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ON THE STRENGTH OF TWO RECURRENCE THEOREMS

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Abstract. This paper uses the framework of reverse mathematics to investigate the strength of two recurrence theorems of topological dynamics. It establishes that one of these theorems, the existence of an almost periodic point, lies strictly between WKL and ACA (working over RCA_0). This is the first example of a theorem with this property. It also shows the existence of an almost periodic point is conservative over RCA_0 for Π^1_1-sentences.

§**1. Introduction.** Dynamical systems are studied by different branches of mathematics in many different forms. In the simplest setting, a dynamical system (X, T) is comprised of a set X and a transformation $T : X \to X$. By placing different requirements on X and T, structure can be added to the system that will influence its behavior.

Central to the analysis of a dynamical system is the analysis of the orbits of points in the system. If the dynamical system has certain global properties, then this guarantees the existence of points with certain orbits.

THEOREM 1.1 (Birkhoff's recurrence theorem). *Let X be a compact topological space and $T : X \to X$ a continuous transformation. Then there exists $x \in X$ and a sequence $n_1, n_2, \dots,$ such that*

$$\lim_i T^{n_i}(x) \to x.$$

Such an x is called a *recurrent point* of the system (X, T). Comparable results hold if we place a probability measure on the space X and require that T be a measure-preserving transformation.

The standard proof of Birkhoff's recurrence theorem shows the existence of a point x with the following stronger property (see for example [6, Theorem 2.3.4]). For every neighborhood N of x, there is a bound b, such that for all n, there is a $k < b$ with $T^{n+k}(x) \in N$. Such a point x is called an *almost periodic point* of the system (X, T).

The objective of this paper is to analyze the reverse mathematical strength of the existence of recurrent points and almost periodic points. The motivation for this work lies not just in the intrinsic interest of Birkhoff's recurrence theorem but in the fact that this is the simplest of a family of recurrence theorems that have

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widespread applications. In this respect, Birkhoff's recurrence theorem is similar to Ramsey's theorem and the reverse mathematical study of Ramsey's theorem has been remarkable fruitful. Two examples will illustrate the importance of recurrence theorems. Firstly, Furstenberg's multiple recurrence theorem, a theorem of measure-preserving systems can be used to prove Szemeredi's theorem. A second example, which has been studied from a reverse mathematical perspective, is the Auslander–Ellis theorem. This theorem states that if X is a compact metric space, with metric d, and $T : X \to X$ is a continuous transformation, then for any point x, there exists a point y such that:

(i) y is an almost periodic point of the system.
(ii) $(\forall \epsilon)(\exists n)(d(T^n(x), T^n(y)) < \epsilon)$.

Blass, Hirst, and Simpson have shown that ACA_0^+ proves the Auslander–Ellis theorem [2]. It is an open question as to whether or not it follows from ACA_0 [4]. The Auslander–Ellis theorem can be used to prove Hindman's theorem. Hindman's theorem states that if the integers are colored with finitely many colors, then there exists an infinite set S such that $\{n : n \text{ is a finite sum of elements of } S\}$ is homogenous. Blass, Hirst, and Simpson also showed that the strength of Hindman's Theorem lies between ACA_0^+ and ACA_0 [2]. Recent work, particularly of Towsner, has shed further light on the difficult question of calibrating the strength of Hindman's theorem [1, 7–9]. The reverse mathematics of topological dynamical systems has also been studied by Friedman, Simpson, and Yu [3]. Friedman, Simpson, and Yu investigated the strength of Sharkovsky's theorem. We will make use of their results on the orbits of points in models of 2^{nd} order arithmetic.

In this paper, we will investigate topological dynamical systems where X is a closed subset of Cantor space and T is a continuous transformation. This is a very important class of topological dynamical systems because subsets of natural numbers can be coded as elements of Cantor space. The proof of Hindman's theorem via the Auslander–Ellis theorem uses such systems. In the next section we will develop and formalize two principles.

(i) RP: Every topological dynamical system on Cantor space contains a recurrent point.
(ii) AP: Every topological dynamical system on Cantor space contains an almost periodic point.

In Section 3, we will show that over RCA_0, RP is equivalent to WKL. This is perhaps a little surprising because the set of recurrent points of a system may not be closed. The principal AP is more unusual. In Section 4, we analyze the standard proof of the existence of an almost periodic point and show this requires ACA_0 to carry out. However, from the perspective of reverse mathematics, this proof is not optimal. In fact, over RCA_0, the principle AP lies *strictly* between WKL and ACA. This is the first natural example of a principle with this property. The separation between AP and WKL is established in Section 5, and the separation between ACA and AP is established in Section 6. Note that this also separates AP from ACA in terms of proof-theoretic strength.

Harrington proved that WKL_0 is conservative for Π^1_1-sentences over RCA_0. In Section 7, we show that $\mathsf{RCA}_0 + \mathsf{AP}$ also has this property.

The PA degrees are those Turing degrees that contain a complete extension of Peano Arithmetic. This is a very well-studied upwards-closed class of Turing degrees. From a computability-theoretic perspective, the proof that the principle AP lies strictly between WKL and ACA establishes the existence of another interesting upwards-closed strict subclass of the PA degrees. In Section 8, we conclude with a number of open questions.

1.1. Notation. Let $\sigma, \tau \in 2^{<\mathbb{N}}$ and let $X \in 2^{\mathbb{N}}$. The length of the string σ is denoted by $|\sigma|$. We will write $\sigma \preceq \tau$ if σ is an initial segment of τ and $\sigma \prec \tau$ if σ is a strict initial segment of τ. We will write $\sigma \prec X$ if σ is an initial segment of X. We will denote the empty string by λ. The set of extensions of σ in $2^{\mathbb{N}}$ is denoted by $[\![\sigma]\!]$. If $U \subseteq 2^{<\mathbb{N}}$ then we will denote by $[\![U]\!]$ the set $\bigcup_{\sigma \in U} [\![\sigma]\!]$. For each n, $\{0,1\}^n$ will denote all strings in $2^{<\mathbb{N}}$ of length n. The string $\sigma\tau$ is the concatenation of σ with τ. If $i \in \mathbb{N}$, then σ^i denotes the string σ repeated i times, e.g., $\sigma^3 = \sigma\sigma\sigma$. $X \upharpoonright_i$ is the string of length i that is an initial segment of X. By $\sigma^{\mathbb{N}}$ we mean the element of $2^{\mathbb{N}}$ obtained by repeating σ infinitely many times i.e., $\lim_i \sigma^i$.

§2. Topological dynamics in RCA_0.

A standard definition of a topological dynamical system on Cantor space is the following.

Definition 2.1. A pair (C, F) is a *topological dynamical system* on $2^{\mathbb{N}}$ if C is a nonempty closed subset of $2^{\mathbb{N}}$, and $F : C \to C$ is a continuous transformation.

Sometimes F is required to be a homeomorphism but we will not consider that possibility here. From now on, we will often refer to a topological dynamical system as simply a system.

We need to consider how we encode a system inside a model of 2nd order arithmetic. The standard approach to encoding a closed set is to regard it as the set of infinite paths through a tree in $2^{<\mathbb{N}}$. For now, we will denote a tree in $2^{<\mathbb{N}}$ by a capital roman letter e.g., C. The set of infinite paths in C will be denoted by $[C]$. The approach we take for encoding a continuous transformation is also standard.

Definition 2.2. Let C be a tree in $2^{<\mathbb{N}}$. A function $f : C \to C$ *encodes a continuous partial transformation of* $[C]$ if

(i) f is total.
(ii) f is order preserving.

We will also make the assumption that if $f : C \to C$ encodes a continuous partial transformation of $[C]$, then $f(\lambda) = \lambda$ and for all $\sigma \in 2^{<\mathbb{N}} \setminus \{\lambda\}$, $|f(\sigma)| < |\sigma|$. This assumption results in no loss of generality and will simplify some proofs.

Definition 2.3. A function $f : C \to C$ *encodes a continuous transformation of* $[C]$ if

(i) f encodes a continuous partial transformation of $[C]$.
(ii) For all $X \in [C]$, for all l, there exists an m, such that $|f(X \upharpoonright_m)| > l$.

Let f encode a continuous transformation of $[C]$. We will denote by $F : [C] \to [C]$ the function encoded by f i.e., for all $X \in [C]$, $F(X) = \lim_{m \in \mathbb{N}} f(X \upharpoonright_m)$.

