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Equilibrium turbulent boundary layers with lateral streamline convergence or divergence

T. B. NICKELS †
Department of Engineering, Cambridge University, Trumpington Street, Cambridge CB2 1PZ, UK

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The constraints necessary for equilibrium solutions of the boundary layer equations are explored for turbulent boundary layers subject to lateral convergence and divergence and with longitudinal pressure gradients. It is shown that in addition to the well-known equilibrium solutions for two-dimensional boundary layers there are additional *possible* equilibrium states for boundary layers with these extra rates-of-strain acting. The necessary constraints for equilibrium are derived and discussed.

1. Introduction

This analytical study was motivated by the results of two experimental studies conducted at the University of Melbourne examining the effects of streamline divergence and convergence on developing turbulent boundary layers (Saddoughi & Joubert 1991, Panchapakesan *et al.* 1997). In these experiments, a developed turbulent boundary layer on a flat plate was subjected to streamline divergence in the one case and convergence in the other, by linearly diverging (or converging) the sidewalls of the wind tunnel while maintaining a zero pressure gradient by the adjustment of a flexible 'ceiling' above the plate. Measurements of the spanwise variation of pressure and skin friction suggest that these flows may be considered as axisymmetric, i.e. homogeneous along circles centred at the point where the mean streamlines meet. Although the internal angle between the sidewalls was 10° in both cases, the behaviour of the boundary layer in the two cases was markedly different. The diverging case showed very little change in the mean flow and turbulence and appeared to be close to, and approaching, an equilibrium state. The converging case, however, showed significant changes in the flow structure and appeared to be in the state far from equilibrium. This prompted the question as to why the two flows are so markedly different and, more generally, under what conditions is it possible for boundary layers with streamline divergence or convergence to achieve a state of equilibrium?

Equilibrium boundary layers are special solutions of the boundary layer equations where the shapes of the mean velocity and turbulence profiles remain self-similar when non-dimensionalized appropriately. Clauser (1954) was one of the first researchers to study equilibrium solutions for turbulent boundary layers. By analogy with laminar flows, he proposed the possibility of turbulent equilibrium boundary layers with pressure gradients acting. Proceeding by physical arguments he postulated a pressure gradient parameter which, if held constant, could lead to equilibrium solutions. He further showed experimentally two flows in which this equilibrium was achieved. The equilibrium solutions of Clauser (1954) are solutions in which the mean

† Email address for correspondence: tbn22@eng.cam.ac.uk

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velocity profiles in defect form are self-similar over most of the layer. These are self-similar solutions of the Reynolds averaged boundary layer equations in which the viscous terms have been neglected and hence they will be referred to as approximate equilibrium flows. These solutions do not ensure exact equilibrium since the Reynolds numbers of the flows are changing and hence the flow very near the wall, which is affected by viscosity, is not in equilibrium. In Clauser's flows similarity of the defect does not extend all the way to the wall, although it does apply over a large region of the boundary layer profile. These solutions are then appropriate for high Reynolds numbers and may be precise at infinite Reynolds number. Following on from this earlier work, Clauser (1956) gave a general discussion of the turbulent boundary layer in which he considered broader issues and also showed that for his class of approximate equilibrium flows that the shear stress could also be nearly self-similar. He also examined the appropriate form for the pressure gradient parameter for approximate equilibrium. Bradshaw (1967) established flows of this nature and studied the turbulence structure experimentally.

Townsend (1956) and Rotta (1962) examined the conditions required for equilibrium more rigorously than Clauser and established a set of conditions under which exact equilibrium is possible. The equilibrium is considered exact if the defect similarity applies across the whole layer so that the viscous terms are included in the similarity formulation. In the case of rough walls the similarity solution would extend close to the height of the roughness elements (above the roughness sublayer). They also considered the conditions necessary for the equilibrium of the stress profiles. They both found that for smooth walls the only exact equilibrium layer that could occur was the favourable pressure gradient sink flow. Coles (1957) and Perry (1968) considered flows where the conditions for exact equilibrium are relaxed in order to account for experimental evidence that an empirical equilibrium state is possible for flows which do not meet the stringent conditions derived by Rotta (1962) and Townsend (1956). Both authors derived parameters which, if kept constant, could lead to empirical equilibrium flows despite the fact that these flows do not satisfy the conditions required for exact equilibrium.

