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New examples of non-Fourier–Mukai functors

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Abstract

A celebrated result by Orlov states that any fully faithful exact functor between the bounded derived categories of coherent sheaves on smooth projective varieties is of geometric origin, i.e. it is a Fourier–Mukai functor. In this paper we prove that any smooth projective variety of dimension ≥ 3 equipped with a tilting bundle can serve as the source variety of a non-Fourier–Mukai functor between smooth projective schemes.

1. Introduction

Throughout we fix a base field k and all constructions are linear over k. In 1997, Orlov proved the following result.

Theorem 1.1 [Orl97, Theorem 2.2]. *Let $X/k, Y/k$ be smooth projective schemes. Then every fully faithful exact functor $\Psi : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ is isomorphic to a Fourier–Mukai functor associated with an object of $D^b(\mathrm{coh}(X \times_k Y))$, the Fourier–Mukai kernel.*

This result is of seminal importance because it allows for such a functor Ψ to be analysed by means of a geometric study of the kernel.

1.1 Non-Fourier–Mukai functors

The first example of a non-Fourier–Mukai functor between bounded derived categories of smooth projective schemes was given by the second and third authors, and can be found in [RVdBN19] together with an appendix by Amnon Neeman improving on one of the key results. The functor is of the form

$$D^b(\mathrm{coh}(Q)) \to D^b(\mathrm{coh}(\mathbb{P}^4)),$$

where Q is a three-dimensional smooth quadric and $\mathbb{P}^4$ is its ambient projective space. The construction proceeds in two steps.

(i) First a prototypical non-Fourier–Mukai functor is constructed between certain non-geometric DG-categories.

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(ii) Then, using a quite involved argument, this functor is turned into a geometric one.

In step (i), for a smooth projective variety X, and given a Hochschild cohomology class $0 \neq \eta \in \mathrm{HH}^{2\dim_k(X)}(X, \omega_X^{\otimes 2})$, a functor

$$L : D^b(\mathrm{coh}(X)) \to D_\infty(\mathcal{X}_\eta)$$

is constructed, where $D_\infty(\mathcal{X}_\eta)$ is the derived category of an A_∞-category $\mathcal{X}_\eta$ which can be thought of as a generalized deformation of X in the η-direction (see § 5 for more details). This functor is not Fourier–Mukai in a generalized sense, see Definition 2.2.

In step (ii), one needs to move from the non-geometric category $D_\infty(\mathcal{X}_\eta)$ to an honest derived category of coherent sheaves. In [RVdBN19, § 11] this is achieved by showing that the inclusion $f : X \to Y$ of a smooth quadric $X = Q$ of maximal isotropy index in $Y = \mathbb{P}^4$ annihilates η (in the sense that $f_*\eta = 0$), which allows for the construction of the functor Ψ as a composition of L with a pushforward to $D^b(\mathrm{coh}(\mathbb{P}^4))$, back to the geometric world. The composition with the pushforward yields the required functor

$$D^b(\mathrm{coh}(Q)) \to D^b(\mathrm{coh}(\mathbb{P}^4)),$$

but the drawback is the need to again check that the composition is non-Fourier–Mukai. This is achieved using an obstruction theory that quickly gets hard to control as the dimension of X grows, and indeed the original paper [RVdBN19] only gave one concrete example of a non-Fourier–Mukai functor despite the very general initial setup.

In this paper we show that, as long as one is not worried about keeping $\dim_k(Y)$ small, it is possible to bypass this intricate analysis and construct non-Fourier–Mukai functors starting from L in a different way.

Remark 1.2. In a short note [Vol19], Vologodsky shows that over a field of positive characteristic non-Fourier–Mukai functors arise quite naturally. Given a smooth projective scheme X over $\mathbb{Z}_p$, one considers the embedding $i : X \hookrightarrow Z$ of the special fiber. The main result is a criterion for the endofunctor $Li^* \circ i_*$ of $D^b(\mathrm{coh}(X))$ not to be of Fourier–Mukai type. This criterion is satisfied, in particular, for X the flag variety of GL_n, for $n > 2$. However, the functors obtained in this way do admit a $\mathbb{Z}$-linear DG-lift.

1.2 New examples

In this paper we show that if one is not interested in 'small' examples the second part of the construction can be simplified, giving rise to many more examples of non-Fourier–Mukai functors.

Recall that if X is a scheme, then a *tilting bundle* T on X is a vector bundle on X such that $\mathrm{Ext}_X^{>0}(T, T) = 0$ and such that T generates $D_{\mathrm{Qch}}(\mathcal{O}_X)$. The following is our main result.

Theorem 1.3 (see § 5). *Let X be a smooth projective scheme of dimension $m \geq 3$ which has a tilting bundle. Then there is a non-Fourier–Mukai functor*

$$D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y)), \tag{1}$$

where Y is a smooth projective scheme.

As a concrete example, we may for instance take $X = \mathbb{P}^m$, $m \geq 3$, which has the Beilinson tilting bundle $T = \bigoplus_{i=0}^m \mathcal{O}_X(i)$.

1.3 Geometric realizations

To prove Theorem 1.3, we combine results from [Orl16] with ideas from [Orl20]. There are again two main steps involved.

(i) In a first step, we construct a fully faithful functor Aus

$$D^b(\mathrm{coh}(X)) \xrightarrow{L} \mathcal{A} \xhookrightarrow{\mathrm{Aus}} \mathcal{T}$$

from (the thick envelope of) the essential image $\mathcal{A}$ of L to a triangulated category $\mathcal{T}$ with a full exceptional collection. This construction is based on a version of the Auslander(–Dlab–Ringel) algebra for filtered A_∞-algebras, which is explained in § 3.

(ii) We then invoke Orlov's gluing result [Orl16, Theorem 4.15], which implies the existence of a fully faithful functor

$$\mathcal{T} \xhookrightarrow{\mathrm{Geom}} D^b(\mathrm{coh}(Y)),$$

for some smooth and projective scheme Y (this is often referred to as a *geometric realization*, not to be confused with the geometric realization of a simplicial set).

One can then show that the composed functor

$$D^b(\mathrm{coh}(X)) \xrightarrow{L} \mathcal{A} \xhookrightarrow{\mathrm{Aus}} \mathcal{T} \xhookrightarrow{\mathrm{Geom}} D^b(\mathrm{coh}(Y))$$

is still non-Fourier–Mukai, thus proving Theorem 1.3.