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It is impossible to discuss recurrent points and almost periodic points without discussing orbits. Given a function $f : C \to C$ encoding a continuous partial transformation of $[C]$, define $f : C \times \mathbb{N} \to C$ by

$$f(\sigma, k) = \begin{cases} \sigma & \text{if } k = 0, \\ f(f(\sigma, k-1)) & \text{if } k > 0. \end{cases}$$

When convenient, we will write $f^k(\sigma)$ for $f(\sigma, k)$. Observe that $f^1 = f$. The function f^k is total because it has been defined by primitive recursion from f. It is order preserving by Π_1-induction on k for the formula

$$(\forall k)(\forall \sigma, \tau, \rho, \pi)((\sigma \preceq \tau \wedge \rho = f^k(\sigma) \wedge \pi = f^k(\tau)) \to \rho \preceq \pi).$$

Hence RCA_0 proves that if $f : C \to C$ encodes a continuous partial transformation of $[C]$, then for all k, $f^k : C \to C$ encodes a continuous partial transformation of $[C]$. However, the following theorem establishes that the analogous result *does not* hold for continuous transformations.

Theorem 2.4 (RCA_0 – Friedman, Simpson and Yu [3]). *The following are equivalent*:

(i) *The disjunction of* Σ^0_2*-induction and* WKL.
(ii) *For all* $k \in \mathbb{N}$ *and all continuous transformations* $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, F^k *is a continuous transformation from* $2^{\mathbb{N}}$ *to* $2^{\mathbb{N}}$.

Given this result, the following definition is perhaps the most natural way to formalize a dynamical system in RCA_0.

Definition 2.5 (RCA_0). A pair (C, f), where C is a tree in $2^{<\mathbb{N}}$ and $f : C \to C$ is a function, encodes a system if

(i) $[C] \neq \emptyset$.
(ii) For all k, f^k encodes a continuous transformation of $[C]$.

Condition (ii) of Definition 2.5 is ensured if f encodes a uniformly continuous transformation of $[C]$.

Definition 2.6. A function $f : C \to C$ *encodes a uniformly continuous transformation of* $[C]$ if

(i) f encodes a continuous partial transformation of $[C]$.
(ii) For all l, there exists an m, such that for all $\sigma \in \{0,1\}^m \cap C$, $|f(\sigma)| > l$.

Lemma 2.7 (RCA_0 – Friedman, Simpson and Yu [3]). *If* $f : C \to C$ *encodes a uniformly continuous transformation of* $[C]$, *then for all* $k \in \mathbb{N}$, $f^k : C \to C$ *encodes a uniformly continuous transformation of* $[C]$.

Lemma 2.8 (RCA_0). *If* (C, f) *encodes a system then for all* k *and all* $X \in [C]$, $F(F^k(X)) = F^{k+1}(X)$.

Proof. Fix a k and $X \in [C]$. For any l, there exists some m_l such that $F^k(X) \restriction_l \preceq f^k(X \restriction_{m_l})$. Hence

$$F(F^k(X)) = \lim_l f(F^k(X) \restriction_l) = \lim_l f(f^k(X \restriction_{m_l})) = \lim_l f^{k+1}(X \restriction_{m_l}) = F^{k+1}(X). \quad \dashv$$

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The orbit of X under F is the sequence $\langle F^k(X) : k \in \mathbb{N}\rangle$. Note that this is uniform and hence $\oplus_{k\in\mathbb{N}} F^k(X)$ exists by recursive comprehension in any model of RCA_0 that includes X and f.

§3. Recurrent points. We call X a *recurrent point* of a topological dynamical system (C, f), if $X \in [C]$ and

$$(\forall n, c)(\exists k)(F^{n+k}(X) \succeq X\restriction_c).$$

We call X an *almost periodic point* of a topological dynamical system (C, f), if $X \in [C]$ and

$$(\forall c)(\exists b)(\forall n)(\exists k < b)(F^{n+k}(X) \succeq X\restriction_c).$$

This leads to two principles. First RP is the principle that every topological dynamical system on $2^{\mathbb{N}}$ contains a recurrent point. The second AP is the principle that every topological dynamical system on $2^{\mathbb{N}}$ contains an almost periodic point. Over RCA_0 we have the obvious implication that AP implies RP because an almost periodic point is a recurrent point.

The following theorem would be trivial if condition (i) in Definition 2.5, was replaced by requiring the tree C to be infinite. Given such a C, we could simply take (C, f) be our system where f encodes the identity transformation. Any recurrent point of this system would have to be an element of $[C]$ hence proving WKL.

THEOREM 3.1. *Over* RCA_0, RP *implies* WKL.

PROOF. Let T be an infinite computable tree in $2^{<\mathbb{N}}$. We will regard T as a computable tree in $3^{<\mathbb{N}}$ (i.e., a computable subtree of $3^{<\mathbb{N}}$ such that no node of T contains a 2). We will define a system on $3^{\mathbb{N}}$ such that any recurrent point of the system is a path on T. As $3^{\mathbb{N}}$ is computably homeomorphic to $2^{\mathbb{N}}$, this is sufficient to prove the theorem. The idea behind the following definition of f is that if $X \in 3^{\mathbb{N}}$ is a path on T, then $F(X) = X$. If X is not a path on T, then the orbit of X moves in increasing lexicographical order searching for a path on T, looping around if it extends 2. The extra branching of $3^{\mathbb{N}}$ allows us to move the orbit of $F(X)$ if X is not a path on T.

Define the following function $f : 3^{<\mathbb{N}} \to 3^{<\mathbb{N}}$. First $f(\lambda) = \lambda$. Second if $|\sigma| > 0$, let $n = |\sigma| - 1$. If $\sigma \in T$, let $f(\sigma) = \sigma\restriction_n$. If $\sigma \notin T$, then let π be the shortest initial segment of σ such that $\pi \notin T$. Because T is a subtree of $2^{<\mathbb{N}}$, if π contains a 2, then π must end with 2. Define

$$f(\sigma) = \begin{cases} \rho 1 0^{\mathbb{N}}\restriction_n & \text{if } \pi = \rho 0 \vee \pi = \rho 02, \\ \rho 2 0^{\mathbb{N}}\restriction_n & \text{if } \pi = \rho 1 \vee \pi = \rho 12, \\ 0^n & \text{if } \pi = 2. \end{cases}$$

Because f is uniformly continuous, it is not difficult to verify that $(3^{\mathbb{N}}, f)$ is a system. Let $\leq_{lex}$ be the lexicographical ordering on finite strings. (Recall that under this ordering $\sigma \leq_{lex} \tau$ if $\sigma \preceq \tau$ or $\sigma(i) < \tau(i)$ for the least i where these strings differ.)

The following claim establishes that if X is not a path in T, then the only way for X to be recurrent is for some element in the orbit of X to extend the string 2.

CLAIM. *Let* $n > 0$. *Let* $\sigma_0, \sigma_1, \dots, \sigma_n$ *be a finite sequence of strings such that* $\sigma_0 \succ \sigma_n$, *for all* $i < n$, $f(\sigma_i) = \sigma_{i+1}$ *and* $\sigma_n \notin T$. *Then for some* $k < n$, $2 \preceq \sigma_k$.

PROOF. Consider $S = \{i \leq n : \sigma_i \leq_{lex} \sigma_n \wedge \sigma_i \notin T\}$. The set S is not empty as it contains n. Hence S has a least element l. Now $l \neq 0$ as σ_0 is a strict extension of σ_n. Let $k = l - 1$. First $\sigma_k \notin T$ as otherwise $f(\sigma_k) = \sigma_l \in T$. By minimality of l we have that $\sigma_k \not\leq_{lex} \sigma_n$ and in particular $\sigma_k \not\leq_{lex} \sigma_l$. Now because $\sigma_k \notin T$, the definition of f implies that $\sigma_k \succeq 2$ (if $\sigma_k \not\succeq 2$, then $\sigma_l = f(\sigma_k) \geq_{lex} \sigma_k$). ⊣

The next claim means that if $X \in [T]$ and Y is lexicographically less than X, then $f(Y)$ is also lexicographically less than X.

Claim. *If* $|\tau| > |\sigma|$, $\sigma <_{lex} \tau$, *and* $\tau \in T$ *then* $f(\sigma) <_{lex} \tau$.