The above studies of equilibrium flow all apply for the nominally two-dimensional boundary layer that occurs on a plate where the streamlines are parallel when viewed in a direction normal to the wall. It does not seem to be generally known that equilibrium solutions can also exist for flows with streamline convergence or divergence with or without streamwise pressure gradients. In this paper the analysis for equilibrium is extended to these cases.

2. Definition of equilibrium

In this paper two definitions of equilibrium are used. Approximate equilibrium boundary layers are boundary layers where the mean velocity defect profile is self-similar over most of the layer and the Reynolds shear stress profile is also self-similar over most of the layer. The phrase 'over most of the layer' means we neglect the region near the wall where viscosity becomes important. If the Reynolds number of the flow is sufficiently high this region is small and makes only a small contribution to the momentum and mass flux. These solutions are derived by assuming that the viscous terms are negligible which is true over most of the layer when the Reynolds number of the flow is sufficiently large. The equilibrium flows of Clauser fall into this category. Exact equilibrium boundary layers are solutions of the boundary layer equations in which the similarity applies across the whole layer and may be derived

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from approximate equilibrium solutions by ensuring that the local Reynolds number of the flow is constant. In these layers the velocity-defect similarity applies all the way to the wall as does the Reynolds shear stress similarity since the viscous terms have not been neglected. Exact equilibrium boundary layers are a subclass of approximate equilibrium layers with additional restrictions imposed.

3. The Reynolds shear stress

In the following analysis the flow will be considered axisymmetric with U the mean velocity in the radial (r) direction, V the mean velocity in the wall normal (y) direction and u and v the velocity fluctuations. This is consistent with the experiments mentioned in § 1 which are (to a very good approximation) homogeneous along circles centred at the point where the mean streamlines intersect. The momentum equation for boundary layers in cylindrical coordinates may then be written as

$$U\frac{\partial U}{\partial r}+V\frac{\partial U}{\partial y}=\frac{-1}{\rho}\frac{\partial P}{\partial r}+\frac{1}{\rho}\frac{\partial(-\overline{uv})}{\partial y}+\nu\frac{\partial^2 U}{\partial y^2}, \tag{3.1}$$

where the flow is assumed axisymmetric in the mean and the assumption that the point of interest is not too close to the origin ($r=0$) where extra terms in the viscous stress become important (in particular where $\partial^2 U/\partial y^2 \gg U/r^2$) is implicit. The other terms neglected are in line with the usual boundary layer assumptions including the streamwise (radial) derivative of the normal stresses. Checks on the effect of including these extra terms show them to be negligible in the momentum balance and show that they do not affect the conclusions of the analysis (for the experiments discussed in § 1, inclusion of the extra terms changes the streamwise momentum by less than 2 %). Using the mean continuity equation

$$\frac{1}{r}\frac{\partial Ur}{\partial r}+\frac{\partial V}{\partial y}=0 \tag{3.2}$$

and multiplying both sides of (3.1) by δ/U_τ^2, after some algebra it is possible to show that

$$g'_{12}=-\frac{\delta}{U_\tau}\frac{\mathrm{d}U_1}{\mathrm{d}r}(2f-\eta f'-f^2/S+I_1 f'/S)+\delta\frac{\mathrm{d}S}{\mathrm{d}r}(f-f^2/S+I_1 f'/S)$$
$$+S\frac{\mathrm{d}\delta}{\mathrm{d}r}(\eta f'-I_1 f'/S)-S\frac{\delta}{r}(\eta f'-I_1 f'/S)+\frac{1}{\delta^+}f'', \tag{3.3}$$

where δ is the boundary layer thickness and U_τ is the wall shear velocity ($U_\tau=\sqrt{\tau_w/\rho}$ where τ_w is the shear stress at the wall and ρ is the density of the fluid). In these equations the following definitions have been used:

