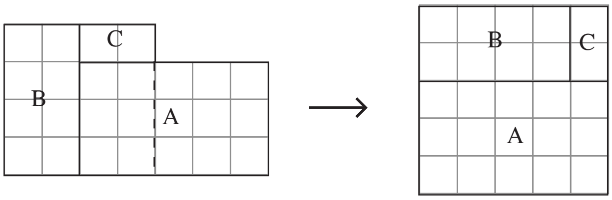


Diagram 5



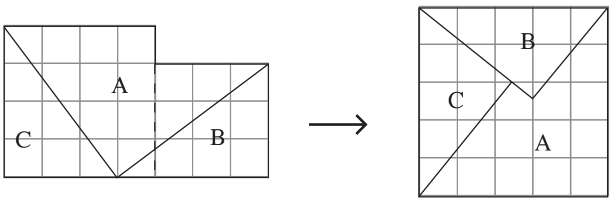
As previously, we can add the constraint that all the pieces are rectangular and still achieve a three-piece dissection as follows:

Diagram 6



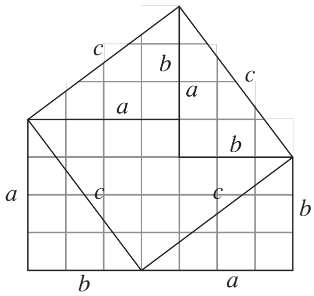
But there is a rather beautiful dissection in this case where the dissection lines are no longer parallel to the sides of the original figure:

Diagram 7



This dissection is universal in its application – take any right-angled triangle with sides of length a and b (not necessarily integers) and hypotenuse of length c; in fact some would claim that this constitutes a dissection 'proof' of the theorem of Pythagoras:

Diagram 8



If we attach a five-square to a twelve-square and use the fact that $12^2 + 5^2 = 13^2$, we get the following three-piece dissections:

Diagram 9

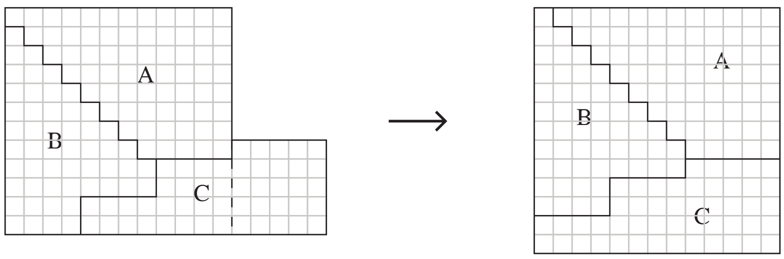
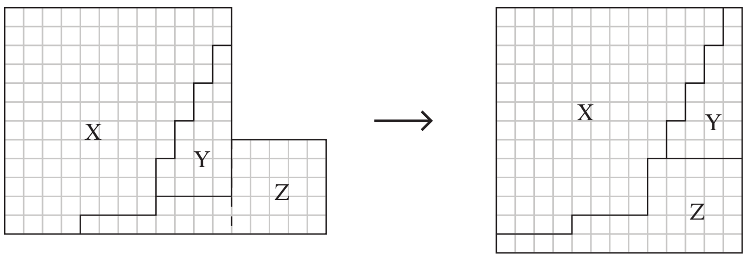


Diagram 10



We note that the above two dissections are translational, without rotation of the pieces.

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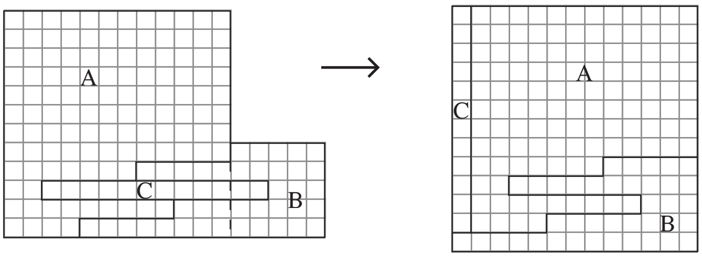
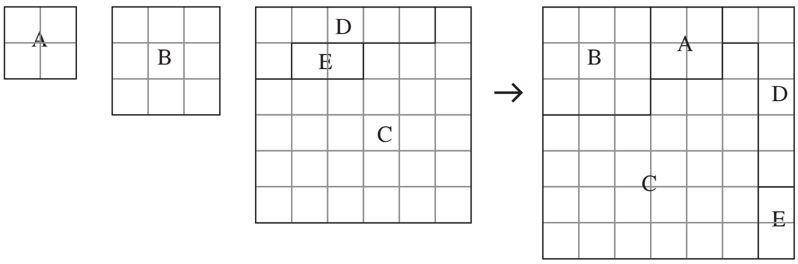
Diagram 11



