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The zero-level centralizer in endomorphism algebras

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For an endomorphism $\varphi \in \mathrm{End}_R(M)$ of a left R-module ${}_RM$, we investigate the structure and the polynomial identities of the zero-level centralizer $\mathrm{Cen}_0(\varphi)$ and the factor $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$. A double zero-centralizer theorem for $\mathrm{Cen}_0(\mathrm{Cen}_0(\varphi))$ is also formulated.

1. Introduction

If S is a ring (or algebra), then the centralizer $\mathrm{Cen}(s) = \{u \in S \mid us = su\}$ of an element $s \in S$ is a subring (subalgebra) of S. We have $\mathrm{Cen}(s) = \bigcup_{c\in \mathrm{LCen}(s)} \mathrm{Cen}_c(s)$, where $\mathrm{Cen}_c(s) = \{u \in S \mid us = su = c\}$ is called the c-level centralizer and $\mathrm{LCen}(s) = \{c \in S \mid \mathrm{Cen}_c(s) \neq \varnothing\}$ is a subring of $\mathrm{Cen}(s)$. The zero-level centralizer $\mathrm{Cen}_0(s) = \{u \in S \mid us = su = 0\}$ (or the two-sided annihilator) of s is an ideal of $\mathrm{Cen}(s)$ and $u+\mathrm{Cen}_0(s) \mapsto us$ is a natural $\mathrm{Cen}(s)/\mathrm{Cen}_0(s) \to \mathrm{LCen}(s)$ isomorphism of the additive abelian groups.

The aim of this paper is to investigate the zero-level centralizer $\mathrm{Cen}_0(\varphi)$ and the factor $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$ for an element φ in the endomorphism ring $\mathrm{End}_R(M)$ of a left R-module ${}_RM$. Our treatment follows the arguments of [1] and is heavily based on the results of [1,6]. Thus, we restrict our attention to the case of a finitely generated semi-simple ${}_RM$. First we focus on a nilpotent φ, and then we shall see that, for a non-nilpotent φ, the study of $\mathrm{Cen}_0(\varphi)$ can be reduced to the nilpotent case.

We were unable to find related results in the literature, in spite of the fact that the objects of our investigations arise very naturally. Surprisingly, the dimension formula for the zero-level centralizer of a square matrix has not yet appeared in linear algebra books (see, for example, [2,4,5,7]).

In § 2 we consider a fixed nilpotent Jordan normal base of ${}_RM$ with respect to a given nilpotent $\varphi \in \mathrm{End}_R(M)$ and present all the necessary prerequisites from [1,6].

Section 3 is devoted entirely to the nilpotent case. Theorem 3.3 gives a complete characterization of $\mathrm{Cen}_0(\varphi)$ and $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$. If the base ring is local, then a more accurate description of these algebras can be found in theorem 3.4. Using theorem 3.4 and the identities of certain subalgebras of a full matrix algebra over R/J, where J is the Jacobson radical of R, in theorems 3.6 and 3.8 we exhibit explicit polynomial identities for $\operatorname{Cen}_0(\varphi)$ and $\operatorname{Cen}(\varphi)/\operatorname{Cen}_0(\varphi)$, respectively.

In § 4 we deal with the non-nilpotent case. A complete description of $\operatorname{Cen}_0(\varphi)$ (as a particular ideal of an algebra of certain invariant endomorphisms) can be found in theorem 4.1. If $A \in M_n(F)$ is an $n \times n$ matrix over a field F, then the mentioned dimension formula $\dim_F \operatorname{Cen}_0(A) = [\dim_F(\ker(A))]^2$ is an immediate corollary of theorem 4.1. Theorems 4.5 and 4.6 deal with the containment relation $\operatorname{Cen}_0(\varphi) \subseteq \operatorname{Cen}_0(\sigma)$, where $\sigma \in \operatorname{End}_R(M)$ is another endomorphism. Since this containment is equivalent to $\sigma \in \operatorname{Cen}_0(\operatorname{Cen}_0(\varphi))$, theorems 4.5 and 4.6 can be considered as double zero-centralizer theorems.

2. Prerequisites

In order to provide a self-contained treatment, we collect some notation, definitions and statements from [1, 6]. Let $Z(R)$ and $J = J(R)$ denote the centre and the Jacobson radical of a ring R (with identity). Let $(z^k) \lhd R[z]$ denote the ideal generated by z^k in the ring $R[z]$ of polynomials of the commuting indeterminate z.

For an R-endomorphism $\varphi\colon M \to M$ of a (unitary) left R-module ${}_RM$, a subset

$$\{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\} \subseteq M$$

is called a nilpotent Jordan normal base of ${}_RM$ with respect to φ if each R-submodule $Rx_{\gamma,i} \leqslant M$ is simple,

$$\bigoplus_{\gamma\in\Gamma,1\leqslant i\leqslant k_\gamma} Rx_{\gamma,i} = M$$

is a direct sum, $\varphi(x_{\gamma,i}) = x_{\gamma,i+1}$, $\varphi(x_{\gamma,k_\gamma}) = 0$ for all $\gamma \in \Gamma$, $1 \leqslant i \leqslant k_\gamma$, and the set $\{k_\gamma \mid \gamma \in \Gamma\}$ of integers is bounded. Now Γ is called the set of (Jordan) blocks and the size of the block $\gamma \in \Gamma$ is the integer $k_\gamma \geqslant 1$.

Theorem 2.1. *Let $\varphi \in \operatorname{End}_R(M)$ be an R-endomorphism of a left R-module ${}_RM$. Then the following are equivalent.*

(i) *${}_RM$ is a semi-simple left R-module and φ is nilpotent of index n.*

(ii) *There exists a nilpotent Jordan normal base $X = \{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\}$ of ${}_RM$ with respect to φ such that $n = \max\{k_\gamma \mid \gamma \in \Gamma\}$.*

Theorem 2.2. *Let $\varphi \in \operatorname{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. If*

$$\{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\} \quad \text{and} \quad \{y_{\delta,j} \mid \delta \in \Delta,\ 1 \leqslant j \leqslant l_\delta\}$$

are nilpotent Jordan normal bases of ${}_RM$ with respect to φ, then Γ is finite and there exists a bijection $\pi\colon \Gamma \to \Delta$ such that $k_\gamma = l_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. Thus, the sizes of the blocks of a nilpotent Jordan normal base are unique up to a permutation of the blocks. We also have $\ker(\varphi) = \bigoplus_{\gamma\in\Gamma} Rx_{\gamma,k_\gamma}$, and hence $\dim_R(\ker(\varphi)) = |\Gamma|$.