$$\eta=\frac{y}{\delta}, \tag{3.4}$$

$$S=\frac{U_1}{U_\tau}, \tag{3.5}$$

$$f(\eta)=\frac{U_1-U}{U_\tau}, \tag{3.6}$$

$$I_1=\int_0^\eta f\,\mathrm{d}\eta, \tag{3.7}$$

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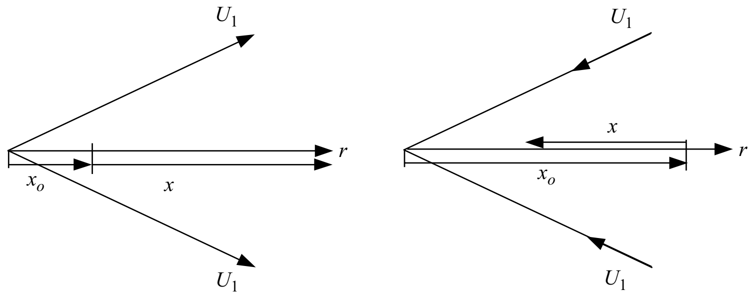

FIGURE 1. Co-ordinate system.

$$C_1 = \int_0^\infty f \, \mathrm{d}\eta \tag{3.8}$$

$$\delta = \frac{\delta^* S}{C_1}, \tag{3.9}$$

$$\delta^+ = \frac{\delta U_\tau}{\nu} \tag{3.10}$$

$$g_{12}(\eta) = \frac{-\overline{uv}}{U_\tau^2}, \tag{3.11}$$

where U_1 is the velocity outside the boundary layer and is assumed positive in the positive r direction – i.e. away from the origin. δ^* is the displacement thickness of the boundary layer. The dash denotes differentiation of the function of η with respect to η. In order to more easily discuss both diverging flows (flow out from the origin) and converging flows (flow towards the origin) in a coordinate system that is more natural for comparison with experiments, we consider the simple coordinate transformations

$$r = x + x_o, \quad x > 0, \tag{3.12}$$

for diverging flow and

$$r = x_o - x, \quad 0 < x < x_o, \tag{3.13}$$

for converging flow. These flows then start from some initial radius $r = x_o$, develop in the positive x-direction which is away from the origin in the diverging case and towards the origin in the converging case as shown in figure 1. This simplifies the interpretation of the terms since, for example a positive value of $\mathrm{d}U_1/\mathrm{d}x$ is a favourable pressure gradient in both flows. The interpretation is then in terms of the usual convention where the flow direction is considered positive.

We will also define a non-dimensional pressure gradient parameter, after Clauser (1956) as

$$\beta_x = \frac{\delta^*}{\tau_o}\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{\delta^*}{U_\tau} S \frac{\mathrm{d}U_1}{\mathrm{d}x} = -C_1 \frac{\delta}{U_\tau}\frac{\mathrm{d}U_1}{\mathrm{d}x} \tag{3.14}$$

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and by analogy following Panchapakesan *et al.* (1997) a divergence parameter which they specified in Cartesian coordinates as

$$\beta_D = -\frac{\delta^*}{U_\tau} S \frac{\mathrm{d}W_1}{\mathrm{d}z} \tag{3.15}$$

and in terms of the axisymmetric coordinate system and our new variable x $\beta_D = -S^2\delta^*/(x + x_o)$ for diverging flow and $\beta_D = S^2\delta^*/(x_o - x)$ for converging flow. With this coordinate system and divergence parameter then the cases of convergence and divergence can be treated using a similar equation in which only the sign of β_D differs, being positive for convergence and negative for divergence. (It is important to note that these flows are axisymmetric and hence in the appropriate polar coordinates there are only two velocity components, this third velocity component only arises when expressing things in Cartesian form as has been done in the experimental papers. These flows are actually (to a very good approximation) two-dimensional since they are homogeneous along circles located at the origin of the streamlines.) Since we will be looking for equilibrium solutions in which the various length scales become proportional then the use of the displacement thickness δ^* is not important but it has been chosen here in order to provide a connection with the work of Clauser on approximate equilibrium solutions in flows with pressure gradients. In terms of these new parameters then the momentum equation may be rewritten as

$$g'_{13} = \frac{\beta_x}{C_1}(2f - \eta f' - f^2/S + I_1 f'/S) + \delta\frac{\mathrm{d}S}{\mathrm{d}x}(f - f^2/S + I_1 f'/S) \\ + S\frac{\mathrm{d}\delta}{\mathrm{d}x}(\eta f' - I_1 f'/S) - \frac{\beta_D}{C_1}(\eta f' - I_1 f'/S) + \frac{1}{\delta^+} f''. \tag{3.16}$$

