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Distinguishing endpoint sets from Erdős space

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Abstract

We prove that the set of all endpoints of the Julia set of $f(z) = \exp(z) - 1$ which escape to infinity under iteration of f is not homeomorphic to the rational Hilbert space $\mathfrak{E}$. As a corollary, we show that the set of all points $z \in \mathbb{C}$ whose orbits either escape to ∞ or attract to 0 is path-connected. We extend these results to many other functions in the exponential family.

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1. *Introduction*

The exponential family $f_a(z) = e^z + a$; $a \in \mathbb{C}$, is the most studied family of functions in the theory of the iteration of transcendental entire functions. For any parameter $a \in \mathbb{C}$, the Julia set $J(f_a)$ is known to be equal to the closure of the escaping set $I(f_a) := \{z \in \mathbb{C} : f_a^n(z) \to \infty\}$ [8]. And when a belongs to the Fatou set $F(f_a)$ (e.g. when $a \in (-\infty, -1]$), the Julia set $J(f_a)$ can be written as a union of uncountably many disjoint curves and endpoints [6, 9]. A point $z \in J(f_a)$ is on a *curve* if there exists an arc $\alpha : [-1, 1] \hookrightarrow I(f_a)$ such that $\alpha(0) = z$. A point $z_0 \in J(f_a)$ is an *endpoint* if z_0 is not on a curve and there is an arc $\alpha : [0, 1] \hookrightarrow J(f_a)$ with $\alpha(0) = z_0$ and $\alpha(t) \in I(f_a)$ for all $t \in (0, 1]$. The set of all endpoints of $J(f_a)$ is denoted $E(f_a)$.

The first to study the surprising topological properties of the endpoints was Mayer in 1988. He proved that ∞ is an explosion point for $E(f_a)$ for all attracting fixed point parameters $a \in (-\infty, -1)$ [16]. That is, $E(f_a)$ is totally separated but its union with ∞ is a connected set. A similarly paradoxical result is due to McMullen 1987 and Karpinska 1999: If $a \in (-\infty, -1)$ then the Hausdorff dimension of $E(f_a)$ is two [17], but the set of curves has Hausdorff dimension one [10].

Alhabib and Rempe extended Mayer's result in 2016 by focusing on the endpoints of $I(f_a)$. They proved that ∞ is an explosion point for the *escaping endpoint set*

$$\dot{E}(f_a) := E(f_a) \cap I(f_a),$$

as well as for $E(f_a)$, for every Fatou parameter $a \in F(f_a)$. The set of non-escaping endpoints $E(f_a) \setminus I(f_a)$ is very different in this regard. Its union with ∞ is totally separated [9] and zero-dimensional [13].

Conjugacy between escaping sets [19] implies that $\dot{E}(f_a)$ and $\dot{E}(f_b)$ are topologically equivalent when a and b are Fatou parameters. The primary aim of this paper is to distinguish the space $\dot{E}(f_a)$; $a \in F(f_a)$, from a certain line-free subgroup of the Hilbert space ℓ^2. For comparison, the entire endpoint set $E(f_a)$ is homeomorphic to *complete Erdős space* $\mathfrak{E}_c := \{\mathbf{x} \in \ell^2 : x_n \in \mathbb{R} \setminus \mathbb{Q}$ for each $n < \omega\}$ for every $a \in (-\infty, -1]$ [11]. The space $\dot{E}(f_a)$ is not homeomorphic to $\mathfrak{E}_c$ [12], but has many of the same topological properties as (the plain) *Erdős space*

$$\mathfrak{E} := \left\{\mathbf{x} \in \ell^2 : x_n \in \mathbb{Q} \text{ for each } n < \omega\right\}.$$

For example, $\dot{E}(f_a)$ and $\mathfrak{E}$ are both one-dimensional, almost zero-dimensional $F_{\sigma\delta}$-spaces which are nowhere $G_{\delta\sigma}$; see [12, 15]. Moreover $\dot{E}(f_a)$ contains a dense copy of $\mathfrak{E}$ in the form of all endpoints that escape to infinity in the imaginary direction [14]. However, in this paper we will show that the two spaces are not equivalent. This provides a negative answer to [12, question 1] and suggests that $\dot{E}(f_a)$ is a fundamental object between the 'rational and irrational Hilbert spaces' $\mathfrak{E}$ and $\mathfrak{E}_c$. It is still unknown whether $\dot{E}(f_a)$ is a topological group, or is at least homogeneous.

The key property distinguishing $\dot{E}(f_a)$ from $\mathfrak{E}$ will involve the notion of a C-set. A *C-set* in a topological space X is an intersection of clopen subsets of X. Note that for every rational number $q \in \mathbb{Q}$ the set $\{\mathbf{x} \in \mathfrak{E} : x_0 = q\}$ is a nowhere dense C-set in $\mathfrak{E}$ because the ℓ^2-norm topology on $\mathfrak{E}$ is finer than the zero-dimensional topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$. Thus $\mathfrak{E}$ can be written as a countable union of nowhere dense C-sets. On the other hand:

THEOREM 1·1. *If $a \in F(f_a)$ then $\dot{E}(f_a)$ cannot be written as a countable union of nowhere dense C-sets.*

COROLLARY 1·2. *If $a \in F(f_a)$ then $\dot{E}(f_a) \not\simeq \mathfrak{E}$.*

Our proof of Theorem 1·1 will involve constructing simple closed (Jordan) curves in $\mathbb{C}$ whose intersections with $J(f_{-1})$ are contained in $I(f_{-1})$.