If $\varphi \in \mathrm{End}_R(M)$ is an arbitrary R-endomorphism of the left R-module ${}_RM$, then, for $u \in M$ and $f(z) = a_1 + a_2z + \cdots + a_{n+1}z^n \in R[z]$ (an unusual use of indices), the multiplication

$$f(z) * u = a_1u + a_2\varphi(u) + \cdots + a_{n+1}\varphi^n(u)$$

defines a natural left $R[z]$-module structure on M. This left action of $R[z]$ on M extends the left action of R on ${}_RM$. For any R-endomorphism $\psi \in \mathrm{End}_R(M)$ with $\psi \circ \varphi = \varphi \circ \psi$, we have $\psi(f(z) * u) = f(z) * \psi(u)$, and hence $\psi\colon M \to M$ is an $R[z]$-endomorphism of the left $R[z]$-module ${}_{R[z]}M$. On the other hand, if $\psi\colon M \to M$ is an $R[z]$-endomorphism of ${}_{R[z]}M$, then

$$\psi(\varphi(u)) = \psi(z * u) = z * \psi(u) = \varphi(\psi(u))$$

implies that $\psi \circ \varphi = \varphi \circ \psi$. Now

$$\mathrm{Cen}(\varphi) = \{\psi \mid \psi \in \mathrm{End}_R(M) \text{ and } \psi \circ \varphi = \varphi \circ \psi\}$$

is a $Z(R)$-subalgebra of $\mathrm{End}_R(M)$, and the argument above gives that $\mathrm{Cen}(\varphi) = \mathrm{End}_{R[z]}(M)$.

Henceforth, ${}_RM$ is semi-simple and we consider a fixed nilpotent Jordan normal base

$$X = \{x_{\gamma,i} \mid \gamma \in \varGamma, 1 \leqslant i \leqslant k_\gamma\} \subseteq M$$

with respect to a given nilpotent $\varphi \in \mathrm{End}_R(M)$ of index $n = \max\{k_\gamma \mid \gamma \in \varGamma\}$.

The $\varGamma$-copower $\coprod_{\gamma\in\varGamma} R[z]$ is an ideal of the $\varGamma$-direct power ring $(R[z])^\varGamma$ comprising all elements $\boldsymbol{f} = (f_\gamma(z))_{\gamma\in\varGamma}$ with a finite set $\{\gamma \in \varGamma \mid f_\gamma(z) \neq 0\}$ of non-zero coordinates. The copower (power) has a natural $(R[z], R[z])$-bimodule structure. For an element $\boldsymbol{f} = (f_\gamma(z))_{\gamma\in\varGamma}$ with $f_\gamma(z) = a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,n_\gamma+1}z^{n_\gamma}$, the formula

$$\begin{aligned}\varPhi(\boldsymbol{f}) &= \sum_{\gamma\in\varGamma, 1\leqslant i\leqslant k_\gamma} a_{\gamma,i}x_{\gamma,i} \\ &= \sum_{\gamma\in\varGamma}\biggl(\sum_{1\leqslant i\leqslant k_\gamma} a_{\gamma,i}\varphi^{i-1}(x_{\gamma,1})\biggr) \\ &= \sum_{\gamma\in\varGamma} f_\gamma(z) * x_{\gamma,1}\end{aligned}$$

defines a function $\varPhi\colon \coprod_{\gamma\in\varGamma} R[z] \to M$.

Lemma 2.3. *The function $\varPhi$ is a surjective left $R[z]$-homomorphism. We then have $\varphi(\varPhi(\boldsymbol{f})) = \varPhi(z\boldsymbol{f})$ for all $\boldsymbol{f} \in \coprod_{\gamma\in\varGamma} R[z]$ and the kernel*

$$\coprod_{\gamma\in\varGamma} J[z] + (z^{k_\gamma}) \subseteq \ker(\varPhi) \lhd_l \prod_{\gamma\in\varGamma} R[z]$$

is a left ideal of the power (and hence of the copower) ring. If R is a local ring (R/J is a division ring), then $\coprod_{\gamma\in\varGamma}(J[z] + (z^{k_\gamma})) = \ker(\varPhi)$.

Hereafter, we also require that ${}_RM$ be finitely generated, $m = \dim_R(\ker(\varphi))$, $\varGamma = \{1, 2, \ldots, m\}$ and we assume that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1$ for the block sizes.

Now $\coprod_{\gamma\in\Gamma} R[z] = (R[z])^{\Gamma}$, and an element $\boldsymbol{f} = (f_\gamma(z))_{\gamma\in\Gamma}$ of $(R[z])^{\Gamma}$ is a $1\times m$ matrix (row vector) over $R[z]$. For an $m\times m$ matrix $\boldsymbol{P} = [p_{\delta,\gamma}(z)]$ in $M_m(R[z])$, the matrix product

$$\boldsymbol{f}\boldsymbol{P} = \sum_{\delta\in\Gamma} f_\delta(z)\boldsymbol{p}_\delta$$

of $\boldsymbol{f}$ and $\boldsymbol{P}$ is a $1\times m$ matrix over $(R[z])^{\Gamma}$, where $\boldsymbol{p}_\delta = (p_{\delta,\gamma}(z))_{\gamma\in\Gamma}$ is the δth row vector of $\boldsymbol{P}$ and

$$(\boldsymbol{f}\boldsymbol{P})_\gamma = \sum_{\delta\in\Gamma} f_\delta(z) p_{\delta,\gamma}(z).$$

Consider the following subsets of $M_m(R[z])$:

$$\begin{aligned}
\mathcal{M}(X) &= \{\boldsymbol{P}\in M_m(R[z]) \mid \boldsymbol{f}\boldsymbol{P}\in\ker(\Phi), \forall \boldsymbol{f}\in\ker(\Phi)\},\\
\mathcal{I}(X) &= \{\boldsymbol{P}\in M_m(R[z]) \mid \boldsymbol{P} = [p_{\delta,\gamma}(z)],\ p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma}),\ \forall \delta,\gamma\in\Gamma\}\\
&= \begin{bmatrix} J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m})\\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m})\\ \vdots & \vdots & \ddots & \vdots\\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m})\end{bmatrix},\\
\mathcal{N}(X) &= \{\boldsymbol{P}\in M_m(R[z]) \mid \boldsymbol{P} = [p_{\delta,\gamma}(z)],\ z^{k_\delta}p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma}),\ \forall \delta,\gamma\in\Gamma\}.
\end{aligned}$$

