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The Classification of Pin_4-Bundles over a 4-Complex

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Abstract. In this paper we show that the Lie-group Pin_4 is isomorphic to the semidirect product $(\mathrm{SU}_2 \times \mathrm{SU}_2)$ $\mathrm{Z}/2$ where $\mathrm{Z}/2$ operates by flipping the factors. Using this structure theorem we prove a classification theorem for Pin_4-bundles over a finite 4-complex X.

1 Introduction

Let G be a compact Lie group. The set of isomorphism classes of principal G-bundles over a topological space X is in one-to-one correspondence to free homotopy classes from X to BG. The homotopy type of BG is determined by being the orbit space of a free G action on a contractible space EG. This means that knowing the homotopy type of the $(k+1)$-skeleton of BG translates the classification of principal G-bundles over a finite k-complex X into calculations in obstruction theory.

We now specialize to X being a finite 4-complex. The case $G = \mathrm{SU}_2$ is very easy: The fact that the 5-skeleton of $B\,\mathrm{SU}_2$ is S^4 and Hopf's classification theorem for $[X, S^4]$ imply that SU_2-bundles over a four-complex X are in 1–1 correspondence to $H^4(X;\mathrm{Z})$, the isomorphism given by the second Chern class.

A. Dold and H. Whitney clarified the case $G = \mathrm{SO}_n$. In [DW59] a general classification theorem for SO_n-bundles is given in terms of obstruction theory. The three dimensional case takes a particular nice form: SO_3-bundles over a 4-complex are classified by the second Stiefel-Whitney class w_2 and the first Pontrijagin class p_1. Moreover, every pair (w_2, p_1) satisfying $\mathcal{P}w_2 \equiv p_1 \bmod 4$, where $\mathcal{P}$ is the Pontrijagin square, is realized as classifying pair for some SO_3-bundle P over X.

Let's move on to the case of a disconnected structure group. The case of $G = O_3$ follows from the SO_3-case since $O_3 = \mathrm{SO}_3 \times \mathrm{Z}/2$ and therefore $BO_3 = B\,\mathrm{SO}_3 \times B\mathrm{Z}/2$. In this paper we want to look at the Lie group Pin_4, a double cover of O_4, and will give a classification theorem for Pin_4-bundles over a 4-complex.

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2 Clifford Algebras, Pin and Spin

General Setup We'll give a brief review of the basics. A more detailed reference is Chapter I in [LM89]. Let (V, q) be a real vector space with quadratic form q. The Clifford algebra $\mathrm{Cl}(V, q)$ is the algebra generated by all $v \in V$ and 1 subject to the relations $v \cdot v = -q(v) \cdot 1$. We are particularly interested in the case $V = \mathrm{R}^n$ and $q^{\pm}(v) = \mp|v|^2$, and will write $\mathrm{Cl}_n^{\pm}$ for $\mathrm{Cl}(\mathrm{R}^n, q^{\pm})$.

$\mathrm{Pin}_n^{\pm}$ is the subgroup of the multiplicative group of $\mathrm{Cl}_n^{\pm}$ generated by elements $v \in S^{n-1}$. Conjugation by an element $v \in \mathrm{R}^n \subseteq \mathrm{Pin}_n^{\pm}$ leaves $\mathrm{R}^n \subseteq \mathrm{Cl}_n^{\pm}$ invariant and preserves q. Therefore we get a map

$$\begin{aligned} \widetilde{\mathrm{Ad}}\colon \mathrm{Pin}_n^{\pm} &\to O_n \\ \phi &\mapsto \big(y \mapsto \alpha(\phi) y \phi^{-1}\big), \end{aligned}$$

where $y \in \mathrm{R}^n$ and α is the endomorphism of $\mathrm{Cl}_n^{\pm}$ which extends $v \mapsto -v$ on R^n. $\widetilde{\mathrm{Ad}}$ is a twofold cover, [LM89, I.2.10]. For $v \in \mathrm{R}^n$ $\widetilde{\mathrm{Ad}}(v)$ is just the reflection at the hyperplane perpendicular to v.