Proof. If $\sigma \prec \tau$, then $\sigma \in T$ and so $f(\sigma)$ is an initial segment of σ and the result holds. Otherwise let ξ be the longest common initial segment of σ and τ. So $\xi 0 \preceq \sigma$. Because ξ is on the tree, either $f(\sigma)$ extends $\xi 0$ or $f(\sigma) = \xi 10^j$ for some j. However as $|\tau| > |\sigma| \geq |f(\sigma)|$ this implies that in either case $f(\sigma) <_{lex} \tau$. ⊣

We now use these claims to establish a contradiction. Essentially if R is a recurrent point not in $[T]$ then the orbit of R has to loop around (1st claim) but this orbit cannot pass any path in $[T]$ (2nd claim). Let R be a recurrent point for this system. Assume $R \notin [T]$. Take $\sigma \prec R$ such that $\sigma \notin T$. As R is a recurrent point, there exists a sequence $\sigma_0, \dots, \sigma_n$ such that $\sigma \preceq \sigma_n \preceq \sigma_0 \prec R$ and $f(\sigma_i) = \sigma_{i+1}$. By the 1st claim for some $k < n$, $2 \preceq \sigma_k$.

Let $\tau \in T$ such that for all $i \leq n$, $|\tau| > |\sigma_i|$. Now σ_{k+1} is a string of all 0's, and $|\sigma_{k+1}| < |\tau|$. Hence $\sigma_{k+1} <_{lex} \tau$. It follows that $\sigma_n <_{lex} \tau$ by inducting over the 2nd claim. Now as $\sigma_0 \succeq \sigma_n$ and $\sigma_n \not\preceq \tau$ because $\sigma_n \notin T$, this implies that $\sigma_0 <_{lex} \tau$.

But this is impossible. If $\sigma_0 <_{lex} \tau$ then again by inducting over the 2nd claim, for all i, $\sigma_i <_{lex} \tau$ and so $\sigma_i \not\succeq 2$. This contradicts the fact that $\sigma_k \succeq 2$. Hence $R \in [T]$. ⊣

Theorem 3.2. *Over* RCA_0, WKL *implies* RP.

Proof. Let (C, f) be a system. We can define the set of recurrent points of this system, $\mathcal{R}$, as follows.

$$\mathcal{R} = \{X \in [C] \colon (\forall c)(\exists n, l > c)(f^n(X \restriction_l) \succeq X \restriction_c)\}.$$

This shows that $\mathcal{R}$ is Π^0_2 in $C \oplus f$. In order to prove that WKL implies RP, it is sufficient to show there is a nonempty $\Pi^0_1(C \oplus f)$ class contained in $\mathcal{R}$. We will construct a sequence of finite sets of strings $\langle U_i \colon i \in \mathbb{N}\rangle$ and let $\bigcap_i [\![U_i]\!] \cap [C]$ be our $\Pi^0_1(C \oplus f)$ class. We will ensure that if $X \in [\![U_i]\!] \cap [C]$, then for some $n, l > i$, $f^n(X \restriction_l) \succeq X \restriction_i$, hence if $X \in \bigcap_i [\![U_i]\!] \cap [C]$, then $X \in \mathcal{R}$ and so X is a recurrent point of (C, f).

The difficulty with defining U_i, is that it is possible that $[\![U_i]\!] \cap [C]$ might be empty. To avoid this occuring, we will ensure that for all i, there is an s_i such that, $\bigcup_{n<s_i} F^{-n}([\![U_i]\!]) \supseteq [C]$. This means that $[\![U_i]\!]$ cannot be removed entirely from $[C]$ because otherwise $[C]$ would either be empty or, for some $X \in [C]$, there would be some n such that $F^n(X) \notin [C]$. In either case (C, f) would not be a system. In the construction, the sets $[\![V_i]\!]$ are an approximation to the preimages of $[\![U_i]\!]$. The idea is to wait until $[\![V_i]\!]$ covers the whole space.

Let $U_0 = V_0 = \{\lambda\}$ and $s_i = 0$. We will assume that we are given U_i and V_i, both finite sets of strings and s_i a number such that:

(i) $(\forall \tau \in U_i)(\exists n)(i \leq n \leq s_i \wedge f^n(\tau) \succeq \tau \restriction_i)$.
(ii) $(\forall \sigma \in V_i)(\exists n \leq s_i)(\exists \tau \in U_i)(f^n(\sigma) \succeq \tau)$.
(iii) $[\![V_i]\!] \supseteq [C]$.

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These conditions hold trivially for the case $i = 0$. We inductively define $U_{i+1}[s]$ and $V_{i+1}[s]$ as follows.

$$U_{i+1}[s] = \{\sigma \in 2^{<\mathbb{N}} \colon ((\exists \tau \in U_i)(\sigma \succ \tau)) \wedge (\exists n)((i \leq n \leq s) \wedge (f^n(\sigma) \succeq \sigma \restriction_i))\}.$$
$$V_{i+1}[s] = \{\sigma \in 2^{<\mathbb{N}} \colon (\exists n \leq s)(\exists \tau \in U_{i+1}[s])(f^n(\sigma) \succeq \tau)\}.$$

These definitions imply that:

(i) $[\![U_{i+1}[s]]\!] \subseteq [\![U_i]\!]$.
(ii) $[\![U_{i+1}[s]]\!] \subseteq [\![U_{i+1}[s+1]]\!]$.
(iii) $[\![V_{i+1}[s]]\!] \subseteq [\![V_{i+1}[s+1]]\!]$.

Claim. $\bigcup_s [\![V_{i+1}[s]]\!] \supseteq [C]$.

Proof. By applying bounding, there is some $h > \max\{|\tau| \colon \tau \in V_i\}$ such that

$$(\forall \sigma \in \{0,1\}^h \cap C)(\forall m \leq s_i)(|f^m(\sigma)| \geq i+1).$$

Take $X \in [C]$. By the pigeon-hole principle there is some $\sigma \in \{0,1\}^h$ and $j, k \in \mathbb{N}$ such that $F^j(X) \in [\![\sigma]\!]$ and $F^{j+k}(X) \in [\![\sigma]\!]$. We can also ensure that $k \geq i+1$. Now applying WKL, we know that σ extends some element of V_i. From the definition of V_i, this means that for some $m \leq s_i$, $f^m(\sigma)$ extends some $\tau \in U_i$. Let $Y = F^{j+m}(X)$ so $Y \succ f^m(\sigma) \succeq \tau$. Further $F^k(Y) = F^{j+k+m}(X) \succ f^m(\sigma) \succeq \tau$ as well. Finally, $Y \restriction_{i+1} = F^k(Y) \restriction_{i+1}$ because $|f^m(\sigma)| \geq i+1$. Take l such that $f^k(Y \restriction_l) \succeq f^m(\sigma)$. Thus $Y \restriction_l \in U_{i+1}[\max\{k, l\}]$ and $X \in [\![V_{i+1}[s]]\!]$ where $s > \max\{j+m, k, l\}$ is large enough such that $f^{j+m}(X \restriction_s) \succeq Y \restriction_l$. ⊣

Hence by compactness, there is some least s_{i+1} such any string of length s_{i+1} in C extends some element of $V_{i+1}[s_{i+1}]$. We define $U_{i+1} = U_{i+1}[s_{i+1}]$ and $V_{i+1} = V_{i+1}[s_{i+1}]$. Note that $[\![U_{i+1}]\!]$ is a closed set in Cantor Space. Fix i. Take any $X \in [C]$. We know that $X \in [\![V_i]\!]$ and hence there exists some $n \leq s_i$ such that $F^n(X) \in [\![U_i]\!]$. Hence $[\![U_i]\!] \cap [C] \neq \emptyset$. Thus $\langle [\![U_i]\!] \cap [C] \colon i \in \mathbb{N} \rangle$ is a nested sequence of nonempty closed sets and so by WKL contains an element R, which is a recurrent point of (C, f). ⊣

§4. Minimal systems.

We now investigate the principle AP. The standard proof that every topological dynamical system has an almost periodic point uses the existence of minimal systems. In order to define a minimal system, we need to introduce the notion of a subsystem. Let (C, f) be a system and let D be a subtree of C. If $(D, f \restriction_D)$ is a system, then we call $(D, f \restriction_D)$ a subsystem of (C, f). We call a system (C, f) *minimal*, if for any subsystem $(D, f \restriction_D)$ of (C, f), we have that $[D] = [C]$.

The following lemma shows that this definition of subsystem is not restrictive.

