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WEAKLY REDUCTIVE SEMIGROUPS WITH ATOMISTIC CONGRUENCE LATTICES

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Abstract

The structure of semigroups with atomistic congruence lattices (that is, each congruence is the supremum of the atoms it contains) is studied. For the weakly reductive case the problem of describing the structure of such semigroups is solved up to simple and congruence free semigroups, respectively. As applications, all commutative, finite, completely semisimple semigroups, respectively, with atomistic congruence lattices are described.

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1. Introduction and Preliminaries

A lattice is atomistic if each element is the supremum of the atoms it contains. Examples are the chain of two elements, the power set lattice of a set or the partition lattice of some set. In [3] it is shown that a semilattice has an atomistic congruence lattice if and only if it is a locally finite tree. In this paper we study the structure of semigroups whose congruence lattices are atomistic. Examples are congruence free semigroups (as a trivial case), left (right) zero semigroups, null semigroups, rectangular bands and semigroups whose congruence lattice is Boolean. In the second section we obtain *necessary* conditions on a semigroup in order that its congruence lattice be atomistic. The main tool for investigating the structure of such semigroups S is to consider the decomposition of S into its $\mathcal{J}$-classes. We introduce

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the construction of trees of 0-simple semigroups and show that each globally idempotent semigroup with an atomistic congruence lattice can be so constructed (Theorem 1). The structure of such a semigroup can be described by a locally finite tree X, 0-simple semigroups I_α indexed by the elements of X and partial homomorphisms between the nonzero parts of the semigroups I_α. Furthermore, if an arbitrary semigroup has an atomistic congruence lattice then it is an inflation of such a semigroup.

In Section 3 we study the problem for weakly reductive semigroups and obtain necessary and sufficient conditions in order that the congruence lattice be atomistic (Theorem 2). Using this characterization we are able to characterize all commutative, finite and completely semisimple semigroups with atomistic congruence lattices (up to locally finite trees and simple groups). This will be done in Section 4. Furthermore, we observe that the properties "atomistic" and "Boolean", "complemented modular" and "relatively complemented" for the congruence lattice of a weakly reductive semigroup are strongly connected.

For the remainder of this part we collect some definitions and results which are basic for our considerations (for further details see [4] or [6]).

A semilattice is a (*locally finite*) *tree* if each interval $[x, y] = \{z : x \leq z \leq y\}$ is a (finite) chain. For a semigroup S, $S^* = S$ if S has no zero and $S^* = S\backslash\{0\}$ if 0 is the zero of S; and $S^1 = S$ if S has an identity and $S^1 = S \cup \{1\}$ such that $1 \notin S$ and $s1 = 1s = s$ for all $s \in S$ otherwise. Green's relation J is defined by $a\ J\ b$ if and only if $J(a) = J(b)$, where $J(x)$ is the principal ideal generated by x (that is, $J(x) = S^1xS^1$). The J-class containing a is denoted by J_a. The set $I(a) = J(a)\backslash J_a = \{x \in J(a) : J(x) \neq J(a)\}$ is an ideal in $J(a)$ (or empty). The semigroup $J(a)/I(a)$ is called a *principal factor*. Each principal factor is either simple, 0-simple or null (see [4]). A semigroup is (*completely*) *semisimple* if each principal factor is (completely) (0)-simple. Let S be a subsemigroup of a semigroup T. Then T is an inflation of S if there exists a function $f : T \to S$ such that $f|S = \mathrm{id}_S$ and $ab = (af)(bf)$ for all $a, b \in T$. In this case f is the *inflation function*. A semigroup S is *weakly reductive* if for $a, b \in S$, $za = zb$ and $az = bz$ for all $z \in S$ imply $a = b$. A semigroup S is *globally idempotent* if $S^2 = S$. The lattice of all congruences on a semigroup S is denoted by $\mathrm{Con}\, S$. The identical and the universal relations are denoted by $\varepsilon_S = \varepsilon$ and $\omega_S = \omega$, respectively. A congruence ρ on S is an *atom* if it *covers* ε, to be denoted by $\rho \succ \varepsilon$, that is, $\varepsilon < \rho$ and $[\varepsilon, \rho] = \{\varepsilon, \rho\}$. The set of all atoms of $\mathrm{Con}\, S$ is denoted by $\mathrm{At}\, S$. For an arbitrary relation R on S, R^* is the congruence on S which is generated by R.

For 0-simple semigroups we have the following result.

RESULT 1 [5]. *A 0-simple semigroup is congruence free if and only if for any two distinct elements* $x, y \in S$ *there exist* $u, v \in S$ *such that* $uxv = 0$ *and* $uyv \neq 0$ *or* $uxv \neq 0$ *and* $uyv = 0$.

2. Trees of 0-simple semigroups

In this section we obtain necessary conditions on an arbitrary semigroup S in order that its congruence lattice be atomistic. For this purpose we study the decomposition of S into its J-classes and the ordered set S/J. We introduce the construction of trees of 0-simple semigroups and show that each globally idempotent semigroup S whose congruence lattice is atomistic can be so constructed.

LEMMA 1. *Let* $\rho \in \operatorname{At} S$ *and* $a\rho b$ *for* $J_a > J_b$. *Then* $J_x < J_a$ *implies that* $J_x \leq J_b$. *In this case* $J(a) = J_a \cup J(b)$.

