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THE WIGNER PROPERTY FOR CL-SPACES AND FINITE-DIMENSIONAL POLYHEDRAL BANACH SPACES

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Abstract We say that a map f from a Banach space X to another Banach space Y is a phase-isometry if the equality

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\}$$

holds for all $x, y \in X$. A Banach space X is said to have the Wigner property if for any Banach space Y and every surjective phase-isometry $f : X \to Y$, there exists a phase function $\varepsilon : X \to \{-1, 1\}$ such that $\varepsilon \cdot f$ is a linear isometry. We present some basic properties of phase-isometries between two real Banach spaces. These enable us to show that all finite-dimensional polyhedral Banach spaces and CL-spaces possess the Wigner property.

Keywords: the Wigner property; phase-isometry; CL-space; polyhedral space

2020 Mathematics subject classification Primary 46B20;
Secondary 46B04

1. Introduction

Throughout this paper, we consider the spaces all over the real field. Let X and Y be Banach spaces. We say that a map $f : X \to Y$ is a *phase-isometry* if it satisfies

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in X).$$

Two maps $f, g : X \to Y$ are called *phase equivalent* if there exists a phase function $\varepsilon : X \to \{-1, 1\}$ such that $g = \varepsilon \cdot f$. In the present paper, it is the connection between phase-isometries and isometries that will concern us. Under the onto assumption, the well-known Mazur–Ulam theorem [16] states that an isometry f of a real Banach space X onto another Banach space Y with $f(0) = 0$ is linear. Thus, it is natural to ask

Question 1.1. *Is it true that every surjective phase-isometry f between two Banach spaces X and Y is phase equivalent to a linear isometry?*

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Another important result which is related to linear isometries is Wigner's theorem [25] and its generalizations (see [2, 4, 5, 17, 19]). Rätz's result [19, Corollary 8(a)] showed that every map f between two real Hilbert spaces H and K with the property that

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \quad (x, y \in H),$$

is phase equivalent to a linear isometry. Recently, Maksa and Páles [12] revisited this real version of Wigner's theorem, and they used this to obtain that every phase-isometry between real Hilbert spaces is phase equivalent to a linear isometry. Namely, Question 1.1 has a positive answer when X and Y are real Hilbert spaces. In our paper [8], we provide an affirmative answer to Question 1.1 with X and Y being $l_p(\Gamma)$ spaces $(0 < p < \infty)$. The first author and Jia presented in [9] a similar result for $\mathcal{L}_\infty(\Gamma)$-type spaces. Recall that a Banach space X is said to have the *Wigner property* if Question 1.1 has an affirmative answer for an arbitrary target Y. It should be mentioned that the previous results for real Hilbert spaces, $l_p(\Gamma)$ spaces or $\mathcal{L}_\infty(\Gamma)$-type spaces are not enough to establish that these spaces have the Wigner property since the target spaces are not arbitrary. Very recently, Question 1.1 for general Banach space has started in our paper [21], in which we proved that smooth Banach spaces, $\mathcal{L}_\infty(\Gamma)$-type spaces and $l_1(\Gamma)$-spaces enjoy this property.

Although this manuscript is a continuation of our paper [21], the techniques we use in this note are totally different from those in [21]. We use a new method to construct the desired linear isometries. It is also a key step to solve Question 1.1. Actually, although our main result is about polyhedral spaces and CL-spaces, more properties of phase-isometries are shown to hold in general Banach spaces for the sake of possible applications. In §2, we introduce a notion of 'star points' to study the maximal convex subsets (i.e., facets) of the unit sphere. This notion enables us to show that every phase-isometry between two Banach spaces maps star points to star points in §3. By this, we can build a required isometry between two cones generated by the maximal convex subsets of corresponding unit spheres. We apply this result in §4 to establish that finite-dimensional polyhedral spaces (i.e., those spaces whose unit sphere is a polyhedron) and CL-spaces which include many classical Banach spaces have the Wigner property. It is known that CL-spaces are very famous and important in the development of the geometry of Banach spaces. Recall from [15] that

Definition 1.2. A Banach space X is said to be *a CL-space (an almost CL-space)* if $B_X = \mathrm{co}(M \cup -M)$ ($B_X = \overline{\mathrm{co}}(M \cup -M)$, respectively) for every maximal convex set M of S_X.

The notion of CL-spaces was first introduced in 1960 by R. Fullerton [3]. Examples of real CL-spaces are $L_1(\mu)$ spaces for any measure μ and its isometric preduals, in particular, $C(K)$, where K is a compact Hausdorff space (see [10, §3]). The c_0 and l^1 sums of CL-spaces are also proved to be CL-spaces (see [15]). For more information on CL-spaces, we refer to [1, 6, 11, 14, 20, 22, 23]. Let us mention that the key part in handling Question 1.1 is how to construct or find a proper linear isometry with the desired property. Although we can provide an affirmative answer to Question 1.1 for CL-spaces, we do not know whether this is true for almost-CL-spaces.

Throughout what follows, we shall freely use without explicit mention notation '$\pm$' to mean that either '$+$' or '$-$' holds. For a Banach space X, B_X, S_X and X^* will stand for

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its unit ball, its unit sphere and its dual space, respectively. A subset M of S_X is said to be a *maximal convex subset of* S_X if it is not properly contained in any other convex subset of S_X.

2. Star points on the unit sphere of Banach spaces

In this section, we introduce the concept of star points in the unit sphere of Banach spaces. This concept plays an important role in showing main lemma and some propositions. Geometric characterizations of such points are also given for further results in next section.