Lemma 4.1 (RCA_0). *Let (C, f) and (D, g) be systems. Suppose that $[D] \subseteq [C]$ and for all $X \in [D]$, $G(X) = F(X)$. Then there is an $E \subseteq C$ such that $(E, f \restriction_E)$ is a subsystem of (C, f), and $[D] = [E]$.*

Proof. Let $E = \{\sigma \in D \cap C \colon (\forall k < |\sigma|)(f^k(\sigma) \in D \cap C)\}$. Note that if $X \in [D]$, then for all k, $F^k(X) = G^k(X) \in [D]$. Hence $X \in [E]$ and so $[D] = [E]$. This also shows that $[E]$ is not empty. Finally if $\sigma \in C \setminus E$, then for some $k < |\sigma|$ we have that $f^k(\sigma) \notin D$ (if $\sigma \notin D$, then take $k = 0$). Now if $f(\tau) = \sigma$ then $|\tau| > |\sigma|$

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and so $|\tau| > k+1$. As $f^{k+1}(\tau) \notin D$, it follows that $\tau \notin E$ and so the range of $f \restriction_E$ is contained in E. ⊣

The following lemma and proposition are effective versions of standard results. They are implicit in Lemmas 5.9 and 5.10 of Blass, Hirst, and Simpson [2]. Blass, Hirst, and Simpson's results are more general and apply to arbitrary compact metric spaces.

Lemma 4.2 (WKL_0). *Any point in a minimal system is almost periodic.*

Proof. Let (D, f) be a minimal system and take any $X \in [D]$. Assume X is not an almost periodic point. If so there exists some $\tau \prec X$ such that

$$(\forall b)(\exists n)(\forall k \leq b) F^{n+k}(X) \not\succ \tau. \tag{4.1}$$

Define $E = \{\sigma \in D \colon (\forall k \leq |\sigma|)(f^k(\sigma) \not\succeq \tau)\}$. Now as $F^0(X) = X \succ \tau$ we know $X \notin [E]$ and hence $[E] \subsetneq [D]$. By the argument used in Lemma 4.1, we have that for all $\sigma \in E$, $f(\sigma) \in E$. Now $(E, f \restriction_E)$ cannot be a system because this would contradict the minimality of (D, f). This means that $[E] = \emptyset$. Applying WKL, there exists a b, such that E contains no string of length b. Hence for all $Z \in [D]$ there is some $k < b$ with $F^k(Z) \succ \tau$. This contradicts (4.1) and hence our assumption that X is not an almost periodic point is incorrect. ⊣

Classically, Zorn's lemma to used to construct a minimal subsystem. This is not necessary for systems in Cantor space because Cantor space contains a computable basis of open sets. This allows us to show that ACA_0 implies that any system contains a minimal subsystem. To find a minimal subsystem simply enumerate the basis and ask in order can any element be removed. In particular, let $\{\sigma_i\}_{i\in\mathbb{N}}$ enumerate the finite strings. Given a system (C, f) let $C_0 = C$. If (C_i, f) has been defined, let C_{i+1} be equal to $\{\tau \in C_i \colon (\forall n \leq |\tau|)(f^n(\tau) \not\succeq \sigma_i)\}$ if the later set is infinite. Otherwise let $C_{i+1} = C_i$. After noting that C_{i+1} is computable in $f \oplus C_i$, it is not difficult to verify that $(\bigcap_i C_i, f)$ is a minimal subsystem of (C, f). This gives us the following result.

Proposition 4.3. ACA_0 *proves that any system contains an almost periodic point.*

For the remainder of this section, and in the following section, we will work with models of WKL_0. In order to simplify the exposition of the proofs, we will work with Π^0_1 classes of reals in Cantor space, as opposed to trees in $2^{<\mathbb{N}}$. Recall that C is a Π^0_1 class in Cantor space if for some Σ^0_0 formula φ, we have that $C = \{X \in 2^{\mathbb{N}} \colon (\forall n)\varphi(X \restriction_n)\}$. We will denote by $C[s]$ the set $\{X \in 2^{\mathbb{N}} \colon (\forall n \leq s)\varphi(X \restriction_n)\}$.

Theorem 4.4. *Over* WKL_0, ACA *is equivalent to statement that every system contains a minimal subsystem.*

Proof. The argument proceeding Proposition 4.3 shows that ACA_0 proves that every system contains a minimal subsystem. To show the other direction we will work over WKL_0 as our base system.

First we will show how to encode a single bit of $\emptyset'$ into a system. Let f be the left-shift i.e., the mapping $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $f(X)(i) = X(i+1)$. Fix n, we will define a Π^0_1 class C such that given any minimal subsystem $(D, f \restriction_D)$ of (C, f) the set $[\![01]\!] \cap D$ is empty if and only if $n \notin \emptyset'$. In particular, if $n \notin \emptyset'$, then

$$C = \{0^i 1^{\mathbb{N}} \colon i \in \mathbb{N}\} \cup \{1^i 0^{\mathbb{N}} \colon i \in \mathbb{N}\}.$$

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Observe that in this case, if $(D, f \upharpoonright_D)$ is a minimal subsystem of (C, f), then $D = \{0^{\mathbb{N}}\}$ or $D = \{1^{\mathbb{N}}\}$.

Let $S_i = \{F^n((0^i 1^i)^{\mathbb{N}}) : n \in \mathbb{N}\}$. Each S_i is a minimal system with $2 \cdot i$ elements. For example,

$$S_2 = \{(0011)^{\mathbb{N}}, 011(0011)^{\mathbb{N}}, 11(0011)^{\mathbb{N}}, 1(0011)^{\mathbb{N}}\}.$$

If $n \in \emptyset'$, then we will define C to be equal to S_i for some i compatible with our definition of C at the stage n enters $\emptyset'$. Formally, let t be ∞ if $n \notin \emptyset'$ and let t be the least s such that that $n \in \emptyset'[s]$ otherwise. Let $E_s = \{X \in 2^{\mathbb{N}} : (\exists l \leq s)(0^l 1^{s-l} \prec X \vee 1^l 0^{s-l} \prec X)\}$. Define

$$C = \begin{cases} \bigcap_s E_s & t = \infty, \\ S_i & t = i < \infty. \end{cases}$$

Observe that $S_i \subseteq \bigcap_{s<i} E_s$. Hence C is a Π^0_1 class. Let (D, f) be a minimal subsystem of (C, f). To determine if n is in $\emptyset'$ wait until a stage s such that either $n \in \emptyset'$ or $[\![01]\!] \cap D[s] = \emptyset$ (the existence of such an s when $n \notin \emptyset'$ requires WKL).

In order to code all elements of $\emptyset'$, we use the uniformity in the definition above to build a product system. Observe that if (C, f) and (D, g) are systems then so is $(C \times D, f \times g)$ where $f \times g(x, y) = (f(x), g(y))$. Further if (x, y) is an almost periodic point of $(C \times D, f \times g)$, then x is an almost periodic point of (C, f) and y is an almost periodic point of (D, g). Given a minimal subsystem of $(C \times D, f \times g)$, we know that every point is almost periodic. Hence the projection of these points on the first coordinate are almost periodic points of C. Now let (C, f) be the example above and let (D, g) be arbitrary. We know that when $n \notin \emptyset'$, then $C \cap [\![01]\!]$ contains no almost periodic points. Hence we know that the projection of a minimal subsystem of $(C \times D, f \times g)$ on the first coordinate has empty intersection with $[\![01]\!]$. If, on the other hand n does enter $\emptyset'$, then take any almost periodic point (x, y) in $(C \times D, f \times g)$. Because x is almost periodic in (C, f) for some k we have that $f^k(x) \succ [\![01]\!]$ and $(f \times g)^k(x, y)$ projects on the first coordinate to $f^k(x)$. Hence the projection of a minimal subsystem of $(C \times D, f \times g)$ on the first coordinate does not have empty intersection with $[\![01]\!]$. This arguments above hold for an infinite product and so we can code all elements of $\emptyset'$ as detailed below.

For all n, let C_n be the set defined by the above construction. Let $C = \Pi_n C_n$ (i.e., $X \in C$ if and only if for all n, $X^{[n]} \in C_n$ where $X^{[n]}$ denotes the nth column of X). Let f be the mapping produced by applying the left-shift to each column. Now if $(D, f \upharpoonright_D)$ is a minimal subsystem of (C, f) then we have that $n \notin \emptyset'$ if and only if the set $\{X \in D : X^{[n]} \succ [\![01]\!]\}$ is empty. Using WKL_0, this set is empty if and only if the associated tree is finite and we have provided a Σ^0_1 definition of the complement of $\emptyset'$. By relativizing this argument, we can give a $\Sigma^0_1(Z)$ definition of the complement of Z' for any Z in our model. This shows that any model of WKL_0 plus "every system contains a minimal subsystem" is a model of ACA. ⊣

§5. Separating AP **from** WKL**.** We have seen that ACA_0 proves AP. Further RCA_0 + AP proves WKL because any almost periodic point is a recurrent point. In this section we will separate AP from WKL. We will show that there is a model of WKL_0 that is not a model RCA_0 + AP. The natural numbers in this model will

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be the true natural numbers and so we will work with full induction. We will also regard our closed sets as Π^0_1 classes, as this simplifies the exposition.