THEOREM 1·3. *If $a \in (-\infty, -1]$ then each point of $\mathbb{C}$ can be separated from ∞ by a simple closed (Jordan) curve in $F(f_a) \cup I(f_a)$.*

Note that the curve in Theorem 1·3 avoids all non-escaping endpoints of $J(f_a)$.

COROLLARY 1·4. *If $a \in (-\infty, -1]$ then $F(f_a) \cup I(f_a)$ is path-connected.*

We observe that $F(f_{-1}) \cup I(f_{-1})$ is simply the set of all points $z \in \mathbb{C}$ such that $f^n_{-1}(z) \to 0$ or $f^n_{-1}(z) \to \infty$.

2. *Preliminaries*

2.1. *Outline of paper*

We will prove Theorem 1·1 and Corollary 1·2 for the particular mapping

$$f(z) := f_{-1}(z) = e^z - 1.$$

The proof for all other Fatou parameters will then follow from the conjugacy [19, theorem 1·2]. Theorem 1·3 and Corollary 1·4 will essentially follow from the case $a = -1$ as well.

Instead of working directly in the Julia set $J(f) \subset \mathbb{C}$, we will work in a topologically equivalent subset of $\mathbb{R}^2$ known as a brush. In this section we will give the definition of a brush and its connection to exponential Julia sets. We will then define a specific brush which is homeomorphic to $J(f)$, together with a mapping which models f.

In Section 3 we will prove three easy lemmas for the model mapping. In Section 4 we will show that certain endpoint sets in brushes cannot be written as countable unions of nowhere dense C-sets, given the existence of certain Jordan curves. In Section 5 we will prove Theorem 1·1 by showing that such curves exist with respect to our model of $\dot{E}(f)$. Theorem 1·1 is strengthened by Remark 6·2 in Section 6. Finally, the proofs of Theorem 1·3 and Corollary 1·4 are given in Section 7.

2.2. *Brushes and Cantor bouquets*

Let $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$. A *brush* is a closed subset of $\mathbb{R}^2$ of the form

$$B = \bigcup_{y \in Y} [t_y, \infty) \times \{y\},$$

where $Y \subset \mathbb{P}$ and $t_y \in \mathbb{R}$. The *endpoints* of B are the points $\langle t_y, y \rangle$.

When $a \in (-\infty, -1]$, the Julia set $J(f_a)$ is a *Cantor bouquet* [1, 2] which is homeomorphic to a brush with a dense set of endpoints. For all other Fatou parameters, $J(f_a)$ is a *pinched Cantor bouquet* that is homeomorphic to a brush modulo a closed equivalence relation on its set of endpoints; see [19, corollary 9·3] and [3]. The relation establishes a one-to-one correspondence between endpoints of $J(f)$ and equivalence classes of endpoints of B [13, section 3·2].

2.3. *Brush model of J(f)*

We will now define a mapping $\mathcal{F}$ and a brush $J(\mathcal{F})$ that model f and $J(\mathcal{F})$. These objects were studied extensively in [2, 19].

Let $\mathbb{Z}^\omega$ denote the space of integer sequences $\underline{s} = s_0 s_1 s_2 \cdots$ in the product (or lexicographic order) topology. Define $\mathcal{F} : [0, \infty) \times \mathbb{Z}^\omega \to \mathbb{R} \times \mathbb{Z}^\omega$ by

$$\langle t, \underline{s} \rangle \longmapsto \langle F(t) - 2\pi |s_0|, \sigma(\underline{s}) \rangle,$$

where $F(t) = e^t - 1$ and σ is the shift map on $\mathbb{Z}^\omega$; i.e.

$$\sigma(s_0 s_1 s_2 \cdots) = s_1 s_2 s_3 \cdots .$$

For each $x = \langle t, \underline{s} \rangle \in [0, \infty) \times \mathbb{Z}^\omega$ put $T(x) = t$ and $\underline{s}(x) = \underline{s}$. The integer sequence $\underline{s}(x)$ is called the *external address* of x.

Define

$$J(\mathcal{F}) = \left\{ x \in [0, \infty) \times \mathbb{Z}^\omega : T\left(\mathcal{F}^n(x)\right) \geq 0 \text{ for all } n \geq 0 \right\}.$$

Continuity of $\mathcal{F}$ implies that $J(\mathcal{F})$ is closed in $[0, \infty) \times \mathbb{Z}^\omega$. Now let

$$\mathbb{S} = \left\{ \underline{s} \in \mathbb{Z}^\omega : \text{ there exists } t \geq 0 \text{ such that } \langle t, \underline{s} \rangle \in J(\mathcal{F}) \right\},$$

and for each $\underline{s} \in \mathbb{S}$ put $t_{\underline{s}} = \min\{t \geq 0 : \langle t, \underline{s} \rangle \in J(\mathcal{F})\}$. Then

$$J(\mathcal{F}) = \bigcup_{\underline{s} \in \mathbb{S}} [t_{\underline{s}}, \infty) \times \{\underline{s}\}.$$

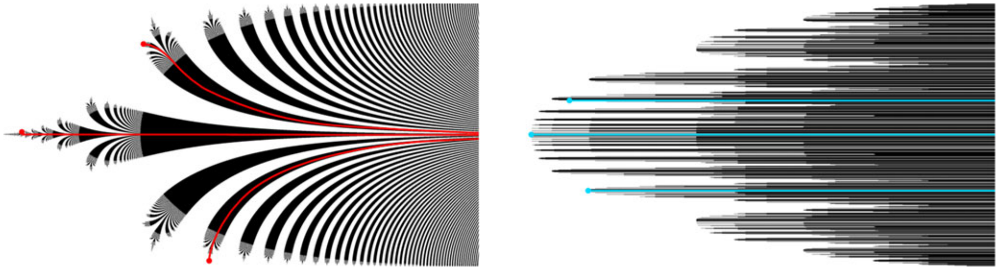

Fig. 1. Three endpoints and curves in $J(f)$ (left) and in $J(\mathcal{F})$ (right).