PROOF. Let $x = sat$ for $s, t \in S^1$. Then $x = sat\rho sbt$. Since J_x and J_b are contained in $I(a)$ we obtain that $sat = sbt$. Otherwise $(sat, sbt)^*$ is a proper congruence which is strictly contained in ρ.

Conversely, if $\operatorname{Con} S$ is atomistic any two of such neighbours in S/J are "linked" by an atom.

LEMMA 2. *Let* $J_b < J_a$ *and assume that* $J_x < J_a$ *implies that* $J_x \leq J_b$. *If* $\operatorname{Con} S$ *is atomistic then there exists* $v \in J_b$ *and an atom* ρ *such that* $a \rho v$.

PROOF. We consider the congruence $(a, b)^*$. Since $\operatorname{Con} S$ is atomistic there exist $\rho_1, \ldots, \rho_n \in \operatorname{At} S$ such that $a = a_0\rho_1 a_1 \cdots \rho_n a_n = b$ for certain $a_i \in S$ and $\rho_i \subseteq (a, b)^*$. Since $\{a, b\} \subseteq J(a)$, all a_i are contained in $J(a)$, that is, $J_{a_i} \leq J_a$. Let i be the smallest index such that $J_{a_i} < J_{a_{i-1}} = J_a$. The assumption on J_b then implies that $J_{a_i} \leq J_b$. Since $J_a = J_{a_{i-1}}$ and $a_{i-1}\,\rho_i\,a_i$, by Lemma 1, we get that $J_{a_i} \geq J_b$ and thus $J_{a_i} = J_b$. Also aJa_{i-1} implies that $a = sa_{i-1}t$ for some $s, t \in S^1$ and thus $a = sa_{i-1}t\,\rho_i\,sa_it$. Now $J_{sa_it} \leq J_b$ and Lemma 1 imply that $J_{sa_it} = J_b$.

LEMMA 3. *If* $\operatorname{Con} S$ *is atomistic then* S/J *is a locally finite tree.*

PROOF. S/J is directed so it suffices to show that each interval in S/J is a finite chain. Let $J_a > J_b$; there exist atoms $\rho_1, \ldots, \rho_n$ such that $a = b_0\,\rho_1\,b_1 \cdots \rho_n\,b_n = b$ and $b_i \in J(a)$ for all i. Let $g(a, b) = n$ be the shortest possible length of all such sequences and

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$$h(J_a, J_b) = \min\{g(x, y): x\,J\,a, y\,J\,b\}.$$

By induction on $h(J_a, J_b)$ we show that each interval in S/J is a finite chain. If $h(J_a, J_b) = 1$ then an immediate consequence of Lemma 1 is that $[J_b, J_a] = \{J_b, J_a\}$. Let $J_b < J_a$, $h(J_a, J_b) = n > 1$ and suppose that $[J_x, J_y]$ is a finite chain whenever $h(J_y, J_x) < n$. Let $u = a_0\,\rho_1\,a_1 \cdots \rho_n\,a_n = v$ for certain $\rho_i \in \operatorname{At} S$ such that $u\,J\,a$, $v\,J\,b$ and $a_i \in J(a)$ for all i. In particular, $J_{a_i} \leq J_a$. If $J_{a_i} = J_a$ then $h(J_a, J_b) = h(J_{a_i}, J_b) < n$, a contradiction. Therefore $J_{a_i} < J_a$. Also, since $J(a) = J_a \cup J(a_1)$ then $a_i \in J(a_1)$. Therefore, $J_{a_1} < J_a$ and $a_k \in J(a_1)$ for all $k \geq 1$. In particular $h(J_{a_1}, J_b) < n$ and our assumption applies: $[J_b, J_{a_1}]$ is a finite chain. If $J_x \in [J_b, J_a]$ then $J_x = J_a$ or $J_x \leq J_{a_1}$. Therefore $[J_b, J_a] = [J_b, J_{a_1}] \cup \{J_a\}$ is a finite chain.

In the next statements let us assume that $\operatorname{Con} S$ is atomistic.

LEMMA 4. *Let $J_a \succ J_b$. Then there exists a partial homomorphism $f: J_a \to J_b$ so that $xy = (xf)y$ and $yx = y(xf)$ for all $x \in J_a$ and $y \in S$ such that $xy, yx \in J(b)$, respectively. In particular, if $xy \in J(b)$ for $x, y \in J_a$ then $xy = (xf)(yf)$.*

PROOF. By Lemma 2, there exists an atom ρ such that $a\,\rho\,u$ for some $u \in J_b$. Let $x \in J_a$; $x = sat$ for some $s, t \in S^1$. Then $x = sat\,\rho\,sut$. By Lemma 1, $sut \in J_b$. If $x\,\rho\,v$ for some $v \in J_b$ then $v = sut$ since $\rho|J(b) = \varepsilon$. Thus for each $x \in J_a$ there exists a unique element in J_b to be denoted by xf such that $x\,\rho\,xf$. Let $y \in S$ such that $xy \in J(b)$. Then $xy\,\rho\,(xf)y$. Then $\rho|J(b) = \varepsilon$ implies that $xy = (xf)y$. Now let $x, y \in J_a$ such that $xy \in J_a$. Then $x\,\rho\,xf$, $y\,\rho\,yf$ and the definition of f imply that $(xy)f\,\rho\,xy$ and $xy\,\rho\,(xf)(yf)$. Since $\rho|J(b) = \varepsilon$ we get that f is a partial homomorphism.