The key to the separation is the following technical lemma. Let (C, f) be a system. A point $X \in C$ is called a *periodic point* of (C, f) if for some n, $F^n(X) = X$. Let $Orb(X)$ be the orbit of X. Note that if X is a periodic point then $Orb(X)$ is a finite set.

LEMMA 5.1. *Let f be the left-shift on Cantor space. Let $P \subseteq 2^{\mathbb{N}}$ be a Π^0_1 class. There is a Π^0_1 class C, computable uniformly in an index for P such that (C, f) is a system and either*:

(i) $C \cap P = \emptyset$; *or*
(ii) *There is a nonempty Π^0_1 class $\widehat{P} \subseteq P$ with the property that no element of $\widehat{P}$ is an almost periodic point of (C, f).*

PROOF. The definition of C is simple. Let $\{X_i\}_{i\in\mathbb{N}}$ be an enumeration of the periodic points in $(2^{\mathbb{N}}, f)$. Such an enumeration exists because any periodic point is of the form $\sigma^{\mathbb{N}}$ for some finite string σ. We let $C = 2^{\mathbb{N}}$ unless at some least stage s, we have that $Orb(X_i) \cap P = \emptyset$ for some $i < s$. If so we let $C = Orb(X_i)$ for the least i for which this holds at stage s. The definition of C is uniform because we can refine C to $Orb(X_i)$ at any point.

If $C = Orb(X_i)$ for some periodic point X_i, then $C \cap P = \emptyset$ and condition (5.1) is meet. Hence we will consider the case that $C = 2^{\mathbb{N}}$. If there is some computable point $X \in P$ such that X is not almost periodic, then condition (5.1) holds by defining $\widehat{P} = \{X\}$. Hence we will assume that any computable point in P is almost periodic.

We inductively define a sequence of finite strings $\sigma_1, \sigma_2, \dots$. The strings will have the following properties. If $i < j$ then $\sigma_i \preceq \sigma_j$. For all i, 1^i is a substring of σ_{i+1} but 1^{i+1} is not. The string $\sigma_1 = 0^n$ for some $n > 0$.

(i) The sequence $10^{\mathbb{N}}$ is computable and not almost periodic. Hence $10^{\mathbb{N}} \notin P$ and so there exists some $n_1 > 0$ such that $[\![10^{n_1}]\!] \cap P = \emptyset$. Let $\sigma_1 = 0^{n_1}$.
(ii) The sequence $11(\sigma_1 1)^{\mathbb{N}}$ is computable and not almost periodic (the subsequence 11 only occurs once). Hence $11(\sigma_1 1)^{\mathbb{N}} \notin P$, and so there exists some $n_2 > 0$ such that $[\![11(\sigma_1 1)^{n_2}]\!] \cap P = \emptyset$. Let $\sigma_2 = (\sigma_1 1)^{n_2}$.
(iii) Similarly $111(\sigma_2\sigma_1 11)^{\mathbb{N}} \notin P$, and so there exists some n_3 such that $[\![111(\sigma_2\sigma_1 11)^{n_3}]\!] \cap P = \emptyset$. Let $\sigma_3 = (\sigma_2\sigma_1 11)^{n_3}$.
(iv) In general we define $\sigma_{i+1} = (\sigma_i\sigma_{i-1}\dots\sigma_1 1^i)^{n_i}$ such that

$$[\![1^{i+1}(\sigma_i\sigma_{i-1}\dots\sigma_1 1^i)^{n_i}]\!] \cap P = \emptyset.$$

Consider the periodic systems generated by $(\sigma_i)^{\mathbb{N}}$. Because $C = 2^{\mathbb{N}}$, for all i, there is some $Y_i \in Orb((\sigma_i)^{\mathbb{N}}) \cap P$.

CLAIM. *For all i, $Y_i(0) = 0$.*

PROOF. Take any Y_i. Let $k \in \mathbb{N}$ be the largest number such that 1^k is an initial segment of Y_i. First $k < i$ because any substring of 1's in $(\sigma_i)^{\mathbb{N}}$ has length less than i. Further, by construction if $1^k 0$ forms an initial sequence of Y_i then $1^k\sigma_k$ forms an initial sequence of Y_i, but σ_k was chosen so that $[\![1^k\sigma_k]\!] \cap P = \emptyset$. Note here we are using the fact that if $i < j$ then $\sigma_i \preceq \sigma_j$. Hence $Y_i(0) = 0$. ⊣

CLAIM. *Fix k. Let $c_k = 2k + \sum_{s=1}^{k} |\sigma_s|$. Then for all $i > k$, $Y_i \restriction_{c_k}$ contains 1^k as a substring.*

PROOF. Let $\tau = \sigma_k \dots \sigma_1$. We will show by induction that for $i > k$, σ_i is a string of the form $\tau 1^{n_1} \tau 1^{n_2} \tau 1^{n_3} \dots \tau 1^{n_l}$ where each $n_j \geq k$ for $j \in \{1, \dots, l\}$. First $\sigma_{k+1} = (\sigma_k \sigma_{k-1} \dots \sigma_1 1^k)^{n_k} = (\tau 1^k)^{n_k}$ and is clearly of this form. Fix $i \geq k+1$ and assume this holds for all $j \in \{k+1, k+2, \dots, i\}$. Then

$$\sigma_{i+1} = (\sigma_i \sigma_{i-1} \dots \sigma_1 1^i)^{n_i} = (\sigma_i \sigma_{i-1} \dots \sigma_{k+1} \tau 1^i)^{n_i}$$

and so has the desired property by induction. As Y_i is a left-shift of $(\sigma_i)^{\mathbb{N}}$ and $c_k = |\tau| + 2k$, the claim holds. ⊣

Let Y be an accumulation point of $\{Y_i : i \in \mathbb{N}\}$. Hence Y is an element of P as P is closed. The sequence Y has the property that $Y(0) = 0$ and for all k, the initial segment $Y \restriction_{c_k}$ contains a subsequence of 1^k. Observe that the sequence $\{c_i\}$ is computable. Now define $\widehat{P} \subseteq P$ to be the following Π^0_1 class

$$\{Z \in P : Z(0) = 0 \wedge (\forall k)(1^k \text{ is a substring of } Z \restriction_{c_k})\}.$$

If all the assumptions are met until this point, $\widehat{P}$ is nonempty and no element of $\widehat{P}$ is an almost periodic point. Hence condition (5.1) is met. ⊣

In the proof of the following theorem we will make use of the fact that if $P \subseteq 2^{\mathbb{N}}$ is a Π^0_1 class and $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a total computable function, then both $f(P)$ and $f^{-1}(P)$ are Π^0_1 classes.

THEOREM 5.2. WKL_0 *does not prove* AP.

PROOF. Let f be the left-shift. Let $\{Q_i\}_{i \in \mathbb{N}}$ be a enumeration of all Π^0_1 classes. It follows from the uniformity of Lemma 5.1, that we can build a product system

$$(C, g) = \prod_{e \in \mathbb{N}} (C_e, f)$$

such that if Q is the e^{th} Π^0_1 class then either

(i) $\pi_e(Q) \cap C_e = \emptyset$ or
(ii) There is a nonempty Π^0_1 class $\widehat{Q} \subseteq Q$ such that no element of $\pi_e(\widehat{Q})$ is almost periodic,

where π_e is the projection on the e^{th} coordinate. See Theorem 4.4 for an example of how to encode such a product system. While Lemma 5.1 guarantees the existence of a nonempty Π^0_1 subset of $\pi_e(Q)$, no element of which is almost periodic, this can be pulled-back along π_e to obtain $\widehat{Q}$. We will show that there is a set of PA degree that does not compute an almost periodic point of (C, f).

CONSTRUCTION. At stage 0, let P_0 be a nonempty Π^0_1 class of sets of PA degree. At stage $s+1$, let Φ_s be the s^{th} Turing functional. If for some n the set $\{X \in P_s : \Phi_s^X(n) \uparrow\}$ is not empty, then let P_{s+1} be this set for the least such n.