2. Preliminaries on A_∞-categories

Fix an arbitrary base field k.[1] Our general reference for A_∞-algebras and A_∞-categories will be [Lef03]. For a good reference in English, consult [ELO10]. Sometimes we silently use notions for categories which are only introduced for algebras (i.e. categories with one object) in [ELO10]. We assume that all A_∞-notions are *strictly unital*. Unless otherwise specified we use cohomological grading.

Remark 2.1. We rely throughout on the fact that the homotopy categories of A_∞-categories and DG-categories are equivalent. See [COS19]. This implies, in particular, that we can freely use Orlov's gluing results in [Orl16] in the A_∞-context.

Definition 2.2. Let $\mathfrak{a}$, $\mathfrak{b}$ be pretriangulated A_∞-categories [BLM17] and put $\mathcal{A} = H^0(\mathfrak{a})$, $\mathcal{B} = H^0(\mathfrak{b})$. We say that an exact functor $F : \mathcal{A} \to \mathcal{B}$ is *Fourier–Mukai* if there is an A_∞-functor $f : \mathfrak{a} \to \mathfrak{b}$ such that $F \cong H^0(f)$ as graded functors.

Often $\mathfrak{a}$, $\mathfrak{b}$ are uniquely determined by $\mathcal{A}$, $\mathcal{B}$ (see [CS18, LO10]) or else implicit from the context, and then we do not specify them.

Remark 2.3. If X, Y are smooth projective varieties and $F : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ is a traditional Fourier–Mukai functor which means that it can be written as $R\,\mathrm{pr}_{2*}(\mathcal{K} \overset{L}{\otimes}_{X\times Y} L\,\mathrm{pr}_1^*(-))$ for $\mathcal{K} \in D^b(\mathrm{coh}(X \times Y))$, then it is Fourier–Mukai in our sense. This follows from the easy part of [Toë07, Theorem 8.15] combined with Remark 2.1.

For an A_∞-category $\mathfrak{a}$ we denote by[2] $\mathcal{D}_\infty(\mathfrak{a})$ the DG-category of left A_∞-modules. The A_∞-Yoneda functor

$$\mathfrak{a} \to \mathcal{D}_\infty(\mathfrak{a}^\circ) : X \mapsto \mathfrak{a}(-, X) \tag{2}$$

is quasi-fully faithful [Lef03, Lemma 7.4.0.1]. The corresponding homotopy category $D_\infty(\mathfrak{a}) := H^0(\mathcal{D}_\infty(\mathfrak{a}))$ is a compactly generated triangulated category [Kel06, § 4.9] with compact generators

[1] Although the reference [RVdBN19] is written with the blanket assumption of characteristic zero, that hypothesis is not needed for the parts of the paper that are used here.
[2] $\mathcal{D}_\infty(\mathfrak{a})$ is denoted by $\mathcal{C}_\infty(\mathfrak{a})$ in [Lef03], and by $A - \mathrm{mod}_\infty$ in [ELO10, § 3.1].

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$\mathfrak{a}(X,-)$ for $X \in \mathrm{Ob}(\mathfrak{a})$. We write $\mathcal{P}\mathrm{erf}(\mathfrak{a})$ for the full DG-subcategory of $\mathcal{D}_\infty(\mathfrak{a})$ spanned by the compact objects in $D_\infty(\mathfrak{a})$ and we also put $\mathrm{Perf}(\mathfrak{a}) = H^0(\mathcal{P}\mathrm{erf}(\mathfrak{a}))$.

If $\mathcal{A}$ is a triangulated category and $S \subset \mathrm{Ob}(\mathcal{A})$, then the category *classically generated by* S [BVdB03, § 1] is the smallest thick subcategory of $\mathcal{A}$ containing S. It is denoted by $\langle S \rangle$. By [Kel94, § 5.3],[Nee92, Lemma 2.2] $\mathrm{Perf}(\mathfrak{a})$ is classically generated by the objects $\mathfrak{a}(X,-)$.

If $f : \mathfrak{a} \to \mathfrak{b}$ is an A_∞-functor, then we may view $\mathfrak{b}$ as an A_∞-$\mathfrak{b}$-$\mathfrak{a}$-bimodule. Hence, we have a 'standard' DG-functor

$$\mathfrak{b} \overset{\infty}{\otimes}_{\mathfrak{a}} - : \mathcal{D}_\infty(\mathfrak{a}) \to \mathcal{D}_\infty(\mathfrak{b})$$

which (for algebras) is introduced in [Lef03, § 4.1.1]. We recall the following basic result.

LEMMA 2.4. *For A_∞-categories $\mathfrak{a}, \mathfrak{b}$ and a quasi-fully faithful A_∞-functor $f : \mathfrak{a} \to \mathfrak{b}$, the induced functor $\mathfrak{b} \overset{\infty}{\otimes}_{\mathfrak{a}} - : D_\infty(\mathfrak{a}) \to D_\infty(\mathfrak{b})$ is fully faithful. Moreover, this functor restricts to a fully faithful Fourier–Mukai functor* $\mathrm{Perf}(\mathfrak{a}) \to \mathrm{Perf}(\mathfrak{b})$.

Proof. By the same argument as in the proof of [Lef03, Lemme 4.1.1.6] there is a quasi-isomorphism

$$\mathfrak{b} \overset{\infty}{\otimes}_{\mathfrak{a}} \mathfrak{a}(X,-) \to \mathfrak{b}(fX,-) \tag{3}$$

for $X \in \mathrm{Ob}(\mathfrak{a})$, functorial in X. In other words there is a pseudo-commutative diagram

$$\begin{array}{ccc} H^0(\mathfrak{a})^\circ & \xrightarrow{H^0(f)} & H^0(\mathfrak{b})^\circ \\ \downarrow & & \downarrow \\ D_\infty(\mathfrak{a}) & \xrightarrow[\mathfrak{b} \overset{\infty}{\otimes}_{\mathfrak{a}} -]{} & D_\infty(\mathfrak{b}) \end{array}$$

where the vertical arrows are the Yoneda embeddings $X \mapsto \mathfrak{a}(X,-)$, $Y \mapsto \mathfrak{b}(Y,-)$. The full faithfulness of the lower arrow follows by dévissage. The claim about Perf follows immediately from (3). $\square$

The following lemma is a variant on Lemma 2.4 and could have been deduced from it.

