Harmonic Coordinates on Fractals with Finitely Ramified Cell Structure

Alexander Teplyaev

Abstract. We define sets with finitely ramified cell structure, which are generalizations of post-critically finite self-similar sets introduced by Kigami and of fractafolds introduced by Strichartz. In general, we do not assume even local self-similarity, and allow countably many cells connected at each junction point. In particular, we consider post-critically infinite fractals. We prove that if Kigami's resistance form satisfies certain assumptions, then there exists a weak Riemannian metric such that the energy can be expressed as the integral of the norm squared of a weak gradient with respect to an energy measure. Furthermore, we prove that if such a set can be homeomorphically represented in harmonic coordinates, then for smooth functions the weak gradient can be replaced by the usual gradient. We also prove a simple formula for the energy measure Laplacian in harmonic coordinates.

Introduction

There is a well-developed theory of Dirichlet (energy, resistance) forms, and corresponding random processes, on the class of post-critically finite (pcf for short) self-similar sets, which are finitely ramified [2, 19, 22, 36, 39]. Also, many piecewise and stochastically self-similar fractals have been considered [7, 11, 12, 37]. The general non self-similar energy forms on the Sierpiński gasket were studied in [35]. In all the mentioned works the fractals considered have finitely ramified cell structure. In this paper we will extend some aspects of this theory to a class of spaces which may have no self-similarity in any sense and may have infinitely many cells connected at every junction point. Throughout this paper we extensively and substantially use the general theory of resistance forms [23]. The existence of such forms is a delicate question even in the self-similar pcf case [13,23,33]. To prove our results we use some methods introduced in [41]. In the present paper we give the basic background information, and the reader may find all the details in [23, 41].

In Section 1 we give the definition of a resistance form in the sense of Kigami [23]. In Section 2 we define sets with finitely ramified cell structures. Examples of such fractals are pcf self-similar sets introduced by Kigami [19, 22], fractafolds introduced by Strichartz [38], random fractals [7, 11, 12], and non self-similar Sierpiński gaskets [35, 42]. The key topological assumption is that there is a cell structure such that every cell has finite boundary, but we do not assume any self-similarity.

The terminology we use can be explained as follows. The term *post-critically infinite* in this context means that every junction point can be an intersection of countably infinite number of cells with pairwise disjoint interior, that is, every cell can be

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linked to countably many other cells. The term *finitely ramified* means that every cell is joined with its complement in a finite number of points. A good example of an infinitely ramified fractal is the Sierpiński carpet. There exists a self-similar diffusion and corresponding Dirichlet form on the Sierpiński carpet [3–5, 31], but its uniqueness has not been proved.

In Section 3 we prove that Kigami's resistance form is a local regular Dirichlet form under appropriate conditions. In Section 4 we prove that if the resistance form satisfies certain non degeneracy assumptions, then there exists a weak Riemannian metric, defined almost everywhere, such that the energy can be expressed as the integral of the norm of a weak gradient with respect to an energy measure. This generalizes earlier results by Kusuoka [30] and the author [41]. Furthermore, in Section 5 we prove that if the finitely ramified fractal can be homeomorphically represented in harmonic coordinates, then the weak gradient can be replaced by the usual gradient for smooth functions, which generalizes an earlier result by Kigami in [20]. In Section 6 we prove a simple formula for the energy measure Laplacian in harmonic coordinates. This formula was announced in [42], in the case of the standard energy form on the Sierpiński gasket, without a proof. In a sense, the formula for the energy Laplacian is the second derivative with respect to the generalized Riemannian metric. In the case of the standard energy form on the Sierpiński gasket, it is proved by Kusuoka in [29] that this generalized Riemannian metric has rank one almost everywhere. This can be interpreted as that in harmonic coordinates on the Sierpiński gasket the energy Laplacian is the one dimensional second derivative in the tangential direction. We conjecture that this is the case for any finitely ramified fractal considered in this paper. The main tool we use in this theorem is approximating the finitely ramified fractal by a sequence of so-called quantum graphs [27, 28]. In Section 7 we discuss self-similar finitely ramified fractals, and the existence of self-similar resistance forms in particular. In Section 8 we give several examples of finitely ramified fractals for which our theory can be applied. Among them are factor-spaces of pcf self-similar sets, and post-critically infinite analogs of the Sierpiński gasket.

In the case of the standard energy form on the Sierpiński gasket, it is proved by Kigami in [26] that the heat kernel with respect to the energy measure has Gaussian asymptotics in harmonic coordinates (a weaker version was obtained in [34]). Recently, powerful machinery was developed to obtain heat kernel estimates on various "rough" spaces, including many fractals [6, 25]. It is not unlikely that this theory is applicable to many, if not all, finitely ramified fractals in harmonic coordiantes. Also, some results about the singularity of the energy measure with respect to product measures [8, 14, 15] are valid in the case of finitely ramified self-similar fractals under suitable extra assumptions.

1   Kigami's Resistance Forms

Below we restate the definition of a resistance form in [23].

Definition 1.1   A pair $(\mathcal{E}, \text{Dom}\,\mathcal{E})$ is called a *resistance form* on a countable set V_* if it satisfies the following conditions.

(RF1) Dom $\mathcal{E}$ is a linear subspace of $\ell(V_*)$ containing constants, $\mathcal{E}$ is a nonnegative symmetric quadratic form on Dom $\mathcal{E}$, and $\mathcal{E}(u, u) = 0$ if and only if u is constant on V_*.

(RF2) Let $\sim$ be the equivalence relation on Dom $\mathcal{E}$ defined by $u \sim v$ if and only if $u - v$ is constant on V_*. Then $(\mathcal{E}/\sim, \text{Dom } \mathcal{E})$ is a Hilbert space.

(RF3) For any finite subset $V \subset V_*$ and for any $v \in \ell(V)$, there exists $u \in \text{Dom } \mathcal{E}$ such that $u|_V = v$.

(RF4) For any $p, q \in V_*$

$$\sup\left\{ \frac{(u(p) - u(q))^2}{\mathcal{E}(u, u)} : u \in \text{Dom } \mathcal{E}, \mathcal{E}(u, u) > 0 \right\} < \infty.$$

This supremum is denoted by $R(p, q)$ and called the *effective resistance* between p and q.

(RF5) For any $u \in \text{Dom } \mathcal{E}$ we have the $\mathcal{E}(\bar{u}, \bar{u}) \leqslant \mathcal{E}(u, u)$, where

$$\bar{u}(p) = \begin{cases} 1 & \text{if } u(p) \geqslant 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leqslant 1. \end{cases}$$

Property (RF5) is called the *Markov property*.

Note that the effective resistance R is a metric on V_*, and that any function in Dom $\mathcal{E}$ is R-continuous. Let Ω be the R-completion of V_*. Then any $u \in \text{Dom } \mathcal{E}$ has a unique R-continuous extension to Ω.

For any finite subset $U \subset V_*$ the finite dimensional Dirichlet form $\mathcal{E}_U$ on U is defined by

$$\mathcal{E}_U(f, f) = \inf\{\mathcal{E}(g, g) : g \in \text{Dom } \mathcal{E}, g|_U = f\},$$

which exists by [23], and moreover there is a unique g for which the inf is attained. The Dirichlet form $\mathcal{E}_U$ is called the trace of $\mathcal{E}$ on U, and denoted $\mathcal{E}_U = \text{Trace}_U(\mathcal{E})$. By the definition, if $U_1 \subset U_2$ then $\mathcal{E}_{U_1}$ is the trace of $\mathcal{E}_{U_2}$ on U_1, that is, $\mathcal{E}_{U_1} = \text{Trace}_{U_1}(\mathcal{E}_{U_2})$.

Theorem (Kigami [23]) *Suppose that V_n are finite subsets of V_* and that $\bigcup_{n=0}^\infty V_n$ is R-dense in V_*. Then $\mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}_{V_n}(f, f)$ for any $f \in \text{Dom } \mathcal{E}$, where the limit is actually non-decreasing. Is particular, $\mathcal{E}$ is uniquely defined by the sequence of its finite dimensional traces $\mathcal{E}_{V_n}$ on V_n.*

Theorem (Kigami [23]) *Suppose that V_n are finite sets, for each n there is a resistance form $\mathcal{E}_{V_n}$ on V_n, and this sequence of finite dimensional forms is compatible in the sense that each $\mathcal{E}_{V_n}$ is the trace of $\mathcal{E}_{V_{n+1}}$ on V_n, where $n = 0, 1, 2, \ldots$. Then there exists a resistance form $\mathcal{E}$ on $V_* = \bigcup_{n=0}^\infty V_n$ such that $\mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}_{V_n}(f, f)$ for any $f \in \text{Dom } \mathcal{E}$, and the limit is actually non-decreasing.*

2   Finitely Ramified Fractals

Definition 2.1   A *finitely ramified fractal F* is a compact metric space with a *cell structure* $\mathcal{F} = \{F_\alpha\}_{\alpha \in \mathcal{A}}$ and a *boundary (vertex) structure* $\mathcal{V} = \{V_\alpha\}_{\alpha \in \mathcal{A}}$ such that the following conditions hold.