Note that $\mathcal{I}(X)$ and $\mathcal{N}(X)$ are $(R[z],R[z])$-sub-bimodules of $M_m(R[z])$ in a natural way. For $\delta,\gamma\in\Gamma$, let $k_{\delta,\gamma} = k_\gamma - k_\delta$ when $1\leqslant k_\delta < k_\gamma \leqslant n$, and $k_{\delta,\gamma}=0$ otherwise. It can be verified that the condition $z^{k_\delta}p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma})$ in the definition of $\mathcal{N}(X)$ is equivalent to $p_{\delta,\gamma}(z)\in J[z]+(z^{k_{\delta,\gamma}})$ and so

$$\mathcal{N}(X) = \begin{bmatrix} R[z] & R[z] & R[z] & \cdots & R[z]\\ J[z]+(z^{k_1-k_2}) & R[z] & R[z] & \cdots & R[z]\\ J[z]+(z^{k_1-k_3}) & J[z]+(z^{k_2-k_3}) & R[z] & \cdots & R[z]\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ J[z]+(z^{k_1-k_m}) & J[z]+(z^{k_2-k_m}) & J[z]+(z^{k_3-k_m}) & \cdots & R[z]\end{bmatrix}.$$

Lemma 2.4. *$\mathcal{I}(X) \lhd_l M_m(R[z])$ is a left ideal, $\mathcal{N}(X)\subseteq M_m(R[z])$ is a subring, $\mathcal{I}(X)\lhd\mathcal{N}(X)$ is an ideal and $\mathcal{M}(X)$ is a $Z(R)$-subalgebra of $M_m(R[z])$. The ideal $zM_m(R[z])\lhd M_m(R[z])$ is nilpotent modulo $\mathcal{I}(X)$ with $z^nM_m(R[z])\subseteq\mathcal{I}(X)$. If R is a local ring, then $\mathcal{N}(X)=\mathcal{M}(X)$.*

Theorem 2.5. *Let $\varphi\in\mathrm{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. For $\boldsymbol{P}\in\mathcal{M}(X)$ and $\boldsymbol{f}=(f_\gamma(z))_{\gamma\in\Gamma}$ in $(R[z])^{\Gamma}$, the formula*

$$\psi_{\boldsymbol{P}}(\Phi(\boldsymbol{f})) = \Phi(\boldsymbol{f}\boldsymbol{P})$$

properly defines an R-endomorphism $\psi_{\boldsymbol{P}}\colon M\to M$ of ${}_RM$ such that $\psi_{\boldsymbol{P}}\circ\varphi = \varphi\circ\psi_{\boldsymbol{P}}$, and the assignment $\Lambda(\boldsymbol{P}) = \psi_{\boldsymbol{P}}$ gives an $\mathcal{M}(X)^{\mathrm{op}}\to\mathrm{Cen}(\varphi)$ homomorphism of $Z(R)$-algebras. If $\psi\circ\varphi=\varphi\circ\psi$ holds for some $\psi\in\mathrm{End}_R(M)$, then there exists an $m\times m$ matrix $\boldsymbol{P}\in\mathcal{M}(X)$ such that $\psi(\Phi(\boldsymbol{f}))=\Phi(\boldsymbol{f}\boldsymbol{P})$ for all $\boldsymbol{f}=(f_\gamma(z))_{\gamma\in\Gamma}$ in $(R[z])^{\Gamma}$. Thus, $\Lambda\colon\mathcal{M}(X)^{\mathrm{op}}\to\mathrm{Cen}(\varphi)$ is surjective.

LEMMA 2.6. $\mathcal{I}(X) \subseteq \ker(\Lambda)$ *(Λ is defined in theorem 2.5). If R is a local ring, then* $\mathcal{I}(X) = \ker(\Lambda)$.

3. The zero-level centralizer of a nilpotent endomorphism

We retain all settings from § 2 and define the subsets of $M_m(R[z])$ as follows:

$$\mathcal{M}_0(X) = \{\boldsymbol{P} \in \mathcal{M}(X) \mid z\boldsymbol{f}\boldsymbol{P} \in \ker(\Phi), \forall \boldsymbol{f} \in (R[z])^{\Gamma}\},$$
$$\mathcal{N}_0(X) = \{\boldsymbol{P} \in M_m(R[z]) \mid \boldsymbol{P} = [p_{\delta,\gamma}(z)],\ p_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma - 1}), \forall \delta, \gamma \in \Gamma\}.$$

Since $p_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma-1})$ and $zp_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma})$ are equivalent, we have

$$\mathcal{N}_0(X) = \begin{bmatrix} J[z] + (z^{k_1-1}) & J[z] + (z^{k_2-1}) & \cdots & J[z] + (z^{k_m-1}) \\ J[z] + (z^{k_1-1}) & J[z] + (z^{k_2-1}) & \cdots & J[z] + (z^{k_m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z] + (z^{k_1-1}) & J[z] + (z^{k_2-1}) & \cdots & J[z] + (z^{k_m-1}) \end{bmatrix}.$$

LEMMA 3.1. $\mathcal{I}(X) \subseteq \mathcal{N}_0(X)$, $z^{n-1}M_m(R[z]) \subseteq \mathcal{N}_0(X)$, $\mathcal{N}_0(X) \lhd_l M_m(R[z])$ *is a left ideal and* $\mathcal{N}_0(X) \lhd \mathcal{N}(X)$ *is an ideal. If R is a local ring, then* $\mathcal{N}_0(X) = \mathcal{M}_0(X)$.

Proof. The containment $\mathcal{I}(X) \subseteq \mathcal{N}_0(X)$ obviously holds and $z^{n-1}M_m(R[z]) \subseteq \mathcal{N}_0(X)$ is a consequence of $(z^{n-1}) \subseteq (z^{k_\gamma-1})$. Since the γth column of the matrices in $\mathcal{N}_0(X)$ comes from a (left) ideal $J[z] + (z^{k_\gamma-1})$ of $R[z]$, we can see that $\mathcal{N}_0(X)$ is a left ideal of $M_m(R[z])$.