Before our conclusions, we need some notation and concepts most of which come from [7]. By the Hahn–Banach and Krein–Milman theorems, we can see that every maximal convex subset M of S_X has the form

$$M = \{x \in S_X : x^*(x) = 1\}.$$

for some extreme point x^* of B_{X^*}. We denote by $\mathfrak{M}(X)$ the set made up of all maximal convex subsets of S_X, i.e.,

$$\mathfrak{M}(X) := \{M \subset S_X : M \text{ is a maximal convex subset of } S_X\}.$$

Given $x \in S_X$, the *star of* x with respect to S_X is defined by

$$St(x) := \{y \in S_X : \|y + x\| = 2\}.$$

It is easy to check that $\|x + y\| = 2$ if and only if $[x, y] \subset S_X$ for every $x, y \in S_X$. Then we can rewrite

$$St(x) = \{y \in S_X : [x, y] = \{tx + (1 - t)y : t \in [0, 1]\} \subset S_X\}.$$

Clearly, for every $x \in S_X$, there is a maximal convex subset of S_X containing x. Thus, $St(x)$ is precisely the union of all maximal convex subsets of S_X containing x, i.e.,

$$St(x) = \bigcup\{M \in \mathfrak{M}(X) : x \in M\}. \tag{2.1}$$

It is interesting to give a result demonstrating the relationship between the maximal convex subsets and the star sets of unit sphere. The proof here is based on an idea of [18, Proposition 2.3], and given only for the sake of completeness.

Proposition 2.1. *Let X be a Banach space, and let $M \in \mathfrak{M}(X)$. Then M is precisely the intersection of all $St(x)$ with $x \in M$, that is, $M = \bigcap_{x \in M} St(x)$.*

Proof. Note first that $M \subset St(x)$ for every $x \in M$ implies that

$$M \subset \bigcap_{x \in M} St(x).$$

To prove the converse, choose $z \in \bigcap_{x \in M} St(x)$. Then $\|z + x\| = 2$ for all $x \in M$. Set two open convex sets

$$S_1 = M + \mathrm{int} B_X \quad \text{and} \quad S_2 = -z + \mathrm{int} B_X,$$

where $\mathrm{int} B_X$ denotes the interior points of B_X. Obviously, $S_1 \cap S_2 = \emptyset$. By the separation theorem, there is a linear functional $x^* \in S_{X^*}$ and a real number c such that $x^*(z_1) >$

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$c > x^*(z_2)$ for every $z_1 \in S_1$ and every $z_2 \in S_2$. It follows that

$$\sup x^*(-z + \text{int}B_X) \leq c \leq \inf x^*(x + \text{int}B_X)$$

for every $x \in M$. Consequently,

$$-x^*(z) + 1 \leq c \leq x^*(x) - 1$$

for every $x \in M$. This shows that $c = 0$ and $x^*(z) = x^*(x) = 1$ for every $x \in M$. By the maximality of M, we have $M = \{x \in S_X : x^*(x) = 1\}$, and hence $z \in M$. The proof is complete. □

We will provide a concept which turns out to be a useful tool for analysing maximal convex subsets of unit sphere of Banach spaces.

Definition 2.2. Let X be a Banach space. A point $x \in S_X$ is called a *star point* of S_X if $St(x)$ is a maximal convex subset of S_X.

We can see that an element $x \in S_X$ is a star point if and only if $St(x)$ is convex since that $St(x)$ is convex is sufficient to show that $St(x)$ is a maximal subset of S_X. For better understanding of star points, we now give two characterizations of star points on the unit sphere which will be of use later.

Proposition 2.3. *Let X be a Banach space and $x \in S_X$. Then the following statements are equivalent*:

(1) x is a star point of S_X.

(2) For all y, $z \in St(x)$, we have $\|y + z\| = 2$.

(3) There is a unique maximal convex set of S_X containing x.

Proof. That (1) implies (2) is obvious. To prove that (2) implies (3), suppose that there are two sets M_1, $M_2 \in \mathfrak{M}(X)$ containing x. Clearly, $M_1 \cup M_2 \subset St(x)$. Now, take an arbitrary $z \in M_1$. Since (2) holds, it follows that $[z, y] \subset S_X$ for every $y \in M_2$. Hence we have $M_2 \subset \text{co}(M_2, z) \subset S_X$. Then the maximality of M_2 ensures us that $\text{co}(M_2, z) = M_2$, and thus $z \in M_2$. It follows that $M_1 \subset M_2$, and so $M_1 = M_2$. Finally, (3) implies (1) following from the identity (2.1). □

By the previous proposition, we can present an example here that draws upon [7, Exercise 2.18], which shows that every maximal convex set of the unit sphere in a separable Banach space has star points.

Example 2.4. If X is a separable Banach space, then S_X has star points. Indeed, let $M \in \mathfrak{M}(X)$. Choose $\{x_n\}$ to be a dense subset of M. Then the point $x_0 = \sum 2^{-n} x_n$ is a star point of S_X. Actually, for every $M_1 \in \mathfrak{M}(X)$ containing x_0, there exists a functional $x^* \in S_{X^*}$ such that $M_1 = \{x \in S_X : x^*(x) = 1\}$. In particular, $x^*(x_0) = 1$ implies that $x^*(x_n) = 1$ for all $n \in \mathbb{N}$. This and the maximality of M show that $M = M_1$. So there is a unique maximal convex set of S_X containing x_0. This completes the proof by Proposition 2.3.

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The notion of star points relates to the familiar notion of smooth points in Banach spaces.

Remark 2.5. Recall that $x \in S_X$ is called a *smooth* point if there exists only one support functional $x^* \in S_{X^*}$ at x. Proposition 2.3 and the Hahn–Banach theorem establish that if x is a smooth point of S_X, then it is a star point of S_X. It should be remarked that the smooth points of S_X are dense in S_X, whenever X is a separable Banach space [7, p. 171]. Therefore, for any separable Banach space X, the set consisting of star points is dense in S_X.

By the above results, one may wonder whether every Banach space possesses a star point. The following easy example shows that it is not true.

Example 2.6. Let Γ be an uncountable set. Note that for every $x \in l_1(\Gamma)$, $St(x)$ is not a convex subset of $S_{l_1(\Gamma)}$. It follows that the unit sphere of $l_1(\Gamma)$ does not have star points.

Let X be a Banach space and $M \in \mathfrak{M}(X)$. Recall that a point $x \in M$ is called a *non-proper support point* of M if any $x^* \in S_{X^*}$ with $x^*(x) = 1$, we have $x^*(z) = 1$ for all $z \in M$. It is routine to verify that there is only a unique maximal convex set M of S_X containing x. So a non-proper support point of M is a star point. Now we may conclude a direct relation from the above arguments that for every $x \in S_X$,

$$x \text{ is a smooth point} \Rightarrow x \text{ is non-proper support point} \Rightarrow x \text{ is a star point.}$$

However, none of the converse is true, even for the finite-dimensional spaces.