Otherwise, we have that Φ_s is total on all elements of P_s. Let $Q = \Phi_s(P_s)$. Now Q is a Π^0_1 class because there is a total functional that agrees with Φ_s on the elements of P_s. Let e be an index for Q as a Π^0_1 class. There are two possible outcomes. First $\pi_e(Q) \cap C_e = \emptyset$ in which case let $P_{s+1} = P_s$ and note that no element of P_{s+1} computes an element of C via Φ_s let alone an almost periodic element. The other possible outcome is that there is some nonempty $\widehat{Q} \subseteq Q$ such that no element of $\pi_e(\widehat{Q})$ is almost periodic in C_e (and hence no element of $\widehat{Q}$ is almost periodic in C). For this outcome let $P_{s+1} = \{X \in P_s : \Phi_s^X \in \widehat{Q}\}$. In this case, P_{s+1} is a nonempty Π_1^0 class, no element of which computes an almost periodic point in C via Φ_s (again we are making use of the fact that the projection of an almost periodic point is an almost periodic point as discussed in Theorem 4.4).

By compactness there is some $X \in \bigcap_i P_i$. This set X is of PA degree and X does not compute an almost periodic point of (C, g). Now it is standard result that there is a model of WKL_0, such that all sets in this model are Turing below X. This model does not contain an almost periodic point for the system (C, g) and shows that WKL_0 does not imply AP. ⊣

§**6. Separating** ACA **from** AP**.** In this section, we will show that there exists a model of $\mathsf{RCA}_0 + \mathsf{AP}$ that is not a model of ACA. To achieve this, we will prove that every topological dynamical system on Cantor space has an almost periodic point that is low relative to the system. Because the main theorem of this section is a separation result, we could make use of full induction. However, we will restrict ourselves to Σ_1^0 induction so that we can make use of these results in Section 7.

The objective is to construct an almost periodic point of a system while forcing the jump. Let (C, f) be a system and let U be a c.e. set of strings. If there is a subsystem $(D, f \restriction_D)$ of (C, f) such that $[D] \cap [\![U]\!] = \emptyset$, then we can replace our original system with $(D, f \restriction_D)$. Any almost periodic point in $(D, f \restriction_D)$ is an almost periodic point of (C, f) and we know that such a point cannot meet U.

If we cannot find such a subsystem, then we will show that for some b, for all $X \in [C]$, there exists some $k < b$ with $F^k(X) \in [\![U]\!]$. We will use this fact to build a new system (D, g) such that $[D] \subseteq [C] \cap [\![U]\!]$ and for all $X \in [D]$, $G(X) = F^k(X)$ for some $k < b$. We will show that this gives us a certain recurrence property that allows us to build an almost periodic point of (C, f) that meets U.

Let us look at a simple example that illustrates the idea. Consider the following permutation on 6 elements (12)(3456). This is a topological dynamical system when $\{1, 2, 3, 4, 5, 6\}$ is given the discrete topology. Now suppose at some stage the points 2, 3, and 6 are enumerated into some open set U. Our new system becomes (2)(36). We form the new orbits by simply skipping over those elements not in U. In the following definition we use $f(\sigma, k)$ instead of $f^k(\sigma)$ to aid readability.

Definition 6.1.

(i) Let f and g encode continuous transformations of C and D respectively with $D \subseteq C$. Call g a *piece-wise combination of iterates of* f if for some l, b there is a function $j : \{0, 1\}^l \to \{1, \dots, b\}$ such that for all $\sigma \in D$ with $|\sigma| \geq l$, $g(\sigma) = f(\sigma, j(\sigma \restriction_l))$.

(ii) Let $(C, f), (D, g)$ be systems. We say that (D, g) *refines* (C, f), written $(D, g) \leq (C, f)$ if:
  (a) $D \subseteq C$.
  (b) g is a piece-wise combination of iterates of f.

Clearly if $(D, f \restriction_D)$ is a subsystem of (C, f) then $(D, f \restriction_D) \leq (C, f)$.

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LEMMA 6.2. (WKL_0) *Let* $(C, f), (D, g)$ *be systems such that* $(D, g) \le (C, f)$. *If* $X \in [D]$, *then* $(\exists b)(\forall n)(\exists k \le b)F^{n+k}(X) \in [D]$.

PROOF. Assume this fails for some $X \in [D]$. Let b witness that $(D, g) \le (C, f)$ i.e., $\{1, \dots, b\}$ is range of the function j. Consider the set of n such that

$$\{F^{n+1}(X), F^{n+2}(X), \dots, F^{n+b}(X)\} \cap [D] = \emptyset.$$

This set is c.e. in X and nonempty by assumption. Hence it contains a least element l. By minimality and the fact that $F^0(X) = X \in [D]$ we must have that $F^l(X) \in [D]$ hence for some $k \in \{1, \dots, b\}$, $G(F^l(X)) = F^k(F^l(X))$ and so $F^{l+k}(X) = G(F^l(X)) \in [D]$ contradicting our assumption. ⊣

LEMMA 6.3. (WKL_0) *The refinement relation is transitive.*

PROOF. Let $(E, h) \le (D, g) \le (C, f)$. Clearly $E \subseteq C$. Let $j_1 : \{0,1\}^{l_1} \to \{1, \dots, b_1\}$ and $j_1 : \{0,1\}^{l_2} \to \{1, \dots, b_2\}$ be such that for all σ, if $|\sigma| \ge \max\{l_1, l_2\}$ then

(i) $g(\sigma) = f^{j_1(\sigma\restriction_{l_1})}(\sigma)$.
(ii) $h(\sigma) = g^{j_2(\sigma\restriction_{l_2})}(\sigma)$.

Let $b_3 = b_1 \cdot b_2$. Let $l_3 > l_2$ be sufficiently large such that for all $\sigma \in \{0,1\}^{l_3}$, for all $n < b_2$, $|g^n(\sigma)| > l_1$.

Take any string σ such that $|\sigma| \ge l_3$. Let $m = j_2(\sigma \restriction_{l_2})$. Then $h(\sigma) = g^m(\sigma)$. Further

$$\begin{aligned} g^m(\sigma) &= g \circ g^{m-1}(\sigma) \\ &= f^{j_1(g^{m-1}(\sigma)\restriction_{l_1})} \circ g^{m-1}(\sigma) \\ &= f^{j_1(g^{m-1}(\sigma)\restriction_{l_1})} \circ f^{j_1(g^{m-2}(\sigma)\restriction_{l_1})} \circ \dots \circ f^{j_1(g^{0}(\sigma)\restriction_{l_1})}(\sigma). \end{aligned}$$

Hence $h(\sigma) = f^n(\sigma)$, where $n = \sum_{i=0}^{m-1} j_1(g^i(\sigma) \restriction_{l_1})$. Because $m \le b_2$, we have that n only depends on $\sigma \restriction_{l_3}$. Further $n \le b_3$. Hence h is a piece-wise combination of iterates of f and so $(E, h) \le (C, f)$. ⊣

LEMMA 6.4. (WKL_0) *Let* (C, f) *be a system and* U *a c.e. set. There is a system* (D, g) *refining* (C, f) *such that either*:

(i) $[D] \cap [\![U]\!] = \emptyset$; *or*
(ii) $[D] \subseteq [\![U]\!]$.

PROOF. Define

$$D_0 = \{\sigma \in C : (\forall n \le |\sigma|)(\forall \tau \in U[|\sigma|])(f(\sigma, n) \not\succeq \tau)\}.$$

To establish that D_0 is a tree, let σ and σ' be any two strings such that $\sigma \preceq \sigma'$. Assume $\sigma \notin D_0$. If $\sigma \notin C$ then because C is a tree $\sigma' \notin D_0$. If $\sigma \in C$ then for some $n \le |\sigma|$ and $\tau \in U[|\sigma|]$, $f(\sigma, n) \succeq \tau$. Hence $f(\sigma', n) \succeq \tau$ and $\tau \in U[|\sigma'|]$. Thus $\sigma' \notin D_0$.

CLAIM. *For all* $\sigma \in D_0$, $f(\sigma) \in D_0$.