Finally, $\mathbb{Z}^\omega$ can be identified with $\mathbb{P}$ using an order isomorphism between $\mathbb{Z}^\omega$ in the lexicographic ordering and $\mathbb{P}$ in the real ordering [2, observation 3·2]. Under this identification, $J(\mathcal{F}) \subset \mathbb{R} \times \mathbb{P}$. Hence $J(\mathcal{F})$ is a brush. See Figure 1.

The restrictions $\mathcal{F} \restriction J(\mathcal{F})$ and $f \restriction J(f)$ are topologically conjugate [19, section 9]. Hence $J(\mathcal{F})$ is homeomorphic to $J(f)$, and the action of $\mathcal{F}$ on $J(\mathcal{F})$ captures the essential dynamics of f on $J(f)$; see also [2, p.74].

3. *Lemmas for $\mathcal{F}$*

In order to prove Theorem 1·1 we need a few basic lemmas regarding the model above. Define $F^{-1}(t) = \ln(t+1)$ for $t \geq 0$, so that F^{-1} is the inverse of F. For each $n \geq 1$, the n-fold composition of F^{-1} is denoted F^{-n}.

LEMMA 3·1 (Iterating between double squares) *Let $x \in J(\mathcal{F})$. If $k \geq 5$ and*

$$T\left(\mathcal{F}^{2k^2}(x)\right) \geq F^{k^2}(1),$$

then $T(\mathcal{F}^n(x)) \geq F^k(1)$ for all $n \in [2(k-1)^2, 2k^2]$.

Proof. Suppose $k \geq 5$ and $T(\mathcal{F}^{2k^2}(x)) \geq F^{k^2}(1)$. Let $n \in [2(k-1)^2, 2k^2]$. Then there exists $i \leq 4k-2$ such that $n = 2k^2 - i$. We have

$$F^i\big(T(\mathcal{F}^n(x))\big) = F^i\left(T\left(\mathcal{F}^{2k^2-i}(x)\right)\right) \geq T\left(\mathcal{F}^i\left(\mathcal{F}^{2k^2-i}(x)\right)\right) = T\left(\mathcal{F}^{2k^2}(x)\right) \geq F^{k^2}(1).$$

Applying F^{-i} to each side of the inequality shows that

$$T\big(\mathcal{F}^n(x)\big) \geq F^{k^2-i}(1) > F^k(1),$$

where we used the fact $k^2 - i > k$ for all $k \geq 5$ and $i \leq 4k-2$.

LEMMA 3·2 (Forward stretching). *Let $x, y \in J(\mathcal{F})$. If $\underline{s}(y) = \underline{s}(x)$ and $T(y) > T(x)$, then for every $n \geq 1$ we have $T(\mathcal{F}^n(y)) \geq F^n(T(y) - T(x))$.*

Proof. Suppose $\underline{s}(y) = \underline{s}(x)$ and $T(y) > T(x)$. Let $\varepsilon = T(y) - T(x)$ and $\delta = T(\mathcal{F}^n(y)) - T(\mathcal{F}^n(x))$. Note that $\delta > 0$. So $T(y) \leq T(x) + F^{-n}(\delta)$ by [2, observation 3·9]. Therefore $\delta \geq F^n(\varepsilon)$. We have

$$T\big(\mathcal{F}^n(y)\big) \geq T\big(\mathcal{F}^n(y)\big) - T\big(\mathcal{F}^n(x)\big) = \delta \geq F^n(\varepsilon) = F^n(T(y) - T(x))$$

as desired.

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Lemma 3·1 implies that $T(\mathcal{F}^n(x)) \to \infty$ if the double square iterates $T(\mathcal{F}^{2k^2}(x))$ increase at a sufficient rate. Lemma 3·2 will be used to show that $T(\mathcal{F}^{2k^2}(x))$ increases at that rate for certain points $x \in J(\mathcal{F})$.

LEMMA 3·3 (Logarithmic orbit). *$F^{-n}(1) < 3/n$ for all $n \geq 1$.*

Proof. The proof is by induction on n. If $n = 1$ then we have $F^{-n}(1) = F^{-1}(1) = \ln(2) < 3 = 3/n$. Now suppose the inequality holds for a given n. Then

$$F^{-(n+1)}(1) = F^{-1}(F^{-n}(1)) < F^{-1}(3/n) = \ln\left(\frac{3}{n} + 1\right) < \frac{3}{n+1}$$

by calculus.

Although the series $\sum_{k=1}^{\infty} F^{-k}(1)$ diverges, Lemma 3·3 implies that $\sum_{k=1}^{\infty} F^{-k^2}(1)$ converges (to something less than $\pi^2/2$).