Example 2.7. (1) For each $0 < t < 1$, let us write $X_t = (\mathbb{R}^2, \|\cdot\|)$, where the unit sphere is given by

$$S_{X_t} = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1, |a| > t\} \cup \{(a, \pm\sqrt{1-t^2}) \in \mathbb{R}^2 : |a| \leq t\}.$$

Then the point $(t, \sqrt{1-t^2})$ is a star point but not a non-proper support point.

(1) Define an equivalent norm $\|\cdot\|$ on l_1 by

$$\|x\| = \frac{\|x\|_{l_1} + \|x\|_{l_2}}{2}, \quad \forall x \in l_1,$$

where $\|\cdot\|_{l_1}$ and $\|\cdot\|_{l_2}$ denote the usual l_1-norm and l_2-norm, respectively. Then the unit vectors $\{e_n\}_{n\geq 1} \subset X = (l_1, \|\cdot\|)$ are all non-proper support points but not smooth points. Indeed, X is a Banach space with a strict convex norm. Therefore, every maximal convex subset of S_X is a singleton. It follows that all points of the unit sphere are non-proper support points, whereas there are more than one support functional in S_{X^*} for each e_n with $n \geq 1$. For example, $e_n^* = (a_i)_{i\geq 1}$ where $a_i \in \mathbb{R}$ with $a_n = 1$ and $a_i = 0$ for $i \neq n$. Then for each $j \neq n$, $x_n^* = e_n^* + \frac{1}{2}e_j^* \in S_{X^*}$ is a support functional at e_n. Thus e_n is not a smooth point. One can easily check that this conclusion also holds for the $l_1^{(m)}$ which is l_1-spaces in dimension m with $m \geq 2$.

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Remark 2.8. Although three conceptions are quite different in general Banach spaces, they are the same in some classical Banach spaces, such as $L_p(\mu)$ (μ is a measure and $1 \leq p \leq \infty$) and $C(K)$ (K is a compact Hausdorff space).

3. The properties of phase-isometries on Banach spaces

In this section, we start the study of surjective phase-isometries between two Banach spaces X and Y. We start this section with a basic property of surjective phase-isometry which was verified in [21, Lemma 2.1].

Lemma 3.1 (Tan and Huang [21, Lemma 2.1]). *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Then f is a injective norm-preserving map and $f(-x) = -f(x)$ for all $x \in X$.*

An elementary result is presented to show that every phase-isometry carries star points to star points and we do not know whether this is true for smooth points and non-proper support points.

Lemma 3.2. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Then for every $x \in S_X$, $St(x) \in \mathfrak{M}(X)$ if and only if $St(f(x)) \in \mathfrak{M}(Y)$.*

Proof. Since the inverse of f denoted by f^{-1} is also a surjective phase-isometry, we only need to prove that if $St(x) \in \mathfrak{M}(X)$ then $St(f(x)) \in \mathfrak{M}(Y)$. Suppose that it is not true. Then Proposition 2.3 allows the existence of two distinct elements $y_1, y_2 \in St(f(x))$ such that $\|y_1 + y_2\| < 2$. We set $z_1 := \frac{1}{2}(y_1 + f(x))$ and $z_2 := \frac{1}{2}(y_2 + f(x))$. It is easy to see that $z_1, z_2 \in St(f(x))$ satisfy

$$\|z_1 + z_2\| < 2 \quad \text{and} \quad \|z_2 - z_1\| < 2.$$

Since f is surjective, there are $x_1, x_2 \in S_X$ such that $f(x_1) = z_1$ and $f(x_2) = z_2$. Observe from the fact that f is a phase-isometry that

$$x_1, x_2 \in St(x) \cup St(-x).$$

Thus $\|x_1 + x_2\| = 2$ or $\|x_1 - x_2\| = 2$, which leads to a contradiction that $\|z_1 + z_2\| = 2$ or $\|z_1 - z_2\| = 2$. The proof is complete. □

We shall take up to construct maps concerning the maximal convex sets of the unit sphere, which is also a key step to obtain a desired linear isometry from X onto Y.

Definition 3.3. Let X be a Banach space and $M \in \mathfrak{M}(X)$. Denote by C_M^X the cone generated by M, i.e., $C_M^X = \cup_{\lambda \geq 0} \lambda M$. We see that $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in C_M^X$. We shall freely use this observation without explicit mention throughout what follows.

Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that $M \in \mathfrak{M}(X)$ and $x_0 \in M$. We define a map $f_M : C_M^X \to Y$ with respect to

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x_0 given by

$$f_M(x) = \begin{cases} f(x) & \text{if } \|f(x)+f(x_0)\| = \|x+x_0\|; \\ -f(x) & \text{if } \|f(x)-f(x_0)\| = \|x+x_0\| \neq \|x-x_0\|. \end{cases}$$

We can also define a phase-isometry $F_M : X \to Y$ given by

$$F_M(x) = \begin{cases} f_M(x) & \text{if } x \in C_M^X; \\ -f_M(-x) & \text{if } x \in -C_M^X; \\ f(x) & \text{if } x \notin C_M^X \cup -C_M^X. \end{cases}$$

Remark 3.4. It is easy to see that for every $x \in C_M^X$, we have

$$\|f_M(x) + f_M(x_0)\| = \|x + x_0\| = \|x\| + \|x_0\|.$$

Moreover, we can see that the map $F_M : X \to Y$ is an odd mapping, since $f(-x) = -f(x)$ for every $x \in X$ by Lemma 3.1. It follows that $F_M : X \to Y$ is surjective phase-isometry. Then, Lemma 3.1 again establishes that F_M is a bijection and $f_M : C_M^X \to Y$ is injective.

We now deal with the case that maximal convex set M has a star point. An interesting property of f_M is presented below.

