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On the bifurcation of limit cycles in a dynamic model of a small open economy

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In this paper a four-dimensional macroeconomic model of a small open economy, describing the development of income, capital stock, interest rate and money stock, which was constructed in [5] (Makovínyiová, K. & Zimka, R. (2009) On stability in generalized Schinasi's macroeconomic model under fixed exchange rates. *Tatra Mt. Math. Publ.* **43**, 115–122), is analysed. Sufficient conditions for the existence of one pair of purely imaginary eigenvalues and two eigenvalues with negative real parts in the linear approximation matrix of the model are found. Formulae for the calculation of the bifurcation coefficients of the model are derived. A statement about the existence of limit cycles is made. A numerical example is given illustrating the results.

Key words: Macroeconomic dynamic model; Equilibrium; Stability; Normal form of differential equations on centre manifold; Bifurcation equation; Limit cycle

1 Introduction

In the paper we analyse a model of a small open economy:

$$\begin{aligned} \dot{Y} &= \alpha \ \left[I(Y,K,R) + G - S(Y^D,R) - T(Y) + J(Y,\rho) \right], \\ \dot{K} &= I(Y,K,R), \\ \dot{R} &= \beta \ [L(Y,R) - M], \\ \dot{M} &= J(Y,\rho) + \gamma \ (R - R_f), \end{aligned} \tag{1}$$

where Y = net real national income, K = real physical capital stock, R = nominal rate of interest of domestic country, M = nominal money stock, I = net real private investment expenditure on physical capital, G = real government expenditure (fixed), S = savings, T = real income tax, J = net exports in real terms, ρ = exchange rate (value of a unit of foreign currency in terms of domestic currency), L = liquidity preference function, R_f = nominal rate of interest of foreign country, α, β, γ = positive parameters, t = time and

$$Y^D = Y - T(Y), \dot{Y} = \frac{dY}{dt}, \dot{K} = \frac{dK}{dt}, \dot{R} = \frac{dR}{dt}, \dot{M} = \frac{dM}{dt}.$$

For the analysis of model (1) we need to express the basic economic properties of the functions involved in the model by equivalent mathematical expressions. We know, for example, that increasing national income Y creates good conditions for the increase of investments $I(Y,K,R)$, and, on the other hand, with increasing interest rate R, firms have harder conditions to gain money for investments. These facts can also be expressed in words in the following way: The partial derivative of investment $I(Y,K,R)$ with respect to income Y is positive and the partial derivative of investment $I(Y,K,R)$ with respect to interest rate R is negative. Analogously we could also describe by mathematical expressions the basic properties of other functions in model (1). These are as follows:

$$\frac{\partial I(Y,K,R)}{\partial Y} > 0, \frac{\partial I(Y,K,R)}{\partial K} < 0, \frac{\partial I(Y,K,R)}{\partial R} < 0, \frac{\partial S(Y^D,R)}{\partial Y^D} > 0,$$

$$\frac{\partial S(Y^D,R)}{\partial R} > 0, \frac{\partial T(Y)}{\partial Y} > 0, \frac{\partial J(Y,\rho)}{\partial Y} < 0, \frac{\partial J(Y,\rho)}{\partial \rho} > 0, \tag{2}$$

$$\frac{\partial L(Y,R)}{\partial Y} > 0, \frac{\partial L(Y,R)}{\partial R} < 0.$$

Model (1) was introduced in [5]. It deals with a small open economy, which means that economic processes in this economy have negligible influence on the ones in the region with which it is connected through inter-regional trade and inter-regional capital movement (this region we call a foreign country). The first equation in (1) expresses the reaction of the income Y to the fluctuations of demand. The second equation says that the investment I contributes to the physical capital stock K. The third equation shows that the interest rate R reacts to differences between the money demand L and the money stock M. The fourth equation expresses the fact that the net export J and differences between the interest rates of domestic and foreign countries contribute to the money stock M. Parameter α can take small values as the income Y reacts rather slowly to the changes of the functions involved in the model, while parameters β and γ can take large values as the interest rate and the money supply react to the changes of the functions very quickly. This knowledge about possible values of parameters α, β and γ will be used in the analysis of the model.

In this paper we consider the case of fixed exchange rate regime, which means that ρ is constant. With a fixed exchange rate, the basic macroeconomic processes are influenced primarily by changes in interest rate R, which brings results quickly, and also by structural changes in an economy, the result of which is displayed over a longer time interval. For example, business connections between Japan and the United States in 1950s and 1960s were realized under a fixed exchange rate regime. Also, trade relations among the states in Eurozone are done under a fixed exchange rate, this can be looked at as the extreme form of fixed exchange rate regime.