4. *Jordan curve lemma*

The following topological lemma will also be required to prove Theorem 1·1. Recall that a *C-set* in a space X is an intersection of clopen subsets of X.

LEMMA 4·1. *Let*

$$B = \bigcup_{y \in Y} [t_y, \infty) \times \{y\}$$

be a brush, and let $E = \{\langle t_y, y\rangle : y \in Y\}$ denote the set of endpoints of B. Suppose $\tilde{E} \subset E$ and $\tilde{E} \cup \{\infty\}$ is connected. If there exists an open set $U \subset \mathbb{R}^2$ such that:

(i) *$U \cap \tilde{E} \neq \varnothing$;*

(ii) *∂U is a simple closed curve; and*

(iii) *$\partial U \cap E \subset \tilde{E}$,*

then $\tilde{E}$ cannot be written as a countable union of nowhere dense C-sets.

Proof. Suppose that $\tilde{E}$, U, and $\beta := \partial U$ satisfy all of the hypotheses. Let

$$A := \{y \in Y : \beta \text{ contains an interval of } [t_y, \infty) \times \{y\}\}.$$

Since every collection of pairwise disjoint arcs of a simple closed curve is countable, A is countable. So $\tilde{E} \cap (\mathbb{R} \times A)$ is countable. Observe also that since B is closed in $\mathbb{R} \times Y$ and $Y \subset \mathbb{P}$, the space E has a neighbourhood basis of C-sets of the form $E \cap (-\infty, t] \times W$, where $t \in \mathbb{R}$ and W is clopen in Y. By [4, theorem 4·7] and the assumption that $\tilde{E} \cup \{\infty\}$ is connected, we see that $\tilde{E} \setminus (\mathbb{R} \times A) \cup \{\infty\}$ is connected.

Now aiming for a contradiction, suppose that $\tilde{E} = \bigcup\{C_n : n < \omega\}$ where each C_n is a nowhere dense C-set in $\tilde{E}$. Let V be the unbounded component of $\mathbb{R}^2 \setminus \beta$. Since $\tilde{E} \setminus (\mathbb{R} \times A)$ $\cup\{\infty\}$ is connected,

$$\tau := \overline{U \cap \tilde{E}} \cap \overline{V \cap \tilde{E}} \cap \tilde{E} \setminus (\mathbb{R} \times A)$$

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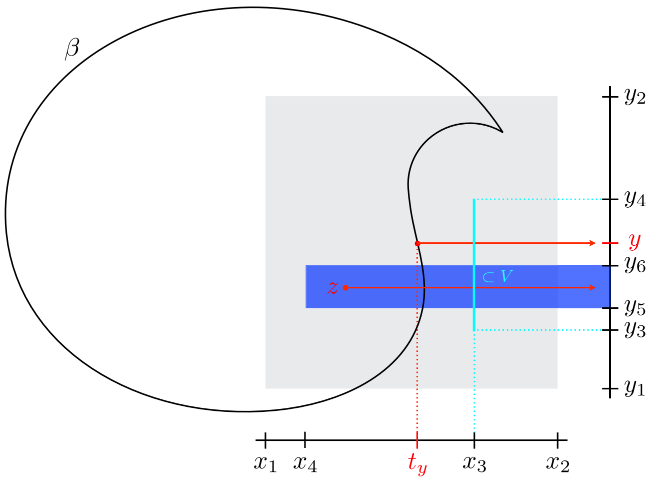

Fig. 2. Construction of clopen set in Lemma 4·1.

is non-empty. Note also that τ is a relatively closed subset of

$$\beta \cap \tilde{E} \setminus (\mathbb{R} \times A) = \beta \cap E \setminus (\mathbb{R} \times A),$$

which is a G_δ-subset of E. And E is a G_δ-subset of $\mathbb{R} \times \mathbb{P}$ because B is closed and $B \setminus E$ is the union of countably many closed sets $B + \langle 1/n, 0\rangle$. Therefore τ is completely metrisable. By Baire's theorem there is an open rectangle $(x_1, x_2) \times (y_1, y_2)$ and $n < \omega$ such that $\varnothing \neq \tau \cap ((x_1, x_2) \times (y_1, y_2)) \subset C_n$. Let

$$\langle t_y, y\rangle \in \tau \cap ((x_1, x_2) \times (y_1, y_2)).$$

See Figure 2.

Since $y \notin A$, there exist $x_3 \in (t_y, x_2)$ and $y_3, y_4 \in (y_1, y_2) \cap \mathbb{Q}$ such that $y_3 < y < y_4$ and $\{x_3\} \times [y_3, y_4] \subset \mathbb{R}^2 \setminus \beta$. Without loss of generality, assume $\{x_3\} \times [y_3, y_4] \subset V$. Note that $U \cap \tilde{E} \cap ((x_1, x_3) \times (y_3, y_4)) \neq \varnothing$ because

$$\langle t_y, y\rangle \in \overline{U \cap \tilde{E}} \cap ((x_1, x_3) \times (y_3, y_4)).$$

Further, $C_n \cup (\tilde{E} \cap (\mathbb{R} \times A))$ is nowhere dense in $\tilde{E}$, so there exists

$$z \in U \cap \tilde{E} \cap ((x_1, x_3) \times (y_3, y_4)) \setminus (C_n \cup (\mathbb{R} \times A)).$$

