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WINNER PLAYS COMPETITION MODELS

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Suppose there are n players in an ongoing competition, with player i having value v_i, and suppose that a game between i and j is won by i with probability $v_i/(v_i+v_j)$. Consider the winner plays competition where in each stage two players play a game, and the winner keeps playing in the next game. We consider two models for choosing its opponent, analyze both models as Markov chains, and determine their stationary probabilities as well as other quantities of interest.

Keywords: winner plays, markov chain, stationary probabilities, time reversible

1. INTRODUCTION

Suppose there are n players in an ongoing competition, with player i having a value $v_i, i=1,\ldots,n$. We suppose that at each stage two of the players play a game, and that, independently of what has previously occurred, a game between i and j is won by i with probability $v_i/(v_i+v_j)$. Supposing that the winner of a game plays in the next game, we consider two different models for choosing its opponent. The first model supposes that the $n-2$ players not involved in a game are waiting in an ordered line and that with probability $p(j,k)$, $\sum_{j=1}^{n-2}\sum_{k=1}^{n-2}p(j,k)=1$, the player currently in queue position j is the next opponent and, with the relative positions of the other $n-3$ players in queue remaining as they are, the loser of the just finished game is put in queue position k. Our second model supposes that each player has both a value and a queue weight, with player i having queue weight w_i. It supposes that if $i_1,\ldots,i_{n-2}$ are in queue, then the loser of the current game joins the queue and i_j is chosen as the next opponent with probability $w_{i_j}/\sum_{k=1}^{n-2}w_{i_k}$. We analyze both models as Markov chains and determine their stationary probabilities as well as other quantities of interest.

Aside from its inherent interest, the special case of the first model that always takes the player who has been waiting in queue the longest as the next opponent can be applied to the selection problem where we are interested in determining, under a constraint on the number of games that can be played, which player has the largest value. One policy of interest is to utilize the preceding to choose successive game opponents, and to then select the player who has the most wins after the prescribed number of games have been played. (See Azizi, Cao, and Ross [2] for numerical results concerning this method.)

Models that assume that each player has a value and that one player will beat another with a probability equal to its value divided by the sum of their values are known as Bradley-Terry type models. Such models were originally proposed by Zermelo [13] and later popularized by Bradley and Terry [3]. Such models have often been applied to sports. For example, the World Chess Federation and the European Go Federation successfully adopted this model for ranking players (see Hastie and Tibshirani [8]). Cattelan, Varin, and Firth [6] developed a dynamic Bradley–Terry model to analyze the abilities of teams in sports tournaments, and McHale and Morton [10] presented a Bradley–Terry model for forecasting match results of men's tennis. The Bradley–Terry model has also been assumed when analyzing knockout tournaments, which are tournaments in which a single loss eliminates a player and the tournament winner is the last player without a loss. Adler et al. [1] assumed a Bradley–Terry model in analyzing a random knockout tournament, where the number of matches in each round is fixed and the choice and pairings of the players in the round is randomly determined from those still remaining; Cao and Ross [4] used such a model when analyzing a winner plays knockout tournament in which players are initially randomly lined up. The first two in line play a game, with the winner of a game then playing against the next player in line. In both [1] and [4], bounds on the player's tournament win probabilities are obtained.

Statistical estimation of the player values, based on the results of previously played games, has also been considered. For instance, Hunter [9] proposed an iterative procedure for finding the maximum likelihood estimators. Bayesian estimation approaches have also been studied. Guiver and Snelson [7] presented an Expectation–Propagation method to approximate a posterior distribution, while Caron and Doucet [5] proposed a Gibbs sampler for a Bayesian inference based on a suitable set of latent variables. Ross and Zhang [12] utilized a post-stratification approach to explicitly determine posterior means.

2. QUEUE POSITION MODEL

Suppose that the n players are initially randomly put in a linear order, and the first two play a game. After each game, with probability $p(j,k)$ the player in queue position j leaves the queue, then the loser joins queue position k and the relative positions of the other players remain the same, where $j,k = 1,2,\ldots,n-2$ and $\sum_{j=1}^{n-2}\sum_{k=1}^{n-2} p(j,k) = 1$.