(i) $\mathcal{A}$ is a countable index set.
(ii)   Each F_α is a distinct compact connected subset of F.
(iii) Each V_α is a finite subset of F_α with at least two elements.
(iv)   If $F_\alpha = \bigcup_{j=1}^k F_{\alpha_j}$, then $V_\alpha \subset \bigcup_{j=1}^k V_{\alpha_j}$.
(v)   There exists a filtration $\{\mathcal{A}_n\}_{n=0}^\infty$ such that

    (a) $\mathcal{A}_n$ are finite subsets of $\mathcal{A}$, $\mathcal{A}_0 = \{0\}$, and $F_0 = F$;
    (b) $\mathcal{A}_n \cap \mathcal{A}_m = \varnothing$ if $n \neq m$;
    (c) for any $\alpha \in \mathcal{A}_n$ there are $\alpha_1, \ldots, \alpha_k \in \mathcal{A}_{n+1}$ such that $F_\alpha = \bigcup_{j=1}^k F_{\alpha_j}$.

(vi) $F_{\alpha'} \cap F_\alpha = V_{\alpha'} \cap V_\alpha$ for any two distinct $\alpha, \alpha' \in \mathcal{A}_n$.
(vii) For any strictly decreasing infinite cell sequence $F_{\alpha_1} \supsetneq F_{\alpha_2} \supsetneq \cdots$ there exists $x \in F$ such that $\bigcap_{n \geqslant 1} F_{\alpha_n} = \{x\}$.

If these conditions are satisfied, then $(F, \mathcal{F}, \mathcal{V}) = (F, \{F_\alpha\}_{\alpha \in \mathcal{A}}, \{V_\alpha\}_{\alpha \in \mathcal{A}})$ is called a *finitely ramified cell structure*.

Notation 2.2   We denote $V_n = \bigcup_{\alpha \in \mathcal{A}_n} V_\alpha$. Note that $V_n \subset V_{n+1}$ for all $n \geqslant 0$ by Definition 2.1. We say that F_α is an *n-cell* if $\alpha \in \mathcal{A}_n$.

Remark 2.3   By the definition every cell in a finitely ramified fractal has a boundary consisting of isolated points (see Proposition 2.10), which implies the name *point connected*. In particular, any pcf self-similar set is a finitely ramified fractal. However, every vertex $v \in V_*$ of a finitely ramified fractal can be an intersection of countably many cells with pairwise disjoint interior (see Example 8.9). Hence even if a finitely ramified fractal is self-similar, it does not have to be a pcf self-similar set.

Remark 2.4   In this definition the vertex boundary V_0 of $F_0 = F$ can be arbitrary, and in general may have no relation with the topological structure of F. However, Assumptions (WN) and (HC) made below will *de facto* impose restrictions on the choice of V_0. In particular, the energy measure, gradient and the energy measure Laplacian all depend on the choice of V_0.

This is somewhat different from the theory of pcf self-similar sets in [19, 21, 22], where V_0 is uniquely determined as the post-critical set of the pcf self-similar structure. Note, however, that the same topological fractal F can have different self-similar structures, and different post-critical sets in particular.

Note that every pcf self-similar set is a finitely ramified fractal, see [19, 22] and Section 7.

Remark 2.5   In general a filtration is not unique for a finitely ramified fractal. For example, the filtration $\widetilde{\mathcal{A}}_k = \mathcal{A}_{2k}$ satisfies all the conditions of Definition 2.1. However, the results of this paper do not depend on the choice of the filtration. In particular, Kigami's resistance forms, energy measures, *etc.* are independent of the filtration.

If the fractal is self-similar, then changing the self-similar structure sometimes is very useful, and changing filtration is one of the results of changing it. Moreover, if the finitely ramified fractal is not self-similar, then it may not be clear what the natural filtration is.

Assumption 1 In what follows we assume that some filtration is fixed.

Proposition 2.6 *For any $x \in F$ there is a strictly decreasing infinite sequence of cells satisfying condition* (vii) *of the definition. The diameter of cells in any such sequence tend to zero.*

Proof Suppose $x \in F$ is given. We choose $F_{\alpha_1} = F$. Then if F_{α_n} is chosen, we choose $F_{\alpha_{n+1}}$ to be a proper sub-cell of F_{α_n} which contains x. Suppose for a moment that the diameter of cells in such a sequence does not tend to zero. Then for each n there is $x_n \in F_{\alpha_n}$ such that $\liminf_{n \to \infty} d(x_n, x) = \varepsilon > 0$. By compactness there is $y \in \bigcap_{n \geqslant 1} F_{\alpha_n}$ such that $d(y, x) \geqslant \varepsilon$. This is a contradiction with the property (vii) of Definition 2.1. ∎

Proposition 2.7 *The topological boundary of F_α is contained in V_α for any $\alpha \in \mathcal{A}$.*

Proof For any closed set A we have $\partial A = A \cap \mathrm{Closure}(A^c)$, where A^c is the complement of A. If $A = F_\alpha$ is an n-cell, then $\mathrm{Closure}(A^c)$ is the union of all n-cells except F_α. Then the proof follows from property (vi) of Definition 2.1. ∎

Proposition 2.8 *The set $V_* = \bigcup_{\alpha \in \mathcal{A}} V_\alpha$ is countably infinite, and F is uncountable.*

Proof The set V_* is a countable union of finite sets, and every cell is a union of at least two smaller sub-cells. Then each cell is uncountable by properties (ii) and (iii) of Definition 2.1. ∎

Proposition 2.9 *For any distinct $x, y \in F$, there is $n(x, y)$ such that if $m \geqslant n(x, y)$, then any m-cell cannot contain both x and y.*

Proof Let $\mathcal{B}_m(x, y)$ be the collection of all m-cells that contain both x and y. By definition any cell in $\mathcal{B}_{m+1}(x, y)$ is contained in a cell which belongs to $\mathcal{B}_m(x, y)$. Therefore, if there are infinitely many nonempty collections $\mathcal{B}_m(x, y)$, then there is an infinite decreasing sequence of cells that contains both x and y. This is a contradiction with property (vii) of Definition 2.1. ∎

Proposition 2.10 *For any $x \in F$ and $n \geqslant 0$, let $U_n(x)$ denote the union of all n-cells that contain x. Then the collection of open sets $\mathcal{U} = \{U_n(x)^\circ\}_{x \in F, n \geqslant 0}$ is a countable fundamental sequence of neighborhoods. Here B° denotes the topological interior of a set B.*

Moreover, for any $x \in F$ and open neighborhood U of x there exist $y \in V_*$ and n such that $x \in U_n(x) \subset U_n(y) \subset U$. In particular, the smaller collection of open sets $\mathcal{U}' = \{U_n(x)^\circ\}_{x \in V_*, n \geqslant 0}$ is a countable fundamental sequence of neighborhoods.

Proof   Note that the collection $\mathcal{U}'$ is countable because V_* is countable by Proposition 2.8. The collection $\mathcal{U}$ is countable because if x and y belong to the interior of the same n-cell, then $U_n(x) = U_n(y)$.

First, suppose $x \in V_*$. Then we have to show that for any open neighborhood U of x there exists $n \geqslant 0$ such that $U_n(x) \subset U$. Suppose for a moment that such n does not exist. Then for any n the set $U_n(x) \backslash U$ is a nonempty compact set. Moreover, the sequence of sets $\{U_n(x) \backslash U\}_{n \geqslant 0}$ is decreasing and so has a nonempty intersection. Then we can choose $z \in \bigcap_{n \geqslant 0} U_n(x) \backslash U$, and for any n there is an n-cell that contains both x and z. This is a contradiction with Proposition 2.9.

Now suppose $x \notin V_*$. Then for any $n > 0$ there exists $y_n \in V_n$ such that $x \in U_n(y_n) \subset U_{n-1}(x)$. Moreover, we can assume also that $U_n(y_n) \subset U_{n-1}(y_{n-1})$ for any $n > 1$. Then we have to show that for any open neighborhood U of x there exist $n > 0$ such that $U_n(y_n) \subset U$. Suppose for a moment that such n does not exist. Then the set $U_n(y_n) \backslash U$ is a nonempty compact set. Moreover, the sequence of sets $\{U_n(y_n) \backslash U\}_{n \geqslant 1}$ is decreasing and so has a nonempty intersection. Then we can choose $z \in \bigcap_{n \geqslant 1} U_n(y_n) \backslash U$, and for any $n > 1$ there is an $(n-1)$-cell that contains both x and z. This is a contradiction with Proposition 2.9.   ∎

3   Resistance Forms on Finitely Ramified Fractals

We assume that there is a resistance form on V_* in the sense of Kigami [22, 23], see Definition 1.1. For convenience we will denote $\mathcal{E}_n(f, f) = \mathcal{E}_{V_n}(f, f)$, see Section 1. Recall that $\mathcal{E}(f, f) = \lim_{n \to \infty} \mathcal{E}_n(f, f)$ for any $f \in \mathrm{Dom}\,\mathcal{E}$, where the limit is actually non-decreasing.

Definition 3.1   A function is harmonic if it minimizes the energy for the given set of boundary values.