Since z is an endpoint and B is closed in $\mathbb{R} \times Y$, we can find $y_5, y_6 \in (y_3, y_4) \cap \mathbb{Q}$ and $x_4 > x_1$ such that $y_5 < y_6$, $z \in (x_4, \infty) \times (y_5, y_6)$, and

$$(\{x_4\} \times [y_5, y_6]) \cup ([x_4, \infty) \times \{y_5, y_6\}) \cap B = \varnothing.$$

Further, since $z \notin C_n$ there is a relatively clopen subset O of $\tilde{E} \setminus (\mathbb{R} \times A)$ such that $z \in O$ and $C_n \cap O = \varnothing$. Then

$$U \cap O \cap ([x_4, x_3] \times [y_5, y_6])$$

is a non-empty bounded clopen subset of $\tilde{E} \setminus (\mathbb{R} \times A)$. This contradicts the previously established fact that $\tilde{E} \setminus (\mathbb{R} \times A) \cup \{\infty\}$ is connected.

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5. *Proof of Theorem 1·1*

As indicated in Section 2.2, we will identify $\mathbb{Z}^\omega$ with $\mathbb{P}$, so that

$$J(\mathcal{F}) \subset \mathbb{R} \times \mathbb{P} \subset \mathbb{R}^2.$$

Let $E(\mathcal{F}) = \{\langle t_{\underline{s}}, \underline{s}\rangle : s \in \mathbb{S}\}$ denote the set of endpoints of $J(\mathcal{F})$. Define

$$I(\mathcal{F}) = \{x \in J(\mathcal{F}) : T(\mathcal{F}^n(x)) \longrightarrow \infty\} \text{ and}$$
$$\tilde{E}(\mathcal{F}) = I(\mathcal{F}) \cap E(\mathcal{F}).$$

The conjugacy [19, theorem 9·1] shows that $\tilde{E}(\mathcal{F}) \simeq \dot{E}(f)$, and $\tilde{E}(\mathcal{F}) \cup \{\infty\}$ is connected by [2, theorem 3·4]. Thus to reach the conclusion that $\dot{E}(f)$ cannot be written as a countable union of nowhere dense C-sets, by Lemma 4·1 we only need to find a simple closed curve $\beta \subset \mathbb{R}^2$ such that:

(i) if U is the bounded component of $\mathbb{R}^2 \setminus \beta$ then $U \cap \tilde{E}(\mathcal{F}) \neq \varnothing$, and

(ii) $\beta \cap J(\mathcal{F}) \subset I(\mathcal{F})$ (in particular, $\beta \cap E(\mathcal{F}) \subset \tilde{E}(\mathcal{F})$).

The β that we construct will essentially be the boundary of a union of rectangular regions in $\mathbb{R}^2$. We recursively define the collections of rectangles (or "boxes") as follows. Choose n sufficiently large so that $(-n, n)^2 \cap \tilde{E}(\mathcal{F}) \neq \varnothing$. For the sake of simplicity, let us assume $n = 1$. Let $\mathscr{B}_0 = \{[-1, 1]^2\}$.

CLAIM 5·1. *There is a sequence of finite collections of boxes* $\mathscr{B}_1, \mathscr{B}_2, \ldots$ *such that for every* $k \geq 1$ *and* $B \in \mathscr{B}_k$:

(i) $B = [a, b] \times [c, d]$ *for some* $a, b \in \mathbb{R}$ *and* $c, d \in \mathbb{Q}$ *with* $a < b$ *and* $c < d$*;*

(ii) $b - a = F^{-k^2}(1)$*;*

(iii) $d - c \leq F^{-k^2}(1)$*;*

(iv) *there exists* $B' \in \mathscr{B}_{k-1}$ *such that* $\{a\} \times [c, d] \subset \{b'\} \times [c', d']$;

(v) $\{a\} \times [c, d] \cap J(\mathcal{F}) \neq \varnothing$*;*

(vi) $B \cap B^* = \varnothing$ *for all* $B^* \in \mathscr{B}_k \setminus \{B\}$*;*

(vii) *if* $\underline{s}, \hat{\underline{s}} \in [c, d] \cap \mathbb{Z}^\omega$ *then* $\underline{s} \restriction 2k^2 = \hat{\underline{s}} \restriction 2k^2$ *(i.e.* $s_i = \hat{s}_i$ *for all* $i < 2k^2$*); and*

(viii) *for every* $B' \in \mathscr{B}_{k-1}$ *and* $x \in J(\mathcal{F}) \cap \{b'\} \times [c', d']$ *there exists* $B \in \mathscr{B}_k$ *such that* $x \in B$*.*

Proof. Suppose $\mathscr{B}_{k-1}$ has already been defined. Let

$$K = \bigcup_{B' \in \mathscr{B}_{k-1}} J(\mathcal{F}) \cap \{b'\} \times [c', d'].$$

Since $J(\mathcal{F})$ is closed in $\mathbb{R}^2$ [2, theorem 3·3] and $\mathscr{B}_{k-1}$ is finite, K is compact. Given a segment of integers $\langle s_0, s_1, ..., s_{2k^2-1}\rangle$ of length $2k^2$, observe that the set $\{\underline{s} \in \mathbb{Z}^\omega : \underline{s} \restriction 2k^2 = \langle s_0, s_1, ..., s_{2k^2-1}\rangle\}$ is clopen and convex in the lexicographic ordering on $\mathbb{Z}^\omega$, and thus corresponds to the intersection of $\mathbb{P}$ with an interval whose endpoints are in $\mathbb{Q}$. By compactness of K, the projection

$$\pi_1[K] = \{y \in \mathbb{R} : [0, \infty) \times \{y\} \cap K \neq \varnothing\}$$