We can analyze the preceding as a Markov chain by letting the state be the sequence of the players in queue. Because there are $\binom{n}{2}$ choices for the 2 contestants and $(n-2)!$ possible orderings for the $n-2$ in queue, there are $\binom{n}{2}(n-2)!$ states. For example, for $i_1, i_2, \ldots, i_n$ being a permutation of $1,2,\ldots,n$, the state $(i_1, i_2, \ldots, i_{n-2})$ represents that players i_{n-1}, i_n are in the current game, and i_j is jth in line in queue, $j = 1,\ldots,n-2$. We assume that the $p(j,k)$ are such that the Markov chain is irreducible.

We first show that the stationary probability of each state is proportional to the sum of strengths of the players in the current game. Let $v = \sum_{j=1}^{n} v_j$.

Proposition 2.1: *For $(i_1, i_2, \ldots, i_n)$ being a permutation of $(1,2,\ldots,n)$, the stationary probabilities are*

$$\pi_{i_1,i_2,\ldots,i_{n-2}} = c(v_{i_{n-1}} + v_{i_n})$$

where $c = 1/((n-1)!v)$.

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PROOF: The hypothesized stationary probabilities sum to 1 and satisfy the stationarity equations

$$\begin{aligned}\pi_{i_1,i_2,\ldots,i_{n-2}} &= \sum\sum\nolimits_{j<k}\pi_{i_1,\ldots i_{j-1},i_{n-1},i_j,\ldots,i_{k-1},i_{k+1},\ldots,i_{n-2}}\frac{v_{i_n}}{v_{i_n}+v_{i_k}}p(j,k)\\ &+\sum_k \pi_{i_1,\ldots i_{k-1},i_{n-1},i_{k+1},\ldots,i_{n-2}}\frac{v_{i_n}}{v_{i_n}+v_{i_k}}p(k,k)\\ &+\sum\sum\nolimits_{j>k}\pi_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_j,i_{n-1},i_{j+1},\ldots,i_{n-2}}\frac{v_{i_n}}{v_{i_n}+v_{i_k}}p(j,k)\\ &+\sum\sum\nolimits_{j<k}\pi_{i_1,\ldots i_{j-1},i_n,i_j,\ldots,i_{k-1},i_{k+1},\ldots,i_{n-2}}\frac{v_{i_{n-1}}}{v_{i_{n-1}}+v_{i_k}}p(j,k)\\ &+\sum_k \pi_{i_1,\ldots i_{k-1},i_n,i_{k+1},\ldots,i_{n-2}}\frac{v_{i_{n-1}}}{v_{i_{n-1}}+v_{i_k}}p(k,k)\\ &+\sum\sum\nolimits_{j>k}\pi_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_j,i_n,i_{j+1},\ldots,i_{n-2}}\frac{v_{i_{n-1}}}{v_{i_{n-1}}+v_{i_k}}p(j,k)\end{aligned}$$

which proves the result. ■

COROLLARY 2.2: *In the stationary state,*

(a) *given the players in a game, all possible orderings of the $n-2$ in queue are equally likely;*

(b) *given the winner of a game, its opponent in that game was equally likely to be any of the other $n-1$ players.*

PROOF: The proof of (a) is immediate from Proposition 2.1. To prove (b), let A_{ij} be the event that the game is played by i and j, $i \neq j$. Then, with B_i being the event that the game is won by i,

$$P(A_{ij}|B_i) = \frac{P(B_i|A_{ij})P(A_{ij})}{\sum_{k\neq i} P(B_i|A_{ik})P(A_{ik})} = \frac{\frac{v_i}{v_i+v_j}(v_i+v_j)}{\sum_{k\neq i}\frac{v_i}{v_i+v_k}(v_i+v_k)} = \frac{1}{n-1}. \quad ■$$

COROLLARY 2.3: *Let $P_{i,j}, P_i^w, P_i^l$, and P_i be, respectively, the steady-state probabilities that the current game is between i and j, that the current game is won by i, that the current game is lost by i, and that i is playing in the current game.*

(a) $P_{i,j} = (v_i + v_j)/((n-1)v)$

(b) $P_i^w = v_i/v$.

(c) $P_i^l = (v - v_i)/((n-1)v)$.

(d) $P_i = (v + (n-2)v_i)/((n-1)v)$.