LEMMA 2.5. *Assume that $\mathfrak{a}$ is a pre-triangulated A_∞-category [BLM17] such that $H^0(\mathfrak{a})$ is Karoubian and classically generated by $T \in \mathrm{Ob}(\mathfrak{a})$. Put $\mathsf{R} = \mathfrak{a}(T,T)$. The A_∞-functor*

$$f : \mathfrak{a} \to \mathcal{D}_\infty(\mathsf{R}^\circ) : X \mapsto \mathfrak{a}(T,X)$$

defines a quasi-equivalence

$$\mathfrak{a} \to \mathcal{P}\mathrm{erf}(\mathsf{R}^\circ)$$

or, equivalently, an equivalence of triangulated categories

$$H^0(\mathfrak{a}) \cong \mathrm{Perf}(\mathsf{R}^\circ). \tag{4}$$

Proof. We must prove (4). We have $H^0(f)(T) = \mathsf{R}$. By hypothesis $H^0(\mathfrak{a})$ is classically generated by T and by the previous discussion $\mathrm{Perf}(\mathsf{R}^\circ)$ is classically generated by R. Moreover, because the Yoneda functor is quasi-fully faithful, $H^0(f)$ is fully faithful when restricted to T. The rest follows by dévissage. $\square$

3. Geometric realization of a filtered A_∞-algebra

Let (R, m_*) denote a finite-dimensional A_∞-algebra equipped with a (decreasing) filtration $F^* := \{F^p\mathsf{R}\}_{p \geq 0}$. This means that $\{F^p\mathsf{R}\}_{p \geq 0}$ is a decreasing filtration of the underlying

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(finite-dimensional) graded vector space of R satisfying the compatibility conditions

$$m_p(F^{i_1}\otimes\cdots\otimes F^{i_p})\subset F^{i_1+\cdots+i_p} \tag{5}$$

for all p and all $i_1,\dots,i_p$.

Assume $F^n\mathsf{R}=F^n=0$ for some $n\geq 0$. In this case we may define the *(modified) Auslander A_∞-category* $\Gamma=\Gamma_{\mathsf{R},F^*}$ of (R,F^*). The objects of Γ are the integers $0,\dots,n-1$ and we set

$$\Gamma(j,i):=F^{\max(j-i,0)}/F^{n-i}. \tag{6}$$

By setting $\Gamma_{i,j}=\Gamma(j,i)$, we can represent Γ schematically via the matrix

$$(\Gamma_{i,j})=\begin{pmatrix} \mathsf{R} & F^1 & F^2 & \cdots & F^{n-1}\\ \mathsf{R}/F^{n-1} & \mathsf{R}/F^{n-1} & F^1/F^{n-1} & \cdots & F^{n-2}/F^{n-1}\\ \mathsf{R}/F^{n-2} & \mathsf{R}/F^{n-2} & \mathsf{R}/F^{n-2} & \cdots & F^{n-3}/F^{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathsf{R}/F^1 & \mathsf{R}/F^1 & \mathsf{R}/F^1 & \mathsf{R}/F^1 & \mathsf{R}/F^1 \end{pmatrix} \tag{7}$$

so that composition is given by matrix multiplication.

The grading on R induces a grading on Γ. Because of condition (5), the higher multiplications on R also induce higher multiplications on Γ. Indeed,

$$\max(i_{p+1}-i_1,0)\leq\max(i_2-i_1,0)+\cdots+\max(i_{p+1}-i_p,0), \tag{8}$$

so

$$m_p(F^{\max(i_2-i_1,0)}\otimes\cdots\otimes F^{\max(i_{p+1}-i_p,0)})\subset F^{\max(i_{p+1}-i_1,0)}. \tag{9}$$

In addition,

$$\begin{aligned}&\max(i_2-i_1,0)+\cdots+\max(i_k-i_{k-1})+(n-i_k)\\&\quad+\max(i_{k+2}-i_{k+1},0)+\cdots+\max(i_{p+1}-i_p,0)\\&\quad\geq\max(i_2-i_1,0)+\cdots+\max(i_{k-1}-i_{k-2})+(n-i_{k-1})\geq n-i_1,\end{aligned} \tag{10}$$

so m_p passes to the quotients

$$m_p^\Gamma:\Gamma_{i_1,i_2}\otimes\Gamma_{i_2,i_3}\otimes\cdots\otimes\Gamma_{i_{p-1},i_p}\otimes\Gamma_{i_p,i_{p+1}}\to\Gamma_{i_1,i_{p+1}} \tag{11}$$

making Γ into an A_∞-category.

Remark 3.1. The same construction also yields the A_∞-algebra $\bigoplus_{i,j}\Gamma_{i,j}$, which encodes the same data as Γ. The above construction is similar in spirit to [KL15, § 5]. If R is concentrated in degree 0 and F is the radical filtration, we obtain a subalgebra of Auslander's original algebra [Aus99], which is nowadays often referred to as the Auslander–Dlab–Ringel algebra (see, for example, [CE18]).

As $\Gamma_{0,0}=\mathsf{R}$, by thinking of R as an A_∞-category with one object we have a fully faithful strict A_∞-functor

$$\mathsf{R}\to\Gamma$$

whence we obtain the following result by Lemma 2.4.