PROOF OF CLAIM. If $f(\sigma) \notin D_0$, then let $\sigma' = f(\sigma)$. There is some $n \le |\sigma'|$ and $\tau \in U[|\sigma'|]$ such that $f(\sigma', n) \succeq \tau$. Hence $f(\sigma, n + 1) \succeq \tau \in U[|\sigma|]$. We have that $\sigma \notin D_0$ because $n + 1 \le |\sigma|$ (here we use our assumption that $|f(\sigma)| < |\sigma|$ if $\sigma \ne \lambda$). ⊣

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This claim establishes that if D_0 is infinite, then $(D_0, f\restriction_{D_0})$ refines (C, f), and by the definition of D_0 with $n = 0$, we have that $[D_0] \cap [\![U]\!] = \emptyset$.

Now consider the case that D_0 is finite. Let s be least such that D_0 contains no string of length s. Define

$$D_1 = \{\sigma \in C : (|\sigma| < s) \vee (\exists \tau \in U[s](\sigma \succeq \tau))\}.$$

We will show that D_1 is infinite. Take any $X \in [C]$ and let $\sigma = X\restriction_s$. As $\sigma \notin D_0$, there is a $k \le s$ such that $f^k(\sigma) \succeq \tau \in U[s]$. Hence for all n, $f^k(X\restriction_n) \in D_1$.

Let l be such that if $|\sigma| = l$, then $|f(\sigma)| \ge s$. Define $j : \{0,1\}^l \to \{1, \dots, s+1\}$ by

$$j(\sigma) = \begin{cases} 1 & \text{if } \sigma \notin C, \\ k & \text{where } k \ge 1 \text{ is least such that } (\exists \tau \in U[s]) f(\sigma, k) \succeq \tau \text{ if } \sigma \in C. \end{cases}$$

Note that $j(\sigma)$ is well-defined because for $\sigma \in \{0,1\}^l$ we have that $|f(\sigma)| \ge s$ and so $f(\sigma) \notin D_0$. Hence for some $k \in \{0, \dots, s\}$ we have that $f(f(\sigma), k)$ extends some element of $U[s]$. Define a function $g : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by

$$g(\sigma) = \begin{cases} \lambda & \text{if } |\sigma| < l, \\ f(\sigma, j(\sigma\restriction_l)) & \text{otherwise.} \end{cases}$$

It follows from the definition of j and D_1, that if $\sigma \in C$, then $g(\sigma) \in D_1$. Hence $(D_1, g\restriction_{D_1})$ is a system. Clearly, g is a piece-wise combination of iterates of f and hence $(D_1, g\restriction_{D_1}) \le (C, f)$. Finally $[D_1] \subseteq [\![U[s]]\!]$. ⊣

The proof given of the proceeding lemma provides some more information that we will make use of in Section 7. We state this as the following lemma.

Lemma 6.5. (WKL_0) *Consider the set*

$$\{\sigma \in C : (\forall n \le |\sigma|)(\forall \tau \in U[|\sigma|])(f(\sigma, n) \not\succeq \tau)\}.$$

Case (i) *of Lemma* 6.4 *holds if this set is infinite. Case* (ii) *of Lemma* 6.4 *holds if this set is finite, and further there is a* (D, g) *refining* (C, f) *such that for all* $X \in [C]$ *there is a* k *with* $F^k(X) \in [D]$.

We make use of full induction for the following lemma.

Lemma 6.6. *Any system* (C, f) *contains an almost periodic point* X *such that* $X' \le_T (C \oplus f)'$.

Proof. We define a sequence of systems $\{(C_e, f_e)\}_{e\in\mathbb{N}}$ such that for all e, $(C_{e+1}, f_{e+1}) \le (C_e, f_e)$. Let $(C_0, f_0) = (C, f)$. At stage $e+1$, let $U_e = \{\sigma \in 2^{<\mathbb{N}} : \Phi_e^\sigma(e)\downarrow\}$. Let (C_{e+1}, f_{e+1}) refine (C_e, f_e) such that either $[C_{e+1}] \cap [\![U_e]\!] = \emptyset$ or $[C_{e+1}] \subseteq [\![U_e]\!]$.

An examination of the proof of Lemma 6.4 shows that this sequence can be constructed below $(C \oplus f)'$. In Lemma 6.4, D_0, D_1, and g are defined uniformly from C and f (the definition of D_1 and g depend on D_0 being finite). Further $(C \oplus f)'$, can determine whether or not D_0 is finite and hence decide how to refine (C, f).

By compactness, $\bigcap_e [C_e]$ is not empty. In fact $\bigcap_e [C_e]$ contains a unique point X, because every finite set occurs as infinitely many c.e. sets U_e. Now $X' \le_T (C \oplus f)'$ because whether $\Phi_e^X(e)$ halts can be determined at stage e of the construction.

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We show that X is an almost periodic point of C. Fix $\sigma \prec X$. Now for some e, $[\![\sigma]\!] \supseteq [C_e]$. Thus by Lemmas 6.2 and 6.3, there is some bound b, such that for all n, there is some $k \leq b$ such that $F^{n+k}(X) \in [C_e] \subseteq [\![\sigma]\!]$. Hence X is an almost periodic point of (C, f). ⊣

Using the standard approach, the previous proposition can be used to build an ω-model of RCA_0 and AP such that every real in the model is low. Hence we obtain the following theorem.

THEOREM 6.7. *There is an ω-model of* RCA_0 *and* AP *that is not a model of* ACA.

§7. A conservation result. The goal for this section is to show that $\mathsf{RCA}_0 + \mathsf{AP}$ is conservative over RCA_0 for Π^1_1-sentences. The following lemma is the key to doing this. Instead of preserving Σ^0_1-induction directly, it is easier to preserve the least number principle. This next lemma allows us to refine (C, f) to a system (D, g) such that every $X \in [D]$ preserves the least number principle for a fixed Σ^0_0 formula.

LEMMA 7.1 (WKL_0). *Let (C, f) be a system, P be a real, and φ be a Σ^0_0 formula. There is a system (D, g) refining (C, f) such that for all $X \in [D]$, the following formula holds*

$$(\exists n)(\exists s)(\varphi(n, X \restriction_s, P)) \rightarrow \\ (\exists l)((\exists s)(\varphi(l, X \restriction_s, P) \wedge (\forall m < l)(\neg(\exists s)(\varphi(m, X \restriction_s, P))))).$$

PROOF. Let $U = \{\tau \colon (\exists n)\varphi(n, \tau, P)\}$. Let (D, g) be a refinement of (C, f) guaranteed by Lemma 6.4. If $[D] \cap [\![U]\!] = \emptyset$, then we can take the system (D, g) to satisfy the conclusion of this lemma. Otherwise, $[D] \subseteq [\![U]\!]$. By applying compactness there is some bound b, such that

$$[D] \subseteq [\![\{\tau \colon (\exists n \leq b)\varphi(n, \tau, P)\}]\!].$$

For all n, define the following sets

$$\begin{aligned}
O(\leq n, \varphi) &= \{\tau \colon (\exists m \leq n)\varphi(m, \tau, P)\},\\
O(< n, \varphi) &= \{\tau \colon (\exists m < n)\varphi(m, \tau, P)\},\\
C(\leq n, \varphi) &= \{\sigma \in D \colon (\forall i \leq |\sigma|)(\forall \tau \in O(\leq n))(g(\sigma, i) \not\succeq \tau)\},\\
C(< n, \varphi) &= \{\sigma \in D \colon (\forall i \leq |\sigma|)(\forall \tau \in O(< n))(g(\sigma, i) \not\succeq \tau)\}.
\end{aligned}$$

Intuitively, $O(\leq n, \varphi)$ is the the open set of reals that add something less than or equal to n, to the set defined by φ. The set $C(\leq n, \varphi)$ is the closed set of reals (represented as a tree) whose orbit never adds something less than or equal to n. Similarly for $O(< n, \varphi)$ and $C(< n, \varphi)$. These definitions are designed to be used with Lemma 6.5. Because of the uniformity in the above definitions, we have that the following set is c.e. in $P \oplus D \oplus g$

$$S = \{n \colon C(\leq n, \varphi) \text{ is finite}\}.$$

Note that by finite, it is meant that the tree defined by $C(\leq n, \varphi)$ is finite. As $b \in S$, it follows by Σ^0_1-induction that S has a least element l.

By Lemma 6.5, there is a system (E, h) such that (E, h) refines (D, g) and $[E] \subseteq [\![O(\leq l, \varphi)]\!]$. Further we have that for all $X \in [D]$ there is a k such that $G^k(X) \in [E]$.