PROOF: The proof of (a) follows from Proposition 2.1 by summing the stationary probabilities over all $(n-2)!$ possible orderings of those in queue when i plays j. To prove (b) and (c), note that $P_{i,j}(v_i/(v_i+v_j)) = v_i/((n-1)v)$ is the probability that i beats j, and $P_{i,j}v_j/(v_i+v_j) = v_j/((n-1)v)$ is the probability that i loses to j, in the current game; the results then follow upon summing this over all $j \neq i$. Part (d) follows either by using that $P_i = \sum_{j:j\neq i} P_{i,j}$, or by using that $P_i = P_i^w + P_i^l$. ■

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A player alternates between waiting in queue and playing games. Let Q^i_j denote the amount of time player i spends in queue, after losing for the jth time, before it again plays a game. Also, let G^i_j be the number of games that i plays until it loses, on its jth entrance to a game from queue. Thus, for instance, the first time i begins to play it plays G^i_1 games, winning the first $G^i_1 - 1$ of them and then losing the next, and then waits in queue for Q^i_1 games, then plays G^i_2 games, and so on.

PROPOSITION 2.4: *For fixed i, $Q^i_j, j = 1, 2, \ldots$ are independent and identically distributed. The distribution of Q^i_j is the same for all j, and $E[Q^i_j] = n - 2$.*

PROOF: Since the selection from queue only depends on the queue positions, rather than the players in those positions, it is easy to see that the time that a player spends in queue after a loss is independent of its earlier waiting times in queue, has the same distribution for all players, and this distribution does not depend on $v_1, \ldots, v_n$. So let us assume that $v_1 = v_2 = \cdots = v_n$. Now, in this case, the proportion of games that are won by each player is $1/n$. But each time player i loses a game is a renewal, and so by renewal reward process theory the proportion of games won by i is the expected number of games it wins when it starts playing divided by the expected number of games between losses. Because it wins each game it plays with probability $1/2$, the mean number of games it plays before returning to queue is 2, giving that

$$\frac{1}{n} = P^w_i = \frac{1}{2 + E[Q^i_j]}$$

which proves the result. ■

Now, let $\bar{G}^i = \lim_{m\to\infty}(\sum_{j=1}^m G^i_j/m)$. (It follows from renewal reward theory that $\bar{G}^i$ exists and is almost surely constant.)

PROPOSITION 2.5:

$$\bar{G}^i = \frac{(n-2)v_i + v}{v - v_i}$$

PROOF: If we say that a new cycle begins each time i leaves the queue, then the proportion of games won by i during the first m cycles is $\sum_{j=1}^m (G^i_j - 1)/\sum_{j=1}^m (G^i_j + Q^i_j)$. Because this converges to P^w_i as $m \to \infty$, we see upon dividing numerator and denominator by m and using Proposition 4 that $v_i/v = (\bar{G}^i - 1)/(\bar{G}^i + n - 2)$, which proves the result. ■

Remark 2.6:

(1) The result that $E[Q^i_j] = n - 2$ also follows from Little's formula of queuing theory, (see [11]) which yields that the average number of players in queue (namely, $n-2$) is equal to the average arrival rate to queue (namely, 1) times the average time that a player spends in queue.

(2) If the protocol is that the loser (rather than the winner) plays in the next game, then because the probability that i would lose in a game with j is $1/v_i/(1/v_i + 1/v_j)$, it follows that all results remain true when "winner" is replaced by "loser" and v_i by $1/v_i$. For instance, the proportion of games in which i is the loser is $1/v_i/(\sum_j 1/v_j)$.

(3) We have supposed that the $p(j, k)$ are such that the resulting Markov chain is irreducible. If we supposed that after each game the player in queue position j plays next with probability p_j and that, independently of which position is chosen the loser

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of the game joins queue position k with probability q_k (and the relative positions of the other players remain the same), then the Markov chain will be irreducible if and only if $(p_1 + q_1)(p_{n-2} + q_{n-2}) > 0$.

3. QUEUE WEIGHT MODEL

As noted earlier, this model supposes that player i has a weight w_i, $i = 1, \ldots, n$, and that if $i_1, i_2, \ldots, i_{n-2}$ are in queue, then i_j will be selected to play in the next game with probability $w_{i_j} / \sum_{k=1}^{n-2} w_{i_k}$.

This model can be analyzed as a Markov chain by letting the state be the 2 players in the current game. We say that the state is (i, j), if i and j are playing and $i < j$. We first show that the Markov chain is time reversible and gives the stationary probabilities. Let $v = \sum_{j=1}^{n} v_j$ and $w = \sum_{j=1}^{n} w_j$.