This result can be extended to any comparable J-classes.

LEMMA 5. *Let $J_a > J_b$. Then there exists a partial homomorphism $f: J_a \to J_b$ such that $xz = (xf)z$ or $zx = z(xf)$ for all $x \in J_a$, $z \in S$ such that xz or $zx \in J(b)$, respectively.*

PROOF. The interval $[J_b, J_a]$ is a finite chain so there exist unique J_{a_i} such that $J_a = J_{a_0} \succ J_{a_1} \succ \cdots \succ J_{a_n} \succ J_b$. Let $f_i: J_{a_{i-1}} \to J_{a_i}$ be the mapping considered in Lemma 4. Let $f = f_1 f_2 \cdots f_n$. Then f is a partial homomorphism and for $x \in J_a$, $z \in S$ such that $xz \in J(b)$, by Lemma 4 we obtain that $xz = (xf_1)z = (xf_1f_2)z = \cdots = (xf_1f_2\cdots f_n)z = (xf)z$. The analogous argument for zx completes the proof.

PROPOSITION 1. *Each non-maximal J-class of S is the non-zero part of a 0-simple semigroup. In particular, S^2 is semisimple.*

PROOF. Let $J_b < J_a$; $b = sat$ for some $s, t \in S^1$. Since $J_b \leq J_s, J_a, J_t$ we obtain that $b = (sf)(ag)(th)$ where f, g, h denote the mappings constructed in Lemma 5 so that $sf, ag, th \in J_b$.

LEMMA 6. *The semigroup S is an inflation of S^2.*

PROOF. The case where $S = S^2$ is trivial. Let $a \in S \backslash S^2$ and let J_b denote the unique $\mathcal{J}$-class which is covered by J_a. Let $f_a: J_a \to J_b$ be the partial homomorphism constructed in Lemma 4. Let $z \in S$; a cannot be written as a product so $J_{az} < J_a$ and thus $J_{az} \leq J_b$. By Lemma 4, $az = (af_a)z$ and by analogy, $za = z(af_a)$. Now define $f: S \to S^2$ by $xf = xf_x$ if $x \in S \backslash S^2$ and $xf = x$ otherwise. Then f is an inflation function.

Since inflations are trivial from an algebraic point of view we consider the semisimple semigroup S^2 rather than S itself. The results so far motivate the following construction.

CONSTRUCTION. Let X be a locally finite tree, to each $\alpha \in X$ associate a 0-simple semigroup I_α ($\neq \{0\}$) so that $I_\alpha \cap I_\beta = \varnothing$ if $\alpha \neq \beta$. For $\alpha \in X^*$ let $f_\alpha: I^*_\alpha \to I^*_{\alpha^+}$ be a partial homomorphism where α^+ denotes the unique element of X such that $\alpha \succ \alpha^+$. Let $f_{\alpha,\alpha} = \mathrm{id}_{I^*_\alpha}$ and $f_{\alpha,\beta}$ be defined by $f_{\alpha,\beta} = f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}$ where the α_i's are defined by $\alpha = \alpha_1 \succ \alpha_2 \cdots \alpha_n \succ \beta$. We suppose that for arbitrary $a \in I^*_\alpha$ and $b \in I^*_\beta$ the set

$$D(a, b) = \{\gamma \in X: (af_{\alpha,\gamma})(bf_{\beta,\gamma}) \text{ is defined in } I^*_\gamma\}$$

is not empty. Let $\delta(a, b)$ denote the greatest element of $D(a, b)$. Let $S = \bigcup(I^*_\alpha: \alpha \in X)$ and define a multiplication $*$ on S by the rule

$$a * b = (af_{\alpha,\delta(a,b)})(bf_{\beta,\delta(a,b)}) \qquad (a \in I^*_\alpha,\ \beta \in I^*_\beta)$$

where the right hand side product is defined in $I^*_{\delta(a,b)}$.

DEFINITION. The groupoid S is a *tree of* 0*-simple semigroups*, to be denoted by $S = (X; I_\alpha, f_{\alpha,\beta})$. If each I_α, $\alpha \in X$, is congruence free (with zero and not the null semigroup of order two) then S is a *tree of congruence free semigroups*.

If X has a least element μ then by definition I^*_μ is closed under multiplication and thus is a simple semigroup. If, in addition, S is a tree of congruence free semigroups then the congruence freeness of $I^*_\mu \cup \{0\}$ implies that I^*_μ consists of exactly one element. A straightforward verification shows that S is a semigroup. Similar constructions appear in [1], [2], [7], [9], [12].