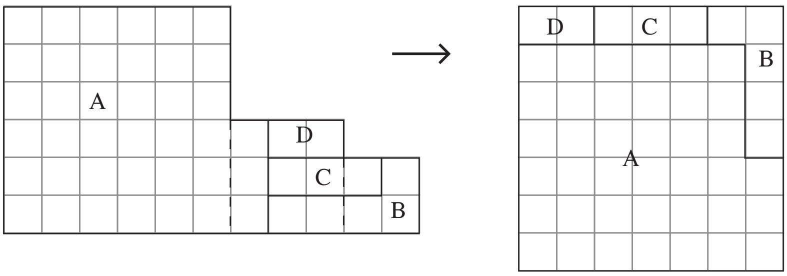
Moving on to integer squares which are the sum of three integer squares, we consider $2^2 + 3^2 + 6^2 = 7^2$. In this case, the optimal number of pieces that has been achieved is five, as follows (see [1]):

Diagram 12



However, by suitably attaching the three smaller squares, we can reduce the number of pieces to four, as follows:

Diagram 13



In fact, by attaching the smaller squares in a different way, we can reduce the number of pieces to three, as follows:

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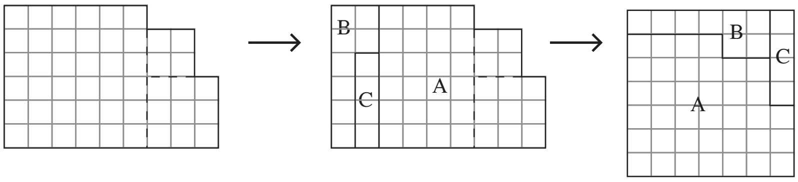
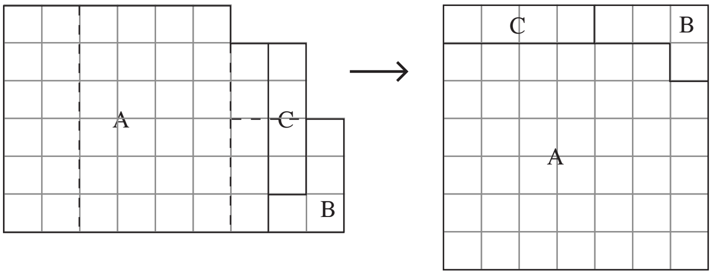
Diagram 14



Or alternatively:

Diagram 15



Next, we consider the equation $1^2 + 4^2 + 8^2 = 9^2$. Several five-piece dissections exist (see [1]) and these are believed to be optimal. For example, we have:

Diagram 16

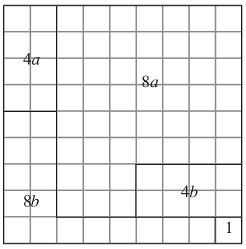


Diagram 17

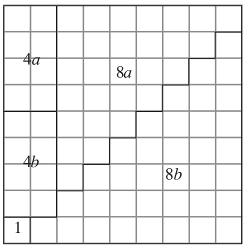
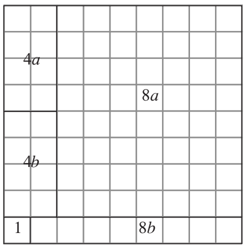


Diagram 18



We note that the figure in Diagram 17 is a step dissection, while that in Diagram 18 is a rectangular dissection. However, if we attach the three smaller squares suitably, we get a remarkable three-piece dissection as follows:

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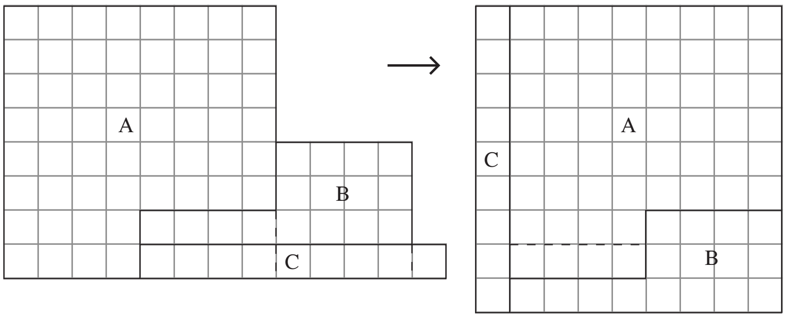
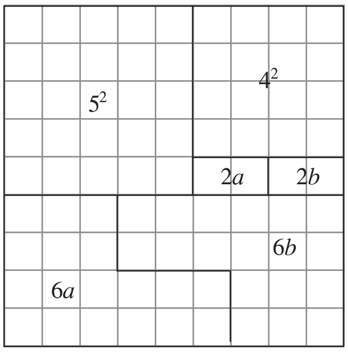
Diagram 19