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can be covered by finitely many of those intervals. Let $\mathcal{I}$ be a finite cover of $\pi_1[K]$ consisting of such intervals. Let

$$-1 = q_0 < q_1 < q_2 < \cdots < q_n = 1$$

be the increasing enumeration of all c's and d's used in $\mathscr{B}_{k-1}$, together with the endpoints of all intervals in $\mathcal{I}$. If necessary, we can insert a few more rationals to obtain $q_{i+1} - q_i \leq F^{-k^2}(1)$ for each $i < n$. Since $\pi_1[K]$ is a compact set missing $\mathbb{Q}$, we can also guarantee that for every $i < n-1$, $[q_i, q_{i+1}] \cap \pi_1[K] = \varnothing$ or $[q_{i+1}, q_{i+2}] \cap \pi_1[K] = \varnothing$.

Note that b' is the same for each box $B' \in \mathscr{B}_{k-1}$ (in fact, $b' = 1 + \sum_{i=1}^{k-1} F^{-i^2}(1)$). If $\{b'\} \times [q_i, q_{i+1}] \cap J(\mathcal{F}) \neq \varnothing$ then add the box $[b', b' + F^{-k^2}(1)] \times [q_i, q_{i+1}]$ to the collection $\mathscr{B}_k$. It can be easily shown that defining $\mathscr{B}_k$ in this manner will satisfy conditions (i) through (viii).

Let $\mathscr{B} = \bigcup\{\mathscr{B}_k : k < \omega\}$. For each $B = [a, b] \times [c, d] \in \mathscr{B}$ let

$$h_B : \{a\} \times [c, d] \longrightarrow ([a, b] \times \{c, d\}) \cup (\{b\} \times [c, d])$$

be a homeomorphism with fixed points $\langle a, c\rangle$ and $\langle a, d\rangle$.

CLAIM 5·2. *There is a continuous mapping*

$$g : [0, 1] \longrightarrow [1, \infty) \times [-1, 1]$$

such that $g(0) = \langle 1, -1\rangle$, $g(1) = \langle 1, 1\rangle$, *and* $g[0, 1] \cap J(\mathcal{F}) \subset I(\mathcal{F})$.

Proof. The mapping g will be the pointwise limit of a uniformly Cauchy sequence of continuous functions. To begin, define $g_0(t) = \langle 1, 2t - 1\rangle$ for all $t \in [0, 1]$. Now suppose $k \geq 1$ is given and g_{k-1} has been defined. Let $t \in [0, 1]$. We set

$$g_k(t) = g_{k-1}(t)$$

if $g_{k-1}(t)$ is not in any member of $\mathscr{B}_k$. Otherwise, by item (vi) there is a unique box $B = [a, b] \times [c, d] \in \mathscr{B}_k$ which contains $g_{k-1}(t)$. Inductively $g_{k-1}(t)$ is contained in some element of $\mathscr{B}_0 \cup \ldots \cup \mathscr{B}_{k-1}$, so by (iv) we have $g_{k-1}(t) \in \{a\} \times [c, d]$. Put

$$g_k(t) = h_B(g_{k-1}(t)).$$

The mapping g_k defined in this manner is easily seen to be continuous.

Note that the image of g_k is contained in $[1, \infty) \times [-1, 1]$, $g_k(0) = \langle 1, -1\rangle$, and $g_k(1) = \langle 1, 1\rangle$. To see that the sequence (g_k) is uniformly Cauchy, fix $\varepsilon > 0$. Let N be such that $\sum_{k=N}^{\infty} 1/k^2 < \varepsilon/5$. For any $t \in [0, 1]$, if $g_k(t) \neq g_{k-1}(t)$ then $g_k(t)$ and $g_{k-1}(t)$ belong to the same element of $\mathscr{B}_k$. Hence, by items (ii) and (iii), the distance between $g_k(t)$ and $g_{k-1}(t)$ is at most $\sqrt{2}F^{-k^2}(1)$. Combined with Lemma 3·3 we have

$$|g_k(t) - g_{k-1}(t)| \leq \sqrt{2}F^{-k^2}(1) < \sqrt{2}\frac{3}{k^2} < \frac{5}{k^2}$$

for every $k < \omega$ and $t \in [0, 1]$. Thus for any $i > j \geq N$ and $t \in [0, 1]$,

$$|g_i(t) - g_j(t)| \leq \sum_{k=j}^{i-1} |g_{k+1}(t) - g_k(t)| < \sum_{k=N}^{\infty} \frac{5}{k^2} < \varepsilon.$$

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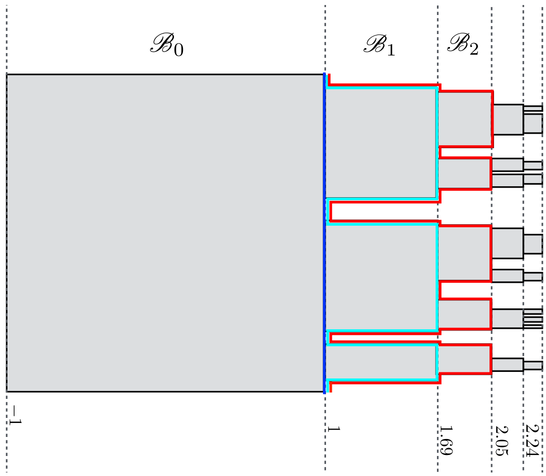

Fig. 3. Illustration of g_0, g_1 and g_2.