From the above equation it may be seen that for approximate similarity to be possible for the Reynolds shear stress, the following conditions must apply

$$S = \text{constant}, \tag{3.17}$$

i.e. the skin friction coefficient C'_f is not a function of x

$$\frac{\mathrm{d}\delta}{\mathrm{d}x} = \text{constant}, \tag{3.18}$$

$$\beta_D = \text{constant} \quad \text{and} \tag{3.19}$$

$$\beta_x = \text{constant}. \tag{3.20}$$

Implicit in this analysis is the choice to scale the mean velocity defect and the turbulence quantities using the wall shear velocity and a suitably defined, boundary layer thickness. In this sense it is in line with the classical analysis of conditions for equilibrium solutions of the equations. However, since S is constant is a condition of the solutions then scaling with the wall shear velocity and the external flow velocity are equivalent. Note that the boundary layer thickness used here is proportional to the Clauser–Rotta length scale Δ (which in this notation is $\Delta = \delta^* S = C_1\delta$) and hence is a well-defined integral length scale. It is only chosen here rather than the Clauser–Rotta length scale since under most circumstances its value will be close to the more commonly used experimental measurements of boundary layer thickness (such as the '99 % thickness'). While this is not necessary for the analysis it may be helpful for some readers.

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4. The momentum integral equation

Integrating the momentum equation across the boundary layer in the vertical direction leads to the momentum integral equation for flows with simple convergence or divergence (see also Kehl 1943; Head & Patel 1968).

In terms of the parameters defined earlier this may be written as

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{S^2}\left[1 + \beta_x + \frac{2\beta_x + \beta_D}{H}\right], \tag{4.1}$$

where θ is the momentum thickness of the layer. It is not difficult to show that this reduces to the usual form for boundary layers with no lateral straining ($\beta_D = 0$) and an applied pressure gradient. In the zero pressure gradient case with an extra lateral straining the equation further reduces to

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{S^2}\left(1 + \frac{\beta_D}{H}\right). \tag{4.2}$$

5. Approximate equilibrium

Now in the case of approximate equilibrium solutions, which will be considered first, the last term is neglected, the assumption being that if the Reynolds number (defined here as δ^+) is sufficiently large this term may be neglected over most of the flow. It should be noted that these solutions may also be good approximations to the real behaviour in the case where the variation of δ^+ with streamwise distance is very small. These might be considered quasi-equilibrium solutions. We will then consider exact equilibrium solutions in which these assumptions are unnecessary.

Hence the analysis suggests that approximate equilibrium solutions may be possible for flows with streamline convergence or divergence if the above conditions are satisfied. The conditions are similar to those previously derived for two-dimensional boundary layers with the extra condition given by (3.19).

The momentum integral equation imposes extra constraints which will here be used to decide which of the flows satisfying the above conditions are possible. We consider first the zero pressure gradient cases which inspired the analysis. It may be recalled that in the experiments the case with diverging flow appeared to be close to an equilibrium solution whereas the converging case was far from equilibrium.

5.1. *Zero pressure gradient*

Here we consider first the case with zero streamwise (radial) pressure gradient. In the converging streamline case from the definition of β_D we have

$$\beta_D = S^2\delta^*/(x_o - x), \;\; \beta_D > 0. \tag{5.1}$$

Rearranging this equation and differentiating we find an expression for $\mathrm{d}\delta^*/\mathrm{d}x$,

$$\frac{\mathrm{d}\delta^*}{\mathrm{d}x} = \frac{-\beta_D}{S^2} \tag{5.2}$$

and since both β_D and S are greater than zero then

$$\frac{\mathrm{d}\delta^*}{\mathrm{d}x} < 0. \tag{5.3}$$

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The momentum integral equation for zero pressure gradient flow with streamline divergence or convergence can be rewritten

$$\frac{d\delta^*}{dx} = \frac{1}{S^2}(H + \beta_D), \tag{5.4}$$

using the fact that $H = \delta^*/\theta$ is constant if the flow is in equilibrium. Hence for the converging flow where $\beta_D > 0$ we have

$$\frac{d\delta^*}{dx} > 0 \tag{5.5}$$

which is in direct contradiction with (5.3). Hence, a converging zero pressure gradient flow cannot be in equilibrium – a result that is consistent with the experimental observations.