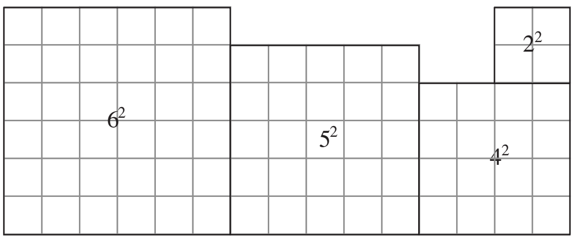
Next, we look at the equation $9^2 = 6^2 + 5^2 + 4^2 + 2^2$. Based on this, at least four dissections exist, all of which have six pieces, for example: (see [1])

Diagram 20



However, if we attach the four smaller squares as follows

Diagram 21



and dissect the resulting figure, as indicated, we get

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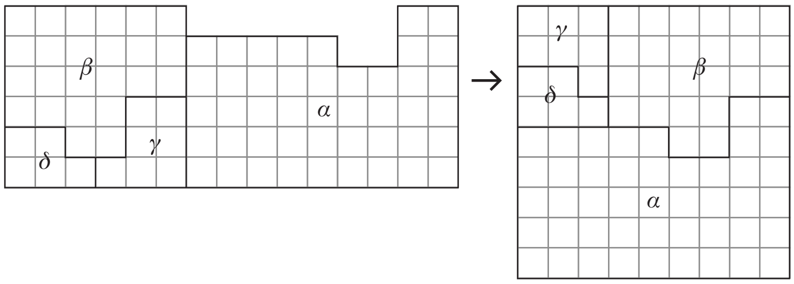
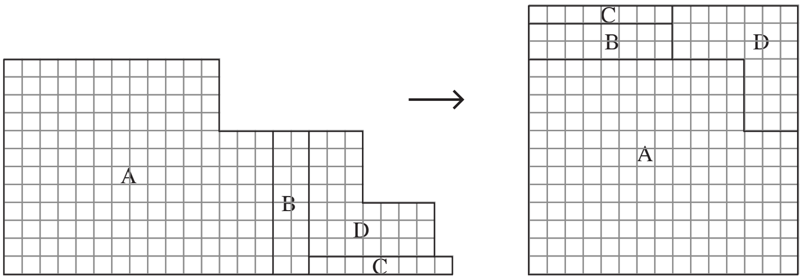
Diagram 22



which is a four-piece dissection.

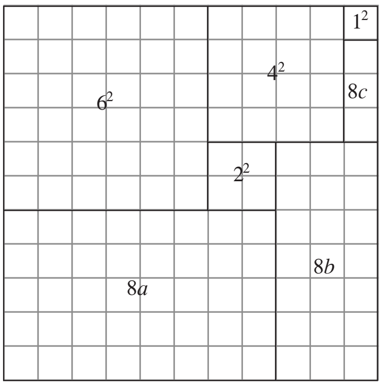
For $12^2 + 8^2 + 4^2 + 1^2 = 15^2$ we have a neat four-piece dissection

Diagram 23



Moving on one more step, we consider the equation $11^2 = 8^2 + 6^2 + 4^2 + 2^2 + 1^2$. For this case we produce a new rectangular dissection, with seven pieces, which is believed to be optimal.

Diagram 24



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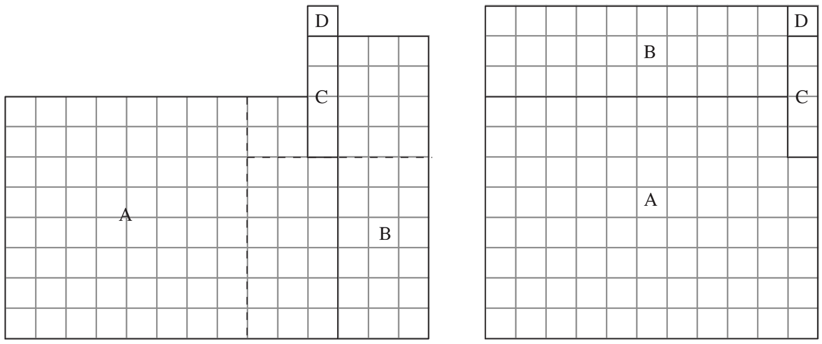
However, if we attach the smaller squares suitably, we can achieve a four-piece dissection as follows:

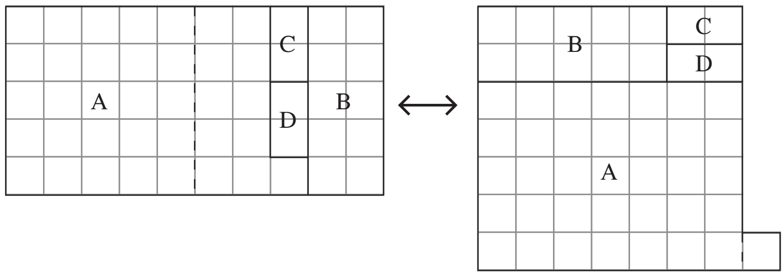
Diagram 25



Clearly, we have merely scratched the surface of this interesting area of dissecting attached squares and indeed equilateral triangles and other regular polygons. See [1] and [3] for a myriad of potential questions to be asked and answered.

One of these is a related problem that has not received a great deal of attention. We have $5^2 + 5^2 = 7^2 + 1^2 = 50$. Can we dissect attached squares to illustrate this equation?

Diagram 26



This is a four-piece dissection and we ask if it can be improved to three-pieces.

Next we consider the equation $7^2 + 4^2 = 8^2 + 1^2 = 65$. Here is a four-piece dissection by which each figure can be reassembled to form the other:

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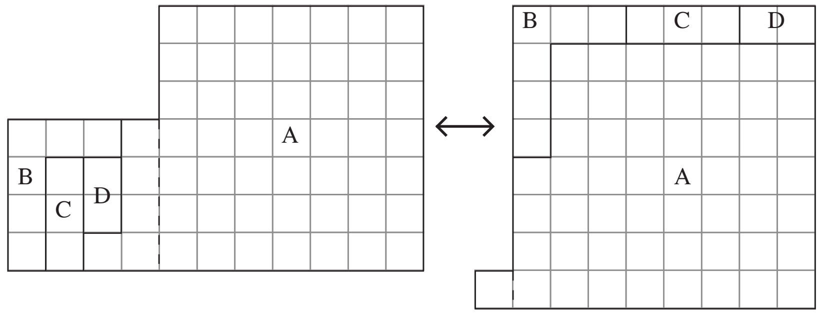
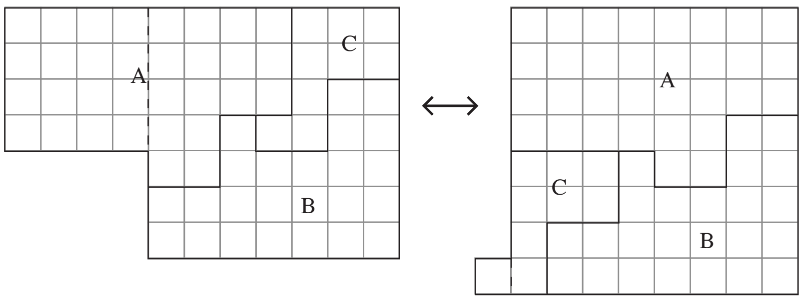
Diagram 27



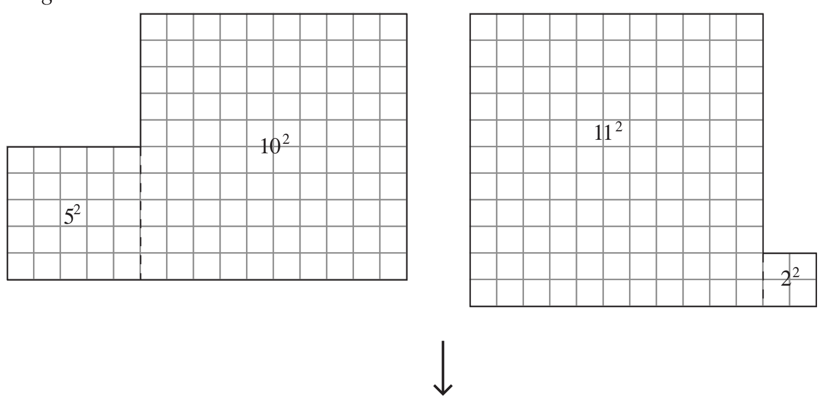
However, there is a rather extraordinary three-piece dissection in this case which is also translational.