Model (1) has the classical IS/LM (Investment–Saving/Liquidity preference–Money supply) structure, enriched by the development of the physical capital stock and the money stock. Model (1) can be looked at as an expansion of Asada's three-dimensional model of a small open economy, which was introduced and analysed in [1].

It is assumed in Asada's model that parameter β in the equation for the interest rate, $\dot{R} = \beta(L(Y,R)-M)$, is infinite so that the interest rate adjusts instantaneously to preserve the equilibrium $M = L(Y,R)$ on the money market (as in [1] we suppose throughout the paper that the price level is constant and is normalized to the value 1). This assumption is rather strong and not very realistic. In model (1) the equation for the rate of interest is present. It enlarges the dimension of the model, which makes it more complex to analyse, but on the other hand it offers further possibilities for the investigation of the dynamics of variables involved in the model.

The questions for the existence of an equilibrium of the model and its stability were investigated in [5].

In the present paper the question of the existence of limit cycles is analysed. Stable limit cycles can appear in the case of the linear approximation matrix of model (1) having at least one pair of purely imaginary eigenvalues with the others having negative real parts. In the present paper we investigate the case of one pair of purely imaginary eigenvalues. In Section 2 we derive sufficient conditions on parameters α, β and γ which guarantee this situation (such parameters are called critical parameters). This result is formulated in Theorem 1. Theorem 2 in this section recalls the main result from [5] on the stability of an equilibrium. Whether limit cycles arise or not at some values of parameters α, β and γ depends on the types of the first non-zero resonant terms in the partial normal model form on centre manifold. We suppose that the first two resonant terms have non-zero real parts. These real parts create the bifurcation equation of the model. The transformation of the model to its partial normal form on centre manifold and the formulae for the calculation of the first resonant terms are stated in Theorem 3 in Section 3. Theorem 4 concerns the existence of limit cycles. Section 4 presents a numerical example illustrating the results. The conclusions in Section 5 reassess the main results of the paper and show how they could be utilized in practice.

2 Analysis of the model

2.1 Model specification

We assume the following form of the functions in model (1):

$$
\begin{aligned}
I(Y,K,R) &= f_1(Y) - i_2K - i_3R + i_0,\\
S(Y^D,R) &= f_2(Y^D) + s_3R + s_0,\\
T(Y) &= t_1Y - t_0,\\
L(Y,R) &= f_3(Y) - l_3R + l_0,\\
J(Y,\rho) &= J(Y),\ \ \rho \text{ is constant,}
\end{aligned}
\tag{3}
$$

where $f_1(Y), f_2(Y^D), f_3(Y), J(Y)$ are non-linear functions with respect to Y of the type C^6.

After substituting (3) into model (1) we get the model

$$
\begin{aligned}
\dot{Y} &= \alpha\left[f_1(Y) - i_2K - i_3R + i_0 + G - f_2(Y^D) - s_3R - s_0 - t_1Y + t_0 + J(Y)\right],\\
\dot{K} &= f_1(Y) - i_2K - i_3R + i_0,\\
\dot{R} &= \beta\left[f_3(Y) - l_3R + l_0 - M\right],\\
\dot{M} &= J(Y) + \gamma\,(R - R_f).
\end{aligned}
\tag{4}
$$

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Suppose model (4) has a unique positive equilibrium $E^*(\gamma) = (Y^*(\gamma), K^*(\gamma), R^*(\gamma), M^*(\gamma))$, $Y^*(\gamma) > 0$, $K^*(\gamma) > 0$, $R^*(\gamma) > 0$, $M^*(\gamma) > 0$.

Remark 1 Sufficient conditions for the existence of an equilibrium $E^*(\gamma) = (Y^*(\gamma), K^*(\gamma), R^*(\gamma), M^*(\gamma))$ were found in [5].