Proposition 3.5. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that $M \in \mathfrak{M}(X)$ contains a star point x_0. Let $f_M : C_M^X \to Y$ be a map with respect to x_0. Then $M_1 = f_M(M) \in \mathfrak{M}(Y)$ and $f_M(C_M^X)$ is a cone in Y generated by M_1. Furthermore, $f_M : C_M^X \to C_{M_1}^Y$ is a surjective isometry.*

Proof. By Lemma 3.2 and Remark 3.4, it is apparent that $St(f(x_0)) \in \mathfrak{M}(Y)$ and $f_M(M) \subset St(f(x_0))$. For every $y \in St(f(x_0))$ with $f(x) = y$ for some $x \in S_X$. Since f is a phase-isometry, we see that

$$\|x + x_0\| = \|f(x) + f(x_0)\| = 2 \quad \text{or} \quad \|x - x_0\| = \|f(x) + f(x_0)\| = 2.$$

It follows that x or $-x \in St(x_0) = M$ by Proposition 2.3. As a consequence, we get $y = f_M(x)$ with $x \in M$ or $y = f_M(-x)$ with $-x \in M$, which implies that $f_M(M) = St(f(x_0))$.

Set $M_1 := f_M(M)$. It remains to verify that $f_M(C_M^X) = C_{M_1}^Y$. By the definition of f_M, for every $0 \neq x \in C_M^X$, we have

$$\|f_M(x) + f_M(x_0)\| = \|x + x_0\| = \|x\| + \|x_0\| = \|f_M(x)\| + \|f_M(x_0)\|.$$

This allows us to conclude that

$$\|\frac{f_M(x)}{\|f_M(x)\|} + f_M(x_0)\| = 2.$$

An immediate conclusion from this is that $f_M(x)/\|f_M(x)\| \in St(f_M(x_0)) = M_1$, and so $f_M(x) \in C_{M_1}^Y$. For every $0 \neq y \in C_{M_1}^Y$ with $f(x) = y$ for some $x \in X$. Note that $\|f(x) +$

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$f(x_0)\| = \|f(x)\| + \|f(x_0)\|$, and since f is a phase-isometry, we have either

$$\|x\| + \|x_0\| = \|f(x) + f(x_0)\| = \|x + x_0\|$$

or

$$\|x\| + \|x_0\| = \|f(x) + f(x_0)\| = \|x - x_0\|.$$

It follows that $\pm x/\|x\| \in St(x_0) = M$. As a consequence, $f(x) = f_M(x)$ with $x \in C_M^X$ or $f(x) = f_M(-x)$ with $-x \in C_M^X$. Both cases show that $y = f(x) \in f_M(C_M^X)$. Thus, we finish the proof of $f_M(C_M^X) = C_{M_1}^Y$.

Finally, as seen from the above for all x, $y \in C_M^X$, we have

$$\|f_M(x) + f_M(y)\| = \|f_M(x)\| + \|f_M(y)\| = \|x\| + \|y\| = \|x + y\|.$$

That f_M is a phase-isometry leads to

$$\|f_M(x) - f_M(y)\| = \|x - y\|$$

as required, which finishes the proof. □

One may have a question whether the definition of f_M depends on the choice of x_0. It may have something to do with the choice of x_0 in some degree.

Lemma 3.6. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that $M \in \mathfrak{M}$ has a star point. Then $f_M = \pm f'_M$, where f_M and f'_M are defined with respect to two distinct points x_0 and x'_0 of M, respectively.*

Proof. We can assume that $x_0 \in M$ is a star point. Then $f(x_0)$ is also a star point of S_Y by Lemma 3.2. Since f is a phase-isometry, we have

$$2 = \|x_0 + x'_0\| \in \{\|f(x_0) + f(x'_0)\|, \|f(x_0) - f(x'_0)\|\}.$$

If $\|f(x_0) + f(x'_0)\| = \|x_0 + x'_0\| = 2$, then $f(x'_0) \in St(f(x_0))$. We conclude from this and f_M's definition that for every $0 \neq x \in C_M^X$ and $\theta \in \{-1, 1\}$, $f_M(x) = \theta f(x)$ implies that $f'_M(x) = \theta f(x)$. Namely, $f_M = f'_M$. Another possibility is that

$$\|f(x_0) - f(x'_0)\| = \|x_0 + x'_0\| = 2 > \|x_0 - x'_0\|.$$

Applying a similar argument as above shows that $f'_M = -f_M$. □

We still need more lemmas to investigate the property of these f_M's. The first one bases completely on the idea of [24, Lemma 3.4] (we apply almost the same proof). For our use and convenience, we present here a generalized version which also establishes that all surjective phase-isometries are separably determined in some sense. For every subset A of a Banach space, $[A]$ stands for its closed linear span.

Lemma 3.7. *Let X and Y be two Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Then for every separable subset $A \subset X$, there are separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $A \subset X_0$ and $f(X_0) = Y_0$.*

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Proof. Let $X_1 = [A]$ and $Y_1 = [f(X_1)]$. Choose $\{x_n\} \subset X_1$ which is dense in X_1. To see that Y_1 is separable, it suffices to show that the set $\{f(x_n)\} \cup \{-f(x_n)\}$ is dense in $f(X_1)$. By Lemma 3.1, it is clear that $\{f(x_n)\} \cup \{-f(x_n)\} \subset f(X_1)$. Now for every $y \in f(X_1)$ with $f(x) = y$ for some $x \in X_1$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges to x in norm. For every $k \geq 1$, choose $y_k \in Y$ satisfying

$$y_k = \begin{cases} f(x_{n_k}) & \text{if } \|f(x_{n_k}) - f(x)\| = \|x_{n_k} - x\| \neq \|x_{n_k} + x\|; \\ -f(x_{n_k}) & \text{if } \|f(x_{n_k}) - f(x)\| = \|x_{n_k} + x\|. \end{cases}$$

Thus, it is obvious that $\|y_k - y\| = \|y_k - f(x)\| = \|x_{n_k} - x\|$. This means that $\{f(x_n)\} \cup \{-f(x_n)\}$ is dense in $f(X_1)$ as desired. Define inductively separable subspaces $X_n \subset X$ and $Y_n \subset Y$ such that for all $n \geq 2$,

$$X_n = [f^{-1}(Y_{n-1})] \quad \text{and} \quad Y_n = [f(X_n)].$$

Then it is easy to verify that $X_0 = \bigcup_{n=1}^{\infty} X_n$ and $Y_0 = \bigcup_{n=1}^{\infty} Y_n$ are just the subspaces that we need. □

We will follow the line of the proof of [24, Lemma 3.5] to obtain a similar conclusion for phase-isometries instead of isometries. By Lemmas 3.6 and 3.7, we are now able to generalize Proposition 3.5 to the general case where the assumption on star points can be removed.