This proves that (g_k) is uniformly Cauchy. Therefore (g_k) converges uniformly to a continuous function $g : [0, 1] \to [1, \infty) \times [-1, 1]$ which satisfies $g(0) = \langle 1, -1 \rangle$ and $g(1) = \langle 1, 1 \rangle$.

It remains to show that $g[0, 1] \cap J(\mathcal{F}) \subset I(\mathcal{F})$. Let $x = g(t) \in g[0, 1] \cap J(\mathcal{F})$. By Lemma 3·1 and the fact $F^k(1) \to \infty$, to prove $x \in I(\mathcal{F})$ it suffices to show $T(\mathcal{F}^{2j^2}(x)) \geq F^{j^2}(1)$ for all $j \geq 1$. Fix $j \geq 1$. To prove $T(\mathcal{F}^{2j^2}(x)) \geq F^{j^2}(1)$, by continuity of $T \circ \mathcal{F}^{2j^2}$ we only need to show that every neighbourhood of x contains a point w such that $T(\mathcal{F}^{2j^2}(w)) \geq F^{j^2}(1)$.

Let W be a neighbourhood of x. Observe that since c and d are rational in item (i), the top and bottom edges of all boxes miss $J(\mathcal{F})$. So if $g_k(t) \in J(\mathcal{F})$, then $g_k(t)$ belongs to the right edge of an element of $\mathscr{B}_k$. Then by the construction of g_{k+1} and items (iv) and (viii) we get $g_{k+1}(t) \neq g_k(t)$. Since $g_k(t) \to x$, it follows that the sequence $g_0(t), g_1(t), \ldots$ is not eventually constant. So for every k there exists $B \in \mathscr{B}_k$ such that $g_k(t) \in B$. By (ii) and (iii), the diameter of each box in $\mathscr{B}_k$ is at most $\sqrt{2}F^{-k^2}(1)$, which goes to 0 as $k \to \infty$ (e.g. by Lemma 3·3). Since $g_k(t) \to x$ and x belongs to the interior of W, there exists $k > j$ such that W contains a box $B(k) \in \mathscr{B}_k$. By (v) there exists $w \in B(k) \cap W \cap J(\mathcal{F})$.

By (iv), for each $i < k$ there exists a unique $B(i) \in \mathscr{B}_i$ such that $\pi_1[B(k)] \subset \pi_1[B(i)]$. By (iv) and (v) there exists $y \in B(j) \cap B(j-1) \cap J(\mathcal{F})$. Then y is on the left edge of $B(j)$. Let z be the point on the right edge of $B(j)$ such that $\underline{s}(z) = \underline{s}(y)$ (see Figure 3). Then $z \in J(\mathcal{F})$, and by (i) we have $T(z) - T(y) = F^{-j^2}(1)$. Thus, by Lemma 3·2

$$T\left(\mathcal{F}^{2j^2}(z)\right) \geq F^{2j^2}(T(z) - T(y)) = F^{2j^2}\left(F^{-j^2}(1)\right) = F^{j^2}(1).$$

Note also that $\underline{s}(z) \restriction 2j^2 = \underline{s}(w) \restriction 2j^2$ because z and w are contained in the same horizontal strip where the first $2j^2$ coordinates agree; see items (vii) and (iv). The preceding equations together with $T(z) \leq T(w)$ imply that

$$T\left(\mathcal{F}^{2j^2}(w)\right) \geq F^{j^2}(1),$$

as desired.

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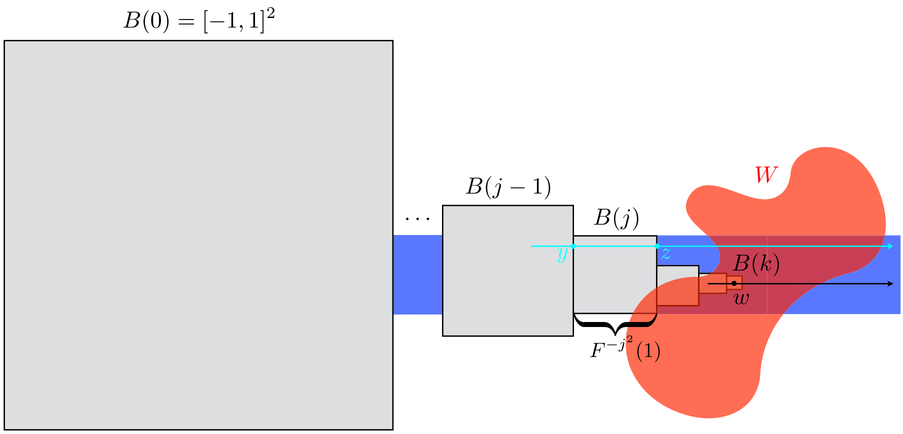

Fig. 4. Proof of Claim 5·2.