Corollary 3.2. *There is a fully faithful functor*

$$\Gamma\overset{\infty}{\otimes}_{\mathsf{R}}-:\mathrm{Perf}(\mathsf{R})\to\mathrm{Perf}(\Gamma).$$

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Proposition 3.3. *Let $\bar{\mathsf{R}} = R/F^1$. There are n quasi-fully-faithful A_∞-functors*

$$\mathcal{P}\mathrm{erf}(\bar{\mathsf{R}}) \to \mathcal{P}\mathrm{erf}(\Gamma)$$

giving rise to a semi-orthogonal decomposition

$$\mathrm{Perf}(\Gamma) = \langle \underbrace{\mathrm{Perf}(\bar{\mathsf{R}}), \dots, \mathrm{Perf}(\bar{\mathsf{R}})}_{n} \rangle.$$

Proof. For $i = 0, \dots, n-1$ let

$$P_i = \Gamma(i, -)$$

and $P_n = 0$. For $i = 0, \dots, n-1$ the element $P_i \in D_\infty(\Gamma)$ corresponds to the $(i+1)$th column in (7) and we have obvious inclusion maps

$$\psi_i : P_{i+1} \to P_i.$$

Put

$$S_i \coloneqq \mathrm{cone}\, \psi_i = \begin{pmatrix} F^i/F^{i+1} \\ F^{i-1}/F^i \\ \vdots \\ \mathsf{R}/F^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{12}$$

(in particular, $S_{n-1} = P_{n-1}$). By the Yoneda lemma we see that

$$\mathrm{Hom}^*_{D_\infty(\Gamma)}(P_j, S_i) = H^*(S_i(j)) = \begin{cases} 0, & \text{if } j > i, \\ H^*(\bar{\mathsf{R}}), & \text{if } j = i. \end{cases} \tag{13}$$

We also find using the long exact sequence for the distinguished triangle

$$P_{i+1} \to P_i \to S_i \to$$

that

$$\mathrm{End}^*_{D_\infty(\Gamma)}(S_i, S_i) = \mathrm{Hom}^*_{D_\infty(\Gamma)}(P_i, S_i) = H^*(\bar{\mathsf{R}}). \tag{14}$$

We now have by (13) semi-orthogonal decompositions

$$\langle P_i, \dots, P_{n-1} \rangle = \langle \langle S_i \rangle, \langle P_{i+1}, \dots, P_{n-1} \rangle \rangle,$$

which, by induction, yield a semi-orthogonal decomposition

$$\mathrm{Perf}(\Gamma) = \langle \langle S_0 \rangle, \dots, \langle S_{n-1} \rangle \rangle.$$

Using (12) and the compatibility conditions (5) for the filtration F^*, we check that the S_i are, in fact, $A_\infty - \Gamma - \bar{\mathsf{R}}$-bimodules. Thus, we have DG functors

$$S_i \overset{\infty}{\otimes}_{\bar{\mathsf{R}}} - : \mathcal{D}_\infty(\bar{\mathsf{R}}) \to \mathcal{D}_\infty(\Gamma)$$

and the corresponding exact functors

$$S_i \overset{\infty}{\otimes}_{\bar{\mathsf{R}}} - : D_\infty(\bar{\mathsf{R}}) \to D_\infty(\Gamma),$$

which send $\bar{\mathsf{R}}$ to S_i and therefore are fully faithful by (14) and Lemma 2.4. Thus, they establish equivalences

$$\mathrm{Perf}(\bar{\mathsf{R}}) \cong \langle S_i \rangle$$

finishing the proof. □

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Let us call an A_∞-algebra A *geometric* if there is a fully faithful Fourier–Mukai functor (in the sense of Definition 2.2) $f : \operatorname{Perf} A \hookrightarrow D^b(\operatorname{coh}(X))$ for X a smooth and projective k-scheme, such that in addition f has a left and a right adjoint.

In the following corollary, we make use of Orlov's powerful gluing result, which in our setting may be formulated as follows (see also Remark 2.1).

Theorem 3.4 [Orl16, Theorem 4.15]. *Given A_∞-algebras A, B, C with C proper and a semi-orthogonal decomposition*

$$\operatorname{Perf} C = \langle \operatorname{Perf} A, \operatorname{Perf} B \rangle.$$

If A and B are geometric, then so is C.

Corollary 3.5 (Geometric realization). *Let R be a finite-dimensional A_∞-algebra equipped with a finite descending filtration such that $\bar{\mathsf{R}} = \mathsf{R}/F^1\mathsf{R}$ is geometric. Then there exists a fully faithful Fourier–Mukai functor $\operatorname{Perf} \mathsf{R} \hookrightarrow D^b(\operatorname{coh}(X))$ where X is a smooth projective k-scheme.*

Proof. Combining Proposition 3.3 with Theorem 3.4 we obtain that there exists a fully faithful Fourier–Mukai functor

$$\operatorname{Perf} \Gamma \hookrightarrow D^b(\operatorname{coh}(X)),$$

where X is a smooth projective k-scheme. Then we pre-compose this functor with the fully faithful Fourier–Mukai functor

$$\operatorname{Perf} \mathsf{R} \hookrightarrow \operatorname{Perf} \Gamma$$

of Corollary 3.2. $\square$

Corollary 3.6. *Assume R is an A_∞-algebra such that $H^*(\mathsf{R})$ is finite dimensional and concentrated in degrees ≤ 0, and moreover $H^0(\mathsf{R})$ is geometric. Then there exists a fully faithful Fourier–Mukai functor $\operatorname{Perf} \mathsf{R} \hookrightarrow D^b(\operatorname{coh}(X))$, where X is a smooth projective k-scheme.*

Proof. Without loss of generality we may assume that R is minimal. We now apply Corollary 3.5 with the filtration $F^p\mathsf{R} = \bigoplus_{i\geq p} \mathsf{R}^{-i}$. $\square$

Remark 3.7. As $H^0(\mathsf{R})$ is assumed to be a finite-dimensional algebra, the following lemma may be helpful for checking geometricity of $H^0(\mathsf{R})$ in order to apply Corollary 3.6:

Lemma 3.8. *Assume that A is a finite-dimensional k-algebra. The following are equivalent:*

(i) *A is geometric;*
(ii) *A is smooth (i.e.* $\operatorname{p\,dim}_{A^e} A < \infty$*);*
(iii) *$A/\operatorname{rad} A$ is separable over k and* $\operatorname{gl\,dim} A < \infty$.

Proof.