Proposition 3.8. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that $M \in \mathfrak{M}(X)$ and $f_M : C_M^X \to Y$ is the map with respect to some $x_0 \in M$. Then $f_M(M) \in \mathfrak{M}(Y)$ and $f_M(C_M^X)$ is a cone in Y generated by $f_M(M)$. Furthermore, f_M is an isometry on C_M^X.*

Proof. We first conclude that this conclusion is true under the assumption that X is a separable Banach space. By Example 2.4, M has a star point. In other words, if the chosen $x_0 \in M$ is not a star point, then by Lemma 3.6, we can replace it by a star point $x_0' \in M$. Proposition 3.5 yields the desired conclusion.

Now we handle the general case. For every finite subset $A \subset M$ with $x_0 \in A$, we apply Lemma 3.7 to F_M to obtain two separable subspaces $X_A \subset X$ and $Y_A \subset Y$ such that $A \subset X_A$ and $F_M(X_A) = Y_A$. Let $M_A \in \mathfrak{M}(X_A)$ such that $M \cap X_A \subset M_A$. The previous argument ensures us that

$$f_{M_A}(M_A) \in \mathfrak{M}(Y_A),$$

where $f_{M_A} : C_{M_A}^{X_A} \to Y_A$ is the induced map with respect to x_0 and M_A. As $A \subset M \cap M_A$, we have $f_M(x) = f_{M_A}(x)$ for every $x \in A$. It follows that

$$\mathrm{co}(f_M(M)) \subset S_Y.$$

So we can choose $M_1 \in \mathfrak{M}(Y)$ such that $f_M(M) \subset M_1$. For the converse, since the inverse f^{-1} of f is also a surjective phase-isometry, there is a map $h_{M_1} : C_{M_1}^Y \to X$ with respect

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to $f(x_0) \in M_1$ and induced from f^{-1}. We can see that

$$h_{M_1} \circ f_M(x) = x$$

for all $x \in C_M^X$. Using the above argument for h_{M_1} and M_1, we also have

$$M \subset h_{M_1}(M_1) \subset \mathrm{co}(h_{M_1}(M_1)) \subset S_X.$$

This and the maximality of M entail that $M = h_{M_1}(M_1)$, and so $f_M(M) = M_1$.

We also need to prove that $f_M(C_M^X) = C_{M_1}^Y$, where $M_1 = f_M(M)$. For all $0 \neq z \in C_M^X$ and $x \in M$, set $B := \{z, x, x_0\}$. Following a similar argument as above, we get two separable subspaces $X_B \subset X$ and $Y_B \subset Y$ such that $B \subset X_B$ and $F_M(X_B) = Y_B$. Let $f_{M_B} : C_{M_B}^{X_B} \to Y_B$ be the induced map with respect to x_0 and M_B. Then $f_{M_B}(M_B) \in \mathfrak{M}(Y_B)$ and $f_{M_B}(C_{M_B}^{X_B}) \subset Y_B$ is a cone generated by $f_{M_B}(M_B)$. It follows that

$$\|f_M(z) + f_M(x)\| = \|f_{M_B}(z) + f_{M_B}(x)\| = \|f_M(z)\| + \|f_M(x)\|.$$

Consequently, we obtain $f_M(z)/\|f_M(z)\| \in St(f_M(x))$ for every $x \in M$. Applying Proposition 2.1, we get $f_M(z)/\|f_M(z)\| \in M_1$, and so $f_M(C_M^X) \subset C_{M_1}^Y$. Conversely, repeating the same argument to $h_{M_1} : C_{M_1}^Y \to X$, we obtain $h_{M_1}(C_{M_1}^Y) \subset C_M^X$. Observe that the map h_{M_1} is injective and

$$h_{M_1} \circ f_M(x) = x$$

for all $x \in C_M^X$. We get $f_M(C_M^X) = C_{M_1}^Y$ as desired. It is simply verified that $f_M : C_M^X \to Y$ is an isometry by noticing that

$$\|f_M(x) + f_M(y)\| = \|x + y\|,$$

for all $x, y \in C_M^X$. The proof is complete. □

Remark 3.9. It should be noted that when E is a subspace of a Banach space X and M is a maximal convex set of S_X, $M_E = M \cap E$ may not be a maximal convex set of S_E by [22, Remark 2.3]. So we can not replace M_A by $M \cap X_A$ to simplify the proof of Proposition 3.8.

Proposition 3.8 also allows us to conclude that Lemma 3.6 still works in the general case. For this reason, in what follows we shall briefly use the notation f_M referring to the map defined in Definition 3.3 without explicit mention x_0 unless it is necessary. We sum up now the result as follows.

Corollary 3.10. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that $M \in \mathfrak{M}(X)$ and $x_0, x_0' \in M$. Then $f_M = \pm f_M'$, where f_M and f_M' are defined with respect to x_0 and x_0' of M, respectively.*

For every Banach space X, since S_X consists of all its maximal convex subsets M, it seems that a demanded isometry F which is phase equivalent to f on the entire space X can be induced from these f_M's. It can not be simply got by defining $F : X \to Y$ by $F(x) = f_M(x)$ for every $M \in \mathfrak{M}(X)$ and $x \in M$. A difficulty should be noted here that for every x in the intersection of the two cones $C_{M_1}^X$ and $C_{M_2}^X$ of distinct sets

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$M_1, M_2 \in \mathfrak{M}(X)$. It may happen that $f_{M_1}(x) \neq f_{M_2}(x)$, for example, $f_{M_1}(x) = f(x)$, while $f_{M_2}(x) = -f(x)$. This means that the definition of F may be actually not valid. We shall present a lemma below as a first step to overcome this difficulty.