Note that any harmonic function is uniquely defined by its restriction to V_0. Moreover, any function on V_0 has a unique continuation to a harmonic function. For any harmonic function h, we have $\mathcal{E}(h, h) = \mathcal{E}_n(h, h)$ for all n by [23]. Also note that for any function $g \in \mathrm{Dom}\,\mathcal{E}$, we have $\mathcal{E}_0(g, g) \leqslant \mathcal{E}(g, g)$, and a function h is harmonic if and only if $\mathcal{E}_0(h, h) = \mathcal{E}(h, h)$.

Let $\mathcal{E}_\alpha(f, f) = (\mathcal{E}_\alpha)_{V_\alpha}(f, f)$, where $\mathcal{E}_\alpha$ is the restriction of $\mathcal{E}$ to F_α. Then

$$\mathcal{E}_n = \sum_{\alpha \in \mathcal{A}_n} \mathcal{E}_{V_\alpha}.$$

Lemma 3.2   *If h is harmonic and continuous, then*

$$\lim_{n \to \infty} \sum_{\substack{\alpha \in \mathcal{A}_n \\ x \in F_\alpha}} \mathcal{E}_\alpha(h|_{V_\alpha}, h|_{V_\alpha}) = 0$$

for any $x \in F$.

Proof Let $\mathcal{E}(h, h) = e > 0$. It is easy to see that the limit under consideration is decreasing and so it exists. Suppose for a moment this limit is equal to $c > 0$.

Without loss of generality we can assume that $h(x) = 0$ and that $|h(y)| \geqslant 1$ for any $y \in V_0 \backslash \{x\}$. By Proposition 2.6 for any $\varepsilon > 0$, there are cells $F_{\alpha_1}, \ldots, F_{\alpha_l}$ such that $|h(x) - h(y)| < \varepsilon$ for any $y \in \bigcup_{j=1}^{l} F_{\alpha_j}$, and $\bigcup_{j=1}^{l} F_{\alpha_j}$ contains a neighborhood of x. Without loss of generality we can assume that $V_0 \cap \left(\bigcup_{j=1}^{l} F_{\alpha_j} \backslash \{x\} \right) = \varnothing$.

Let $V' = \bigcup_{j=1}^{l} V_{\alpha_j}$ and consider the trace of the resistance form on $V_0 \bigcup V'$. Obviously if ε is small, then there is a uniform bound for conductances between points in $V_0 \backslash \{x\}$ and V'. Then consider changing the values of h on V' to zero. Inside of $\bigcup_{j=1}^{l} F_{\alpha_j}$ the energy will be reduced by at least c, since the function is now constant there. On the other hand, outside of $\bigcup_{j=1}^{l} F_{\alpha_j}$ the energy increase will be bounded by a constant times εe. So the total energy will decrease if ε is small enough. This is a contradiction with the definition of a harmonic function, and so $c = 0$.

Note that the proof works even if V' is an infinite set, and so it is applicable to connected spaces with cell structure, such as the Sierpiński carpet, which is not a finitely ramified fractal. ∎

Corollary 3.3 *If h is harmonic and continuous, then there is a unique continuous energy measure ν_h on F defined by $\nu_h(F_\alpha) = \mathcal{E}_\alpha(h\big|_{V_\alpha}, h\big|_{V_\alpha})$ for all $\alpha \in \mathcal{A}$.*

Assumption 2 In what follows we assume that harmonic functions are continuous.

Definition 3.4 We fix a complete, up to constant functions, energy orthonormal set of harmonic functions $h_1, \ldots, h_k$, where $k = |V_0| - 1$, and define the Kusuoka energy measure by $\nu = \nu_{h_1} + \cdots + \nu_{h_k}$.

If $F_{\alpha'} \subset F_\alpha$, then $M_{\alpha,\alpha'} \colon \ell(V_\alpha) \to \ell(V_{\alpha'})$ is the linear map which is defined as follows. If f_α is a function on V_α, then let h_{f_α} be the unique harmonic function on F_α that coincides with f_α on V_α. Then we define $M_{\alpha,\alpha'} f_\alpha = h_{f_\alpha}|_{V_{\alpha'}}$. Thus $M_{\alpha,\alpha'}$ transforms the (vertex) boundary values of a harmonic function on F_α into the values of this harmonic function on $V_{\alpha'}$. We denote $M_\alpha = M_{0,\alpha}$. We denote D_α the matrix of the Dirichlet form $\mathcal{E}_\alpha$ on V_α. By elementary linear algebra we have the following lemma (see [19, 22, 29, 41]).

Lemma 3.5 *If $F_\alpha = \bigcup F_{\alpha_j}$, then*

$$D_\alpha = \sum M_{\alpha,\alpha_j}^* D_{\alpha_j} M_{\alpha,\alpha_j} \quad and \quad \nu(F_\alpha) = \mathrm{Tr}\, M_\alpha^* D_\alpha M_\alpha.$$

In particular ν is defined uniquely in the sense that it does not depend on the choice of the complete energy orthonormal set of harmonic functions.

We denote

$$Z_\alpha = \frac{M_\alpha^* D_\alpha M_\alpha}{\nu(F_\alpha)}$$

if $\nu(F_\alpha) \neq 0$. Then we define matrix valued functions $Z_n(x) = Z_\alpha$ if $\nu(F_\alpha) \neq 0$, $\alpha \in \mathcal{A}_n$, and $x \in F_\alpha \backslash V_\alpha$. Note that $\mathrm{Tr}\, Z_n(x) = 1$ by definition.

Theorem 3.6   *For ν-almost all x there is a limit $Z(x) = \lim_{n\to\infty} Z_n(x)$.*

Proof   One can see, following Kusuoka's original idea [29, 30], that Z_n is a bounded ν-martingale.   ∎

Remark 3.7   In a sense, the matrix valued measure $Zd\nu$ plays the role of a generalized Riemannian metric on the fractal F (see Theorems 5.6 and 6.1). The matrix Z has trace one by definition, but on many fractals it is discontinuous. Moreover, in some examples, such as the Sierpiński gasket, the matrix Z has rank one almost everywhere. Then it can be described as the projection onto the one dimensional tangent space.

One can see that the energy measures ν_h are the same as the energy measures in the general theory of Dirichlet forms [9, 10]. One can also define the matrix Z as the matrix whose entries are the densities

$$Z_{ij} = \frac{d\nu_{h_i,h_j}}{d\nu},$$

using the general theory of Dirichlet forms in [9, 10]. However we give a different description because the pointwise approximation using the cell structure is important in this theorem.

Definition 3.8   A function is *n*-harmonic if it minimizes the energy for the given set of values on V_n.

Note that any *n*-harmonic function is uniquely defined by its restriction to V_n. Moreover, any function on V_n has a unique continuation to an *n*-harmonic function. Also note that for any function $g \in \mathrm{Dom}\,\mathcal{E}$ we have $\mathcal{E}_n(g,g) \leqslant \mathcal{E}(g,g)$, and a function f is *n*-harmonic if and only if $\mathcal{E}_n(f,f) = \mathcal{E}(f,f)$.

Recall that R is the effective resistance metric on V_*, and that any function in $\mathrm{Dom}\,\mathcal{E}$ is R-continuous. Let Ω be the R-completion of V_*. Then any $u \in \mathrm{Dom}\,\mathcal{E}$ has a unique R-continuous extension to Ω. The next theorem generalizes [22, Proposition 3.3.2] for possibly non self-similar finitely ramified fractals.

Theorem 3.9   *Suppose that all n-harmonic functions are continuous. Then any continuous function is R-continuous, and any R-Cauchy sequence converges in the topology of F. Also, there is a continuous injective map $\theta\colon \Omega \to F$ which is the identity on V_*.*

Proof   It is easy to see from the maximum principle that any continuous function can be uniformly approximated by *n*-harmonic functions, which implies that any continuous function is R-continuous. Suppose for a moment that $\{x_k\}$ is an R-Cauchy sequence in V_* which does not converge. By compactness, it must have a limit point, say x. Then, by the results of Section 2, there is n and two disjoint *n*-cells F_α and F_β such that $x \in F_\beta$, but F_α contains an infinite subsequence of $\{x_k\}$, say $\{y_m\}$. There is an *n*-harmonic function f which is identically 1 on F_β and 0 on every point of V_n which is not in F_β. Then for any m we have $R(x, y_m) \geqslant 1/\mathcal{E}(f,f) > 0$, which is a contradiction. Thus, any R-Cauchy sequence converges in the topology

of F. Therefore we can define a continuous map $\theta \colon \Omega \to F$ which is the identity on V_*. Now suppose for a moment that θ is not injective. Then there are two R-Cauchy sequences in V_*, say $\{x_k\}$ and $\{y_k\}$, which have the same limit in F but two different R-limits in Ω, say x and y. By continuity, for any m-harmonic function f we have $f(x) = f(y)$. This is a contradiction since the space of m-harmonic functions separates points of Ω by [23]. Thus, θ is injective. ∎

Remark 3.10   If conditions of Theorem 3.9 are satisfied, then we may consider Ω as a subset of F. Then Ω is the R-closure of V_*. In a sense, Ω is the set where the Dirichlet form $\mathcal{E}$ "lives".