The preimage of SO_n under $\widetilde{\mathrm{Ad}}$ is called $\mathrm{Spin}_n^{\pm}$ and is the subgroup of $\mathrm{Pin}_n^{\pm}$ consisting of products of even numbers of $v \in S^{n-1}$. Since $\alpha(\phi) = \phi$ for $\phi \in \mathrm{Spin}_n$ we see that restricted to Spin_n $\widetilde{\mathrm{Ad}}(\phi)$ is just given by conjugation with $\phi \in \mathrm{Spin}_n$. We will write Ad for the map $\mathrm{Pin}_n^{\pm} \to O_n$ given by conjugation, and therefore $\widetilde{\mathrm{Ad}}_{|\,\mathrm{Spin}_n} = \mathrm{Ad}_{|\,\mathrm{Spin}_n}$.

$\mathrm{Pin}_n^{\pm}$ has a nontrivial one dimensional representation $\chi\colon \mathrm{Pin}_n^{\pm} \to \mathrm{Z}/2$ which is given by extending $V \ni v \mapsto -1$ to all of $\mathrm{Pin}_n^{\pm}$. We see that $\mathrm{Ker}(\chi) = \mathrm{Spin}_n^{\pm}$.

Since SO_n is connected, $\pi_1(\mathrm{SO}_n) = \mathrm{Z}/2$ and both of $\mathrm{Spin}_n^{\pm}$ are nontrivial coverings, we see that Spin_n^{+} and Spin_n^{-} must be isomorphic as groups and coverings of SO_n. Keeping the ambiguity in mind we will from now on drop the superscript and refer only to Spin_n.

Spin_4 ***and Quaternions*** Recall that $\mathrm{H} = \mathrm{R}\langle i, j, k\rangle$ subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. The conjugate of a quaternion $q = a + bi + cj + dk$ is given by $\bar{q} = a - bi - cj - dk$, and $N(q) := q\bar{q}$ defines a norm on H. The group of unit quaternions, *i.e.*, the 3-sphere, can be identified with SU_2, in particular

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Consider the map

$$\begin{aligned} \mu\colon \mathrm{SU}_2 \times \mathrm{SU}_2 &\to \mathrm{GL}_4(\mathrm{R}) \\ (p, q) &\mapsto (x \mapsto pxq^{-1}) \end{aligned}$$

given by quaternionic multiplication. μ maps into SO_4 since $\mu(p, q)$ is norm preserving and has determinant 1. μ defines a double cover of SO_4 and hence there is an isomorphism $\Phi\colon \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{Spin}_4$.

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The Structure of Pin_n The exact sequence of groups

$$1 \to \mathrm{Spin}_n \to \mathrm{Pin}_n^{\pm} \xrightarrow{\chi} \mathbf{Z}/2 \to 1$$

with $n \geq 3$ splits via

$$\sigma(-1) = \begin{cases} e_1 & \text{in the } \mathrm{Pin}_4^+ \text{ case,} \\ e_1 e_2 e_3 & \text{in the } \mathrm{Pin}_4^- \text{ case.} \end{cases}$$

The center of Spin_n is given by

$$C(\mathrm{Spin}_n) = \begin{cases} \mathbf{Z}/2 = \langle -1 \rangle & n \text{ odd,} \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 = \langle \omega \rangle \oplus \langle -\omega \rangle & n \equiv 0 \bmod 4 \\ \mathbf{Z}/4 = \langle \omega \rangle & n \equiv 2 \bmod 4, \end{cases}$$

where $\omega = e_1 \cdots e_n$ is the volume element. We'll always assume that $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbf{R}^n$. Moreover if $\sigma^{\pm}$ is any splitting element of the extension above then $\sigma^{-1} \omega \sigma = (-1)^{n-1} \omega$.

Recall that, unlike in the case where N is abelian, for group extensions of the form

$$1 \to N \to G \to Q \to 1$$

with nonabelian N there may not be a well-defined operation of Q on N given by conjugation with a (set theoretic) section $Q \to G$. However any two choices for a section differ by an element in N which induces an inner automorphism of N. This means that there is a well defined homomorphism from Q to $\mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{Inn}(N)$.