Lemma 3.11. *Let X and Y be Banach spaces, and let $f : X \to Y$ be a surjective phase-isometry. Suppose that M_1, M_2 are in $\mathfrak{M}(X)$ with $M_1 \cap M_2 \neq \emptyset$. If $f_{M_1}(x_0) = \theta f_{M_2}(x_0)$ for some $x_0 \in M_1 \cap M_2$ and some $\theta \in \{-1, 1\}$, then $f_{M_1}(z) = \theta f_{M_2}(z)$ for all $z \in C^X_{M_1} \cap C^X_{M_2}$.*

Proof. We may assume that $\theta = 1$ to simplify the notation, that is $f_{M_1}(x_0) = f_{M_2}(x_0)$. Suppose, on the contrary, that there is a nonzero element $z_0 \in C^X_{M_1} \cap C^X_{M_2}$ such that $f_{M_1}(z_0) = -f_{M_2}(z_0)$. We observe first from Proposition 3.8 that

$$\|x_0 - z_0\| = \|f_{M_1}(x_0) - f_{M_1}(z_0)\| = \|f_{M_2}(x_0) + f_{M_2}(z_0)\| = \|x_0\| + \|z_0\| = \|x_0 + z_0\|.$$

Let u_0 be the mid-point of x_0 and z_0, i.e., $u_0 = (x_0 + z_0)/2$. Obviously, $u_0 \in C^X_{M_1} \cap C^X_{M_2}$. If $f_{M_1}(u_0) = -f_{M_2}(u_0)$, then

$$\|u_0\| = \|x_0 - u_0\| = \|f_{M_1}(x_0) - f_{M_1}(u_0)\| = \|f_{M_2}(x_0) + f_{M_2}(u_0)\| = \|x_0\| + \|u_0\|;$$

If $f_{M_1}(u_0) = f_{M_2}(u_0)$, then

$$\|u_0\| = \|u_0 - z_0\| = \|f_{M_1}(u_0) - f_{M_1}(z_0)\| = \|f_{M_2}(u_0) + f_{M_2}(z_0)\| = \|u_0\| + \|z_0\|.$$

Either case leads to a contradiction. Thus, the proof is complete. □

4. Phase-isometries on finite-dimensional polyhedral Banach spaces and CL-spaces

In this section, we shall apply the preceding results about surjective phase-isometries to the study of finite-dimensional polyhedral Banach spaces (i.e., for those whose unit sphere is a polyhedron) and CL-spaces.

For our result on finite-dimensional polyhedral Banach spaces, we need a result [8, Lemma 2.5] which builds a connection between functionals of dual of respective spaces in terms of a phase-isometry. For a Banach space X, a functional $x^* \in S_{X^*}$ is called a w^*-exposed point of B_{X^*} if x^* is the unique support functional at some smooth point $x \in S_X$.

Lemma 4.1. *Let X and Y be real Banach spaces, and let $f : X \to Y$ be a phase-isometry (not necessarily surjective). Then for every w^*-exposed point x^* of B_{X^*}, there exists a linear functional $\varphi \in Y^*$ of norm one such that $x^*(x) = \pm\varphi(f(x))$ for all $x \in X$.*

The previous result enables us to show that all finite-dimensional polyhedral Banach spaces have the Wigner property.

Theorem 4.2. *Let X be an n-dimensional polyhedral Banach space. Then X has the Wigner property.*

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Proof. Let Y be a Banach space, and let $f : X \to Y$ be a surjective isometry. Given $M_0 \in \mathfrak{M}(X)$, let f_{M_0} be defined as in Definition 3.3. Since X is a polyhedral Banach space of finite dimension, the cone $C^X_{M_0}$ generated by M_0 is a convex body of X. Proposition 3.8 guarantees that $f_{M_0}(C^X_{M_0})$ is also a convex body in Y. Mankiewicz [13] has proved that every isometry between convex bodies is the restriction of an affine onto isometry between the corresponding spaces. By this result, there is a linear isometry $g_{M_0} : X \to Y$ such that its restriction on the cone $C^X_{M_0}$ is f_{M_0}.

We now apply Lemma 4.1 to obtain that for every w^*-exposed point x^* of B_{X^*}, there exists a linear functional $\varphi \in S_{Y^*}$ such that

$$x^*(x) = \pm\varphi(f(x)) = \pm\varphi(f_M(x)) \quad (M \in \mathfrak{M}(X), x \in C^X_M). \tag{4.1}$$

We claim that for every $M \in \mathfrak{M}(X)$, either $x^*(x) = \varphi(f_M(x))$ for all $x \in C^X_M$ or $x^*(x) = -\varphi(f_M(x))$ for all $x \in C^X_M$ holds. It remains to show that the third possibility, that is $x^*(x) = \varphi(f_M(x)) \neq 0$ and $x^*(y) = -\varphi(f_M(y)) \neq 0$ for some $x, y \in C^X_M$, leads to a contradiction. Indeed, it follows from (4.1) that

$$x^*(x + y) = \pm\varphi(f_M(x + y)) = \pm\varphi(f_M(x) + f_M(y)).$$

If $x^*(x + y) = \varphi(f_M(x) + f_M(y))$, this shows that $x^*(y) = -\varphi(f_M(y)) = 0$, a contradiction. If $x^*(x + y) = -\varphi(f_M(x) + f_M(y))$, then obviously $x^*(x) = -\varphi(f_M(x)) = 0$. This is a contradiction which proves the claim. For every w^*-exposed point z^* of B_{X^*}, we can choose $\varphi_{z^*} \in S_{Y^*}$ such that

$$z^*(z) = \varphi_{z^*}(g_{M_0}(z)) \quad \left(z \in C^X_{M_0}\right) \quad \text{and} \quad z^*(x) = \pm\varphi_{z^*}(g_{M_0}(x)) \quad \left(x \in X/C^X_{M_0}\right).$$