If $\boldsymbol{P} \in \mathcal{N}_0(X)$ and $\boldsymbol{Q} \in \mathcal{N}(X)$, then we have $zp_{\delta,\tau}(z) \in J[z] + (z^{k_\tau})$ and $q_{\tau,\gamma}(z) \in J[z] + (z^{k_{\tau,\gamma}})$. Since $k_\tau + k_{\tau,\gamma} \geqslant k_\gamma$, it follows that $zp_{\delta,\tau}(z)q_{\tau,\gamma}(z) \in J[z] + (z^{k_\gamma})$. Thus, $\boldsymbol{PQ} \in \mathcal{N}_0(X)$ and $\mathcal{N}_0(X)$ is an ideal of $\mathcal{N}(X)$.

If R is a local ring, then lemma 2.3 gives that

$$\ker(\Phi) = \coprod_{\gamma \in \Gamma} (J[z] + (z^{k_\gamma})).$$

Let $\mathbf{1}_\delta$ denote the vector with 1 in its δ-coordinate and 0 in all other places. If $\boldsymbol{P} \in \mathcal{M}_0(X)$, then $z\mathbf{1}_\delta\boldsymbol{P} \in \ker(\Phi)$ implies that $zp_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma})$, whence $\boldsymbol{P} \in \mathcal{N}_0(X)$ follows. If $\boldsymbol{P} \in \mathcal{N}_0(X)$ and $\boldsymbol{f} = (f_\gamma(z))_{\gamma\in\Gamma}$ is in $(R[z])^{\Gamma}$, then $zp_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma})$ implies that $zf_\delta(z)p_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma})$ for all $\delta \in \Gamma$. Thus, $z\boldsymbol{f}\boldsymbol{P} \in \ker(\Phi)$ and $\boldsymbol{P} \in \mathcal{M}_0(X)$ follows. □

LEMMA 3.2. $\ker(\Lambda) \subseteq \mathcal{M}_0(X)$ *and, for $\boldsymbol{P} \in \mathcal{M}(X)$, the containments $\boldsymbol{P} \in \mathcal{M}_0(X)$ and $\Lambda(\boldsymbol{P}) \in \mathrm{Cen}_0(\varphi)$ are equivalent. The preimage*

$$\mathcal{M}_0(X) = \Lambda^{-1}(\mathrm{Cen}_0(\varphi)) \lhd \mathcal{M}(X)$$

is an ideal.

Proof. The proof is based on lemma 2.3 and theorem 2.5.

If $\boldsymbol{P} \in \ker(\Lambda)$, then $\Lambda(\boldsymbol{P}) = \psi_{\boldsymbol{P}} = 0$ gives that $\Phi(\boldsymbol{f}\boldsymbol{P}) = \psi_{\boldsymbol{P}}(\Phi(\boldsymbol{f})) = 0$ for all $\boldsymbol{f} \in (R[z])^{\Gamma}$. Since $\Phi\colon \coprod_{\gamma\in\Gamma} R[z] \to M$ is a left $R[z]$-homomorphism, $\Phi(z\boldsymbol{f}\boldsymbol{P}) = z * \Phi(\boldsymbol{f}\boldsymbol{P}) = 0$ implies that $z\boldsymbol{f}\boldsymbol{P} \in \ker(\Phi)$. In view of $\ker(\Lambda) \subseteq \mathcal{M}(X)$, we deduce that $\boldsymbol{P} \in \mathcal{M}_0(X)$.

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If $\boldsymbol{P} \in \mathcal{M}_0(X)$, then $\Lambda(\boldsymbol{P}) = \psi_{\boldsymbol{P}}$ and $\varphi(\psi_{\boldsymbol{P}}(\Phi(\boldsymbol{f}))) = \varphi(\Phi(\boldsymbol{f}\boldsymbol{P})) = \Phi(z\boldsymbol{f}\boldsymbol{P}) = 0$ for all $\boldsymbol{f} \in (R[z])^{\Gamma}$. Thus, $\psi_{\boldsymbol{P}} \circ \varphi = \varphi \circ \psi_{\boldsymbol{P}} = 0$, and hence $\psi_{\boldsymbol{P}} \in \operatorname{Cen}_0(\varphi)$.

If $\Lambda(\boldsymbol{P}) = \psi_{\boldsymbol{P}}$ is in $\operatorname{Cen}_0(\varphi)$, then $\varphi \circ \psi_{\boldsymbol{P}} = 0$ and

$$\begin{aligned}\Phi(z\boldsymbol{f}\boldsymbol{P}) &= \varphi(\Phi(\boldsymbol{f}\boldsymbol{P})) \\ &= \varphi(\psi_{\boldsymbol{P}}(\Phi(\boldsymbol{f}))) \\ &= 0\end{aligned}$$

for all $\boldsymbol{f} \in (R[z])^{\Gamma}$. It follows that $\boldsymbol{P} \in \mathcal{M}_0(X)$.

Obviously, the preimage of the ideal $\operatorname{Cen}_0(\varphi) \lhd \operatorname{Cen}(\varphi)$ is also an ideal. □

THEOREM 3.3. *Let $\varphi \in \operatorname{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. The map $\Lambda\colon \mathcal{M}(X)^{\mathrm{op}} \to \operatorname{Cen}(\varphi)$ induces the following $Z(R)$-isomorphisms for the factor algebras:*

$$\begin{aligned}\mathcal{M}_0(X)^{\mathrm{op}}/\ker(\Lambda) &\cong \operatorname{Cen}_0(\varphi), \\ \mathcal{M}(X)^{\mathrm{op}}/\mathcal{M}_0(X) &\cong \operatorname{Cen}(\varphi)/\operatorname{Cen}_0(\varphi).\end{aligned}$$

Proof. By lemma 3.2, we have

$$\ker(\Lambda \restriction \mathcal{M}_0(X)) = \ker(\Lambda) \quad \text{and} \quad \mathcal{M}_0(X) = \Lambda^{-1}(\operatorname{Cen}_0(\varphi)).$$

Thus, theorem 2.5 ensures that the restricted map $\Lambda \restriction \mathcal{M}_0(X)$ is a surjective $\mathcal{M}_0(X)^{\mathrm{op}} \to \operatorname{Cen}_0(\varphi)$ homomorphism of $Z(R)$-algebras, whence

$$\mathcal{M}_0(X)^{\mathrm{op}}/\ker(\Lambda) \cong \operatorname{Cen}_0(\varphi)$$

follows.