We now are able to formulate

THEOREM 1. *If S is globally idempotent and* Con S *is atomistic then S is a tree of* 0*-simple semigroups.*

PROOF. By Proposition 1 we observe that S is semisimple and hence each principal factor is (0-)simple. Let $X = S/J$. X is a locally finite tree. For $\alpha = J_a$ let $I_\alpha = J(a)/I(a)$. Then $I_\alpha^* = J_a$ and $S = \bigcup\{I_\alpha^* : \alpha \in X\}$. For $\alpha \succ \alpha^+$ let f_α be equal to the mapping $f: J_a \to J_{a^+}$ such that $a \rho af$ for some atom ρ which was obtained in Lemma 4. Let $a, b \in S$, $a \in I_\alpha^* = J_a$ and $b \in I_\beta^* = J_b$ and let $\gamma = J_{ab}$. Let $f_{\alpha,\gamma}$ and $f_{\beta,\gamma}$ be defined according to the rules of the construction. Then by Lemma 5 we have that $(af_{\alpha,\gamma})(bf_{\beta,\gamma}) = ab \in I_\gamma^*$. Therefore $D(a, b)$ is not empty. Also, $\gamma = J_{ab}$ is the greatest element of $D(a, b)$. To see this suppose that $d = (af_{\alpha,\delta})(bf_{\beta,\delta}) \in I_\delta^*$ for some $\delta \geq \gamma$. Then $ab \in J(d)$ and so again by Lemma 5 we obtain that $ab = (af_{\alpha,\delta})(bf_{\beta,\delta})$ which implies that $\gamma = \delta$.

3. Weakly reductive semigroups

We now restrict our investigations to the case when S is weakly reductive. A weakly reductive semigroup S cannot be an inflation of a semigroup $T \neq S$. Therefore, if $\operatorname{Con} S$ is atomistic then weak reductivity of S implies global idempotency and thus we may assume that $S = (X; I_\alpha, f_{\alpha,\beta})$, a tree of 0-simple semigroups. In the next statements we assume that $\operatorname{Con} S$ is atomistic and S is weakly reductive. Lemma 7 is straightforward to prove.

LEMMA 7. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$. Then S is weakly reductive if and only if each principal ideal of S is weakly reductive.*

DEFINITION. Let $\alpha \in X$ and $x, y \in I_\alpha^*$. We define a relation τ_α by

$$x \tau_\alpha y \Leftrightarrow (uxv \in I_\alpha^* \Leftrightarrow uyv \in I_\alpha^* \, \forall u, v \in I_\alpha^*).$$

Then $\tau_\alpha \cup \{(0_\alpha, 0_\alpha)\}$ is the greatest congruence on I_α which saturates I_α^*, that is, in particular, the greatest nonuniversal congruence on I_α.

LEMMA 8. *Let $\alpha \in X^*$. Then the restriction of f_α to an arbitrary τ_α-class is injective.*

PROOF. Let ρ_α denote the greatest congruence on S which saturates I_α^*, in particular,

$$x \rho_\alpha y \Leftrightarrow (uxv \in I_\alpha^* \Leftrightarrow uyv \in I_\alpha^* \, \forall u, v \in S^1).$$

We will prove that $\rho_\alpha | I_\alpha^* = \tau_\alpha$. Obviously we have that $\rho_\alpha | I_\alpha^* \subseteq \tau_\alpha$. Suppose that $x \tau_\alpha y$ but $(x, y) \notin \rho_\alpha$. We may assume there exist $u, v \in S^1$ such that $uxv \in I_\alpha^*$ and $uyv \notin I_\alpha^*$ and $\{u, v\} \cap S \neq \varnothing$. Then $J_u, J_v \geq J_x$ and so $uxv = (uf)x(vg)$ and $uyv = (uf)y(vg)$ such that $uf, vg \in (I_\alpha^*)^1$ and $\{uf, vg\} \cap I_\alpha^* \neq \varnothing$. If $\{uf, vg\} \subseteq I_\alpha^*$ then the proof is finished. If not then we may suppose that $uf \in I_\alpha^*$ and $v \notin S$. Since I_α is 0-simple there exists $w \in I_\alpha^*$ such that $(uf)xw \in I_\alpha^*$. Then $(uf)yw \notin I_\alpha^*$ since $(uf)y \notin I_\alpha^*$. This again is a contradiction to $x \tau_\alpha y$.

Now suppose that $xf_\alpha = yf_\alpha$ for x, y such that $x \tau_\alpha y$. Then $x \rho x f_\alpha = yf_\alpha \rho y$ for some $\rho \in \operatorname{At} S$. ρ does not saturate I_α^* thus $\rho \cap \rho_\alpha \neq \rho$ which implies that $\rho \cap \rho_\alpha = \varepsilon$. Therefore $x = y$.

Lemma 9. *Let $\alpha \in X^*$. Then the restriction of f_α to an arbitrary τ_α-class is constant.*

Proof. Let $x, y \in I_\alpha^*$ with $x \tau_\alpha y$. The congruence ρ_α as defined in Lemma 8, is a supremum of atoms. So $x = a_0 \rho_1 a_1 \cdots \rho_n a_n = y$ for certain elements a_i and atoms $\rho_i \subsetneq \rho_\alpha$. Since ρ_α saturates I_α^*, we observe that $a_i \in I_\alpha^*$ for all i. Let $z \in I_\gamma^*$ for some $\gamma < \alpha$. We have that $\rho_i | I_\delta^* = \varepsilon$ for all $\delta < \alpha$ since ρ_i is an atom. Hence $a_i z = a_{i+1} z$ and $z a_i = z a_{i+1}$ for all i. Multiplying the sequence $x = a_0 \rho_1 a_1 \cdots \rho_n a_n = y$ by z on the left and right, respectively, we obtain that $(xf_\alpha)z = xz = yz = (yf_\alpha)z$ and $z(xf_\alpha) = zx = zy = z(yf_\alpha)$, respectively. Weak reductivity of the semigroup $I = \bigcup(I_\gamma^* : \gamma < \alpha)$ then implies that $xf_\alpha = yf_\alpha$.