Theorem 3.11   *Suppose that all n-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form on Ω (with respect to any measure that charges every nonempty open set).*

Proof   The regularity of $\mathcal{E}$ is proved in [23]. In particular, Dom $\mathcal{E}$ mod (constants) is a Hilbert space in the energy norm. Note that the set of n-harmonic functions is a core of $\mathcal{E}$ in both the original and R-topologies. Also note that if a set is R-compact, then it is compact in the original topology of f by Theorem 3.9. Suppose now f and g are two functions in Dom $\mathcal{E}$ with disjoint compact supports. Then, by the results of Section 2, there is n and a finite number of n-cells $F_{\alpha_1}, \ldots, F_{\alpha_k}$ such that $\bigcup_{i=1}^k F_{\alpha_i}$ contains the support of f but is disjoint with the support of g. Then it is easy to see that for any $m \geqslant n$ we have $\mathcal{E}_m(f, g) = 0$ and so $\mathcal{E}(f, g) = 0$. ∎

4   Generalized Riemannian Metric and Weak Gradient

Definition 4.1   We say that $f \in \mathrm{Dom}\,\mathcal{E}$ is n-*piecewise harmonic* if for any $\alpha \in \mathcal{A}_n$ there is a (globally) harmonic function h_α that coincides with f on F_α.

Note that, by definition, the notion of n-piecewise harmonic functions in general is more restrictive than the more commonly used notion of n-harmonic functions defined in the previous section.

Definition 4.2   We say that the resistance form on a finitely ramified fractal is *weakly nondegenerate* if the space of piecewise harmonic functions is dense in Dom $\mathcal{E}$.

The notion of weakly nondegenerate harmonic structures was studied in [41] in the case of pcf self-similar sets.

Assumption (**WN**)   In what follows we assume that the resistance form is weakly nondegenerate.

Proposition 4.3   *Assumption* (WN) *implies* $\mathrm{supp}(\nu) = F$.

Proof   Our definitions imply that for any cell F_α, there is a function of finite energy with support in this cell. If it can be approximated by piecewise harmonic functions, then $\nu(F_\alpha) > 0$. ∎

By Proposition 2.10 supp$(\nu) = F$ if and only if every cell has a positive measure.

Theorem 4.4   *Let F_ν be the factor-space (quotient) of F obtained by collapsing all cells of zero ν-measure. Then F_ν is a finitely ramified fractal with the cell and vertex structures naturally inherited from F.*

Proof   The only nontrivial condition to verify is that any cell of F_ν has at least two boundary points. The maximum principle implies that a cell F_α has a positive ν-measure if and only if there is a harmonic function which is non constant on V_α. ∎

Definition 4.5   If f is n-piecewise harmonic, then we define its tangent $\mathrm{Tan}_\alpha f$ for $\alpha \in \mathcal{A}_n$ as the unique element of $\ell(V_0)$ that satisfies two conditions:

(i)   if $h_{\alpha,\mathrm{Tan}}$ is the harmonic function with boundary values $\mathrm{Tan}_\alpha f$, then $h_{\alpha,\mathrm{Tan}}$ coincides with f on F_α;

(ii) $h_{\alpha,\mathrm{Tan}}$ has the smallest energy among all harmonic functions h_α such that h_α coincides with f on F_α.

We define L_Z^2 as the Hilbert space of $\ell(V_0)$-valued functions on F with the norm defined by $\|u\|_{L_Z^2}^2 = \int_F \langle u, Zu \rangle \, d\nu$.

Definition 4.6   If f is n-piecewise harmonic, then we define its gradient $\mathrm{Grad}\, f$ as the element of L_Z^2 such that, for ν-almost all x, $\mathrm{Grad}\, f(x) = \mathrm{Tan}_\alpha f$ in the sense of L_Z^2 if $x \in F_\alpha$ and $\alpha \in \mathcal{A}_n$.

Lemma 4.7   *If f is n-piecewise harmonic, then $\mathcal{E}(f, f) = \|\mathrm{Grad}\, f\|_{L_Z^2}^2$.*

Proof   Follows from Lemma 3.5. ∎

Theorem 4.8   *Under Assumption* (WN), Grad *can be extended from the space of piecewise harmonic functions to an isometry* $\mathrm{Grad}\colon \mathrm{Dom}\,\mathcal{E} \to \mathcal{L}_Z^2$, *which is called the* weak gradient.

Proof   The statement follows from Lemma 4.7 and Assumption (WN). ∎

Corollary 4.9   *Under Assumption* (WN), *we have $\nu_f \ll \nu$ for any $f \in \mathrm{Dom}\,\mathcal{E}$.*

Proof   The statement follows from Theorem 4.8. It can also be obtained directly from Assumption (WN) or the general theory of Dirichlet forms [9, 10]. ∎

Conjecture 4.10   We conjecture that the assumption supp$(\nu) = F$ is equivalent to Assumption (WN) for all finitely ramified fractals.

Conjecture 4.11   We conjecture that for any finitely ramified fractal rank $Z(x) = 1$ for ν-almost all x.

The next proposition follows easily from our definitions. It means, in particular, that Conjecture 4.11 implies Conjecture 4.10.

Proposition 4.12 *If* $\operatorname{supp}(\nu) = F$ *and* $\operatorname{rank} Z(x) = 1$ *for ν-almost all x, then Assumption* (WN) *holds.*

5 Gradient in Harmonic Coordiantes

To define harmonic coordinates one needs to choose a complete, up to constant functions, set of harmonic functions $h_1, \ldots, h_k$ and define the coordinate map $\psi\colon F \to \mathbb{R}^k$ by $\psi(x) = (h_1(x), \ldots, h_k(x))$. A particular choice of harmonic coordinates is not important since they are equivalent up to a linear change of variables. Below we fix the most standard coordinates which make the computations simpler.

Definition 5.1 Let $V_0 = \{v_1, \ldots, v_m\}$ and let h_j be the unique harmonic function with boundary values $h_j(v_i) = \delta_{i,j}$. Kigami's harmonic coordinate map $\psi\colon F \to \mathbb{R}^m$ is defined by $\psi(x) = (h_1(x), \ldots, h_m(x))$.

Lemma 5.2
(i) *Any set $\psi(F_\alpha)$ is contained in the convex hull of $\psi(V_\alpha)$.*
(ii) *A set $\psi(F_\alpha)$ has at least two points if and only if $\psi(V_\alpha)$ has at least two points.*
(iii) *If on $F_H = \psi(F)$ we define a cell structure that consists of all sets $\psi(F_\alpha)$ that have at least two points, then conditions* (i)–(v) *and* (vii) *of Definition* 2.1 *are satisfied.*
(iv) *If for all n and for any two distinct $\alpha, \alpha' \in \mathcal{A}_n$ we have*

$$\psi(F_{\alpha'}) \cap \psi(F_\alpha) = \psi(V_{\alpha'}) \cap \psi(V_\alpha),$$

then $F_H = \psi(F)$ is a finitely ramified fractal with the cell structure defined in (iii) *of this lemma.*

Proof The maximum principle implies that $\psi(F_\alpha)$ is contained in the convex hull of $\psi(V_\alpha)$, which implies the other statements. ∎

The next theorem easily follows from this lemma.

Theorem 5.3 $\psi\colon F \to F_H = \psi(F)$ *is a homeomorphism if and only if for any $\alpha \in \mathcal{A}$ the map $\psi|_{V_\alpha}$ is an injection, and $\psi(F_{\alpha'} \cap F_\alpha) = \psi(F_{\alpha'}) \cap \psi(F_\alpha)$ for all $\alpha, \alpha' \in \mathcal{A}$.*

Assumption (HC) In what follows we assume that $\psi\colon F \to F_H = \psi(F)$ is a homeomorphism.

Proposition 5.4 *Assumption (HC) implies Assumption (WN).*

Proof It is easy to see that under Assumption (HC), any cell has positive measure, and that any continuous function can be uniformly approximated by piecewise harmonic functions. The latter is true because all harmonic functions are linear in harmonic coordinates, and the maximum principle implies that $\psi(F_\alpha)$ is contained in the convex hull of $\psi(V_\alpha)$. ∎

Notation 5.5 In what follows, for simplicity, we assume $F = F_H$ and $\psi(x) = x$. Also, we identify $\ell(V_0)$ with $\mathbb{R}^m$ in the natural way.

Theorem 5.6 *Under Assumption* (HC)*, we have that if f is the restriction to F of a $C^1(\mathbb{R}^m)$ function, then $f \in \mathrm{Dom}\,\mathcal{E}$, and such functions are dense in $\mathrm{Dom}\,\mathcal{E}$. Moreover, if $f \in C^1(\mathbb{R}^m)$, then $\mathrm{Grad}\, f = \nabla f$ in the sense of the Hilbert space L_Z^2. In particular we have the* Kigami *formula*

$$\mathcal{E}(f,f) = \|\nabla f\|_{L_Z^2}^2 = \int_F \langle \nabla f, Z \nabla f \rangle \, d\nu$$

for any $f \in C^1(\mathbb{R}^m)$.