(i)⇒(ii) This is [Orl16, Theorem 3.25].
(ii)⇒(iii) The fact that $A/\operatorname{rad} A$ is separable over k is [RR22, Theorem 3.6], which in turn comes from a MathOverflow answer by Rickard [Ric16]. Finite global dimension is classical.
(iii)⇒(i) This is [Orl16, Corollary 5.4]. $\square$

Remark 3.9. If k is algebraically closed, one can even assume that the scheme X in Corollary 3.6 has a full exceptional collection. Indeed, using the radical filtration $F^pH^0(\mathsf{R}) = \operatorname{rad}^p H^0(\mathsf{R})$ for

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the finite-dimensional algebra $H^0(\mathsf{R})$, we see that there is a fully faithful functor

$$\Gamma_{H^0(\mathsf{R}),F^*} \overset{\infty}{\otimes}_{H^0(\mathsf{R})} - : \mathrm{Perf}(H^0(\mathsf{R})) \to \mathrm{Perf}(\Gamma_{H^0(\mathsf{R}),F^*})$$

and a semi-orthogonal decomposition

$$\mathrm{Perf}(\Gamma_{H^0(\mathsf{R}),F^*}) = \langle \mathrm{Perf}(H^0(\mathsf{R})/\operatorname{rad} H^0(\mathsf{R})), \ldots, \mathrm{Perf}(H^0(\mathsf{R})/\operatorname{rad} H^0(\mathsf{R}))\rangle \tag{15}$$

$$= \langle \mathrm{Perf}(k), \ldots, \mathrm{Perf}(k)\rangle. \tag{16}$$

In particular, each copy $\mathcal{A}_i = \mathrm{Perf}(H^0(\mathsf{R}))$ in the semi-orthogonal decomposition

$$\mathrm{Perf}(\Gamma) = \langle \mathrm{Perf}(H^0(\mathsf{R})), \ldots, \mathrm{Perf}(H^0(\mathsf{R}))\rangle = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n\rangle \tag{17}$$

admits a fully faithful functor $\mathcal{A}_i \to \mathsf{C}_i$ for some category C_i with a full exceptional collection. We claim this implies there is also a fully faithful functor

$$\mathrm{Perf}(\Gamma) = \langle \mathrm{Perf}(H^0(\mathsf{R}), \ldots, \mathrm{Perf}(H^0(\mathsf{R})\rangle \to \mathsf{C} \tag{18}$$

for some category C with a full exceptional collection. Indeed, first assume $n = 2$, then we can base change the (perfect) gluing $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule M (responsible for (17)) to a perfect C_1–C_2-bimodule, say M'. The corresponding gluing C of C_1 and C_2 along M' then has a semi-orthogonal decomposition

$$\mathsf{C} = \langle \mathsf{C}_1, \mathsf{C}_2\rangle \tag{19}$$

and, by construction, C has a full exceptional collection. Moreover, one can check that the induced functor

$$\mathrm{Perf}(\Gamma) = \langle \mathcal{A}_1, \mathcal{A}_2\rangle \to \mathsf{C} \tag{20}$$

is still fully faithful. For $n > 2$, we proceed by induction.

Finally, by applying [Orl16, Theorem 5.8], there exists a fully faithful Fourier–Mukai functor

$$\mathrm{Perf}(\Gamma) \to D^b(\mathrm{coh}(X)), \tag{21}$$

for some smooth projective k-scheme X with a full exceptional collection.

4. A_∞-deformations of schemes and objects

In this section we review some material on A_∞-deformations of schemes and the corresponding results for deformations of objects. For the benefit of the reader, we collect the results we need in the rest of the paper, in the generality that we need in this situation. A more general treatment of this can be found in [RVdBN19, §§ 6 and 8].

Definition 4.1. Let $\mathcal{X}$ be a k-linear category and $\mathcal{M}$ be a k-central $\mathcal{X}$-bimodule. The Hochschild complex $\mathbf{C}^\bullet(\mathcal{X}, \mathcal{M})$ is defined as

$$\mathbf{C}^i(\mathcal{X}, \mathcal{M}) = \prod_{X_0,\ldots,X_p \in \mathrm{Ob}(\mathcal{X})} \mathrm{Hom}(\mathcal{X}(X_{p-1}, X_p) \otimes_k \cdots \otimes_k \mathcal{X}(X_0, X_1), \mathcal{M}(X_0, X_p))$$

with the usual differential (see [Mit72]).

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The Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{X},\mathcal{M})$ is the cohomology of the Hochschild complex $\mathbf{C}^\bullet(\mathcal{X},\mathcal{M})$.

Remark 4.2. Let $\mathcal{X}$ be a k-linear category, A be a k-algebra, and $\mathcal{M}$ be a k-central $\mathcal{X}$-bimodule. There is a morphism

$$\mathbf{C}^\bullet(\mathcal{X},\mathcal{M}) \to \mathbf{C}^\bullet(\mathcal{X}\otimes_k A, \mathcal{M}\otimes_k A)$$
$$\eta \mapsto \eta \cup 1$$

where $\eta\cup 1$ is defined by

$$\eta\cup 1(x_1\otimes a_1,\dots,x_n\otimes a_n) = \pm\eta(x_1,\dots,x_n)\otimes a_1\cdots a_n$$

for suitable composable arrows $x_1,\dots,x_n$ in $\mathcal{X}$, where the sign is given by the Koszul convention.

Definition 4.3. Let $\mathcal{X}$ be a k-linear A_∞-category and A be a k-algebra. The tensor product $\mathcal{X}\otimes A$ is the A_∞-category with the same objects as $\mathcal{X}$ and morphisms $\mathcal{X}(-,-)\otimes_k A$. The codifferential $b_{\mathcal{X}\otimes_k A}$ is given by the Taylor coefficients

$$b^1_{\mathcal{X}\otimes_k A}(s(x\otimes a)) = b^1_{\mathcal{X}}(sx)\otimes a,$$
$$b^n_{\mathcal{X}\otimes_k A}(s(x_1\otimes a_1),\dots,s(x_n\otimes a_n)) = \pm b^n_{\mathcal{X}}(sx_1,\dots,sx_n)\otimes a_1\cdots a_n$$

for suitable composable arrows, where the sign is given by the Koszul convention.

For the rest of this section, unless specified otherwise, X denotes a quasi-compact separated k-scheme.

Definition 4.4. If $M\in D(\mathcal{O}_X)$, then the *Hochschild cohomology* of M is defined as

$$\mathrm{HH}^*(X,M) := \mathrm{Ext}^*_{X\times X}(i_{\Delta,*}\mathcal{O}_X, i_{\Delta,*}M)$$

where $i_\Delta : X\to X\times X$ is the diagonal map.