Proposition 2. *If α is not minimal in X then I_α is congruence free.*

Proof. τ_α is the identical relation. Therefore, by Result 1, I_α is congruence free.

Lemma 10. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of 0-simple semigroups. Let $\alpha \geq \beta \geq \gamma \geq \delta \in X$ and $x \rho y$ for some $x \in I_\alpha^*$, $y \in I_\delta^*$ and $\rho \in \operatorname{Con} S$. Then $zf_{\beta,\gamma} \rho z$ for all $z \in I_\beta^*$.*

Proof. See [1, Lemma 9].

Using the following definition, the mapping f_α may be regarded as a binary relation on S:

$$xf_\alpha y \Leftrightarrow x \in I_\alpha^* \quad \text{and} \quad xf_\alpha = y.$$

Lemma 11. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of congruence free semigroups. Then $\rho \in \operatorname{Con} S$ is an atom if and only if $\rho = (f_\alpha \cup \varepsilon_S) \circ (f_\alpha^{-1} \cup \varepsilon_S)$ for some $\alpha \in X^*$.*

Proof. Let $\xi = (f_\alpha \cup \varepsilon_S) \circ (f_\alpha^{-1} \cup \varepsilon_S)$ for some $\alpha \in X$. Then for $u \neq v$ we have $u \xi v$ if and only if $u = v f_\alpha$, $v = u f_\alpha$ or $u f_\alpha = v f_\alpha$. It can be seen easily that ξ is a congruence. Let η, where $\varepsilon \neq \eta \subseteq \xi$, be a congruence. Then $u \eta v$ for some $u \neq v$. If $u = v f_\alpha$ then $z \eta z f_\alpha$ for all $z \in I_\alpha^*$ by Lemma 10 and so $\eta = \xi$. If $u f_\alpha = v f_\alpha$ and $u \neq v$ then there exist $x, y \in I_\alpha^*$ such that $xuy \in I_\alpha^*$ and $xvy \notin I_\alpha^*$, or conversely. Again by Lemma 10, we obtain that $\eta = \xi$. Conversely, let ρ be an arbitrary congruence and $x \rho y$ for $x \neq y$ where $x \in I_\alpha^*$, $y \in I_\beta^*$. If $\alpha \neq \beta$ then we assume that $\alpha\beta < \alpha$. It is easy to see that $z f_{\alpha, \alpha\beta} \rho z$ for all $z \in I_\alpha^*$ and thus $(f_\alpha \cup \varepsilon_S) \circ (f_\alpha^{-1} \cup \varepsilon_S) \subseteq \rho$. If $\alpha = \beta$ and $x \neq y$ then by the same argument as in the first half of the proof we obtain that $(f_\alpha \cup \varepsilon_S) \circ (f_\alpha^{-1} \cup \varepsilon_S) \subseteq \rho$.

Lemma 12. *Let $S = (X; I_\alpha, f_{\alpha,\beta})$ be a tree of congruence free semigroups where X has no least element. If* Con S *is atomistic then for $x, y \in I_\alpha^*$ there exists $\gamma \leq \alpha$ such that $x f_{\alpha,\gamma} = y f_{\alpha,\gamma}$.*

Proof. Let $x, y \in I_\alpha^*$ and $x = x_0 \rho_1 x_1 \cdots \rho_n x_n = y$ for some atoms ρ_i such that all $x_i \in J(x)$. For $x \neq y$ let $g(x, y) = n$ denote the smallest length of such a sequence. We prove the assertion by induction on $g(x, y)$. If $g(x, y) = 1$ then by Lemma 11, $x f_\alpha = y f_\alpha$. Let $g(x, y) = n > 1$ and suppose that the assertion is true whenever $g(u, v) < n$. Let α_i be defined by $x_i \in I_{\alpha_i}^*$. Then $\alpha_i \leq \alpha$ for all i since $x_i \in J(x)$. If $\alpha_i = \alpha$ for all i then $x_0 f_\alpha = x_1 f_\alpha = \cdots = x_n f_\alpha = y f_\alpha$ which is a contradiction to $g(x, y) > 1$. Let j be the first index such that $\alpha_j < \alpha$. Then $x_0 f_\alpha = x_1 f_\alpha = \cdots = x_{j-1} f_\alpha = x_j$. Therefore $j = 1$: $x_1 = x_0 f_\alpha = x f_\alpha$. By the same argument we obtain that $x_{n-1} = x_n f_\alpha = y f_\alpha$. Now there are two alternatives: (i) $x_i \in J(x_1) = J(x_{n-1})$ for all $1 \leq i \leq n-1$ and (ii) there exists i, $1 \leq i \leq n-1$, such that $x_i \in I_\alpha^* = J_{x_0} = J_{x_n}$. Since all $x_i \in J(x)$ only these two cases are possible. For the first case we have that $g(x_1, x_{n-1}) < n$ and so $x_1 f_{\alpha^+,\gamma} = x_{n-1} f_{\alpha^+,\gamma}$ for some $\gamma \leq \alpha^+$. Then $x f_{\alpha,\gamma} = x_1 f_{\alpha^+,\gamma} = x_{n-1} f_{\alpha^+,\gamma} = y f_{\alpha,\gamma}$. In the second case we have $g(x, x_i) < n$ and $g(y, x_i) < n$ and therefore $x f_{\alpha,\gamma} = x_i f_{\alpha,\gamma} = y f_{\alpha,\gamma}$ for some $\gamma < \alpha$.