Now we know that every element of $[E]$ enumerates l into the set defined by φ. However, we need to make sure that l is a minimal element. Now consider the set

$$Z = \{\sigma \in E \colon (\forall i \leq |\sigma|)(\forall \tau \in O(<l,\varphi))(h(\sigma,i) \not\succeq \tau)\}.$$

If this set is finite, then if $X \in [D]$, we have that $G^k(X) \in [E]$ for some k and so for some j, $H^j(G^k(X))$ must extend some element of $O(<l,\varphi)$. As h is a piecewise combination of iterates of g, this implies that $C(<l,\varphi)$ is finite. Thus by compactness, for some m strictly less than l, $C(\leq m,\varphi)$ is finite contradicting the minimality of l. Hence the set Z is infinite and so by Lemma 6.5 there is a a system (F,j) refining (E,h) (and consequently refining (C,f)) such that $[F] \cap [\![O(<l,\varphi)]\!] = \emptyset$. If $X \in [F]$, then $\{m \colon (\exists s)\varphi(m, X\restriction_s, P)\}$ has least element l. ⊣

Our second lemma ensures that the point we create will in fact be an almost periodic point. The proof is complicated by the fact that we have limited induction.

Lemma 7.2 (WKL_0). *Let (C,f) be a system and $i \in \mathbb{N}$. There is a subsystem $(D, f\restriction_D)$ of (C,f) and $b \in \mathbb{N}$ such that for all $X \in [D]$*

$$(\forall n)(\exists k < b)(F^{n+k}(X) \succ X\restriction_i).$$

Proof. Fix i. Let $E_0, E_1, \dots, E_n$ list the subsets of $\{0,1\}^i$ in some computable way, with the property that if $E_k \subseteq E_j$, then $k \leq j$. This implies that E_0 is the empty-set and E_n is all binary strings of length i.

Now enumerate a set W by adding k to W for $k \leq n$ if the following tree is finite

$$\{\sigma \in C \colon (\forall n \leq |\sigma|)(\forall \tau \in E_k)(f(\sigma,n) \not\succeq \tau)\}.$$

As $[C] \neq \emptyset$, $0 \notin W$ but clearly $n \in W$. By Σ^0_1-induction and bounding, there is a maximum element $k+1 \in W$ such that for all j with $k+1 \leq j \leq n$ we have $j \in W$. Consider E_k, it is a maximal sized subset with the property that the following tree is infinite.

$$D = \{\sigma \in C \colon (\forall n \leq |\sigma|)(\forall \tau \in E_k)(f(\sigma,n) \not\succeq \tau)\}.$$

Observe that $[D] \subseteq [\![\{0,1\}^i \setminus E_k]\!]$. Now if $\sigma \in \{0,1\}^i \setminus E_k$, then let s_σ be the least number such that

$$\{\sigma \in C \colon (\forall n \leq |\sigma|)(\forall \tau \in E_k \cup \{\sigma\})(f(\sigma,n) \not\succeq \tau)\}$$

contains no strings of length s_σ. Hence for any $X \in [D]$ there is some $n \leq s_\sigma$ such that $F^n(X) \in [\![\sigma]\!]$. In particular, this includes any element of $[D] \cap [\![\sigma]\!]$. The set $\{(\sigma, s_\sigma) \colon \sigma \in \{0,1\}^i \setminus E_k\}$ is also c.e. and hence by Σ^0_1-bounding, there is some b that bounds all elements of this set. ⊣

Definition 7.3. Let $\mathcal{M}$ and $\widehat{\mathcal{M}}$ be models of 2nd order arithmetic. Call $\mathcal{M}$ *a full first order submodel of* $\widehat{\mathcal{M}}$ if $\mathcal{M}$ is a submodel of $\widehat{\mathcal{M}}$ and they share the same first order part.

Theorem 7.4 (Harrington – unpublished see [5]). *Let $\mathcal{M}$ be a countable model of RCA_0. Then there exists a countable model $\widehat{\mathcal{M}}$ of WKL_0 such that $\mathcal{M}$ is a full first order submodel of $\widehat{\mathcal{M}}$.*

Lemma 7.5. *Let $\mathcal{M}$ be a countable model of WKL_0 and let (C,f) be a system in $\mathcal{M}$. Then there exists a model $\widehat{\mathcal{M}}$ of WKL_0 such that $\mathcal{M}$ is a full first order submodel of $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{M}}$ contains an almost periodic point for the system (C,f).*

PROOF. By Harrington's theorem it is only necessary to find a model $\widehat{\mathcal{M}}$ of RCA_0 such that $\mathcal{M}$ is a full first-order submodel of $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{M}}$ contains an almost periodic point for the system (C, f). Let M be the natural numbers inside $\mathcal{M}$ and let $\mathbb{R}$ be the reals inside $\mathcal{M}$. From outside of the model, let $g : \omega \to M$ and $h : \omega \to \mathbb{R}$ be bijections. Define $C_0 = C$. Now inductively define C_{e+1} as follows.

(i) If $e = 2 \cdot \langle n, m\rangle + 1$, then let C_{e+1} be such that (C_{e+1}, f_{e+1}) refines (C_e, f_e) as per Lemma 7.1 with φ the n^{th} Σ^0_0-formula and $P = h(m)$.
(ii) If $e = 2 \cdot n + 2$ then let C_{e+1} be such that (C_{e+1}, f_{e+1}) refines (C_e, f_e) as per Lemma 7.2 with $i = g(n)$.

Take $R \in \bigcap_e [C_e]$. Let $\widehat{\mathcal{M}}$ be the model obtained by adding all reals computable in $R \oplus Y$ to $\mathcal{M}$ for any $Y \in \mathcal{M}$. The odd stages in the construction of $\widehat{M}$ establish that $\widehat{\mathcal{M}}$ is a model of Σ^0_1-induction because the least number principle is preserved for Σ^0_1-formulas.

We know that R is an almost periodic point of (C, f) because $R \in [C_{e+1}]$ for $e = 2 \cdot n + 2$ establishes that there exists a b such that for all m, there is a $k \leq b$ such that $F^{m+k}(R) \in R\restriction_{g(n)}$. ⊣

We can now iterate the previous lemma using standard arguments to obtain the following theorem.

THEOREM 7.6. *Let $\mathcal{M}$ be a countable model of* RCA_0. *Then there exists a countable model $\widehat{\mathcal{M}}$ of* $\mathsf{RCA}_0 + \mathsf{AP}$ *such that $\mathcal{M}$ is an full first order submodel of $\widehat{\mathcal{M}}$.*

Standard arguments also give us the following corollary (see for example [5, Corollary IX.2.6]).

COROLLARY 7.7. $\mathsf{RCA}_0 + \mathsf{AP}$ *is conservative over* RCA_0 *for Π^1_1-sentences.*

§8. A subclass of PA degrees and open questions. Consider the Turing degrees that can compute an almost periodic point for any computable system. This is an upwards-closed subclass of the PA degrees. By Theorem 5.2, we know that this is a strict subclass of the PA degrees. The following corollary shows that this subclass does not coincide with those PA degrees greater than or equal to $\emptyset'$.

COROLLARY 8.1 (Corollary to Theorem 6.7). *There is a set X of* PA *degree such that $X \not\geq_T \emptyset'$ and X computes an almost periodic point for every computable system.*

PROOF. Let $\{Z_i\}_{i\in\mathbb{N}}$ be a listing of the ideal used to separate ACA from AP over RCA_0. Observe that no finite join of this sequence computes $\emptyset'$.

Construct X by at stage e defining sufficient columns of X to force that $\Phi^X_e \neq \emptyset'$, and then append Z_e to an empty column of X. The set X bounds all elements of the ideal so X is of PA degree. ⊣

The PA degrees have been extensively studied. However, this subclass does not appear to have been encountered before and it merits further investigation.

QUESTION 8.2. *Are there any other characterizations of this subclass*?

A useful answer to Question 8.2 would give some indication as to how this subclass is dispersed in the Turing degrees. As there are computably dominated sets of PA degree, it is natural to ask the following question.

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QUESTION 8.3. *Does this subclass have a computably dominated element?*

For this subclass to have a computably dominated element, it is necessary that the following question has a positive answer.

QUESTION 8.4. *Does every computable system have an almost periodic point that is computably dominated?*

This paper has focused on topological dynamical systems in Cantor space. An obvious next question is the following.

QUESTION 8.5. *Can the results in this paper be generalized to compact computable metric spaces?*

This last question was suggested by André Nies.

QUESTION 8.6. *Does the principle* AP *change for different topological spaces? For example, is* AP *for topological dynamical systems on Cantor space the same as* AP *for topological dynamical systems on the unit interval?*

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