Proof In fact, we will prove this result for a somewhat larger space of functions. We say that f is a piecewise C^1-function if for some n and for all $\alpha \in \mathcal{A}_n$ there is $f_\alpha \in C^1(\mathbb{R}^m)$ such that $f_\alpha\big|_{F_\alpha} = f\big|_{F_\alpha}$. In particular, a piecewise harmonic function is piecewise C^1.

If g is a linear function in R^m, then $g|_{V_0} = \nabla g$ since we identify $\ell(V_0)$ with $\mathbb{R}^m$ in the natural way. Therefore for any piecewise harmonic function f we have $\mathrm{Grad}\, f = \nabla f$ in the sense of the Hilbert space L_Z^2.

Any C^1-function is a piecewise C^1-function, and any piecewise C^1-function can be approximated by piecewise harmonic (that is, piecewise linear) functions in C^1 norm. Thus, to complete the proof we need an estimate of the energy of a function in terms of its C^1 norm, provided by the next simple Lemma 5.7. ∎

Lemma 5.7 *If f is the restriction to F of a $C^1(\mathbb{R}^m)$ function, then*

$$(5.1) \qquad \mathcal{E}_n(f,f) \leqslant \nu(F)\|f\|_{C^1(\mathbb{R}^m)}^2$$

and the same estimate holds for $|\mathcal{E}(f,f)|$.

Proof By definition [19, 22] of $\mathcal{E}_n$, we have that

$$(5.2) \qquad \mathcal{E}_n(f,f) = \sum_{x,y \in V_n} c_{n,x,y}\big(f(x) - f(y)\big)^2$$

$$\leqslant \|f\|_{C^1(\mathbb{R}^m)}^2 \sum_{x,y \in V_n} c_{n,x,y}|x - y|^2 = \|f\|_{C^1(\mathbb{R}^m)}^2 \nu(F). \qquad ∎$$

Remark 5.8 Using Theorem 4.8 one can prove Theorem 5.6 using the general theory of Dirichlet forms in [9, 10] (see Remark 3.7). However, we give a constructive proof which also defines an approximating sequence to the gradient. A similar proof can be made along the lines of the proof of Theorem 6.1 using approximations by quantum graphs.

6 Energy Measure Laplacian in Harmonic Coordinates

The the energy measure Laplacian can be defined as follows. We say that $f \in \mathrm{Dom}\,\Delta_\nu$ if there exists a function $\Delta_\nu f \in L_\nu^2$ such that

$$(6.1) \qquad \mathcal{E}(f,g) = -\int_F g\Delta_\nu f d\nu,$$

for any function $g \in \mathrm{Dom}\,\mathcal{E}$ vanishing on the boundary V_0. By [23], the Laplacian Δ_ν is a uniquely defined linear operator with $\mathrm{Dom}\,\Delta_\nu \subset \mathrm{Dom}\,\mathcal{E}$. In fact $\mathrm{Dom}\,\Delta_\nu$ is $\mathcal{E}$-dense in $\mathrm{Dom}\,\mathcal{E}$, and is also dense in L_ν^2. The Laplacian Δ_ν is self-adjoint with, say, Dirichlet or Neumann boundary conditions. Formula (6.1) is often called the Gauss–Green formula. Extensive information on the relation of a Dirichlet form and its generator, the Laplacian, can be found in [9, 10, 22].

Theorem 6.1 *Under Assumption* (HC)*, we have that if f is the restriction to F of a $C^2(\mathbb{R}^m)$ function, then $f \in \mathrm{Dom}\,\Delta_\nu$, and such functions are $\mathcal{E}$-dense in $\mathrm{Dom}\,\Delta_\nu$. Moreover, ν-almost everywhere $\Delta_\nu f = \mathrm{Tr}(ZD^2 f)$ where $D^2 f$ is the matrix of the second derivatives of f.*

Proof We start with defining a different sequence of approximating energy forms. In various situations these forms are associated with so-called quantum graphs, photonic crystals and cable systems. If $f \in C^1(\mathbb{R}^m)$, then we define

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y \in V_n} c_{n,x,y} \mathcal{E}_{x,y}^Q(f,f)$$

where

$$\mathcal{E}_{x,y}^Q(f,f) = \int_0^1 \left(\frac{d}{dt} f(x(1-t)+ty)\right)^2 dt$$

is the integral of the square of the derivative

$$\frac{d}{dt} f(x(1-t)+ty) = \langle \nabla f(x(1-t)+ty), y-x \rangle$$

of f along the straight line segment connecting x and y. Thus $\mathcal{E}_{x,y}^Q(f,f)$ is the usual one dimensional energy of a function on a straight line segment. If f is linear, then $\mathcal{E}_{x,y}^Q(f,f) = (f(x) - f(y))^2$. Therefore if f is piecewise harmonic, then $\mathcal{E}_n^Q(f,f) = \mathcal{E}_n(f,f)$ for all large enough n. Also $\mathcal{E}_n^Q$ satisfies estimate (5.1). Therefore for any $C^1(R^m)$-function we have $\lim_{n\to\infty} \mathcal{E}_n^Q(f,f) = \mathcal{E}(f,f)$ by Theorem 5.6.

It is easy to see that if g is a $C^1(R^m)$-function vanishing on V_0 and f is a $C^2(R^m)$-function, then

$$\mathcal{E}_n^Q(f,g) = \sum_{x,y \in V_n} c_{n,x,y} \int_0^1 g(x(1-t) + ty)\left(\frac{d^2}{dt^2} f(x(1-t) + ty)\right) dt$$

because, after integration by parts, all the boundary terms are canceled. Then if $\alpha \in \mathcal{A}_n$,

$$\sum_{x,y \in V_\alpha} c_{n,x,y} \frac{d^2}{dt^2} f(x(1-t) + ty)$$

$$= \sum_{x,y \in V_\alpha} c_{n,x,y} \sum_{i,j=1}^m D_{ij}^2 f\big(x(1-t) + ty\big)(y_i - x_i)(y_j - x_j)$$

$$= \mathrm{Tr}\left(M_\alpha^* D_\alpha M_\alpha \big(D^2 f(x_\alpha) + R_n(x, y, t, f, \alpha, x_\alpha)\big)\right),$$

where $x_\alpha \in V_\alpha$ and $\lim_{n \to \infty} |R_n(x, y, t, f, \alpha, x_\alpha)| = 0$ uniformly in $\alpha \in \mathcal{A}_n$, $x, y, x_\alpha \in F_\alpha$, and $t \in [0, 1]$. This completes the proof. Note also that one can obtain an estimate similar to (5.1), as in Corollary 6.4. ∎

Corollary 6.2   *Under Assumption* (HC), $\Delta_\nu f \in L^\infty(F)$ *for any* $f \in C^2(\mathbb{R}^m)$.

Corollary 6.3   *Under Assumption* (HC), *if* $f(x) = \|x\|^2$, *then* $\Delta_\nu f = 1$.

Corollary 6.4   *If* f *is the restriction to* F *of a* $C^2(\mathbb{R}^m)$ *function, and* g *is the restriction to* F *of a* $C^1(\mathbb{R}^m)$ *function vanishing on the boundary, then*

$$|\mathcal{E}_n(f,g)| \leqslant \nu(F)\|g\|_{C(\mathbb{R}^m)}\|f\|_{C^2(\mathbb{R}^m)}$$

and the same estimate holds for $|\mathcal{E}(f,g)|$.

Proof   This estimate follows from the proof of Theorem 6.1. ∎

Remark 6.5   One can also obtain Theorem 6.1 from Theorem 5.6 using the general theory of Dirichlet forms in [9, 10] (see Remark 3.7). However we give a different constructive proof using the approximation by quantum graphs (see [27, 28]).

7   Topologically Self-Similar Finitely Ramified Fractals

Definition 7.1   A compact connected metric space F is called a *finitely ramified self-similar set* if there are injective contraction maps $\psi_1, \ldots, \psi_m \colon F \to F$ and a finite set $V_0 \subset F$ such that $F = \bigcup_{i=1}^m \psi_i(F)$ and for any n and for any two distinct words $w, w' \in W_n = \{1, \ldots, m\}^n$ we have $F_w \cap F_{w'} = V_w \cap V_{w'}$, where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. Here for a finite word $w = w_1 \cdots w_n \in W_n$ we denote

$$\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}.$$

The set V_0 is called the *vertex boundary* of F.

Proposition 7.2 *A finitely ramified self-similar set is a finitely ramified fractal, provided V_0 has at least two elements. We have $\mathcal{A}_n = W_n$ for $n \geqslant 1$ and $\mathcal{A} = \{0\} \cup W_*$, where $W_* = \bigcup_{n \geqslant 1} W_n$.*

Proof All items in Definition 2.1 are self-evident. Note that item (ii) holds because each cell is connected and has at least two elements, and the intersection of two cells is finite. Item (vii) holds because ψ_i are contractions. ∎

Remark 7.3 The question of existence of a "self-similar" metric on self-similar sets was recently studied in detail in [16]. According to [16], our class of self-similar finitely ramified fractals defined above is the same as finitely ramified SSH-fractals (with finite fractal boundary) of [16]. The definition of SSH-fractals in [16] requires fulfillment of a certain set of axioms, one of which is that the maps $\psi_1, \ldots, \psi_m \colon F \to F$ are continuous injections. It is then proved that F can be equipped with a self-similar metric in such a way that the injective maps ψ_j become contractions (as well as local similitudes), but the topology does not change. We use a simplified approach when we assume from the beginning that ψ_i are contractions.