In view of lemma 3.2, the assignment

$$\boldsymbol{P} + \mathcal{M}_0(X) \mapsto \Lambda(\boldsymbol{P}) + \operatorname{Cen}_0(\varphi)$$

is well defined and gives an injective

$$\mathcal{M}(X)^{\mathrm{op}}/\mathcal{M}_0(X) \to \operatorname{Cen}(\varphi)/\operatorname{Cen}_0(\varphi)$$

homomorphism of $Z(R)$-algebras. The surjectivity of this homomorphism is a consequence of the surjectivity of Λ (see theorem 2.5). □

THEOREM 3.4. *Let $\varphi \in \operatorname{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. If R is a local ring, then the zero-level centralizer $\operatorname{Cen}_0(\varphi)$ of φ is isomorphic to the opposite of the factor $\mathcal{N}_0(X)/\mathcal{I}(X)$ as a $Z(R)$-algebra:*

$$\operatorname{Cen}_0(\varphi) \cong (\mathcal{N}_0(X)/\mathcal{I}(X))^{\mathrm{op}} = \mathcal{N}_0(X)^{\mathrm{op}/\mathcal{I}(X)}.$$

We also have an isomorphism

$$\operatorname{Cen}(\varphi)/\operatorname{Cen}_0(\varphi) \cong (\mathcal{N}(X)/\mathcal{N}_0(X))^{\mathrm{op}} = \mathcal{N}(X)^{\mathrm{op}}/\mathcal{N}_0(X)$$

of the factor $Z(R)$-algebras.

Proof. The proof follows directly from lemmas 2.4, 2.6, 3.1 and theorem 3.3. □

Define a left ideal of $M_m(R/J)$ as follows:

$$\mathcal{W}(X) = \{W = [w_{\delta,\gamma}] \mid w_{\delta,\gamma} \in R/J \text{ and } w_{\delta,\gamma} = 0 \text{ if } k_\gamma \geqslant 2\}.$$

The assumption $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1$ ensures that

$$\mathcal{W}(X) = \begin{bmatrix} 0 & \cdots & 0 & R/J & \cdots & R/J \\ 0 & \cdots & 0 & R/J & \cdots & R/J \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & R/J & \cdots & R/J \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & R/J & \cdots & R/J \end{bmatrix}.$$

We shall make use of the ideals

$$\mathcal{L}(X) = (\mathcal{N}_0(X) \cap zM_m(R[z])) + \mathcal{I}(X),$$
$$\mathcal{K}(X) = (\mathcal{N}(X) \cap zM_m(R[z])) + \mathcal{N}_0(X)$$

of $\mathcal{N}(X)$. We have $\mathcal{L}(X) \subseteq \mathcal{K}(X)$ and $\mathcal{L}(X) \subseteq \mathcal{N}_0(X)$ implies that $\mathcal{L}(X) \lhd \mathcal{N}_0(X)$.

Lemma 3.5. *There is a natural ring isomorphism $\mathcal{N}_0(X)/\mathcal{L}(X) \cong \mathcal{W}(X)$, which is also an (R, R)-bimodule isomorphism.*

Proof. If $\boldsymbol{P} = [p_{\delta,\gamma}(z)]$ is in $\mathcal{N}_0(X)$ and $p_{\delta,\gamma}(z)$ has constant term $u_{\delta,\gamma} \in R$, then

$$p_{\delta,\gamma}(z) - u_{\delta,\gamma} \in (J[z] + (z^{k_\gamma - 1})) \cap zR[z]$$

and $k_\gamma \geqslant 2$ implies that $u_{\delta,\gamma} \in J$. Thus, $[u_{\delta,\gamma}] \in M_m(R) \cap \mathcal{N}_0(X)$ and

$$\boldsymbol{P} + \mathcal{L}(X) = [u_{\delta,\gamma}] + \mathcal{L}(X)$$

holds in $\mathcal{N}_0(X)/\mathcal{L}(X)$. The assignment

$$\boldsymbol{P} + \mathcal{L}(X) \mapsto [u_{\delta,\gamma} + J]$$

is well defined and gives an $\mathcal{N}_0(X)/\mathcal{L}(X) \to \mathcal{W}(X)$ isomorphism. □

Theorem 3.6. *Let R be a local ring and let $\varphi \in \operatorname{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. If*

$$f_i(x_1, \ldots, x_r) \in Z(R)\langle x_1, \ldots, x_r\rangle, \quad 1 \leqslant i \leqslant n,$$

and $f_i = 0$ are polynomial identities of the right ideal $\mathcal{W}(X)$ of $M_m^{\mathrm{op}}(R/J)$, then $f_1 f_2 \cdots f_n = 0$ is an identity of $\mathrm{Cen}_0(\varphi)$.

Proof. Theorem 3.4 ensures that $\mathrm{Cen}_0(\varphi) \cong \mathcal{N}_0(X)^{\mathrm{op}}/\mathcal{I}(X)$ as $Z(R)$-algebras. Hence,

$$L = \mathcal{L}(X)/\mathcal{I}(X) \lhd \mathcal{N}_0(X)/\mathcal{I}(X)$$

can be viewed as an ideal of $\mathrm{Cen}_0(\varphi)$. The use of lemma 3.5 gives

$$\mathrm{Cen}_0(\varphi)/L \cong (\mathcal{N}_0(X)^{\mathrm{op}}/\mathcal{I}(X))/L \cong \mathcal{N}_0(X)^{\mathrm{op}}/\mathcal{L}(X) \cong \mathcal{W}(X)^{\mathrm{op}}.$$

It follows that $f_i = 0$ is an identity of $\mathrm{Cen}_0(\varphi)/L$. Thus, $f_i(v_1,\dots,v_r) \in L$ for all $v_1,\dots,v_r \in \mathrm{Cen}_0(\varphi)$, and so

$$f_1(v_1,\dots,v_r)f_2(v_1,\dots,v_r)\cdots f_n(v_1,\dots,v_r) \in L^n.$$

Since $z^n M_m(R[z]) \subseteq \mathcal{I}(X)$ (see lemma 2.4) implies that $L^n = \{0\}$, the proof is complete. □

The assumption $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1$ ensures that

$$\mathcal{U}_0(X) = \{U \in M_m(R/J) \mid U = [u_{\delta,\gamma}] \text{ and } u_{\delta,\gamma} = 0 \text{ if } 1 \leqslant k_\delta < k_\gamma \text{ or } k_\gamma = 1\}$$

is a block upper-triangular subalgebra of $M_m(R/J)$. If $[u_{\delta,\gamma}] \in \mathcal{U}_0(X)$ and $u_{\delta,\gamma} \neq 0$ for some $\delta,\gamma \in \Gamma$, then $2 \leqslant k_\gamma \leqslant k_\delta$. Results regarding the polynomial identities of block upper-triangular matrix algebras can be found in [3].