Since $\mathrm{Spin}_4 \cong S^3 \times S^3$ we see that $\mathrm{Out}(\mathrm{Spin}_4) = \mathbf{Z}/2$. The higher dimensional even Spin groups are simple and looking at their Dynkin diagram we can read of their outer isomorphisms. The list for the dimensions divisible by 4 is

$$\mathrm{Out}(\mathrm{Spin}_{4n}) = \begin{cases} \mathbf{Z}/2 & n = 1 \\ S_3 & n = 2 \\ \mathbf{Z}/2 & n \geq 3. \end{cases}$$

Moreover, in all cases the automorphism class is detected by the induced automorphism of the center.

Extensions $N \to G \to Q$ with fixed homomorphism $\phi\colon Q \to \mathrm{Out}(N)$, if there are any, are in one-to-one correspondence to $H^2(Q; CN)$, where we view CN as a Q module, see [EM47]. Therefore we calculate

$$H^2(\mathbf{Z}/2; C\,\mathrm{Spin}_n) = \begin{cases} \mathbf{Z}/2 & n \text{ odd} \\ 0 & n \equiv 0 \bmod 4 \\ \mathbf{Z}/2 & n \equiv 2 \bmod 4. \end{cases}$$

Putting the information together we see

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Proposition 2.1 *If* $n \equiv 0 \bmod 4$ *then* Pin_n^+ *and* Pin_n^- *are isomorphic as groups.*

In general it might be quite difficult to give a more concrete description of the operation of $\mathbb{Z}/2$ on Spin_n than just saying that it is given by conjugation with a split element. However for Spin_4 this is very easy:

Theorem A Pin_4^+ *and* Pin_4^- *are both isomorphic to the semidirect product*

$$(\mathrm{SU}_2 \times \mathrm{SU}_2) \ \mathbb{Z}/2$$

where -1 *operates by flipping the factors.*

A word of warning: The isomorphism between Pin_4^+ and Pin_4^- is one of Lie groups and is not compatible with the projection to O_4. Therefore the obstructions for the existence of Pin_4^+ and Pin_4^- structures on a given O_4 bundle are different in general. However, since we are only interested in Pin_4 principal bundles and isomorphisms between them, we can drop the superscript again and refer to Pin_4 as given by the semidirect product above.

3 Bundle Theory

3.1 Spin_4 Bundles

The adjoint representation $\mathrm{Ad}\colon \mathrm{Spin}_n \to \mathrm{SO}_n$ defines an associated SO_n bundle P_{SO} for every Spin_n bundle P. We denote the Euler and Pontrijagin classes of P_{SO} by $e(P)$ and $p_i(P)$ respectively. The isomorphism $\mathrm{Spin}_4 \cong \mathrm{SU}_2 \times \mathrm{SU}_2$ implies that $B\,\mathrm{Spin}_4 \simeq B\,\mathrm{SU}_2 \times B\,\mathrm{SU}_2 \simeq \mathbb{H}P^\infty \times \mathbb{H}P^\infty$. Recall that $H^4(\mathbb{H}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$ is generated by $c_2(\gamma)$ where γ is the universal SU_2-bundle. The homotopy class of a map $f\colon X \to B\,\mathrm{Spin}_4$ therefore defines an ordered pair $(a,b) \in H^4(X;\mathbb{Z}^2)$ by pulling back $\big(c_2(\pi_1^*\gamma), c_2(\pi_2^*\gamma)\big)$ where $\pi_{1/2}$ is the projection to the first and second factor. Using the Borel-Hirzebruch formalism for characteristic classes one calculates (see [HH58]).