In addition, it is proved in [16,17] that for every pcf self-similar set defined in [19, 22], there exists a self-similar metric. Therefore our definition of self-similar finitely ramified fractals generalizes the definition of pcf self-similar sets. Our definition allows infinitely many cells to meet at a junction point, which is referred to in [16] as fractals with "infinite multiplicity".

Note that, by definition, each ψ_i maps V_* into itself injectively.

Definition 7.4 A resistance form $\mathcal{E}$ on V_*, in the sense of Section 3, is *self-similar* with *energy renormalization factors* $\rho = (\rho_1, \ldots, \rho_m)$ if for any $f \in \mathrm{Dom}\,\mathcal{E}$ we have

$$(7.1) \qquad \mathcal{E}(f,f) = \sum_{i=1}^{m} \rho_i \mathcal{E}(f_i, f_i).$$

Here we use the notation $f_w = f \circ \psi_w$ for any $w \in W_*$.

The energy renormalization factors, or weights, $\rho = (\rho_1, \ldots, \rho_m)$ are often also called conductance scaling factors because of the relation of resistance forms and electrical networks. They are reciprocals of the resistance scaling factors $r_j = \frac{1}{\rho_j}$.

Definition 7.5 For a set of energy renormalization factors $\rho = (\rho_1, \ldots, \rho_m)$ and any resistance form $\mathcal{E}_0$ on V_0 define the resistance form $\Psi_\rho(\mathcal{E}_0)$ on V_1 by

$$\Psi_\rho(\mathcal{E}_0)(f,f) = \sum_{i=1}^{m} \rho_i \mathcal{E}_0(g_i, g_i),$$

where $g_i = f|_{\psi_i(V_0)} \circ \psi_i^{-1}$. Then $\Lambda(\mathcal{E}_0)$ is defined as the trace of $\Psi_\rho(\mathcal{E}_0)$ on V_0:

$$\Lambda(\mathcal{E}_0) = \mathrm{Trace}_{V_0} \Psi_\rho(\mathcal{E}_0).$$

The next two propositions are essentially proved in [22, 23, 33].

Proposition 7.6   *If $\mathcal{E}$ is self-similar, then $\mathcal{E}_0 = \Lambda(\mathcal{E}_0)$.*

Proposition 7.7   *If $\mathcal{E}_0$ is such that $\mathcal{E}_0 = \Lambda(\mathcal{E}_0)$, then there is a self-similar resistance form $\mathcal{E}$ such that $\mathcal{E}_0$ is the trace of $\mathcal{E}$ on V_0.*

Theorem 7.8   *On any self-similar finitely ramified fractal with a self-similar resistance form, all n-harmonic functions are continuous.*

Proof   By the self-similarity, it is enough to prove that the harmonic functions are continuous. Since all ψ_i are contractions, there is n such that any n-cell contains at most one point of V_0. By the strong maximum principle there is $\varepsilon > 0$ such that for any $w \in W_n$ and any harmonic function h we have

$$\left| \max_{x \in F} h(x) - \min_{x \in F} h(x) \right| \geqslant (1 - \varepsilon) \left| \max_{x \in F_w} h(x) - \min_{x \in F_w} h(x) \right|.$$

Then for any positive integer m and any $w \in W_{mn}$ we have

$$\left| \max_{x \in F} h(x) - \min_{x \in F} h(x) \right| \geqslant (1 - \varepsilon)^m \left| \max_{x \in F_w} h(x) - \min_{x \in F_w} h(x) \right|. \qquad \blacksquare$$

We conjecture that the results of [22, Section 3.3], and many other results of [22, 23] on the topology and analysis on pcf self-similar set hold for finitely ramified self-similar sets as well. The next theorem is one of these results. Following [19, 22], we say that the self-similar resistance form is *regular* if $\rho_i > 1$ for all i.

Theorem 7.9   *If a self-similar resistance form on a self-similar finitely ramified fractal F is regular, then $\Omega = F$.*

Proof   If $\mathrm{diam}_R(\cdot)$ denotes the diameter of a set in the effective resistance metric R, and $\rho_w = \rho_{w_1} \cdots \rho_{w_n}$ for any finite word $w = w_1 \cdots w_n \in W_n$, then

$$\mathrm{diam}_R(F) \geqslant \rho_w \, \mathrm{diam}_R(F_w)$$

by the self-similarity of the resistance form and the definition of the metric R. $\qquad \blacksquare$

Definition 7.10   The group G is said to act on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Proposition 7.11   *If a group G acts on a finitely ramified fractal F, then for each $g \in G$ and each n-cell F_α, $g(F_\alpha)$ is an n-cell.*

Proof   From the results of Section 2 we have that n-cells are connected, have pairwise disjoint interiors, and their topological boundaries are contained in V_n, which by definition is preserved by g. $\qquad \blacksquare$

Theorem 7.12 *Suppose a group G acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights and*

$$(7.2) \qquad \mathcal{E}_0(f, f) = \sum_{x,y \in V_0} (f(x) - f(y))^2.$$

Such a form is unique up to a constant.

Proof It is easy to see that, up to a constant, E_0 is the only G-invariant resistance form on V_0. Let $\rho_1 = (1, \ldots, 1)$. Then $\Lambda(\mathcal{E}_0)$ is also G-invariant and so $\mathcal{E}_0 = c\mathrm{Trace}_{V_0}\Psi_{\rho_1}(\mathcal{E}_0)$ for some c. Then the result holds for $\rho = c\rho_1$ by Proposition 7.6 and Proposition 7.7. ∎

An n-cell is called a boundary cell if it intersects V_0. Otherwise it is called an interior cell. We say that F has connected interior if the set of interior 1-cells is connected, any boundary 1-cell contains exactly one point of V_0, and the intersection of two different boundary 1-cells is contained in an interior 1-cell. The following theorem is proved in [13] for the pcf case, but the proof applies for self-similar finitely ramified fractals without any changes.

Theorem ([13]) *Suppose that F has connected interior, and a group G acts on a self-similar finitely ramified fractal F such that its action on V_0 is transitive. Then there exists a G-invariant self-similar resistance form $\mathcal{E}$.*

Other results in [13] also apply for self-similar finitely ramified fractals.

8 Examples

Example 8.1 *Unit interval:* the usual unit interval is a finitely ramified fractal. In this case V_* can be any countable dense subset of $[0,1]$ which includes $\{0, 1\}$. The usual energy form $\mathcal{E}(f, f) = \int_0^1 |f'(t)|^2 \, dt$ satisfies all the assumptions of our paper. The energy measure is the Lebesgue measure and the Laplacian is the usual second derivative.

Example 8.2 *Quantum graphs:* a quantum graph, a collection of finite number of points in $\mathbb{R}^m$ joined by weighted straight line segments (see [27, 28] and also the proof of Theorem 6.1), is a finitely ramified fractal. The usual energy form on a quantum graph, which is the sum of weighted standard one dimensional forms on each segment, satisfies all the assumptions of our paper.

Example 8.3 *Sierpiński gasket:* the Sierpiński gasket is a finitely ramified fractal. The standard energy form [18,19,22] on the Sierpiński gasket satisfies all the assumptions of our paper. The Sierpiński gasket in harmonic coordiantes (see Figure 1) was first considered in [20], where the statement of Theorem 5.6 was proved in this case.
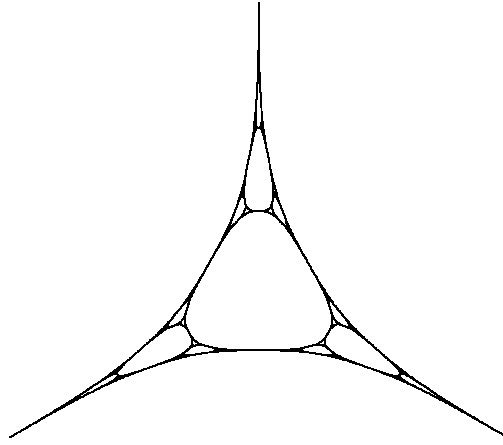
Figure 1: Sierpiński gasket in the standard harmonic coordinates.

The statement of Theorem 6.1 was announced in [42] without a proof. In the case of the standard energy form on the Sierpiński gasket Conjecture 4.11 was proved in [29]. The fact that the energy measure is singular with respect to any product (Bernoulli) measure was proved in [8, 14, 15, 29].
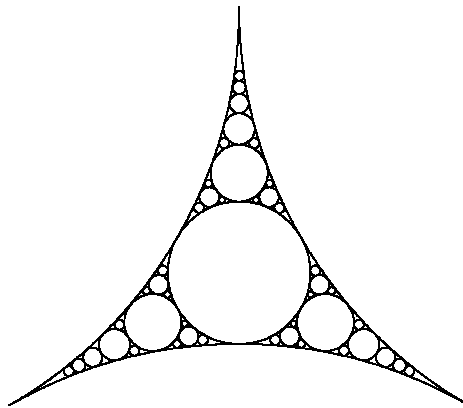

Figure 2: The residue set of the Apollonian packing.