Lemma 3.7. *There is a natural ring isomorphism $\mathcal{N}(X)/\mathcal{K}(X) \cong \mathcal{U}_0(X)$, which is also an (R,R)-bimodule isomorphism.*

Proof. For a matrix $\boldsymbol{P} = [p_{\delta,\gamma}(z)]$ in $\mathcal{N}(X)$, consider the assignment

$$\boldsymbol{P} + \mathcal{K}(X) \mapsto [u_{\delta,\gamma} + J],$$

where $u_{\delta,\gamma} \in R$ is defined as follows. $u_{\delta,\gamma} = 0$ if $k_\gamma = 1$ and $u_{\delta,\gamma}$ is the constant term of $p_{\delta,\gamma}(z)$ if $k_\gamma \geqslant 2$. Clearly, $[u_{\delta,\gamma}] \in M_m(R) \cap \mathcal{N}(X)$, and

$$\boldsymbol{P} + \mathcal{K}(X) = [u_{\delta,\gamma}] + \mathcal{K}(X).$$

In view of the definitions of $\mathcal{N}_0(X)$ and $\mathcal{U}_0(X)$, the above equality ensures that our assignment is a well defined $\mathcal{N}(X)/\mathcal{K}(X) \to \mathcal{U}_0(X)$ map providing the required isomorphism. □

Theorem 3.8. *Let R be a local ring and $\varphi \in \mathrm{End}_R(M)$ be a nilpotent R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. If*

$$f_i(x_1,\dots,x_r) \in Z(R)\langle x_1,\dots,x_r\rangle, \quad 1 \leqslant i \leqslant n-1$$

and $f_i = 0$ are polynomial identities of the $Z(R)$-subalgebra $\mathcal{U}_0(X)$ of $M_m^{\mathrm{op}}(R/J)$, then $f_1 f_2 \cdots f_{n-1} = 0$ is an identity of the factor $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$.

Proof. Theorem 3.4 ensures that $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi) \cong \mathcal{N}(X)^{\mathrm{op}}/\mathcal{N}_0(X)$ as $Z(R)$-algebras; hence,

$$K = \mathcal{K}(X)/\mathcal{N}_0(X) \lhd \mathcal{N}(X)/\mathcal{N}_0(X)$$

can be viewed as an ideal of $\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$. The use of lemma 3.7 gives

$$\begin{aligned}(\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi))/K &\cong (\mathcal{N}(X)^{\mathrm{op}}/\mathcal{N}_0(X))/K \\ &\cong \mathcal{N}(X)^{\mathrm{op}}/\mathcal{K}(X) \\ &\cong \mathcal{U}_0(X)^{\mathrm{op}}.\end{aligned}$$

It follows that $f_i = 0$ is an identity of $(\mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi))/K$. Thus, $f_i(v_1,\dots,v_r) \in K$ for all $v_1,\dots,v_r \in \mathrm{Cen}(\varphi)/\mathrm{Cen}_0(\varphi)$, and so

$$f_1(v_1,\dots,v_r)f_2(v_1,\dots,v_r)\cdots f_{n-1}(v_1,\dots,v_r) \in K^{n-1}.$$

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Since $z^{n-1}M_m(R[z]) \subseteq \mathcal{N}_0(X)$ (see lemma 3.1) implies that $K^{n-1} = \{0\}$, the proof is complete. $\square$

4. The zero-level centralizer of an arbitrary endomorphism

THEOREM 4.1. *Let $\varphi \in \mathrm{End}_R(M)$ be an R-endomorphism of a finitely generated semi-simple left R-module ${}_RM$. Then there exist R-submodules W_1, W_2 and V of M such that $W = W_1 \oplus W_2$ and $M = V \oplus W$ are direct sums,* $\ker(\varphi) \subseteq W$, $\varphi(W) = W_2$, $\varphi(V) = V$, $\dim_R(W_1) = \dim_R(\ker(\varphi))$, $(\varphi \restriction W) \in \mathrm{End}_R(W)$ *is nilpotent and, for the zero-level centralizer of φ, we have* $\mathrm{Cen}_0(\varphi) \cong T$, *where*

$$T = \{\theta \in \mathrm{End}_R(W) \mid \theta(W_1) \subseteq \ker(\varphi) \text{ and } \theta(W_2) = \{0\}\} = \mathrm{Cen}_0(\varphi \restriction W)$$

is a left ideal of

$$\mathrm{End}_R^*(W) = \{\alpha \in \mathrm{End}_R(W) \mid \alpha(\ker(\varphi)) \subseteq \ker(\varphi)\}$$

and a right ideal of

$$\mathrm{End}_R^{**}(W) = \{\alpha \in \mathrm{End}_R(W) \mid \alpha(W_1 + \ker(\varphi)) \subseteq W_1 + \ker(\varphi) \text{ and } \alpha(W_2) \subseteq W_2\}.$$

Proof. The Fitting lemma ensures the existence of an integer $t \geqslant 1$ such that $\mathrm{Im}(\varphi^t) \oplus \ker(\varphi^t) = M$ is a direct sum, where the (left) R-submodules

$$V = \mathrm{Im}(\varphi^t) = \mathrm{Im}(\varphi^{t+1}) = \cdots \quad \text{and} \quad W = \ker(\varphi^t) = \ker(\varphi^{t+1}) = \cdots$$

of ${}_RM$ are uniquely determined by φ. Clearly, $\varphi(V) = V$ and $\varphi(W) \subseteq W$ and the restricted map $(\varphi \restriction W) \in \mathrm{End}_R(W)$ is nilpotent of index $q \geqslant 1$, where $\ker(\varphi^{q-1}) \neq \ker(\varphi^q) = W$. Since ${}_RW$ is also finitely generated and semi-simple, theorem 2.1 provides a nilpotent Jordan normal base $X = \{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\}$ of ${}_RW$ with respect to $\varphi \restriction W$ (we have $x_{\gamma,k_\gamma+1} = 0$ and $q = \max\{k_\gamma \mid \gamma \in \Gamma\}$). Now $W_1 \oplus W_2 = W$ is a direct sum, where

$$W_1 = \bigoplus_{\gamma \in \Gamma} Rx_{\gamma,1} \quad \text{and} \quad W_2 = \bigoplus_{\gamma \in \Gamma, 1 \leqslant i \leqslant k_\gamma} Rx_{\gamma,i+1}.$$

Now we have $\ker(\varphi) \subseteq \ker(\varphi^t) = W$ and $\ker(\varphi) = \ker(\varphi \restriction W) = \bigoplus_{\gamma \in \Gamma} Rx_{\gamma,k_\gamma}$ by theorem 2.2. It follows that

$$\dim_R(W_1) = |\Gamma| = \dim_R(\ker(\varphi)).$$

The definition of the nilpotent Jordan normal base ensures that $\varphi(W) = W_2$.