Now consider the zero pressure gradient, diverging case. In this case $\beta_D < 0$ by definition. Using the definition of this parameter

$$\beta_D = -S^2\delta^*/(x + x_o), \;\; \beta_D < 0. \tag{5.6}$$

Rearranging this equation and differentiating we find an expression for $d\delta^*/dx$,

$$\frac{d\delta^*}{dx} = \frac{-\beta_D}{S^2} \tag{5.7}$$

and since $\beta_D < 0$ and $S > 0$

$$\frac{d\delta^*}{dx} > 0. \tag{5.8}$$

Now returning to the momentum integral equation,

$$\frac{d\delta^*}{dx} = \frac{1}{S^2}(H + \beta_D), \;\; \beta_D < 0, \tag{5.9}$$

and to avoid a contradiction with (5.8) in this case it is necessary that $H + \beta_D > 0$ or, equivalently $\beta_D/H > -1$. This essentially suggests that an equilibrium solution may be possible if the divergence is not too strong.

This analytical result is consistent with the experimental observation that the diverging flow of Saddoughi & Joubert (1991) (with zero pressure gradient) shows little change in the stresses, whereas the converging case presented in Panchapakesan *et al.* (1997) (also with zero pressure gradient) shows large changes and no tendency towards equilibrium. In the case of Saddoughi & Joubert (1991), $\beta_D/H \approx -0.6$, which is also consistent with the constraint given above. The results presented in their paper show the boundary layer length scales are all growing linearly in the latter part of the flow (see their figure 7.), the mean velocity profiles are self-similar (evident from their figure 8.), S is very close to a constant (it varies by only 1.4 % over the last metre of the flow, see their figure 4.). Their β_D also reaches a constant value. Hence all the conditions for similarity are satisfied and the Reynolds stresses are indeed self-similar in the latter part of the flow.

5.2. *Non-zero pressure gradient*

In the case where the pressure gradient is non-zero the conditions are altered. In the diverging case when $\delta^* = A(x + x_o)$ ($A > 0$ is some constant) then $\beta_x =$ constant can be satisfied by any power-law distribution of external velocity, i.e.

$$U_1 = B(x + x_o)^m, \tag{5.10}$$

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where B and m are some arbitrary constants and this leads to

$$\frac{1}{U_1}\frac{dU_1}{dx} = \frac{m}{(x+x_o)} \tag{5.11}$$

and using the definition of β_x and the linear variation of δ^* then

$$\beta_x = -mAS^2, \tag{5.12}$$

where a negative value of m corresponds to an adverse pressure gradient in the direction of the flow. In the converging case the situation is similar and here $\delta^* = A(x_o - x)$, $A > 0$ and hence $\beta_x = mAS^2$ but in this case a negative value of m corresponds to a favourable pressure gradient in the direction of the flow. Using these definitions gives a consistent sign of β_x for the same type of pressure gradient when considering development in the direction of the flow (so, for example, $\beta_x > 0$ always corresponds to an adverse pressure gradient in the flow direction). Care must be taken in interpreting the value of m in terms of the nature of the pressure gradient. In both cases (converging and diverging), it is not difficult to show that

$$\beta_x = m\beta_D. \tag{5.13}$$

The momentum integral equation with non-zero pressure gradient may be written as

$$\frac{d\delta^*}{dx} = \frac{1}{S^2}(H + \beta_x(H+2) + \beta_D). \tag{5.14}$$

In the converging flow as before from the definition of β_D, $d\delta^*/dx < 0$. In this case then

$$\frac{1}{S^2}(H + \beta_x(H+2) + \beta_D) < 0 \tag{5.15}$$

which leads to

$$\beta_x < -(\beta_D + H)/(H+2) \quad \beta_D > 0 \tag{5.16}$$

and since the right-hand side is always negative then a favourable pressure gradient must be applied. This equation also shows that the applied pressure gradient must be sufficiently strong for a given β_D in order to satisfy the equation. Note also in the case for zero divergence, $\beta_D = 0$, then we also find that a favourable pressure gradient is necessary (this is the well-known sink-flow solution).