Lemma 3.1 *The characteristic classes* a, b *and* e, p_1 *of a* Spin_4 *bundle are subject to the relations*

$$e = -a + b \qquad p_1 = 2(a+b).$$

Combining Hopf's classification theorem for $[X, S^4]$ with the above lemma we see

Proposition 3.2

i) *Two* Spin_4 *principal bundles over a compact* 4*-complex* X *are isomorphic iff their characteristic classes* (a,b) *in* $H^4(X;\mathbb{Z}^2)$ *coincide as ordered pairs. Moreover, every ordered pair* (a,b) *can be realized.*
ii) *If* $H^4(X;\mathbb{Z})$ *has no* 2*-torsion, then two* Spin_4 *bundles are isomorphic iff their Euler and first Pontrijagin class coincide.*
iii) *A* Spin_4 *bundle over an oriented* 4*-manifold is characterized by its Euler and Pontrijagin number.*

Moreover, the pairs (e,p) which can be realized are exactly the ones satisfying $2|p$ *and* $4|(p+2e)$.

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3.2 Pin_4 Bundles

Let us start by fixing some handy notation for weakly associated Pin_4-bundles:

Lemma 3.3 *Any Spin_4-principal bundle with characteristic classes (a, b) is isomorphic to $P+Q$, where P and Q are SU_2-bundles with second Chern class equal to a and b, and $P+Q := \Delta^*(P \times Q)$, $\Delta\colon X \hookrightarrow X^2$ is the diagonal. The weakly associated Pin_4-bundle has the form*

$$(P+Q) \times_{\mathrm{Spin}_4} \mathrm{Pin}_4 \cong P+Q \coprod Q+P$$

and right multiplication on the right hand side by an element $(\alpha, \beta; \epsilon) \in \mathrm{Pin}_4$ is given by

$$(p,q)(\alpha,\beta;\epsilon) = \begin{cases} (p\alpha, q\beta) & \epsilon = 1 \\ (q\beta, p\alpha) & \epsilon = -1 \end{cases}$$

$$(q,p)(\alpha,\beta;\epsilon) = \begin{cases} (q\alpha, p\beta) & \epsilon = 1 \\ (p\beta, q\alpha) & \epsilon = -1 \end{cases}$$

Proof A typical element of the LHS looks like $[(p, q, a, b, e)]$ with $p \in P$, $q \in Q$ and $(a, b, e) \in \mathrm{Pin}_4$. Moreover $[(p, q, a, b, e)] = [(ps^{-1}, qt^{-1}, sa, tb, e)]$ for all $(s, t) \in \mathrm{Spin}_4$. An element of the LHS therefore has a unique representative of the form $[(p, q, 1, 1, e)]$ for which we will write $[p, q, e]$.

We now define $\Phi\colon$ LHS $\to$ RHS by

$$\Phi([p,q,a,b,e]) := \begin{cases} (pa, qb) & e = 1, \\ (qb, pa) & e = -1. \end{cases}$$

This is well-defined, and one checks that Φ is equivariant with respect to the right multiplication with elements of Pin_4 on both sides. ■

Lemma 3.4 *The classifying space $B\,\mathrm{Pin}_4$ is homotopy equivalent to the $\mathrm{H}P^\infty \times \mathrm{H}P^\infty$ bundle over $\mathrm{R}P^\infty$ given by the quotient*

$$(\mathrm{H}P^\infty \times \mathrm{H}P^\infty \times S^\infty)_{/(\bar{x},\bar{y},z)\sim(\bar{y},\bar{x},-z)}.$$

Proof It suffices to give a free Pin_4 right operation on a contractible space with quotient as claimed. Think of S^∞ to be the unit sphere in H^∞. Now define an action

$$(S^\infty \times S^\infty \times S^\infty) \times \mathrm{Pin}_4 \to S^\infty \times S^\infty \times S^\infty$$

$$(x,y,z)\cdot(\alpha,\beta,\epsilon) = \begin{cases} (x\alpha^{-1}, y\beta^{-1}, z) & \epsilon = 1 \\ (y\alpha^{-1}, x\beta^{-1}, -z) & \epsilon = -1. \end{cases}$$

One easily checks that this is a free action with the right quotient. ■

Before we can prove our classification result for Pin_4 bundles we need two technical lemmas:

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Lemma 3.5 *Let X and Y be two connected* CW *complexes with basepoints x_0 and y_0. Let $\tilde{Y} \xrightarrow{\pi} Y$ be the universal covering. Identify $\pi_1(Y, y_0)$ with the group of covering transformations of $\tilde{Y}$. Let $\tilde{f}, \tilde{g}\colon X \to \tilde{Y}$, $f := \pi \circ \tilde{f}$ and $g := \pi \circ \tilde{g}$.*

Then: $f \simeq g$ iff $\tilde{f} \simeq \alpha \circ \tilde{g}$ for some $\alpha \in \pi_1 Y$.