If $\theta \in T$, then

$$\theta(\ker(\varphi)) \subseteq \theta(W_1 \oplus W_2) = \theta(W_1) + \theta(W_2) \subseteq \ker(\varphi)$$

implies that T is a left ideal of $\mathrm{End}_R^*(W)$ and a right ideal of $\mathrm{End}_R^{**}(W)$. Clearly, $T = \mathrm{Cen}_0(\varphi \restriction W)$ is a consequence of $\varphi(W) = W_2$ and the fact that $\theta(W) \subseteq \ker(\varphi)$ for all $\theta \in T$.

If $\alpha \in \mathrm{Cen}_0(\varphi)$, then $\alpha \circ \varphi = 0$ implies that $\alpha(V) = \{0\}$ and $\alpha(x_{\gamma,i+1}) = \alpha(\varphi(x_{\gamma,i})) = 0$ for $1 \leqslant i \leqslant k_\gamma - 1$. We also have $\varphi \circ \alpha = 0$, whence $\varphi(\alpha(x_{\gamma,1})) = 0$

and $\alpha(x_{\gamma,1}) \in \ker(\varphi)$ follow. Thus $\alpha(W_2) = \{0\}$, $\alpha(W_1) \subseteq \ker(\varphi)$ and the assignment $\alpha \mapsto \alpha \restriction W$ obviously defines a $\mathrm{Cen}_0(\varphi) \to T$ ring homomorphism.

If $\alpha, \beta \in \mathrm{Cen}_0(\varphi)$ and $\alpha \restriction W = \beta \restriction W$, then $\alpha(V) = \beta(V) = \{0\}$ and $V \oplus W = M$ ensure that $\alpha = \beta$ proving the injectivity of the above map.

If $\theta \in T$ and $\pi_W : V \oplus W \to W$ is the natural projection, then $\theta \circ \pi_W \in \mathrm{Cen}_0(\varphi)$. Indeed, $\varphi \circ \theta \circ \pi_W = 0$ is a consequence of $\theta(W) \subseteq \ker(\varphi)$ and $\theta \circ \pi_W \circ \varphi = 0$ is a consequence of $\varphi(W) = W_2$ and $\theta(W_2) = \{0\}$. Hence, the surjectivity of our assignment follows from $\theta \circ \pi_W \restriction W = \theta$. □

Corollary 4.2. *Let $A \in M_n(F)$ be an $n \times n$ matrix over a field F. Then the F-dimension of the zero-level centralizer of A in $M_n(F)$ is*

$$\dim_F \mathrm{Cen}_0(A) = [\dim_F(\ker(A))]^2.$$

Proof. Now $A \in \mathrm{End}_F(F^n)$ and theorem 4.1 ensures that $\mathrm{Cen}_0(A) \cong T$, where

$$T = \{\theta \in \mathrm{End}_F(W) \mid \theta(W_1) \subseteq \ker(A) \text{ and } \theta(W_2) = \{0\}\}.$$

Our claim follows from the observation that the elements of $\mathrm{Hom}_F(W_1, \ker(A))$ and T can be naturally identified and $\dim_F(W_1) = \dim_F(\ker(A))$. □

Remark 4.3. Theorem 4.1 shows that the determination of the zero-level centralizer can be reduced to the nilpotent case. This reduction depends on the use of the Fitting lemma.

Lemma 4.4. *Let $\varphi, \sigma \in \mathrm{End}_R(M)$ be R-endomorphisms of a finitely generated semisimple left R-module ${}_RM$ such that $\mathrm{Cen}_0(\varphi) \subseteq \mathrm{Cen}_0(\sigma)$. Then $\ker(\varphi) \subseteq \ker(\sigma)$ and $\mathrm{Im}(\sigma) \subseteq \mathrm{Im}(\varphi)$.*

Proof. We use the proof of theorem 4.1. If $\gamma \in \Gamma$ and $\pi_\gamma \in \mathrm{End}_R(M)$ denotes the natural

$$M = V \oplus W = V \oplus \left(\bigoplus_{\delta \in \Gamma, 1 \leqslant i \leqslant k_\delta} R x_{\delta,i} \right) \to R x_{\gamma,k_\gamma}$$

projection, then $\pi_\gamma \circ \varphi^{k_\gamma - 1} \in \mathrm{Cen}_0(\varphi)$. It follows that $\pi_\gamma \circ \varphi^{k_\gamma - 1} \in \mathrm{Cen}_0(\sigma)$. Thus, we obtain that

$$\pi_\gamma \circ \varphi^{k_\gamma - 1} \circ \sigma = \sigma \circ \pi_\gamma \circ \varphi^{k_\gamma - 1} = 0.$$

Since

$$\sigma(x_{\gamma,k_\gamma}) = \sigma(\pi_\gamma(\varphi^{k_\gamma - 1}(x_{\gamma,k_1}))) = 0,$$

we have $x_{\gamma,k_\gamma} \in \ker(\sigma)$ for all $\gamma \in \Gamma$. Thus,

$$\ker(\varphi) = \ker(\varphi \restriction W) = \bigoplus_{\gamma \in \Gamma} R x_{\gamma,k_\gamma} \subseteq \ker(\sigma).$$

The containment $\mathrm{Im}(\sigma) \subseteq \ker(\pi_\gamma \circ \varphi^{k_\gamma - 1})$ is a consequence of $\pi_\gamma \circ \varphi^{k_\gamma - 1} \circ \sigma = 0$, whence we obtain that

$$\mathrm{Im}(\sigma) \subseteq \cap_{\gamma \in \Gamma} \ker(\pi_\gamma \circ \varphi^{k_\gamma - 1}).$$