Diagram 28



In the case of $5^2 + 10^2 = 125 = 11^2 + 2^2$ we have two remarkable three-piece dissections:

Diagram 29



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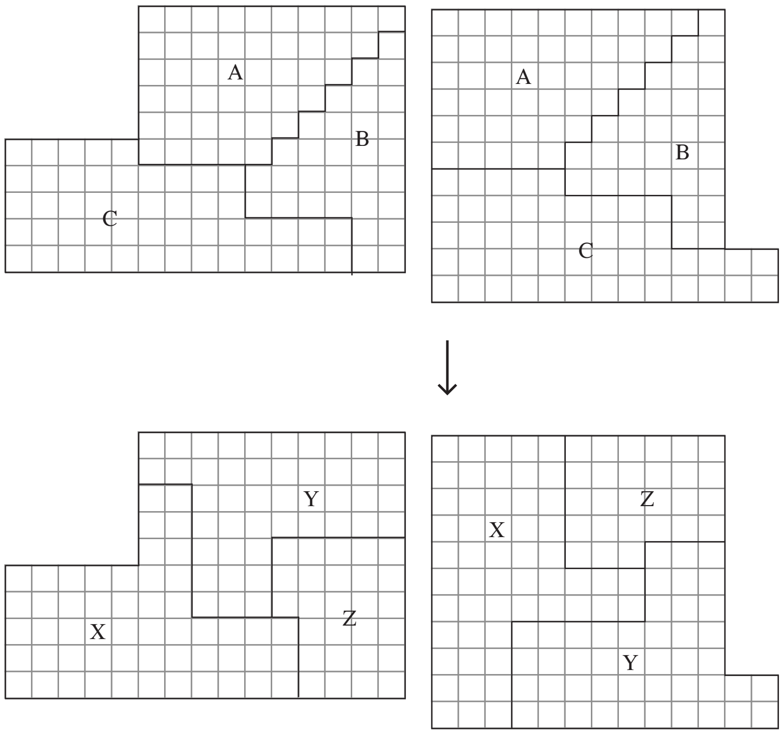
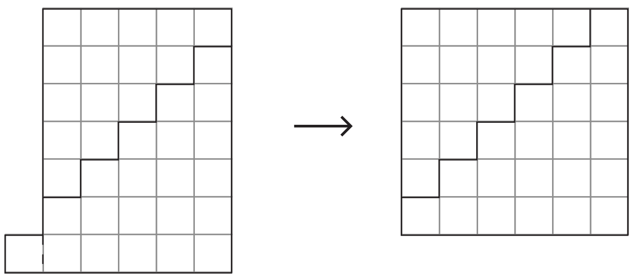


Finally, we mention an ingenious step – dissection which depends on the algebraic identity $n(n + 2) + 1 = (n + 1)^2$. We show the case for $n = 5$, but the method is perfectly general:

Diagram 30



An even simpler two-piece dissection, with both pieces rectangular, based on $(n + 1)^2 = n(n + 2) + 1$, is illustrated for $n = 5$ by:

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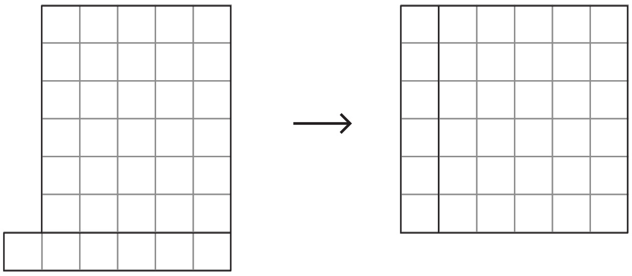
Diagram 31



We remark that if we do not insist that all of our squares have different sizes, considerable reductions in the number of dissected pieces may be achieved by a suitable attachment of squares. The reader may wish to draw diagrams to illustrate the reductions arising from the following equations:

$$3^2 = 2^2 + 2^2 + 1^2 \qquad \text{(3 pieces to 2)}$$
$$4^2 = 3^2 + 2^2 + 1^2 + 1^2 + 1^2 \qquad \text{(6 pieces to 2)}$$
$$5^2 = 3^2 + 3^2 + 2^2 + 1^2 + 1^2 + 1^2 \qquad \text{(7 pieces to 2)}$$

References

1. Joe Kingston and Des MacHale, Dissecting squares, *Math. Gaz.* **85**, (November 2001) pp. 403-430.
2. Joe Kingston and Des MacHale, Some improved dissections of squares, *Math. Gaz.* **87** (July 2003) pp. 280-286.
3. G. N. Frederickson, *Dissections plane and fancy*, Cambridge University Press (1997).

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