This can quickly be proved using covering space theory. On the algebraic side we have:

Lemma 3.6 *Let X be a finite* CW *complex of dimension n, $w\colon G := \pi_1 X \to \mathbb{Z}/2$ a nonzero map, $H := \mathrm{Ker}(w)$ and $\Lambda := \mathbb{Z}[G/H]$ considered as a G-module. Furthermore let $\pi\colon X^w \to X$ be the twofold covering associated to w with covering transformation τ. Then there is an isomorphism*

$$\Phi^*\colon H^*(X; \Lambda) \to H^*(X^w; \mathbb{Z}).$$

Since $\mathrm{Res}_H \Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$ is a trivial H module, the map on cohomology induced by π is given by

$$H^*(X; \Lambda) \xrightarrow{\pi^*} H^*(X^w; \mathrm{Res}_H \Lambda) = H^*(X^w; \mathbb{Z} \oplus \mathbb{Z})$$
$$x \mapsto (\Phi^* x, \tau^* \Phi^* x).$$

In particular π^* is injective.

Proof Recall some facts from algebra: Let $H \subset G$ be a subgroup of finite index, M an H-module and N a G-module, then there is a natural isomorphism

$$\Phi\colon\ \mathrm{Hom}_G\big(N, \underbrace{\mathrm{Hom}_H(\mathbb{Z}G, M)}_{=:\,\mathrm{Coind}_H^G M}\big) \cong \mathrm{Hom}_H(N, M),$$

see [Bro82, p. 64]. In our situation $\Lambda = \mathrm{Coind}_H^G \mathbb{Z}$, where $\mathbb{Z}$ is the trivial H-module. Now let $N_i := C_i(\tilde{X})$ be the chain complex of the universal covering of X, thought of as a G- and H-module. Since the isomorphism Φ is natural it commutes with differentials, and so induces an isomorphism

$$\Phi^*\colon H^*(X; \Lambda) \to H^*(X^w; \mathbb{Z})$$

as claimed.

To see the second claim note that on the cochain level the map induced by π is given by

$$\mathrm{Hom}_G(C_i\tilde{X}, \Lambda) \to \mathrm{Hom}_H(C_i\widetilde{X^w}; \mathbb{Z} \oplus \mathbb{Z})$$
$$\alpha \mapsto (\Phi\alpha, \Phi\alpha \circ \tau).$$

This implies the lemma since the differential on the right hand side respects the direct sum decomposition given by $\mathbb{Z} \oplus \mathbb{Z}$. ■

The one dimensional nontrivial representation of Pin_4 defines a map w_1 from $B\,\mathrm{Pin}_4$ to $\mathbb{R}P^\infty$. The corresponding characteristic class $w_1(P)$ of a Pin_4 bundle P is equal to the first Stiefel-Whitney class of the associated O_4 bundle P_O. Using the cell decomposition of $B\,\mathrm{Pin}_4$ given by 3.4 one calculates that $H^4(B\,\mathrm{Pin}; \mathbb{Z}^-) = \langle\tilde{e}\rangle \cong \mathbb{Z}$. If $f_P\colon X \to B\,\mathrm{Pin}_4$ is the classifying map of P we set $\tilde{e}(P) := f_P^*(\tilde{e}) \in H^4(X; \mathbb{Z}^{w_1 P})$ and call it the twisted Euler class

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of P. Here $H^*(B\,\mathrm{Pin}_4;\mathbb{Z}^-)$ is the twisted cohomology of $B\,\mathrm{Pin}_4$ where $\mathbb{Z}^-$ is the unique non-trivial $\mathbb{Z}/2$-module. Likewise $H^*(X;\mathbb{Z}^{w_1P})$ is the twisted cohomology of X where $\mathbb{Z}^{w_1P}$ are the integers turned into a π_1X-module via the map $w_1P\colon \pi_1X \to \{\pm 1\}$.