Definition 4.5. Let $X=\bigcup_{i=1}^n U_i$ be an affine covering. For $I\subset\{1,\dots,n\}$ let $U_I=\bigcap_{i\in I}U_i$. Let $\mathcal{I}$ be the set $\{I\subset\{1,\dots,n\}\mid I\neq\emptyset\}$. Then $\mathcal{X}$ is defined to be the category with objects $\mathcal{I}$ and Hom-sets

$$\mathcal{X}(I,J) = \begin{cases}\mathcal{O}_X(U_J), & I\subset J,\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

Roughly this allows us to think of $\mathrm{Mod}(\mathcal{X})$ as the category of presheaves associated with an affine covering of X. This construction has many good properties, some of which are summarized in the following.

Lemma 4.6. *There is a fully faithful embedding*

$$w : D(\mathrm{Qch}(X)) \to D(\mathcal{X})$$

and a corresponding fully faithful embedding for bimodules

$$W : D^\delta(\mathrm{Qch}(X)) \to D(\mathcal{X}\otimes_k\mathcal{X}^\circ),$$

where $D^\delta(\mathrm{Qch}(X)) = i_{\Delta,*}D(\mathrm{Qch}(X))$ is the category with the same objects as $D(\mathrm{Qch}(X))$ and morphisms

$$\mathrm{Hom}_{D^\delta(\mathrm{Qch}(X))}(M,N) = \mathrm{Hom}_{D(\mathrm{Qch}(X\times X))}(i_{\Delta,*}M, i_{\Delta,*}N).$$

Moreover, for a quasi-coherent sheaf M on X we have

$$\mathrm{HH}^*(X,M)\cong \mathrm{HH}^*(\mathcal{X},W(M)). \tag{23}$$

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For A a (not necessarily commutative) k-algebra, there exists an A-equivariant version of w:

$$w : D(\mathrm{Qch}(\mathcal{O}_X \otimes_k A)) \to D(\mathcal{X} \otimes_k A),$$

which is also a fully faithful embedding.

Proof. For the construction of the embeddings w and W see [RVdBN19, § 8.3]. The proof of the equivalence (23) is also sketched in [RVdBN19, § 8.3], for a full proof see [LVdB05]. The A-equivariant version is constructed in [RVdBN19, § 8.5]. □

We also need a deformed version of $\mathcal{X}$. We give the definition in this case, but the general construction can be found in [RVdBN19, § 6].

Definition 4.7. Let $\mathcal{M}$ be a k-central $\mathcal{X}$-bimodule and $\eta \in \mathrm{HH}^n(\mathcal{X}, \mathcal{M})$. Let $\tilde{\mathcal{X}}$ be the DG-category $\mathcal{X} \oplus \Sigma^{n-2}\mathcal{M}$: its objects are the objects of $\mathcal{X}$, morphisms are given by $\mathcal{X}(-,-) \oplus \Sigma^{n-2}\mathcal{M}(-,-)$, and composition is coming from the composition in $\mathcal{X}$ and the action of $\mathcal{X}$ on $\mathcal{M}$.

Lift $\eta \in \mathrm{HH}^n(\mathcal{X}, \mathcal{M})$ to a Hochschild cocycle, which we also denote by η. We can think of η as a map $(\Sigma\mathcal{X})^{\otimes n} \to \Sigma(\Sigma^{n-2}\mathcal{M})$ of degree one.

We define as $\mathcal{X}_\eta$ the A_∞-category $\tilde{\mathcal{X}}$ with deformed A_∞-structure given by

$$b_{\mathcal{X}_\eta} := b_{\tilde{\mathcal{X}}} + \eta,$$

where $b_{(-)}$ denotes the codifferential on the corresponding bar construction giving the A_∞-structure, and where we view η as a map of degree one $(\Sigma\mathcal{X})^{\otimes n} \to \Sigma(\Sigma^{n-2}\mathcal{M})$ and extend it to a map $\eta : (\Sigma\mathcal{X}_\eta)^{\otimes n} \to \Sigma\mathcal{X}_\eta$ by making the unspecified component zero. Clearly we have $H^*(\mathcal{X}_\eta) = \tilde{\mathcal{X}}$; the only nontrivial Taylor coefficients of the codifferential $b_{\mathcal{X}_\eta}$ are $b_{\mathcal{X}_\eta,2}$ and $b_{\mathcal{X}_\eta,n}$.

Remark 4.8. For η a cocycle in $\mathbf{C}^\bullet(\mathcal{X}, \mathcal{M})$ and $\eta \cup 1$ the corresponding cocycle in $\mathbf{C}^\bullet(\mathcal{X} \otimes_k A, \mathcal{M} \otimes_k A)$ we have that $(\mathcal{X} \otimes_k A)_{\eta\cup 1} = \mathcal{X}_\eta \otimes_k A$.

Definition 4.9. Let $\mathcal{U} \in \mathrm{Mod}(\mathcal{X})$. A colift of $\mathcal{U}$ to $\mathcal{X}_\eta$ is a pair $(\mathcal{V}, \phi)$, where $\mathcal{V} \in D_\infty(\mathcal{X}_\eta)$ and ϕ is an isomorphism of graded $H^*(\mathcal{X}_\eta)$-modules $H^*(\mathcal{V}) \cong \mathrm{Hom}_{\mathcal{X}}(H^*(\mathcal{X}_\eta), \mathcal{U})$.

Proposition 4.10. *Assume that $\mathcal{M}$ is an invertible $\mathcal{X}$-bimodule and $\mathcal{X}_\eta$ is as in Definition 4.7. The object $\mathcal{U} \in \mathrm{Mod}(\mathcal{X})$ has a colift to $\mathcal{X}_\eta$ if and only if $c_{\mathcal{U}}(\eta) = 0$, where $c_{\mathcal{U}}$ is the characteristic morphism*

$$c_{\mathcal{U}}(\eta) : \mathrm{HH}^n(\mathcal{X}, \mathcal{M}) \to \mathrm{Ext}^n_{\mathcal{X}}(\mathcal{U}, \mathcal{M} \otimes_{\mathcal{X}} \mathcal{U})$$

obtained by interpreting $\eta \in \mathrm{HH}^n(\mathcal{X}, \mathcal{M})$ as a map $\mathcal{X} \to \Sigma^n\mathcal{M}$ in $D(\mathcal{X} \otimes_k \mathcal{X}^\circ)$ and then applying the functor $- \otimes_{\mathcal{X}} \mathcal{U}$ to get a map $U \to \Sigma^n\mathcal{M} \otimes_{\mathcal{X}} \mathcal{U}$.