Of course the condition of Lemma 12 is equivalent to the condition: for any $a \in I_\alpha^*$ and $b \in I_\beta^*$ there exists $\gamma \leq \alpha, \beta$ such that $a f_{\alpha,\gamma} = b f_{\beta,\gamma}$.

Using the following known lemmas, we thus have obtained a characterization of weakly reductive semigroups with atomistic congruence lattices.

Notation. For an arbitrary set X, let $\mathcal{P}(X)$ be the lattice of all subsets of X.

Lemma 13. *Let X be a locally finite tree. Then* Con $X \cong \mathcal{P}(X^*)$.

PROOF. See [3, Lemma 3].

LEMMA 14. *Let* $S = (X; I_\alpha, f_{\alpha,\beta})$ *be a tree of congruence free semigroups* I_α *such that for all* $a \in I_\alpha^*$ *and* $b \in I_\beta^*$ *there exists* γ *satisfying* $af_{\alpha,\gamma} = bf_{\beta,\gamma}$. *Then* $\operatorname{Con} S \cong \operatorname{Con} X$.

PROOF. The lemma is a consequence of the proof of [2, Theorem 8].

LEMMA 15. *Let* $S = (X; I_\alpha, f_{\alpha,\beta})$ *be a tree of* 0*-simple semigroups where* I_α *is congruence free for all* $\alpha \in X^*$ *and* X *has a least element* μ. *Then* $\operatorname{Con} S \cong \operatorname{Con} X \times \operatorname{Con} I_\mu^*$.

PROOF. The lemma is a consequence of the proof of [2, Theorem 8].
Using this and the fact that a product of two lattices is atomistic if and only if each factor is atomistic, we can formulate

THEOREM 2. *Let* S *be a weakly reductive semigroup. Then* $\operatorname{Con} S$ *is atomistic if and only if* S *is isomorphic to one of the following*:

(i) *a simple semigroup* I *such that* $\operatorname{Con} I$ *is atomistic*;

(ii) *a tree of congruence free semigroups* $(X; I_\alpha, f_{\alpha,\beta})$ *such that for each* $x \in I_\alpha^*$, $y \in I_\beta^*$ *there exists* $\gamma \le \alpha, \beta$ *satisfying* $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$;

(iii) *a tree of* 0*-simple semigroups* $(X; I_\alpha, f_{\alpha,\beta})$ *where* X *has a least element* μ *such that* I_μ^* *is a semigroup of type* (i) *and* S/I_μ^* *is a semigroup of type* (ii).

4. Applications

In order to study special classes of semigroups we first need a result for groups.

PROPOSITION 3. *A group has an atomistic congruence lattice if and only if it is a direct sum of simple groups.*

PROOF. For a group we may identify congruences and normal subgroups.

NECESSITY. Suppose that the group G has an atomistic congruence lattice. Let $\{N_i : i \in I\}$ be the set of all atoms of the lattice of normal subgroups of G. Then $G = \bigvee(N_i : i \in I)$. Let $\mathcal{A}$ be defined by

$$\mathcal{A} = \left\{K \subseteq I : \forall i \in K : N_i \cap \bigvee(N_j : j \in K \setminus \{i\}) = \{1\}\right\}.$$

Then $\mathcal{A}$ is not empty. Let $\mathcal{C} \subseteq \mathcal{A}$ be a chain and $J = \bigcup \mathcal{C}$. If $J \notin \mathcal{A}$ then there exists $j \in J$ such that $N_j \subseteq \bigvee(N_i : i \in J, i \neq j)$. Let $n \in N_j$, $n \neq 1$. Then there exist $i_1, \ldots, i_k \in J \backslash \{j\}$ such that $n \in N_{i_1} \vee \cdots \vee N_{i_k}$. Then $N_j \cap (N_{i_1} \vee \cdots \vee N_{i_k}) \neq \{1\}$ and so $N_j \subseteq N_{i_1} \vee \cdots \vee N_{i_k}$ because N_j is an atom. This is a contradiction to the definition of $\mathcal{A}$ because there exists $C \in \mathcal{C}$ which contains the indices i_h as well as j. Therefore $J \in \mathcal{A}$. Now by Zorn's Lemma there exists a maximal element in $\mathcal{A}$, to be denoted by K. If $K = I$ then we obviously have that $G = \sum(N_i : i \in I)$. Now suppose that $K \neq I$. Let $j \in I \backslash K$. If N_j is not contained in $\bigvee(N_k : k \in K)$ then there exists $i \in K$ such that $N_i \subseteq \bigvee(N_k : k \in K, k \neq i) \vee N_j$ because $K \cup \{j\} \notin \mathcal{A}$. Let $N = \bigvee(N_k : k \in K, k \neq i)$. Then we obtain that $\{\{1\}, N_j, N, N \vee N_i, N \vee N_j\}$ forms a non-modular sublattice of the lattice of all normal subgroups of G, a contradiction. Therefore, $N_j \subseteq \bigvee(N_k : k \in K)$ which implies that $G = \sum(N_k : k \in K)$. Then each normal subgroup of some N_i is a normal subgroup of G and therefore all N_i are simple groups.