Now let X be a compact 4-complex and $P \to X$ a Pin_4 principal bundle with $w_1(P) =: w$. Let $\pi\colon X^w \to X$ be the twofold covering corresponding to w, Then π^*P lifts to a Spin_4 bundle $\tilde{P}$ and hence, by Proposition 3.2, there are two classes (a,b) in $H^4(X^w;\mathbb{Z})$ which we will call the classifying pair of P.

Theorem B *Let P and Q be two Pin_4 bundles over a compact 4-complex X with $w_1(P) = w_1(Q) =: w$. Then P is isomorphic to Q iff their classifying pairs (a_P, b_P) and (a_Q, b_Q) coincide as unordered pairs in $H^4(X^w;\mathbb{Z})$.*

- *If $w = 0$ then every unordered pair (a,b) in $H^4(X;\mathbb{Z})$ is realized.*
- *If $w \neq 0$ let $\tau\colon X^w \to X^w$ be the covering transformation. Then every unordered pair (a,τ^*a), $a \in H^4(X^w;\mathbb{Z})$ is realized.*

The classifying classes a, b of P are related to Euler and Pontrijagin classes via

$$e(\tilde{P}) = -a + b, \qquad p_1(\tilde{P}) = 2(a+b).$$

Proof We do the case $w = 0$ first: In this case the classifying maps f_P and f_Q lift to $B\,\mathrm{Spin}$. The model for $B\,\mathrm{Pin}$ implies that the covering transformation ϕ operates as 'flip' on $H^4(B\,\mathrm{Spin};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore if $f\colon X \to B\,\mathrm{Pin}_4$ lifts to $B\,\mathrm{Spin}_4$ the classifying classes coincide as unordered pair for every possible lift of f. Now apply Lemma 3.5 and Proposition 3.2.

Now let w be nonzero. The fibration $B\,\mathrm{Spin}_4 \to B\,\mathrm{Pin}_4 \to \mathbb{R}P^\infty$ implies a commutative diagram

$$\begin{array}{ccccc} & & B\,\mathrm{Pin}_4 & \longleftarrow & B\,\mathrm{Spin}_4 \\ & {\scriptstyle f_P, f_Q} \nearrow & \downarrow {\scriptstyle w_1} & & \\ X & \underset{w_1Q}{\overset{w_1P}{\longrightarrow}} & \mathbb{R}P^\infty. & & \end{array}$$

Since $w_1(P) = w_1(Q)$ there is a homotopy $H\colon X \times I \to \mathbb{R}P^\infty$ connecting $w_1 \circ f_P$ and $w_1 \circ f_Q$. We try to lift this homotopy to $B\,\mathrm{Pin}_4$. The obstructions for doing this on the i-skeleton of X are

$$\mathrm{o}_i \in H^i\big(X, \{\pi_i(B\,\mathrm{Spin}_4)\}\big).$$

Since $\pi_i(B\,\mathrm{Spin}_4) = 0$ for $i \leq 3$ we see that H can be lifted over the 3-skeleton $X^{(3)}$.

Now let X^w be the two fold covering of X associated to w. Since $\pi^*w_1P = \pi^*w_1Q = 0$ there exist lifts $f_{\tilde{P}}$ and $f_{\tilde{Q}}$ of π^*f_P and π^*f_Q to $B\,\mathrm{Spin}$.

$$\begin{array}{ccc} & & B\,\mathrm{Spin}_4 \\ & {\scriptstyle f_{\tilde{P}}, f_{\tilde{Q}}} \nearrow & \downarrow \\ X^w & \underset{\pi^* f_Q}{\overset{\pi^* f_P}{\longrightarrow}} & B\,\mathrm{Pin} \\ \downarrow {\scriptstyle \pi} & & \| \\ X & \xrightarrow{f_P, f_Q} & B\,\mathrm{Pin}\,. \end{array}$$