Proof. This is a combination of [RVdBN19, Lemma 6.4.1] and [RVdBN19, Lemma 6.3.1]. □

5. Proof of Theorem 1.3

We now proceed to give a construction of an exact functor $L : D^b(\mathrm{coh}(X)) \to D_\infty(\mathcal{X}_\eta)$, originally given in [RVdBN19]. We summarize the construction in this particular case for the benefit of the reader. More details in the general setting are in [RVdBN19, § 10].

Construction 5.1. Let X be a smooth projective scheme of dimension $m \geq 3$, which has a tilting bundle. Let $M = \omega_X^{\otimes 2}$; by [RVdBN19, Lemma 9.6.1] we have $\mathrm{HH}^{2m}(X, M) \cong k$, so that we can pick $0 \neq \eta \in \mathrm{HH}^{2m}(X, M)$. View η as an element of $\mathrm{HH}^*(\mathcal{X}, \mathcal{M})$, for $\mathcal{M} = W(M)$, via (23). Construct the A_∞-category $\mathcal{X}_\eta$ as in Definition 4.7.

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We are now in the situation of [RVdBN19, § 10.1] and we can define an exact functor

$$L : D(\mathrm{Qch}(X)) \to D(\mathcal{X}_\eta^{\mathrm{dg}})$$

as in [RVdBN19, (10.3)], where $\mathcal{X}_\eta^{\mathrm{dg}}$ is the unital DG-hull of $\mathcal{X}_\eta$. Then we obtain our exact functor (also denoted by L) as the composition

$$L : D^b(\mathrm{coh}(X)) \hookrightarrow D(\mathrm{Qch}(X)) \xrightarrow{L} D(\mathcal{X}_\eta^{\mathrm{dg}}) \cong D_\infty(\mathcal{X}_\eta).$$

where $D_\infty(\mathcal{X}_\eta) \cong D_\infty(\mathcal{X}_\eta^{\mathrm{dg}}) \cong D(\mathcal{X}_\eta^{\mathrm{dg}})$ by [Lef03, Lemme 4.1.3.8] (cf. also Remark 2.1).

Lemma 5.2. *Let $T \in D^b(\mathrm{coh}(X))$ and $\mathcal{T} = w(T)$. Let L be the functor constructed in Construction 5.1. The following is a distinguished triangle in $D_\infty(\mathcal{X}_\eta)$:*

$$\mathcal{T} \xrightarrow{\alpha} L(T) \xrightarrow{\beta} \Sigma^{-2m+2}\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{T} \to . \tag{24}$$

Proof. This is the distinguished triangle in [RVdBN19, Lemma 10.3] under the equivalence of categories $D(\mathcal{X}_\eta^{\mathrm{dg}}) \cong D_\infty(\mathcal{X}_\eta)$. □

Lemma 5.3. *Let X be a smooth projective scheme of dimension $m \geq 3$ that has a tilting bundle. Then the exact functor*

$$L : D^b(\mathrm{coh}(X)) \to D_\infty(\mathcal{X}_\eta)$$

of Construction 5.1 is non-Fourier–Mukai (see Definition 2.2).

Proof. This proof follows the proof of [RVdBN19, Lemma 11.4] (the argument is written there for the case $m = 3$, but the proof is the same for a general $m \geq 3$). We repeat it here for the benefit of the reader.

Let T be a tilting bundle for X, $A = \mathrm{End}_X(T)$ and $\mathcal{T} = w(T)$. We can think of $\mathcal{T}$ as an element in $\mathrm{Mod}(\mathcal{X} \otimes_k A)$. If L were a Fourier–Mukai functor with an A_∞-lift ℓ such that $H^0(\ell) \cong L$, then by A_∞-functoriality $\ell(T)$ could be viewed as an object in $\mathcal{D}_\infty(\mathcal{X}_\eta \otimes_k A)$ and, consequently, $L(T)$ would be an element in $D_\infty(\mathcal{X}_\eta \otimes_k A)$.

On the other hand, thanks to the distinguished triangle (24) we have

$$H^*(L(T)) = \mathcal{T} \oplus \Sigma^{-2m+2}(\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{T}) = \mathrm{Hom}_{\mathcal{X}}(H^*(\mathcal{X}_\eta), \mathcal{T}).$$

By construction, this isomorphism is compatible with the $H^*(\mathcal{X}_\eta)$- and A-actions. Using Remark 4.8 we obtain that $L(T)$ is a colift of $\mathcal{T} \in \mathrm{Mod}(\mathcal{X} \otimes_k A)$ to $D_\infty((\mathcal{X} \otimes_k A)_{\eta \cup 1}) = D_\infty(\mathcal{X}_\eta \otimes_k A)$.

By Proposition 4.10 the obstruction against the existence of such a colift is the image of $\eta \cup 1$ under the characteristic morphism

$$\mathrm{HH}^{2m}(\mathcal{X} \otimes_k A, \mathcal{M} \otimes_k A) \xrightarrow{c_{\mathcal{T}}} \mathrm{Ext}^{2m}_{\mathcal{X} \otimes_k A}(\mathcal{T}, \mathcal{M} \otimes_k \mathcal{T}).$$

Let $c_{\mathcal{T},A}$ be the composition

$$\mathrm{HH}^{2m}(\mathcal{X}, \mathcal{M}) \xrightarrow{\eta \mapsto \eta \cup 1} \mathrm{HH}^{2m}(\mathcal{X} \otimes_k A, \mathcal{M} \otimes_k A) \xrightarrow{c_{\mathcal{T}}} \mathrm{Ext}^{2m}_{\mathcal{X} \otimes_k A}(\mathcal{T}, \mathcal{M} \otimes_{\mathcal{X}} \mathcal{T}).$$