Example 8.4   *The residue set of the Apollonian packing:* it was proved in [42] that the residue set of the Apollonian packing (see Figure 2) is the Sierpiński gasket in

harmonic coordinates defined by a non self-similar resistance form. This resistance form satisfies all the assumptions of our paper, including Assumption (HC).

Example 8.5 *Random Sierpiński gaskets:* in [35] a family of random Sierpiński gaskets was described using harmonic coordinates. Naturally, the results of this paper apply to these random gaskets, and Assumption (HC) is satisfied due to the way in which these gaskets are constructed. Also, many examples of random fractals in [11, 12] satisfy Assumption (HC), although the harmonic coordinates were not considered explicitly.
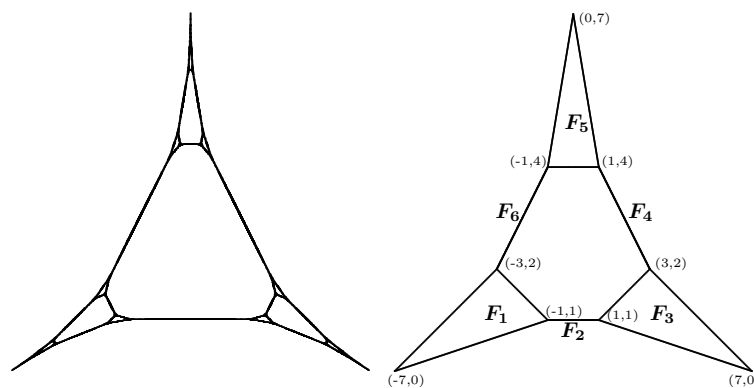
Figure 3: The hexagasket in harmonic coordinates and its first approximation.

Example 8.6 *Hexagasket:* according to [41], the hexagasket satisfies Assumption (WN) but not Assumption (HC). However, by small perturbations of the harmonic coordinates one can construct two functions of finite energy which map the hexagasket into $\mathbb{R}^2$ homeomorphically. Then the conclusion of Theorems 3.6 and 5.6 will hold because of the general theory of Dirichlet forms in [9, 10] (see Remark 3.7). However, Theorem 6.1 will not hold unless these coordinates are in the domain of the energy Laplacian, which is difficult to verify.

Example 8.7 *Quotients of pcf fractals:* if we consider a quotient of a pcf fractal defined by its space of harmonic functions, and conditions of Theorem 5.3 are satisfied (see also Theorem 4.4), then we have a finitely ramified fractal which satisfies Assumption (HC) by definition. In the case of the hexagasket this is illustrated in Figure 3. Note that this set is not self-affine. In harmonic coordinates the hexagasket is represented as a union of a Cantor set and a disjoint union of countably many closed straight line intervals. One can show that the energy measure of this Cantor set is zero, and in fact the energy measure is proportional to the Lebesgue measure on each segment. Note that in the limit no two intervals meet and so it is not a quantum graph, but can be called a generalized quantum graph. In this case a three point

boundary (see [39]) is chosen so that the resulting fractal can be embedded in $\mathbb{R}^2$. For a different choice of the boundary the local structure of the fractal in harmonic coordinates is the same.

Example 8.8　*Vicsek set:* a Vicsek set (see, for instance, [42]) is a finitely ramified fractal which does not satisfy the (WN) and (HC) assumptions. In harmonic coordinates it is represented by four straight line segments which are joined at a point. Therefore in our construction F_H is a quantum graph with five vertices and four edges, which is not homeomorphic to the Vicsek set.

Example 8.9　*Post-critically infinite Sierpiński gasket:* the post-critically infinite Sierpiński gasket is a finitely ramified fractal which has many properties of the Sierpiński gasket, but is not a pcf self-similar set. More exactly, its post-critical set defined in [19, 23] is countably infinite, and each vertex $v \in V_*$ is an intersection of countably many cells with pairwise disjoint interior. This fractal satisfies Definition 7.1 and can be constructed as a self-affine fractal in $\mathbb{R}^2$ using nine contractions, see Figure 4. In Figure 4 we also sketch the first approximation to it in harmonic coordiantes. In particular, Figure 4 shows the values of a symmetric and a skew-symmetric harmonic functions. By Theorem 7.12 one can easily construct a resistance form such that for any n the resistances are equal to $(50/53)^n$ in each triangle with vertices in V_n. The energy renormalization factor is $53/50 = \rho_1 = \cdots = \rho_9$. The fact that this factor is larger than one is significant because it implies that the harmonic structure is regular by Theorem 7.9, that is, $\Omega = F$. By Theorem 5.3, this resistance form satisfies all the assumptions, including Assumption (HC).

Example 8.10　In the end we describe two more examples of post-critically infinite finitely ramified fractals, which are shown in Figures 5 and 6. In these examples for any n there are n-cells which are joined in two points. Both fractals satisfy Definition 7.1 and can be constructed as a self-affine fractal in $\mathbb{R}^2$ using six contractions. In Figures 5 and 6 we also sketch the first approximations to these fractals in harmonic coordiantes. In particular, one can see the values of symmetric and skew-symmetric harmonic functions on each fractal. By Theorem 7.12 one can easily construct resistance forms such that E_0 is given by (7.2). By Theorem 5.3 these resistance forms satisfy Assumption (HC). In the case of the fractal in Figure 5, an elementary calculation shows that the common energy renormalization factor in (7.1) is $5/4$, and so the resistance form is regular. In the case of the fractal in Figure 6, the calculation shows that the common energy renormalization factor in (7.1) is $4/5$, and so the resistance form is non regular.

Remark 8.11　If the assumptions of Theorem 7.12 are satisfied and a Laplacian is defined with respect to the product (Bernoulli) measure that gives equal weight to all n-cells, then one can compute the spectrum of this Laplacian by the so called spectral decimation method of [1,32,40]. In particular, this can be done for the fractals shown in Figures 4, 5 and 6. Note, however, that the results of Section 6 are not applicable to such a Laplacian.
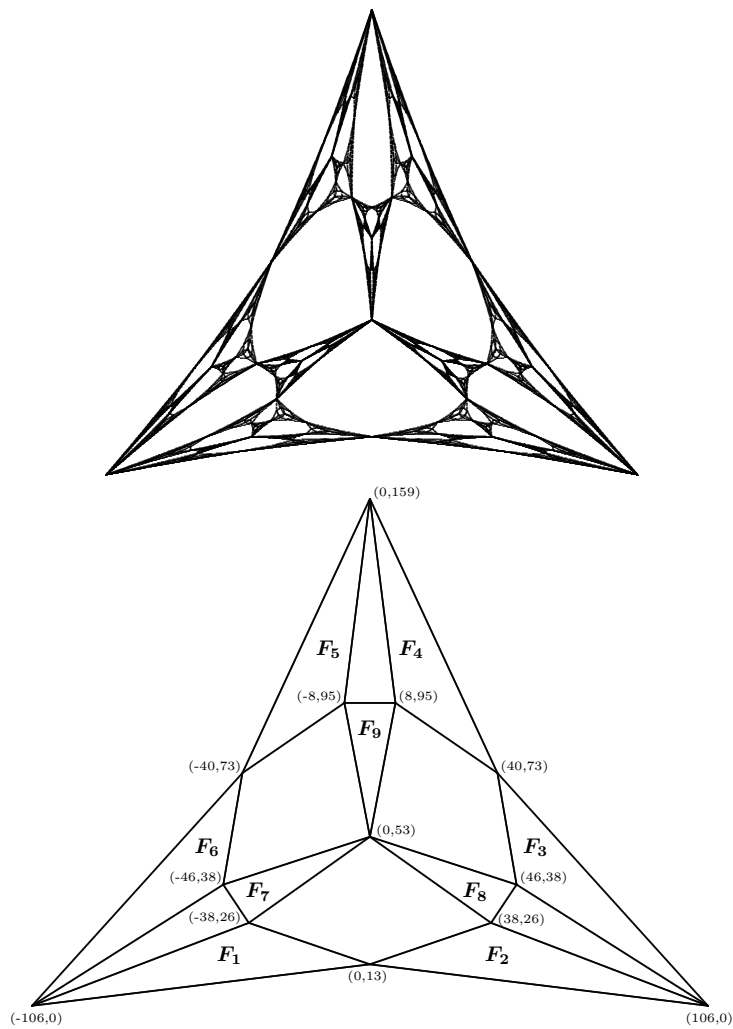
Figure 4: The post-critically infinite Sierpiński gasket in harmonic coordinates and its first approximation.
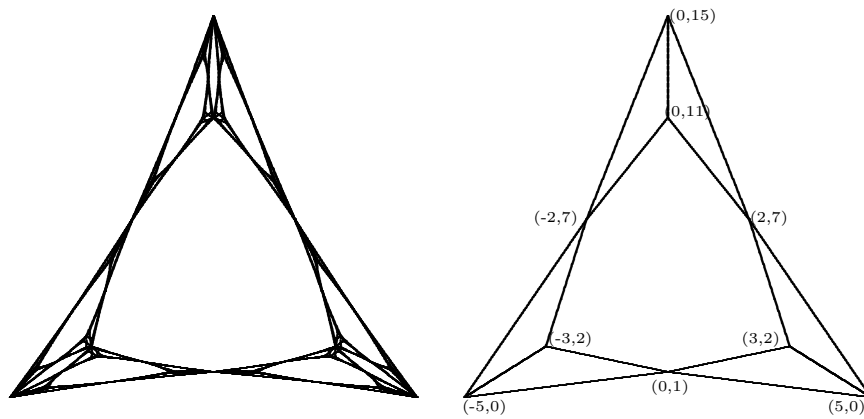
Figure 5: A regular post-critically infinite fractal and its first approximation.
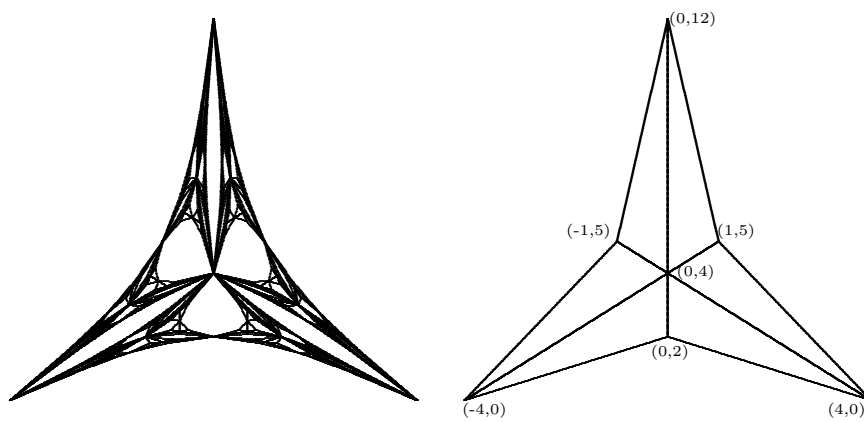