In the diverging case where $\beta_D < 0$ $d\delta^*/dx > 0$ which leads to

$$\beta_x > -(\beta_D + H)/(H+2). \tag{5.17}$$

There are two possible solutions to this equation. If $\beta_D/H < -1$ then $\beta_x > 0$ always, which corresponds to an adverse pressure gradient and if $-1 < \beta_D/H < 0$ then the right-hand side is negative and the pressure gradient may be either a slight favourable pressure gradient or an adverse pressure gradient (or as we saw earlier a zero pressure gradient).

6. Exact equilibrium

In this section the conditions for exact equilibrium will be derived. Exact equilibrium will be used here to describe flows which satisfy the conditions given above for approximate equilibrium with the added restriction that the local Reynolds number of the flow is constant. The consideration of viscosity effectively introduces a new

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length scale and hence an additional condition for equilibrium. In the following analysis the ratio of the viscous length scale to any other length scale is held constant by ensuring that the local Reynolds number of the flow is held constant. An exact equilibrium flow must satisfy the conditions for an approximate equilibrium flow and hence all length scales must be in constant ratio as must all velocity scales so that it suffices to show that for constant kinematic viscosity the product of any velocity with any length is invariant with streamwise distance. Now if U_1 has a power-law distribution then

$$U_1\delta^* = B(x + x_o)^m A(x + x_o) = \text{constant} \tag{6.1}$$

and hence $m = -1$ is the only solution to this equation (the same applies for converging flow). In flow without divergence this corresponds with the classical sink flow solution.

In the case with both pressure gradient and lateral straining effects then for exact equilibrium both β_D and β_x must be constant. The condition for exact equilibrium (given that the conditions for approximate equilibrium are satisfied) may be written as

$$\frac{\mathrm{d}(U_1\delta^*)}{\mathrm{d}x} = 0 \tag{6.2}$$

or, equivalently, as

$$S^2\frac{\mathrm{d}\delta^*}{\mathrm{d}x} = \beta_x. \tag{6.3}$$

Substituting into (5.14) the condition for exact equilibrium is

$$\beta_x = H + H\beta_x + 2\beta_x + \beta_D \tag{6.4}$$

or

$$\beta_x = -(\beta_D + H)/(H + 1). \tag{6.5}$$

For the converging flow β_D is always positive and hence the only way to achieve an exact equilibrium flow is for β_x to be negative which implies the application of a favourable pressure gradient (as was found for the case of approximate equilibrium which is not surprising).

In the case of exact equilibrium we also have the fact that

$$\beta_x = m\beta_D = -1\beta_D \tag{6.6}$$

which leads to the only solution as $\beta_D = 1$, $\beta_x = -1$ corresponding to a converging flow with a favourable pressure gradient applied. It is not possible then to have an exact equilibrium for the diverging flow case, although approximate equilibrium *may* be possible.

In order to relate these conditions to a physical picture of the flow we assume that the free stream velocity at $x = 0$ (which is at some radial distance from the origin) is $U_1(x = 0) = U_o$ and we find that the variation of the velocity is

$$\frac{U_1}{U_o} = \frac{x_o}{(x_o - x)}. \tag{6.7}$$

If we ignore the displacement effect of the boundary layer growth this corresponds with the flow radially inward between two parallel discs. To allow for the boundary layer growth (negative in this case) the upper disc would need to be made slightly conical in practice.

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7. Discussion and conclusions

It has been shown that equilibrium solutions are theoretically possible for boundary layers with pressure gradients and streamline divergence or convergence. Approximate equilibrium layers with pressure gradient and divergence are found to be possible if two parameters are held constant: the Clauser pressure gradient parameter β_x and the divergence parameter β_D. The necessary conditions for these equilibrium layers to exist have been derived and it is found further that exact equilibrium is possible on a smooth wall for the case of streamline convergence with a favourable pressure gradient applied with particular values of the parameters. It must be stressed that the analysis has shown only that certain flows may be in equilibrium in that they satisfy certain conditions. The conditions are necessary but may not be sufficient. The results for approximate equilibrium are consistent with available experimental results in the case of zero streamwise pressure gradient.

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