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By the A-equivariant version of [RVdBN19, (8.14)] we have a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^{2m}(X,M) \arrow[r, "c_{T,A}"] \arrow[d, "W"', "\cong"] & \mathrm{Ext}^{2m}_{\mathrm{Qch}(\mathcal{O}_X\otimes_k A)}(T, M \overset{L}{\otimes}_X T) \arrow[d, "\cong"', "w"] \\
\mathrm{HH}^{2m}(\mathcal{X},\mathcal{M}) \arrow[r, "c_{\mathcal{T},A}"'] & \mathrm{Ext}^{2m}_{\mathcal{X}\otimes_k A}(\mathcal{T}, \mathcal{M} \overset{L}{\otimes}_{\mathcal{X}} \mathcal{T})
\end{tikzcd}$$

of A-equivariant characteristic maps. The rightmost map is an isomorphism by [RVdBN19, (8.13)] and the fact that the A-equivariant version of w is fully faithful (Lemma 4.6). The leftmost map is an isomorphism by (23). By [RVdBN19, Proposition 8.9.2], the upper horizontal map is also an isomorphism. It follows that, because we chose $\eta \neq 0$, its image $c_{\mathcal{T},A}(\eta)$ is also nonzero and provides an obstruction to the existence of a colift. Thus, L cannot be a Fourier–Mukai functor. □

Proof of Theorem 1.3. Let $\mathcal{A}$ be the smallest thick subcategory of $D_\infty(\mathcal{X}_\eta)$ containing the essential image of $D^b(\mathrm{coh}(X))$ under L. It is clear that the corestricted functor

$$L : D^b(\mathrm{coh}(X)) \to \mathcal{A}$$

is still non-Fourier–Mukai.

Let $\mathfrak{a}$ be the full sub-DG-category of $\mathcal{D}_\infty(\mathcal{X}_\eta)$ spanned by $\mathrm{Ob}(\mathcal{A})$ and let T be a tilting bundle for X. Then we have $H^0(\mathfrak{a}) = \mathcal{A}$. Let $\mathsf{R} = \mathfrak{a}(L(T), L(T))$. By Lemma 2.5 we have a quasi-equivalence $\mathfrak{a} \to \mathcal{P}\mathrm{erf}(\mathsf{R}^\circ)$. The composed functor

$$D^b(\mathrm{coh}(X)) \xrightarrow{L} \mathcal{A} \xrightarrow{\cong} \mathrm{Perf}(\mathsf{R}^\circ) \tag{25}$$

is still non-Fourier–Mukai because quasi-equivalences are invertible up to homotopy [Lef03, Théorème 9.2.0.4].

Let $\mathcal{T} = w(T)$ be the left $\mathcal{X}$-module associated with T and let $\mathcal{M} = W(M)$ be the $\mathcal{X}$-bimodule associated with M. By the discussion before [RVdBN19, (11.5)] we have a distinguished triangle of complexes of vector spaces (taking into account that in the current setting the quantity n in [RVdBN19, (11.5)] is equal to $2m$)

$$\mathrm{RHom}_{\mathcal{X}}(\Sigma^{-2m+2}\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{T}, \mathcal{T}) \to \mathrm{RHom}_{\mathcal{X}_\eta}(L(T), L(T)) \to \mathrm{RHom}_{\mathcal{X}}(\mathcal{T}, \mathcal{T}) \to .$$

Using [RVdBN19, Lemma 9.4.1] this becomes

$$\mathrm{RHom}_X(\Sigma^{-2m+2}M^{-1} \otimes_X T, T) \to \mathrm{RHom}_{\mathcal{X}_\eta}(L(T), L(T)) \to \mathrm{RHom}_X(T, T) \to$$

which is equivalent to

$$\Sigma^{2m-2}\,\mathrm{RHom}_X(T, M \otimes_X T) \to \mathrm{RHom}_{\mathcal{X}_\eta}(L(T), L(T)) \to \mathrm{RHom}_X(T, T) \to . \tag{26}$$

The cohomology of $\mathrm{RHom}_X(T, M \otimes_X T)$ is concentrated in degrees $\leq m$. Whence the cohomology of $\Sigma^{2m-2}\,\mathrm{RHom}_X(T, M \otimes_X T)$ is concentrated in degrees $\leq m - (2m-2) < 0$ (as $m \geq 3$). It now follows from (26) that R is an A_∞-algebra such that $H^*(\mathsf{R})$ is finite dimensional and concentrated in degrees ≤ 0 and moreover $H^0(\mathsf{R}) = \mathrm{End}_X(T)$. As $\mathrm{End}_X(T)^\circ$ is tautologically geometric we obtain by Corollary 3.6 a fully faithful Fourier–Mukai functor

$$\mathrm{Perf}(R^\circ) \hookrightarrow D^b(\mathrm{coh}(Y)). \tag{27}$$

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The functor (1) is now the composition of (25) and (27). To see that is non-Fourier–Mukai we factor it as

$$D^b(\mathrm{coh}(X)) \to \mathrm{Perf}(\mathsf{R}^\circ) \cong \mathrm{Perf}(\mathsf{R}^\circ)\tilde{} \subset D^b(\mathrm{coh}(Y)), \tag{28}$$

where $\mathrm{Perf}(\mathsf{R}^\circ)\tilde{}$ is the essential image of (27). Note that because A_∞-quasi-equivalences may be inverted up to homotopy by [Lef03, Théorème 9.2.0.4], the inverse of $\mathrm{Perf}(\mathsf{R}^\circ) \cong \mathrm{Perf}(\mathsf{R}^\circ)\tilde{}$ is also a Fourier–Mukai functor. Now if the composition (28) were Fourier–Mukai, then so would be the corestricted functor $D^b(\mathrm{coh}(X)) \to \mathrm{Perf}(\mathsf{R}^\circ)\tilde{}$. Hence, the composition

$$D^b(\mathrm{coh}(X)) \to \mathrm{Perf}(\mathsf{R}^\circ)\tilde{} \cong \mathrm{Perf}(\mathsf{R}^\circ)$$

would also be a Fourier–Mukai functor; but this composition is equivalent to (25). This is a contradiction. □

Remark 5.4. With a little bit more work one may show that the fact that (25) is non-Fourier–Mukai is also true without the hypothesis that X has a tilting bundle. However the tilting bundle is anyway needed for the rest of the construction.

Remark 5.5. If k is algebraically closed, then using Remark 3.9, one may show that Y can be chosen to have a full exceptional collection.

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