THE ANALYTICITY OF KERNELS

J. DE BARROS-NETO AND F. E. BROWDER

Let V be a paracompact real analytic manifold of dimension $n \geqslant 1$. Following the terminology of the theory of distributions of Schwartz **(4)**, $\mathfrak{D}(V)$ is the linear space of infinitely differentiable functions with compact support in V with the appropriate inductive limit topology, $\mathfrak{E}(V)$ is the Frechet space of infinitely differentiable functions on V, $\mathfrak{D}'(V)$ is the dual space of $\mathfrak{D}(V)$ consisting of the distributions on V, $\mathfrak{E}'(V)$ the dual space of $\mathfrak{E}(V)$ consisting of the distributions with compact support on V. Let $\mathfrak{A}(V)$ be the linear space of real analytic functions on V.

It is our purpose in the present paper to study the nature of the distribution kernels on $V \times V$, in the sense of Schwartz **(5)**, which are analytically very regular. Such kernels, which we shall refer to more briefly as analytic kernels, are obtained by the Schwartz kernel theorem from analytic mappings. The latter are defined as follows: Let L be a continuous linear mapping of $\mathfrak{D}(V)$ into $\mathfrak{E}(V)$. Its transpose (or adjoint) ${}^tL$ is then defined as a continuous linear mapping of $\mathfrak{E}'(V)$ into $\mathfrak{D}'(V)$. Then L is said to be analytic (or bi-analytic) if the following two conditions hold:

(i) ${}^tL$ maps $\mathfrak{D}(V)$ into $\mathfrak{E}(V)$. (It then follows by the closed graph theorem that ${}^tL$ maps $\mathfrak{D}(V)$ continuously into $\mathfrak{E}(V)$, so that by the reflexivity of the space $\mathfrak{E}(V)$ and $\mathfrak{D}(V)$, it follows that ${}^t({}^tL) = L$ maps $\mathfrak{E}'(V)$ continuously into $\mathfrak{D}'(V)$.)

(ii) If ψ lies in $\mathfrak{E}'(V)$ and is analytic on an open subset G of V, then $L\psi$ and ${}^tL\psi$ are both analytic on G.

Associated with each such mapping L by the Schwartz kernel theorem **(5)**, is a distribution $k_{x,y}$ on $V \times V$, infinitely differentiable off the diagonal, such that figuratively

$$(L\psi)(x) = \int k_{x,y}\psi(y)dy,$$

for all ψ in $\mathfrak{D}(V)$, or more precisely, using the pairing $\langle f, \Phi\rangle$ between a distribution f on $V \times V$ and $\Phi \in \mathfrak{D}(V \times V)$ (and similarly on V),

$$\langle L\psi, \psi_1\rangle = \langle k_{x,y}, \psi(y)\psi_1(x)\rangle,$$

for all ψ, ψ_1 in $\mathfrak{D}(V)$.

Our principal result is the following:

THEOREM. *The kernel $k_{x,y}$ of an analytic mapping L is analytic on the complement of the diagonal in $V \times V$.*

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The corresponding result with analyticity replaced by separate analyticity (that is, analyticity in each of x and y separately with the other held fixed) was obtained by de Barros-Neto in **(1)**. An essential tool in the proof of the above Theorem is the criterion for analyticity on $V \times V$ in terms of the analytic properties of a function in the separate variables given by Browder in **(2)**.

Obviously our theorem has important applications to the study of the structure of the fundamental solutions and other kernels obtained from boundary value problems for partial differential operators. We shall discuss these applications elsewhere.

In § 2, we consider some generalizations to general kernels.

The writers are indebted to Professor L. Nachbin for some stimulating discussions of the questions dealt with here.

1. We shall assume as the basis of a great deal of our discussion the theory of topological vector spaces as exposed by Grothendieck in **(3)**. By a standard procedure, we may imbed V in a complex analytic manifold $\tilde{V}$ of dimension n such that at each point x^0 of V, there exists a complex analytic co-ordinate system $\{z_1, \ldots, z_n\}$ in a neighbourhood U of x^0 in $\tilde{V}$ such that $U \cap V \subset \{z: \operatorname{Im}(z_j) = 0 \text{ for } 1 \leqslant j \leqslant n\}$, while $x_j = \operatorname{Re}(z_j)$ are the co-ordinates of a co-ordinate patch in V at x^0. We shall refer to $\tilde{V}$, which we shall fix, as the standard complexification of V.