By [18, corollary 8·17] and [18, theorem 8·23], there is an arc $\alpha : [0,1] \hookrightarrow g[0,1]$ such that $\alpha(0) = g(0) = \langle 1, -1\rangle$ and $\alpha(1) = g(1) = \langle 1, 1\rangle$. Let $\beta = (\{-1\} \times [-1,1]) \cup ([-1,1] \times \{-1,1\}) \cup \alpha[0,1]$. Then β is a simple closed curve, and by Claim 5·2 we have $\beta \cap E(\mathcal{F}) \subset \tilde{E}(\mathcal{F})$. If U is the bounded component of $\mathbb{R}^2 \setminus \beta$ then $(-1,1)^2 \subset U$, hence $U \cap \tilde{E}(\mathcal{F}) \neq \varnothing$. Now we can apply Lemma 4·1 to see that $\tilde{E}(\mathcal{F})$ cannot be written as a countable union of nowhere dense C-sets. This concludes the proof of Theorem 1·1.

6. *Remarks on Theorem 1·1*

Remark 6·1. In the proof of Theorem 1·1 it is possible to show that g is one-to-one, so $\alpha = g$. And the curve β is just the boundary of $\bigcup \mathscr{B}$.

Remark 6·2. The statement of Theorem 1·1 can be strengthened to: *No neighbourhood in $\dot{E}(f)$ can be covered by countably many nowhere dense C-sets of $\dot{E}(f)$.* To see this, let $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$. Since $J(\mathcal{F})$ is closed, there is a box $[a,b] \times [c,d] \subset B(x_0, \varepsilon/2)$ containing x_0 such that

$$(\{a\} \times [c,d] \cup [a,b] \times \{c,d\}) \cap J(\mathcal{F}) = \varnothing.$$

Begin the construction of the $\mathscr{B}_k$'s with $\mathscr{B}_0 = \{[a,b] \times [c,d]\}$, and choose l large enough so that

$$\sum_{k=1}^{\infty} F^{-(l+k)^2}(1) < \frac{\varepsilon}{2}.$$

Construct $\mathscr{B}_k$ with $l+k$ replacing each k in items (ii), (iii) and (vii). By the arguments in Claim 5·2, if $x \in J(\mathcal{F})$ lies in the limit of (g_k) (constructed using the new $\mathscr{B}_k$'s), then $T(\mathcal{F}^{2n^2}(x)) \geq F^{n^2}(1)$ for all $n > l$ and consequently $x \in I(\mathcal{F})$. As in Claim 5·2 we can construct an arc α such that $\beta := (\{a\} \times [c,d] \cup [a,b] \times \{c,d\}) \cup \alpha$ is a simple closed curve in $B(x_0, \varepsilon)$, the bounded component of $\mathbb{R}^2 \setminus \beta$ contains x_0, and $\beta \cap J(\mathcal{F}) \subset I(\mathcal{F})$. Lemma 4·1 now shows that $B(x_0, \varepsilon)$ cannot be covered by countably many nowhere dense C-sets of $\tilde{E}(\mathcal{F})$.

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Remark 6·3. As argued in Section 1, Theorem 1·1 implies that the escaping endpoint set $\dot{E}(f_a)$ is not homeomorphic to Erdős space $\mathfrak{E}$ (Corollary 1·2).

7. *The space* $F(f_a) \cup I(f_a)$ *when* $a \in (-\infty, -1]$

We are now ready to prove Theorem 1·3 and Corollary 1·4. Fix $a \in (-\infty, -1]$.

Proof of Theorem 1·3. Let $z \in \mathbb{C}$. By [19, section 9] and [2, theorem 2·8], there is a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{C}$ such that $\varphi[J(\mathcal{F})] = J(f_a)$ and $\varphi[I(\mathcal{F})] = I(f_a)$. Let $x = \varphi^{-1}(z)$. We have shown that there is a Jordan curve $\beta \subset (\mathbb{R}^2 \setminus J(\mathcal{F})) \cup I(\mathcal{F})$ around x; see Remark 6·2, or simply begin the construction in Section 6 with $\mathscr{B}_0 = \{[-n, n]^2\}$ where $n \in \mathbb{N}$ is such that $x \in (-n, n)^2$. Then $\varphi[\beta] \subset F(f_a) \cup I(f_a)$ is a Jordan curve which separates z from ∞.

In the proof below we will make use of two well-known facts: (a) $F(f_a)$ is path-connected and (b) each component of $J(f_a)$ is contained in $I(f_a)$, with the possible exception of its endpoint.

Proof of Corollary 1·4. Let $z_0, z_1 \in F(f_a) \cup I(f_a)$. We will find a path in $F(f_a) \cup I(f_a)$ from z_0 to z_1. There are three cases to consider.

The case $z_0, z_1 \in F(f_a)$ is trivial since $F(f_a)$ is path-connected.

The next case is that $z_0 \in F(f_a)$ and $z_1 \in I(f_a)$. Let β be a simple closed curve around the point z_1, such that $\beta \cap J(f_a) \subset I(f_a)$ (apply Theorem 1·3). Let γ be the component of $J(f_a)$ containing z_1. There exists $z_2 \in \beta \setminus J(f_a)$ and $z_3 \in \gamma \cap \beta$. There are paths $\alpha_1 \subset F(f_a)$ from z_0 to z_2, $\alpha_2 \subset \beta$ from z_2 to z_3, and $\alpha_3 \subset \gamma$ from z_3 to z_1. It is clear that α_3 can be constructed to avoid the endpoint of γ, so that $\alpha_3 \subset I(f_a)$. Then $\alpha_1 \cup \alpha_2 \cup \alpha_3 \subset F(f_a) \cup I(f_a)$ contains a path from z_0 to z_1.

The third and final case $z_0, z_1 \in I(f_a)$ can be handled by connecting each point z_0 and z_1 to a third point of $F(f_a)$, as was done the second case.

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