SUFFICIENCY. Let $G = \sum G_i$ be a direct sum of simple groups G_i. Let N be a normal subgroup of G and $n \in N$. Then $n = a_1 \cdots a_k b$ where the element a_i belongs to some non-commutative group G_i and b belongs to the centre of G. To each a_i there exists $c_i \in G_i$ such that $a_i c_i \neq c_i a_i$. Then $n c_i n^{-1} c_i^{-1} = a_i c_i a_i^{-1} c_i^{-1} \in N \cap G_i$ and $a_i c_i a_i^{-1} c_i^{-1} \neq 1$. Since G_i is simple $G_i \subseteq N$. In particular, $a_i \in N$ for all i and therefore $b \in N$. The order of b is square free: $o(b) = p_1 \cdots p_s$ for some distinct primes p_j. Let $q_j = p_1 \cdots p_s / p_j$. Then $\langle b^{q_j} \rangle \cong \mathbb{Z}_{p_j}$ and $\langle b^{q_j} \rangle \subseteq N$. The groups G_i and $\langle b^{q_j} \rangle$ are atoms in the lattice of all normal subgroups of G. Then $n \in G_1 \cdots G_k \langle b^{q_1} \rangle \cdots \langle b^{q_s} \rangle \subseteq N$ implies that N is the supremum of the atoms it contains.

4.1. Commutative semigroups.

We first treat the globally idempotent case. A commutative semigroup is 0-simple if and only if it is a commutative group with a zero adjoined. Such a semigroup is congruence free if and only if its non-zero part consists of only one (idempotent) element. So $S = (X; I_\alpha, f_{\alpha,\beta})$, the tree of congruence free semigroups, degenerates to the locally finite tree X. Furthermore, a commutative group has an atomistic lattice of subgroups if and only if it is a direct sum of cyclic groups $\mathbb{Z}_p$ of prime order. So for the globally idempotent case we have exactly the three cases: (i) a direct sum of cyclic groups $\mathbb{Z}_p$ of prime order, (ii) a locally finite tree, (iii) an ideal extension of a semigroup G as (i) by a semigroup X as (ii) with zero.

For the general case we need the following proposition.

PROPOSITION 4. *Let T be an inflation of a semigroup S such that all homomorphic images of S are weakly reductive. The* $\operatorname{Con} T$ *is atomistic if and only if* $\operatorname{Con} S$ *is atomistic and the inflation function f is trivial, that is* $|(T\backslash S)f| = 1$.

PROOF. Suppose that $\operatorname{Con} T$ is atomistic. For each congruence ρ on S, $\rho \cup \varepsilon_T$ is a congruence on T. Therefore, if $\operatorname{Con} T$ is atomistic the same holds for $\operatorname{Con} S$. Let ρ be defined by $x \rho y$ if and only if $x, y \in S$ or $x, y \in T\backslash S$. Let $x, y \in T\backslash S$ and $x = x_0 \rho_1 x_1 \cdots \rho_n x_n = y$ for some atoms $\rho_i \subseteq \rho$. Since the ρ_i's are atoms we have that $\rho_i|S = \varepsilon$. Therefore, $(xf)z = xz = yz = (yf)z$ and $z(xf) = zx = zy = z(yf)$ for all $z \in S$. Weak reductivity of S then implies that $xf = yf$. Conversely, let T be an inflation of a semigroup S such that all homomorphic images of S are weakly reductive, suppose that $\operatorname{Con} S$ is atomistic and $|(T\backslash S)f| = 1$ where f stands for the inflation function. Let $a \in S$ denote the element of S which defines the multiplication of the inflation, that is $xf = a$ for all $x \in T\backslash S$. Let $x \in T\backslash S$, $b \in S$ and suppose that $x \rho b$ for some $\rho \in \operatorname{Con} T$. We obtain that $xz = az \rho bz$ and $zx = za \rho zb$ for all $z \in S$. Since $S/(\rho|S)$ is weakly reductive we have $a \rho b$. Now we may apply Lemma 11 in [2] which proves that under this condition the mapping $\rho \to (\rho|S, \rho|T\backslash S \cup \{a\})$ is an isomorphism between $\operatorname{Con} T$ and $\operatorname{Con} S \times \operatorname{Eq} T\backslash S \cup \{a\}$.

Summarizing these observations we may formulate

THEOREM 3. *A commutative semigroup S has an atomistic congruence lattice if and only if S is isomorphic to one of the following*:

(i) *a direct sum of cyclic groups* $\mathbb{Z}_p$ *of prime order*;

(ii) *a locally finite tree*;

(iii) *an ideal extension of a semigroup of type* (i) *by a semigroup of type* (ii) *with zero*;

(iv) *an inflation of a semigroup of type* (i), (ii) *or* (iii) *with a trivial inflation function.*

We observe that for commutative semigroups S the conditions "$\operatorname{Con} S$ is atomistic" and "$\operatorname{Con} S$ is relatively complemented" are equivalent (see [2, Corollary 14]).