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The preceding case implies that $\pi^* f_P \simeq \pi^* f_Q$ iff the classifying pairs of P and Q coincide as unordered pairs. The assumptions therefore imply that $\pi^* o_4 = 0 \in H^4(X^w; \pi^*\pi_4 B\,\mathrm{Spin}_4)$. But $\pi_4 B\,\mathrm{Spin}_4$ is as $\pi_1 X$-module isomorphic to $\mathrm{Z}[\pi_1 X/\pi_1 X^w]$. Therefore Lemma 3.6 implies that

$$\pi^* \colon H^4(X; \pi_4 B\,\mathrm{Spin}_4) \to H^4(X^w; \pi^*\pi_4 B\,\mathrm{Spin}_4)$$

is injective. Hence $o_4 = 0 \in H^4(X; \pi_4 B\,\mathrm{Spin}_4)$, and therefore $f_P \simeq f_Q$.

To see that $a_P = \tau^* b_P$ observe that the diagram

$$\begin{array}{ccc} X^w & \xrightarrow{f_{\tilde{P}}} & B\,\mathrm{Spin}_4 \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ X^w & \xrightarrow{f_{\tilde{P}}} & B\,\mathrm{Spin}_4, \end{array}$$

where T is the covering transformation of $B\,\mathrm{Spin}_4$ over $B\,\mathrm{Pin}_4$, is commutative since the 2-fold covering $X^w \to X$ is the pullback of $B\,\mathrm{Spin} \to B\,\mathrm{Pin}$ under f_P. Since T^* flips the chosen generators a, b of $H^4(B\,\mathrm{Spin}_4; \mathrm{Z})$ this shows that

$$(\tau^* \tilde{f}^* a, \tau^* \tilde{f}^* b) = (\tilde{f}^* b, \tilde{f}^* a).$$

Since for $w = 0$ the existence follows from the existence part of 3.2 we can restrict ourself to the case $w \neq 0$. To see that every pair $\{a, \tau^* a\}$, $a \in H^4(X^w; \mathrm{Z})$, is realized for some Pin_4-bundle over X let P_a be the SU_2-bundle with $c_2 = a$ and form the Spin_4 bundle $P_a + P_{\tau^* a}$. According to Lemma 3.3 the weakly associated Pin_4 bundle $\hat{P} := (P_a + P_{\tau^* a}) \times_{\mathrm{Spin}_4} \mathrm{Pin}_4$ is equal to $P_a + P_{\tau^* a} \coprod P_{\tau^* a} + P_a$. Since $P_{\tau^* a} = \tau^* P_a$ as SU_2-bundles there is a map $\tau' \colon P_{\tau^* a} \to P_a$ covering τ. To simplify notation we'll write τ' also for $(\tau')^{-1} \colon P_a \to P_{\tau^* a}$.

Having the notation set up we define an involution $\hat{\tau}$ on $\hat{P}$ by

$$\begin{aligned} \hat{\tau} \colon P_a + P_{\tau^* a} \coprod P_{\tau^* a} + P_a &\to P_a + P_{\tau^* a} \coprod P_{\tau^* a} + P_a \\ (p, q) &\mapsto (\tau' p, \tau' q) \\ (q, p) &\mapsto (\tau' q, \tau' p). \end{aligned}$$

This map is a Pin_4 equivariant involution on $\hat{P}$ covering τ on X^w and flipping the components of $\hat{P}$. The quotient $\hat{P}/\hat{\tau}$ is therefore a Pin_4 bundle over X with data $(w, a, \tau^* a)$.

The relation between (a, b) and Euler and Pontrijagin class follows from Lemma 3.1. ■

Corollary 3.7 *Let $H^4(X^w; \mathrm{Z})$ contain no 2-torsion.*

i) Two Spin_4 bundles are isomorphic as Pin_4 bundles iff p_1 coincides and e coincides up to sign.

ii) Two Pin_4 bundles with $w_1 \neq 0$ are isomorphic iff they have the same w_1 and their twisted Euler classes coincide up to sign.

Corollary 3.8 *Pin_4 bundles over a nonorientable 4-manifold X with $w_1 = w_1(X)$ are classified by the absolute value of their Euler number.*

Moreover, every number $k \geq 0$ is realized as Euler number for some Pin_4 bundle P.

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