To see our conclusion, we shall prove that f is phase equivalent to g_{M_0}. Now for every $M \in \mathfrak{M}(X)$, although it may occur that $M \cap M_0 = \emptyset$, there are $M_1, \cdots, M_m \subset \mathfrak{M}(X)$ such that $M_i \cap M_{i+1} \neq \emptyset$ for $i = 0, 1, \cdots, m$, where we use the notation $M_{m+1} = M$. We deduce from Lemma 3.11 and $M_1 \cap M_0 \neq \emptyset$ that there is $\theta_1 \in \{-1, 1\}$ such that $g_{M_0}(x) = \theta_1 f_{M_1}(x)$ for all $x \in C^X_{M_0} \cap C^X_{M_1}$. This and the claim show that

$$\varphi_{z^*}(g_{M_0}(y)) = \varphi_{z^*}(\theta_1 f_{M_1}(y))$$

for all $y \in C^X_{M_1}$. Inductively, there are $\{\theta_i\}_{i=2}^{m+1} \subset \{-1, 1\}$ such that

$$\varphi_{z^*}(g_{M_0}(y)) = \varphi_{z^*}(\theta_1 \cdots \theta_{m+1} f_{M_{m+1}}(y)) = \varphi_{z^*}(\theta_1 \cdots \theta_{m+1} f_M(y))$$

for all $y \in C^X_M$. Set $\theta_M = \theta_1 \cdots \theta_{m+1}$. Then

$$\varphi_{z^*}(g_{M_0}(y)) = \varphi_{z^*}(\theta_M f_M(y)) \tag{4.2}$$

for all $y \in C^X_M$. Let W be the set consisting of all w^*-exposed points of B_{X^*}. By Proposition 3.8, the set $\{\varphi_{z^*}\}_{z^* \in W}$ is a norming set of Y. So (4.2) ensures us that $g_{M_0} = \theta_M f_M$ on C^X_M. This completes the proof. □

The second conclusion of this section is that CL-spaces have the Wigner property. In what follows, to simplify the notation let us make some explanation. X and Y are always used to represent a CL-space and a Banach space, respectively. For every surjective phase-isometry $f : X \to Y$ and every $M \in \mathfrak{M}(X)$, let f_M be defined as in Definition 3.3. The

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unique linear functional $x^* \in S_{X^*}$ such that $M = \{x \in S_X : x^*(x) = 1\}$ will be denoted by x^*_M. By Proposition 3.8, $f_M(M) \in \mathfrak{M}(Y)$. Thus, there is a unique $y^* \in S_{Y^*}$ such that $f_M(M) = \{y \in S_Y : y^*(y) = 1\}$ and we denote it by y^*_M.

For our second result, we present a basic property for CL-spaces, which may be previously known. Its brief proof is given here since we can not find it in literature.

Lemma 4.3. *Let X be a CL-space. For all M_1, $M_2 \in \mathfrak{M}(X)$. If $M_1 \neq -M_2$, then $M_1 \cap M_2 \neq \emptyset$.*

Proof. If $M_1 = M_2$, then the desired conclusion is obviously true. Now we assume that $M_1 \neq M_2$. Choose an element $z \in M_1$ which is not in $M_2 \cup -M_2$. Since $B_X = \mathrm{co}(M_2 \cup -M_2)$, there are $x_1 \in M_2$, $x_2 \in -M_2$ and $\lambda \in (0, 1)$ such that

$$z = \lambda x_1 + (1 - \lambda)x_2.$$

The identity $x^*_{M_1}(z) = 1$ implies that $x_1, x_2 \in M_1$. We conclude from this that $x_1 \in M_1 \cap M_2 \neq \emptyset$. □

Lemma 4.4. *Let X be a CL-space and Y a Banach space, and let $f : X \to Y$ be a surjective phase-isometry. If M_1, $M_2 \in \mathfrak{M}(X)$ with $M_1 \cap M_2 \neq \emptyset$ and $f_{M_1}(x_0) = \theta f_{M_2}(x_0)$ with $\theta \in \{-1, 1\}$ for some $x_0 \in M_1 \cap M_2$, then $y^*_{M_1} \circ f_{M_2} = \theta x^*_{M_1}$ on the cone $C^X_{M_2}$.*

Proof. We may assume that $f_{M_1}(x_0) = f_{M_2}(x_0)$ for some $x_0 \in M_1 \cap M_2$. Namely, the value of θ is chosen to be 1. Moreover, by Corollary 3.10 we can assume that f_{M_1} and f_{M_2} are defined with respect to x_0. If $z \in C^X_{M_2} \cap C^X_{M_1}$, then by Lemma 3.11, we have $f_{M_2}(z) = f_{M_1}(z)$. This gives the desired conclusion in this case. Note that $f(-x) = -f(x)$ for every $x \in X$. Then in the case of $z \in C^X_{M_2} \cap -C^X_{M_1}$, we have $f_{M_2}(z) = -f_{M_1}(-z)$ following from $\|f(z) + f(x_0)\| = \|z + x_0\|$ or $\|f(z) - f(x_0)\| = \|z + x_0\|$. This entails the desired result.

Now we handle the remaining case of $z \in C^X_{M_2}$ and $z \notin C^X_{M_1} \cup -C^X_{M_1}$. Since X is a CL-space, there are $x_1 \in M_1$, $x_2 \in -M_1$ and $\lambda \in (0, 1)$ such that

$$\frac{z}{\|z\|} = \lambda x_1 + (1 - \lambda)x_2.$$

It follows that $x_1, x_2 \in M_2$, and therefore

$$f_{M_2}(\|z\|x_1) = f_{M_1}(\|z\|x_1) \text{ and } f_{M_2}(\|z\|x_2) = -f_{M_1}(-\|z\|x_2).$$

This together with the choice of $y^*_{M_1}$ yields

$$\begin{aligned} 2\|z\| &= y^*_{M_1}(f_{M_2}(\|z\|x_1) - f_{M_2}(\|z\|x_2)) \\ &= y^*_{M_1}(f_{M_2}(\|z\|x_1) - f_{M_2}(z)) + y^*_{M_1}(f_{M_2}(z) - f_{M_2}(\|z\|x_2)) \\ &\leq \|f_{M_2}(\|z\|x_1) - f_{M_2}(z)\| + \|f_{M_2}(z) - f_{M_2}(\|z\|x_2)\| \\ &= \big\|\|z\|x_1 - z\big\| + \big\|z - \|z\|x_2\big\| = 2\|z\|. \end{aligned}$$

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Thus equality holds in the above, i.e., we have

$$y^*_{M_1}(f_{M_2}(\|z\|x_1) - f_{M_2}(z)) = \|f_{M_2}(\|z\|x_1) - f_{M_2}(z)\| = \big\|\|z\|x_1 - z\big\|,$$
$$y^*_{M_1}(f_{M_2}(z) - f_{M_2}(\|z\|x_2)) = \|f_{M_2}(z) - f_{M_2}(\|z\|x_2)\| = \big\|z - \|z\|x_2\big\|.$$