PROPOSITION 3.1: *The Markov chain is time reversible and its stationary probabilities are*

$$\pi_{i,j} = c(v_i + v_j)(w - w_i - w_j)w_i w_j, \quad i < j$$

where $1/c = \sum\sum_{i<j}(v_i + v_j)(w - w_i - w_j)w_i w_j$.

PROOF: The hypothesized stationary probabilities sum to 1 and satisfy the time reversibility equations

$$\pi_{i,j} \frac{v_i}{v_i + v_j} \frac{w_k}{w - w_i - w_j} = \pi_{i,k} \frac{v_i}{v_i + v_k} \frac{w_j}{w - w_i - w_k}$$

which proves the result. ■

We can prove the following corollary in the same manner as we did for Corollary 2.3.

COROLLARY 3.2: *Let P_i^w, P_i^l, and P_i be, respectively, the steady-state probabilities that the current game is won by i, that the current game is lost by i, and that i is playing in the current game.*

(a) $P_i^w = cv_i w_i((w - w_i)^2 - \sum_{j:j \neq i} w_j^2)$.
(b) $P_i^l = cw_i \sum_{j:j \neq i} v_j(w - w_i - w_j)w_j$.
(c) $P_i = cw_i \sum_{j:j \neq i}(v_i + v_j)(w - w_i - w_j)w_j$.

As in the preceding section, let Q_j^i denote the amount of time player i spends in queue, after losing for the jth time, before it again plays a game, and let G_j^i be the number of games that i plays until it loses, on its jth entrance to a game from queue. Also, let $\bar{Q}^i = \lim_{m\to\infty}(\sum_{j=1}^{m} Q_j^i/m)$, and let $\bar{G}^i = \lim_{m\to\infty}(\sum_{j=1}^{m} G_j^i/m)$.

PROPOSITION 3.3:

(a) $\bar{G}^i = P_i/P_i^l$.
(b) $\bar{Q}^i = (1 - P_i)/P_i^l$.

PROOF: Because P_i^l/P_i is the fraction of games played by i that i loses, it follows that P_i/P_i^l is the average number of games played by i between returns to queue. Thus, part (a) is proved. Part (b) now follows because $P_i = \bar{G}^i/(\bar{G}^i + \bar{Q}^i)$. ■

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References

1. Adler, I., Cao, Y., Karp, R., Peköz, E.A., & Ross, S.M. (2017). Random knockout tournaments. *Operations Research* 65(6): 1589–1596.
2. Azizi, M.J., Cao, Y., & Ross, S.M. (2019). Some new selection procedures. Manuscript in preparation.
3. Bradley, R.A. & Terry, M.E. (1952). Rank analysis of incomplete block designs: I. the method of paired comparisons. *Biometrika* 39(3/4): 324–345.
4. Cao, Y. & Ross, S.M. (2019) Winner plays structure in random knockout tournaments. *Probability in the Engineering and Informational Sciences* 33(4): 500–510.
5. Caron, F. & Doucet, A. (2012). Efficient bayesian inference for generalized bradley–terry models. *Journal of Computational and Graphical Statistics* 21(1): 174–196.
6. Cattelan, M., Varin, C., & Firth, D. (2013). Dynamic bradley–terry modelling of sports tournaments. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 62(1): 135–150.
7. Guiver, J. & Snelson, E. (2009). Bayesian inference for plackett-luce ranking models. In *Proceedings of the 26th Annual International Conference on Machine Learning (ICML '09)*. ACM, New York, NY, USA, 377–384.
8. Hastie, T. & Tibshirani, R. (1998). Classification by pairwise coupling. *The Annals of Statistics* 26(2): 451–471.
9. Hunter, D.R. (2004). Mm algorithms for generalized bradley-terry models. *The annals of statistics* 32(1): 384–406.
10. McHale, I. & Morton, A. (2011). A bradley-terry type model for forecasting tennis match results. *International Journal of Forecasting* 27(2): 619–630.
11. Ross, S.M. (2019). *Introduction to probability models* (12th ed). Academic Press.
12. Ross, S.M. & Zhang, Z. (2019). Estimating the strength of sport teams. Manuscript in preparation.
13. Zermelo, E. (1929). Die berechnung der turnier-ergebnisse als ein maximumproblem der wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift* 29(1): 436–460.