If u is a function from $\mathfrak{A}(V)$, there exists a neighbourhood U of V in $\tilde{V}$ such that u may be extended to a holomorphic function on U. We shall denote the linear space of holomorphic functions on U by $\mathfrak{H}(U)$, topologizing the latter space by the usual topology of uniform convergence on compact subsets of U. We then define the topology on $\mathfrak{A}(V)$ as the inductive limit of the topologies induced by the mappings of $\mathfrak{H}(U)$ into $\mathfrak{A}(V)$ obtained by restricting functions from $\mathfrak{H}(U)$ to V (**3**, chapter IV, pp. 255–256**)**, where U runs over the neighbourhoods of V in $\tilde{V}$. Similarly, if A is a compact subset of V, $\mathfrak{A}(A)$ will denote the space of equivalence classes of functions analytic on a neighbourhood of A in $\tilde{V}$, and topologized as the inductive limit of $\mathfrak{H}(U_k)$ where U_k runs through a fundamental sequence of neighbourhoods of A in $\tilde{V}$. It follows that $\mathfrak{A}(A)$ is the inductive limit of a sequence of Frechet spaces. Grothendieck has shown (**3**, Theorem 1, Corollary 2, pp. 268–270**)** that if every bounded closed subset of a generalized $(\mathfrak{LF})$-space E is complete and if $\{E_j\}$ is a sequence of definition for the space E, then each bounded subset B of E is contained and bounded in one of the spaces E_j. By (**3**, 4(6), p. 315**)**, for a compact set A in V, $\mathfrak{A}(A)$ is complete, and hence quasi-complete (that is, bounded closed subsets are complete in the induced uniformity). It follows then that we have:

LEMMA 1. *For each compact subset A of V, if B is a bounded subset of $\mathfrak{A}(A)$, then each function f in B can be extended to a fixed neighbourhood U of A in $\tilde{V}$*

(depending of course upon B), and there exists a constant M such that for the extended function f(z),

$$|f(z)| \leqslant M$$

for z in U.

Proof. The proof follows from the above remarks together with the familiar form of the bounded subsets in $\mathfrak{H}(U_1)$, where U_1 is a neighbourhood containing the compact $\bar{U}$ in its interior.

LEMMA 2. *A necessary and sufficient condition that a subset B of $\mathfrak{A}(A)$ should be bounded in $\mathfrak{A}(A)$, A compact, is that there exists a finite family of co-ordinate patches on A such that in each of the co-ordinate patches there exists a constant C_0 such that*

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x)\right| \leqslant C_0{}^{\Sigma_j \alpha_j}(\pi_j \alpha_j!)$$

Proof. This is an easy consequence of the Cauchy integral formula for polycylinders as far as necessity is concerned, and of an obvious majorization of the power series of f as far as sufficiency is concerned.

By another result of Grothendieck (**3**, first Corollary on p. 36), it follows that if E is a Frechet space which is also a Schwartz space, then its dual E' is a complete bornological space. By (**3**, Proposition 5, p. 336), for every open subset G of the separable manifold V, $\mathfrak{E}(G)$ is a Schwartz space and obviously a Frechet space. Hence $\mathfrak{E}'(G)$ is a complete bornological space. By (**3**, Theorem 2, p. 271), every closed mapping of a complete bornological space E (hence strictly bornological) into a generalized $(\mathfrak{LF})$-space F is continuous. Applying this result to the spaces $E = \mathfrak{E}'(G)$, $F = \mathfrak{A}(A)$ with A compact, we obtain:

LEMMA 3. *Let T be a closed linear mapping of $\mathfrak{E}'(G)$ into $\mathfrak{A}(A)$ with G open in V, A compact. Then T is continuous. In particular, this is true if T is the restriction of a continuous mapping of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$.**

Proof. The first assertion of Lemma 3 was established in the remarks preceding the statement of the Lemma. For the second assertion, it suffices to show that if T is the restriction of a continuous mapping of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$, then T is closed as a mapping of $\mathfrak{E}'(G)$ into $\mathfrak{A}(A)$. However, the topology of $\mathfrak{A}(A)$ is consistent with the induced topology from $\mathfrak{D}'(V)$, so that the last conclusion follows immediately.

LEMMA 4. *Let f be a distribution on $V \times V$ such that f is separately analytic off the diagonal in $V \times V$ (that is, $f(x,.)$ is analytic for fixed x in y for $y \neq x$,*

*By this we mean precisely the following: There exists a continuous mapping T_1 of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$ such that for all u, $T_1 u$ is analytic on a neighbourhood of A and Tu is the equivalence class of $T_1 u$ taken as an element of $\mathfrak{A}(A)$.

while $f(,,y)$ is analytic for fixed y for $x \neq y$). Suppose the mappings Φ_1, and Φ_2, with

$$\Phi_1(x) = f(x,.), \qquad \Phi_2(y) = f(.,y)$$

are continuous mappings of an open set G_1 into $\mathfrak{A}(G_2)$ for another open set G_2 (G_2 into $\mathfrak{A}(G_1)$) in V, $G_1 \cap G_2 = \phi$. Then f is analytic on $G_1 \times G_2$.

Proof. It suffices to consider compact subsets A_1 and A_2 of G_1 and G_2, respectively. Since Φ_1 and Φ_2 are continuous, $\Phi_1(A_1)$ is bounded in $\mathfrak{A}(A_2)$ and $\Phi_2(A_2)$ is bounded in $\mathfrak{A}(A_1)$. Hence, by Lemma 1, there exist neighbourhoods U_1 of A_1 in $\tilde{V}$ and U_2 of A_2 in $\tilde{V}$ such that $f(x,.)$ may be extended for fixed x in A_1 to a function $f(x, z)$ holomorphic for z in A_2 with $|f(x, z)| \leqslant M$, while $f(.,y)$ may be extended to a holomorphic function $f(z, y)$ on U_1 for fixed y in A_2 with $|f(z, y)| \leqslant M$ for all z in U_1, y in A_2. However, these are precisely the hypotheses of Theorem 1 of Browder **(2)** in order that f should be analytic on $G_1 \times G_2$, and thus the Lemma is established.