Figure 6: A non regular post-critically infinite fractal and its first approximation.

References

[1]   N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst and A. Teplyaev, *Vibration modes of 3n-gaskets and other fractals.* To appear in J. Phys. A: Math. Theor. http://www.math.uconn.edu/~teplyaev/research/

[2]   M. T. Barlow, *Diffusions on fractals.* In: Lectures on Probability Theory and Statistics. Lecture Notes in Math. 1690, Springer, Berlin, 1998, pp. 1–121.

[3]   M. T. Barlow and R. F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets.* Canad. J. Math. **51**(1999), no. 4, 673–744.

[4]   ———, *Random walks on graphical Sierpinski carpets.* In: Random Walks and Discrete Potential Theory. Sympos. Math.39, Cambridge University Press, Cambridge, 1999, pp. 26–55.

[5]   ———, *Stability of parabolic Harnack inequalities.* Trans. Amer. Math. Soc. **356**(2004), 1501–1533.

[6]   M. T. Barlow, R. F. Bass and T. Kumagai, *Stability of parabolic Harnack inequalities on metric measure spaces* J. Math. Soc. Japan, **58**(2006), no. 2, 485–619.

[7]   M. T. Barlow and B. M. Hambly, *Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets.* Ann. Inst. H. Poincaré Probab. Statist. **33**(1997), no. 5, 531–557.

[8]   O. Ben-Bassat, R. S. Strichartz, and A. Teplyaev, *What is not in the domain of the Laplacian on Sierpinski gasket type fractals.* J. Funct. Anal. **166**(1999), no. 2, 197–217.

[9]   N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space.* de Gruyter Studies in Mathematics 14, de Gruyter, Berlin, 1991.

[10]   M. Fukushima, Y. Oshima, and M. Takada, *Dirichlet forms and symmetric Markov processes.* de Gruyter Studies in Mathematics 19, de Gruyter, Berlin, 1994.

[11]   B. M. Hambly, *Heat kernels and spectral asymptotics for some random Sierpinski gaskets.* In: Fractal Geometry and Stochastics, II. Progr. Probab. 46, Birkhäuser, Basel, 2000, pp. 239–267.

[12]   ———, *On the asymptotics of the eigenvalue counting function for random recursive Sierpinski gaskets.* Probab. Theory Related Fields **117**(2000), no. 2, 221–247.

[13]   ———, *Self-similar energies on post-critically finite self-similar fractals.* J. London Math. Soc. **74**(2006), no. 1, 93–112.

[14]   M. Hino, *On singularity of energy measures on self-similar sets.* Probab. Theory Related Fields **132**(2005), no. 2, 265–290.

[15]   M. Hino and K. Nakahara, *On singularity of energy measures on self-similar sets. II.* Bull. London Math. Soc. **38**(2006), no. 6, 1019–1032.

[16]   K. Hveberg, *Injective mapping systems and self-homeomorphic fractals.* Ph.D. Thesis, University of Oslo, 2005.

[17]   A. Kameyama, *Distances on topological self-similar sets and the kneading determinants.* J. Math. Kyoto Univ. **40**(2000), 601–672.

[18]   J. Kigami, *A harmonic calculus on the Sierpiński spaces.* Japan J. Appl. Math. **6**(1989), no. 2, 259–290.

[19]   ———, *Harmonic calculus on p.c.f. self-similar sets.* Trans. Amer. Math. Soc. **335**(1993), no. 2, 721–755.

[20]   ———, *Harmonic metric and Dirichlet form on the Sierpiński gasket.* In: Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals 201–218, Pitman Res. Notes Math. Ser. 283, Longman Sci. Tech., Harlow, 1993, pp. 201–218.

[21]   ———, *Effective resistances for harmonic structures on p.c.f. self-similar sets.* Math. Proc. Cambridge Philos. Soc. **115**(1994), no. 2, 291–303.

[22]   ———, *Analysis on Fractals.* Cambridge Tracts in Mathematics 143, Cambridge University Press, 2001.

[23]   ———, *Harmonic analysis for resistance forms.* J. Funct. Anal. **204**(2003), no. 2, 399–444.

[24]   ———, *Local Nash inequality and inhomogeneity of heat kernels.* Proc. London Math. Soc. **89**(2004), no. 2, 525–544.

[25]   ———, *Volume doubling measures and heat kernel estimates on self-similar sets.* To appear in Memoirs of the American Mathematical Society. http://www-an.acs.i.kyoto-u.ac.jp/~kigami/preprints.html

[26]   ———, *Measurable Riemannian goemetry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate.* To appear in Matt. Ann. http://www-an.acs.i.kyoto-u.ac.jp/~kigami/preprints.html

[27] P. Kuchment, *Quantum graphs. I. Some basic structures.* Waves Random Media **14**(2004), no. 1, S107–S128.

[28] ———, *Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs.* J. Phys. A. **38**(2005), no. 22, 4887–4900.

[29] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* Publ. Res. Inst. Math. Sci. **25**(1989), no. 4, 659–680.

[30] ———, *Lecture on diffusion processes on nested fractals.* In: Statistical Mechanics and Fractals. Lecture Notes in Mathematics 1567, Springer-Verlag, Berlin, 1993, pp. 39–98,.

[31] S. Kusuoka and Z. X Yin, *Dirichlet forms on fractals: Poincaré constant and resistance.* Probab. Theory Related Fields **93** (1992), no. 2, 169–196.

[32] L. Malozemov and A. Teplyaev, *Self-similarity, operators and dynamics.* Math. Phys. Anal. Geom. **6**(2003), no. 3, 201–218.

[33] V. Metz, *The cone of diffusions on finitely ramified fractals.* Nonlinear Anal. **55** (2003), no. 6, 723–738.

[34] V. Metz and K.-T. Sturm, *Gaussian and non-Gaussian estimates for heat kernels on the Sierpiński gasket.* In: Dirichlet Forms and Stochastic Processes. de Gruyter, Berlin, 1995, pp. 283–289.

[35] R. Meyers, R. Strichartz, and A. Teplyaev, *Dirichlet forms on the Sierpinski gasket.* Pacific J. Math. **217**(2004), no. 1, 149–174

[36] R. S. Strichartz, *Analysis on fractals.* Notices Amer. Math. Soc **46**(1999), no. 10, 1199–1208.

[37] ———, *Taylor approximations on Sierpiński type fractals.* J. Funct. Anal. **174**(2000), no. 1, 76–127.

[38] ———, *Fractafolds based on the Sierpiński gasket and their spectra.* Trans. Amer. Math. Soc **355**(2003), no. 10, 4019–4043.

[39] ———, *Differential Equations on Fractals: A Tutorial.* Princeton University Press, Princeton, NJ, 2006.

[40] A. Teplyaev, *Spectral Analysis on Infinite Sierpiński Gaskets,* J. Funct. Anal. **159**(1998), no. 2, 537–567.

[41] ———, *Gradients on fractals.* J. Funct. Anal. **174**(2000), no. 1, 128–154.

[42] ———, *Energy and Laplacian on the Sierpiński gasket.* In: Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, Part 1. Proc. Sympos. Pure Math. 72, American Mathematical Society, Providence, RI, 2004, pp. 131–154.

Department of Mathematics
University of Connecticut
Storrs CT 06269-3009
U.S.A.
e-mail: teplyaev@math.uconn.edu