4.2. Finite semigroups.

Again we first treat the globally idempotent case. Put $S = (X; I_\alpha, f_{\alpha,\beta})$, a tree of 0-simple semigroups. Finiteness implies that all I_α are *completely* 0-simple. If α is not minimal in X then I_α is congruence free and therefore $I_\alpha \cong \mathcal{M}^0(I_\alpha, \Lambda_\alpha, P_\alpha)$ where P_α is a $\Lambda_\alpha \times I_\alpha$-matrix of zeros and ones such that each row and each column contain a one and no two rows and no two columns are identical (see [11] or [6]). X has a least element μ and I_μ^* is

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a completely simple semigroup. Therefore, $I_\mu^* \cong \mathcal{M}(I, G, \Lambda, P)$. Suppose that $\operatorname{Con} I_\mu^*$ is atomistic. $\operatorname{Con} I_\mu^*$ is isomorphic to $\mathrm{Ad} = \mathrm{Ad}(I, G, \Lambda, P)$, a sublattice of $\operatorname{Eq} I \times \operatorname{Nor} G \times \operatorname{Eq} \Lambda$, the lattice of *admissible triples* (see [6]) ($\operatorname{Nor} G$ denotes the lattice of all normal subgroups of G). An element $(\xi, N, \eta) \in \operatorname{Eq} I \times \operatorname{Nor} G \times \operatorname{Eq} \Lambda$ is admissible if

$$i \xi j \Rightarrow p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1} \in N \quad \forall \lambda, \mu \in \Lambda$$

and

$$\lambda \eta \mu \Rightarrow p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1} \in N \quad \forall i, j \in I.$$

All elements of the form $(\varepsilon, N, \varepsilon)$ are admissible. Furthermore if (ξ, N, η) is admissible and $\zeta \subseteq \xi$ then (ζ, N, ε) is also admissible. (ω, G, ω) is admissible and hence the supremum of admissible atoms. If (ζ, N, ε) is an atom in $\mathrm{Ad}(I, G, \Lambda, P)$ and $\zeta \neq \varepsilon$ then $N = \{1\}$ and ζ is an atom in $\operatorname{Eq} I$. Let $i, j \in I$ and $\lambda \in \Lambda$. There exist atoms $\rho_1, \ldots, \rho_n$ in Ad such that

$$(i, 1, \lambda) \rho_1 \cdots \rho_n (j, 1, \lambda). \tag{1}$$

Each ρ_k whose first entry is a proper equivalence commutes with each ρ_k whose first entry is the identity. Therefore in (1) we may omit the latter ones. Thus for each $i, j \in I$ there exists $\xi \in \operatorname{Eq} I$ such that $i \xi j$ and $(\xi, \{1\}, \varepsilon)$ is admissible and therefore $(\omega, \{1\}, \varepsilon)$ is admissible. The same holds for $(\varepsilon, \{1\}, \omega)$ and thus $(\omega, \{1\}, \omega)$ is also admissible. We have thus obtained that all triples are admissible and then $I_\mu^* \cong I \times G \times \Lambda$, a rectangular group (see [8]). Since each partition lattice is atomistic $\operatorname{Con} I_\mu^*$ is atomistic if and only if $\operatorname{Nor} G$ is atomistic, that is if and only if G is a direct sum of simple groups. If S is a tree of congruence free semigroups then finiteness implies that X has a least element μ. Then $|I_\mu^*| = 1$ and therefore $x f_{\alpha, \mu} = y f_{\beta, \mu}$ for arbitrary $x \in I_\alpha^*$ and $y \in I_\beta^*$. Since Proposition 4 here also applies, we can formulate

THEOREM 4. *Let S be a finite semigroup. Then* $\operatorname{Con} S$ *is atomistic if and only if S is isomorphic to one of the following*:

(i) *a rectangular group $I \times G \times \Lambda$ such that G is a direct sum of simple groups*;

(ii) *a tree of congruence free semigroups*;

(iii) *a tree of* 0*-simple semigroups $(X; I_\alpha, f_{\alpha,\beta})$ such that I_μ^* is a semigroup of type* (i) (*where μ denotes the least element of X*) *and S/I_μ^* is a semigroup of type* (ii);

(iv) *an inflation of a semigroup of type* (i), (ii) *or* (iii) *with a trivial inflation function.*

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In [4, Theorems 3.11 and 3.14] the partial homomorphisms between non-zero parts of completely 0-simple semigroups are described.

Completely semisimple semigroups can be treated in the same way as the globally idempotent case of finite semigroups, omitting the finiteness conditions. Here it may happen that the locally finite tree X of $S = (X; I_\alpha, f_{\alpha,\beta})$ has no least element and so in (ii) the condition "for $x \in I_\alpha^*$ and $y \in I_\beta^*$ there exists $\gamma \leq \alpha, \beta$ such that $xf_{\alpha,\gamma} = yf_{\beta,\gamma}$" must be added. An example in [1] shows that this is really necessary. In [2, Section 5] a necessary and sufficient condition for this property is given. The present section is closely related to [2, Section 5]. Again for finite and completely semisimple semigroups the properties "$\operatorname{Con} S$ is atomistic" and "$\operatorname{Con} S$ is relatively complemented" are equivalent.

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