LEMMA 5. *Let $k_{x,y}$ be a distribution on $V \times V$, infinitely differentiable off the diagonal. Let G_1 and G_2 be two open subsets of V with disjoint compact closures. Then for each x in G_1, y in G_2,*

$$k_{x,y} = L(\delta_y)(x),$$

$$k_{x,y} = {}^tL(\delta_x)(y).$$

Proof. Since $k_{x,y}$ is infinitely differentiable off the diagonal, k is infinitely differentiable on $G_1 \times G_2$. It follows that

$$L(\delta_y)(x) = \langle k_{x,y_1}, \delta_y(y_1)\rangle = k_{x,y}$$

The other equation follows similarly.

Proof of the Theorem. Let G_1 and G_2 be any two open subsets of V with disjoint compact closures. It suffices to prove that $k_{x,y}$ is analytic on $G_1 \times G_2$. Let $\mathfrak{E}'(G_1)$ be imbedded in $\mathfrak{E}'(V)$ in the natural way. Then for each u in $\mathfrak{E}'(G_1)$, u realized as an element of $\mathfrak{E}'(V)$ is analytic (in fact identically zero) on G_2. Thus ${}^tLu$ restricted to G_2 lies in $\mathfrak{A}(G_2)$. Let T be this restriction. By Lemma 3, T is a continuous mapping of $\mathfrak{E}'(G_1)$ into $\mathfrak{A}(G_2)$. Consider

$$\Phi_1(x) = ({}^tL)(\delta_x).$$

Φ_1 is a continuous mapping of G_1 in $\mathfrak{A}(G_2)$. Similarly, if Φ_2 is given by

$$\Phi_2(y) = L(\delta_y),$$

Φ_2 is a continuous map of G_2 into $\mathfrak{A}(G_1)$. But $k_{x,y} = \Phi_1(x)(y) = \Phi_2(y)(x)$, by Lemma 5.

Hence, by Lemma 4, $k_{x,y}$ is analytic on $G_1 \times G_2$, and the proof of Theorem 1 is complete.

The above proof of Theorem 1 may be applied with purely formal changes to establish the following:

THEOREM 2. *Let V and W be two paracompact real analytic manifolds, L a mapping of* $\mathfrak{E}'(W)$ *into* $\mathfrak{E}(V)$, ${}^tL$ *the dual map of* $\mathfrak{E}'(V)$ *into* $\mathfrak{E}(W)$. *Suppose that* $L(\mathfrak{E}'(W))$ *and* ${}^tL(\mathfrak{E}'(V))$ *are composed of real analytic functions. Then the kernel* $k_{x,y}$ *of L is analytic in* $V \times W$.

2. We state without going into detail a simple generalization of the results of § 1 to general kernels, which may be used in particular for the integral kernels associated with hyperbolic boundary value problems. We shall return to this question in another place.

If A is an open set of the real analytic manifold, V, we designate by $\mathfrak{E}_A'(V)$ the subspace of $\mathfrak{E}'(V)$ of distributions which are analytic on some neighbourhood of A in V.

We shall consider a distribution kernel $k_{x,y}$ on $V \times W$, L and ${}^tL$ the associated mappings, with L mapping $\mathfrak{D}(W)$ into $\mathfrak{D}'(V)$, ${}^tL$ mapping $\mathfrak{D}(V)$ into $\mathfrak{D}'(W)$.

DEFINITION. *Let y be a point of W. The closed subset A of V is said to be a domain of dependence for y with respect to* ${}^tL$ *if there exists a neighbourhood U of y in W such that* ${}^tL$ *can be extended to a continuous mapping of* $\mathfrak{E}_A'(V)$ *into* $\mathfrak{D}'(U)$ *(restricting the values of* ${}^tL$ *to U), such that the image* ${}^tL(\mathfrak{E}_A'(V))$ *is composed of functions analytic in U. A similar definition may be given for a domain of dependence A′ for x in V with respect to L.*

DEFINITION. *The point* (x, y) *in* $V \times W$ *is said to be regular with respect to L provided that there exists a domain of dependence A for y with respect to* ${}^tL$ *which does not contain x and a domain of dependence A′ for x with respect to L which does not contain y.*

THEOREM 3. *Let* $k_{x,y}$ *be a distribution kernel, R the set of points in* $V \times W$ *which are regular with respect to* $k_{x,y}$ *(that is, regular with respect to the corresponding mapping L of which* $k_{x,y}$ *is the kernel). Then R is an open subset of* $V \times W$, *and* $k_{x,y}$ *is analytic on R.*

The proof of Theorem 3 is a simple variant of the proof of Theorem 1.

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Instituto de Matematica Pure e Aplicada
Rio de Janeiro
and
Yale University