It is straightforward to see that $\ker(\pi_\gamma \circ \varphi^{k_\gamma - 1}) = V \oplus W(\gamma)$ and

$$\bigcap_{\gamma\in\Gamma}(V \oplus W(\gamma)) = V \oplus W_2 = \varphi(V) + \varphi(W) = \varphi(V \oplus W) = \operatorname{Im}(\varphi),$$

where

$$W(\gamma) = \bigoplus_{\delta\in\Gamma, 1\leqslant i\leqslant k_\delta, (\delta,i)\neq(\gamma,1)} Rx_{\delta,i}.$$

□

THEOREM 4.5. *Let $\varphi, \sigma \in \operatorname{End}_R(M)$ be R-endomorphisms of a finitely generated semi-simple left R-module ${}_RM$. Then the following are equivalent:*

(i) $\operatorname{Cen}_0(\varphi) \subseteq \operatorname{Cen}_0(\sigma)$*;*

(ii) $\ker(\varphi) \subseteq \ker(\sigma)$ *and* $\operatorname{Im}(\sigma) \subseteq \operatorname{Im}(\varphi)$*.*

Proof. In view of lemma 4.4, it is sufficient to prove (i) $\Longrightarrow$ (ii). For an endomorphism $\tau \in \operatorname{Cen}_0(\varphi)$, we have $\tau \circ \varphi = \varphi \circ \tau = 0$, whence

$$\operatorname{Im}(\sigma) \subseteq \operatorname{Im}(\varphi) \subseteq \ker(\tau) \quad \text{and} \quad \operatorname{Im}(\tau) \subseteq \ker(\varphi) \subseteq \ker(\sigma)$$

follow. Thus, we obtain that $\tau \circ \sigma = \sigma \circ \tau = 0$. In consequence, we have $\tau \in \operatorname{Cen}_0(\sigma)$, and $\operatorname{Cen}_0(\varphi) \subseteq \operatorname{Cen}_0(\sigma)$ follows. □

For a matrix $A \in M_n(F)$, let A^{T} denote the transpose of A.

THEOREM 4.6. *If $A, B \in M_n(F)$ are $n \times n$ matrices over a field F, then the following are equivalent:*

(i) $\operatorname{Cen}_0(A) \subseteq \operatorname{Cen}_0(B)$*;*

(ii) $\ker(A) \subseteq \ker(B)$ *and* $\ker(A^{\mathrm{T}}) \subseteq \ker(B^{\mathrm{T}})$*;*

(iii) $\operatorname{Im}(B) \subseteq \operatorname{Im}(A)$ *and* $\operatorname{Im}(B^{\mathrm{T}}) \subseteq \operatorname{Im}(A^{\mathrm{T}})$*.*

Proof. (i) $\Longrightarrow$ (ii) and (iii). For a matrix $C \in \operatorname{Cen}_0(A^{\mathrm{T}})$, we have $CA^{\mathrm{T}} = A^{\mathrm{T}}C = 0$ and $C^{\mathrm{T}} \in \operatorname{Cen}_0(A)$ is a consequence of

$$AC^{\mathrm{T}} = (A^{\mathrm{T}})^{\mathrm{T}}C^{\mathrm{T}} = (CA^{\mathrm{T}})^{\mathrm{T}} = 0 = (A^{\mathrm{T}}C)^{\mathrm{T}} = C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = C^{\mathrm{T}}A.$$

Thus, $C^{\mathrm{T}} \in \operatorname{Cen}_0(B)$, and a similar argument gives that $C = (C^{\mathrm{T}})^{\mathrm{T}} \in \operatorname{Cen}_0(B^{\mathrm{T}})$. It follows that $\operatorname{Cen}_0(A^{\mathrm{T}}) \subseteq \operatorname{Cen}_0(B^{\mathrm{T}})$. The application of lemma 4.4 for the matrices $A, B, A^{\mathrm{T}}, B^{\mathrm{T}} \in \operatorname{End}_F(F^n)$ gives

$$\ker(A) \subseteq \ker(B),\ \operatorname{Im}(B) \subseteq \operatorname{Im}(A), \qquad \ker(A^{\mathrm{T}}) \subseteq \ker(B^{\mathrm{T}}),\ \operatorname{Im}(B^{\mathrm{T}}) \subseteq \operatorname{Im}(A^{\mathrm{T}}).$$

(ii) $\Longrightarrow$ (i). For a matrix $C \in \operatorname{Cen}_0(A)$, the containment $\operatorname{Im}(C) \subseteq \ker(A)$ is a consequence of $AC = 0$ and $\operatorname{Im}(C^{\mathrm{T}}) \subseteq \ker(A^{\mathrm{T}})$ is a consequence of $A^{\mathrm{T}}C^{\mathrm{T}} = (CA)^{\mathrm{T}} = 0$. Now $\operatorname{Im}(C) \subseteq \ker(B)$ implies that $BC = 0$ and $\operatorname{Im}(C^{\mathrm{T}}) \subseteq \ker(B^{\mathrm{T}})$ implies that $CB = (B^{\mathrm{T}}C^{\mathrm{T}})^{\mathrm{T}} = 0$. Thus, $C \in \operatorname{Cen}_0(B)$ and $\operatorname{Cen}_0(A) \subseteq \operatorname{Cen}_0(B)$ follows.

(iii) $\Longrightarrow$ (i). For a matrix $C \in \operatorname{Cen}_0(A)$, the containment $\operatorname{Im}(A) \subseteq \ker(C)$ is a consequence of $CA = 0$, and $\operatorname{Im}(A^{\mathrm{T}}) \subseteq \ker(C^{\mathrm{T}})$ is a consequence of $C^{\mathrm{T}}A^{\mathrm{T}} = (AC)^{\mathrm{T}} = 0$. Now $\operatorname{Im}(B) \subseteq \ker(C)$ implies that $CB = 0$ and $\operatorname{Im}(B^{\mathrm{T}}) \subseteq \ker(C^{\mathrm{T}})$ implies that $BC = (C^{\mathrm{T}}B^{\mathrm{T}})^{\mathrm{T}} = 0$. Thus, $C \in \operatorname{Cen}_0(B)$ and $\operatorname{Cen}_0(A) \subseteq \operatorname{Cen}_0(B)$ follows. □

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