On the other hand, we get

$$\begin{aligned}2\|z\| &= x^*_{M_1}(\|z\|x_1 - \|z\|x_2) = x^*_{M_1}(\|z\|x_1 - z) + x^*_{M_1}(z - \|z\|x_2)\\ &\le \big\|\|z\|x_1 - z\big\| + \big\|z - \|z\|x_2\big\| = 2\|z\|.\end{aligned}$$

This allows us to assert that

$$x^*_{M_1}(\|z\|x_1 - z) = \big\|\|z\|x_1 - z\big\| = y^*_{M_1}(f_{M_2}(\|z\|x_1) - f_{M_2}(z)).$$

Since we have $y^*_{M_1}(f_{M_2}(\|z\|x_1)) = x^*_{M_1}(\|z\|x_1) = \|z\|$, it follows that $y^*_{M_1}(f_{M_2}(z)) = x^*_{M_1}(z)$ as expected. The proof is complete. □

The previous lemma reveals the connections between distinct maximal convex sets of unit sphere of CL-spaces. We will prove the main result mentioned in the abstract: Every CL-space has the Wigner property.

Theorem 4.5. *Let X be a CL-space. Then X has the Wigner property.*

Proof. Note that $\mathfrak{M}(X)$ is the set consisting of all maximal convex subsets of S_X. One can easily see that

$$X = \bigcup_{M\in\mathfrak{M}(X)} C^X_M.$$

By the axiom of choice, there is a set $\mathfrak{M}(X)^+ \subset \mathfrak{M}(X)$ such that for every $M \in \mathfrak{M}(X)$ either M or $-M$ belongs to $\mathfrak{M}(X)^+$ and $M_1 \cap M_2 \neq \emptyset$ for every $M_1,\ M_2 \in \mathfrak{M}(X)^+$ by Lemma 4.3. Fix $M_0 \in \mathfrak{M}(X)^+$. Lemma 3.11 guarantees that there is a set $\{\theta_M\}_{M\in\mathfrak{M}(X)^+}$ of signs such that $f_{M_0}(x_0) = \theta_M f_M(x_0)$ for some $x_0 \in M \cap M_0$ and each θ_M does not depend on the choices x_0 in $M \cap M_0$. So we can define a phase-isometry $F : X \to Y$ given by

$$F(x) = \begin{cases}\theta_M f_M(x) & \text{if } x \in C^X_M\\ -\theta_M f_M(-x) & \text{if } -x \in C^X_M\end{cases}$$

for all $M \in \mathfrak{M}(X)^+$. To see that F is well defined, it remains to check that

$$\theta_{M_1} f_{M_1}(x) = \theta_{M_2} f_{M_2}(x)$$

for all $x \in C^X_{M_1} \cap C^X_{M_2}$, where $M_1,\ M_2 \in \mathfrak{M}(X)^+$. By Lemma 3.11, it suffices to show that there exists some $z_0 \in M_1 \cap M_2$ such that

$$\theta_{M_1} f_{M_1}(z_0) = \theta_{M_2} f_{M_2}(z_0).$$

Lemma 4.4 ensures us the existence of a corresponding $y^*_{M_0} \in S_{Y^*}$ such that

$$y^*_{M_0}(f_{M_1}(x)) = \theta_{M_1} x^*_{M_0}(x) \quad \text{and} \quad y^*_{M_0}(f_{M_2}(x)) = \theta_{M_2} x^*_{M_0}(x),$$

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for all $x \in M_1 \cap M_2$. Consequently,

$$y^*_{M_0}(\theta_{M_1} f_{M_1}(x)) = y^*_{M_0}(\theta_{M_2} f_{M_2}(x)) = x^*_{M_0}(x) \tag{4.3}$$

for all $x \in M_1 \cap M_2$. Choose $z_0 \in M_1 \cap M_2$. If $x^*_{M_0}(z_0) \neq 0$, then (4.3) and $f_{M_1}(z_0) = \pm f_{M_2}(z_0)$ entail that $\theta_{M_1} f_{M_1}(z_0) = \theta_{M_2} f_{M_2}(z_0)$. If $x^*_{M_0}(z_0) = 0$, then there are $z_1 \in M_0$, $z_2 \in -M_0$ and $\lambda \in (0, 1)$ such that

$$z_0 = \lambda z_1 + (1 - \lambda) z_2.$$

We observe that $z_1 \in M_1 \cap M_2 \cap M_0$ is the desired element. Clearly, F is a surjective phase-isometry, and it is phase equivalent to f. By Lemma 3.1 and Proposition 3.8, we see that F is bijective, $F(M) \in \mathfrak{M}(Y)$ and $F(C^X_M)$ is a cone in Y generated by $F(M)$.

We claim that for every $M \in \mathfrak{M}(X)$, the relation $y^*_M \circ F = \theta_M x^*_M$ holds on X. We can assume that $M \in \mathfrak{M}(X)^+$. By the above observation, we only need to check that

$$y^*_M \circ F(z) = \theta_M x^*_M(z)$$

for every $z \in C^X_{M'}$ with $M' \in \mathfrak{M}(X)^+$. Note that $M \cap M' \neq \emptyset$ and $\theta_M f_M(v) = \theta_{M'} f_{M'}(v)$ for all $v \in M \cap M'$. Lemma 4.4 guarantees that

$$y^*_M \circ F(z) = y^*_M(\theta_{M'} f_{M'}(z)) = \theta_M x^*_M(z)$$

for all $z \in C^X_{M'}$. This finishes the claim.

We now prove that $F : X \to Y$ is a surjective isometry, and so F is just the desired linear isometry by Mazur–Ulam theorem. Indeed, for all $u, v \in X$, it is easily seen that there is an $M \in \mathfrak{M}(X)$ such that $x^*_M(u - v) = \|u - v\|$. The claim implies that

$$\|F(u) - F(v)\| \geq |y^*_M \circ F(u) - y^*_M \circ F(v)| = \|u - v\|.$$

Similarly, we have

$$\|F(u) + F(v)\| \geq \|u + v\|.$$

Since F is a phase-isometry, the previous two inequalities should be equalities. Namely $F : X \to Y$ is an isometry. The proof is complete. □

An immediate conclusion of Theorem 4.5 is the following

Corollary 4.6. *$L_1(\mu)$-spaces and $C(K)$ have the Wigner property, where μ is a positive measure and K is a compact Hausdorff space.*

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