Let us transform the equilibrium $E^*(\gamma)$ onto the origin $E_1^* = (Y_1^* = 0, K_1^* = 0, R_1^* = 0, M_1^* = 0)$ by shifting

$$Y_1 = Y - Y^*,\ K_1 = K - K^*,\ R_1 = R - R^*,\ M_1 = M - M^*.$$

Then model (4) takes the form

$$\begin{aligned}
\dot{Y}_1 &= \alpha\left[f_1(Y_1 + Y^*) - f_2(Y_1^D + (Y^*)^D - t_0) + J(Y_1 + Y^*)\right] \\
&\quad + \alpha\left[-t_1 Y_1 - i_2 K_1 - (i_3 + s_3) R_1\right] \\
&\quad + \alpha\left[-t_1 Y^* - i_2 K^* - (i_3 + s_3) R^* + i_0 - s_0 + t_0 + G\right], \\
\dot{K}_1 &= f_1(Y_1 + Y^*) - i_2 K_1 - i_3 R_1 - i_2 K^* - i_3 R^* + i_0, \\
\dot{R}_1 &= \beta\ \left[f_3(Y_1 + Y^*) - l_3 R_1 - M_1 - l_3 R^* + l_0 - M^*\right], \\
\dot{M}_1 &= J(Y_1 + Y^*) + \gamma\ (R_1 + R^* - R_f).
\end{aligned} \tag{5}$$

The Jacobian matrix $\mathbf{A} = \mathbf{A}(\alpha, \beta, \gamma)$ of model (5) at the equilibrium E_1^* is

$$\mathbf{A}(\alpha, \beta, \gamma) = \begin{pmatrix} -\alpha\ A & -\alpha\ i_2 & -\alpha(i_3 + s_3) & 0 \\ f_{1Y} & -i_2 & -i_3 & 0 \\ \beta f_{3Y} & 0 & -\beta l_3 & -\beta \\ J_Y & 0 & \gamma & 0 \end{pmatrix}, \tag{6}$$

where $A = -f_{1Y} + f_{2Y} - J_Y + t_1$, $f_{1Y} = \frac{df_1(E^*)}{dY}$, $f_{2Y} = \frac{df_2((E^*)^D)}{dY^D}(1 - t_1)$, $f_{3Y} = \frac{df_3(E^*)}{dY}$, $J_Y = \frac{dJ(E^*)}{dY}$.

2.2 Existence of a critical triple

Definition 1 *A triple $(\alpha_0, \beta_0, \gamma_0)$ of parameters α, β, γ is called the critical triple of model (5) if the matrix $\mathbf{A} = \mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\,\omega$, $\mathrm{i}^2 = -1$, and the other two eigenvalues $\lambda_{3,4}$ have negative real parts.*

The eigenvalues of $\mathbf{A} = \mathbf{A}(\alpha, \beta, \gamma)$ are the roots of the characteristic equation of $\mathbf{A}(\alpha, \beta, \gamma)$

$$\lambda^4 + a_1(\alpha, \beta, \gamma)\lambda^3 + a_2(\alpha, \beta, \gamma)\lambda^2 + a_3(\alpha, \beta, \gamma)\lambda + a_4(\alpha, \beta, \gamma) = 0, \tag{7}$$

where

$$\begin{aligned}
a_1 &= \alpha A + \beta l_3 + i_2, \\
a_2 &= \alpha\beta(l_3 A + f_{3Y}(i_3 + s_3)) + \alpha i_2(A + f_{1Y}) + \beta(\gamma + i_2 l_3), \\
a_3 &= \beta(\alpha\gamma A + \alpha(i_2 l_3(A + f_{1Y}) + i_2 s_3 f_{3Y} - J_Y(i_3 + s_3)) + \gamma i_2), \\
a_4 &= \alpha\beta\gamma i_2(A + f_{1Y}) - \alpha\beta i_2 s_3 J_Y.
\end{aligned}$$

Liu's [4] necessary and sufficient conditions for the characteristic equation to have a pair of purely imaginary roots, and the other two roots to have negative real parts are equivalent to the following conditions:

$$a_1 > 0, a_2 > 0, a_4 > 0, \tag{8}$$

$$\Delta_3 = (a_1 a_2 - a_3)a_3 - {a_1}^2 a_4 = 0. \tag{9}$$

On the base of the signs of the partial derivatives (2) and the economic sense of the functions (3) there is

$$\frac{df_1(Y)}{dY} > 0, \frac{df_2(Y^D)}{dY^D} > 0, \frac{df_3(Y)}{dY} > 0, \frac{dJ(Y)}{dY} < 0, \tag{10}$$

$$i_2 > 0,\ i_3 > 0,\ l_3 > 0,\ s_3 > 0,\ 0 < t_1 < 1,$$

and i_0, l_0, s_0, t_0 are constants. If the expression $A = -f_{1Y} + f_{2Y} - J_Y + t_1$ is non-negative, then taking into account that $A + f_{1Y}$ is positive, the coefficients a_1 and a_2 are positive for arbitrary values of positive parameters α, β, γ on the base of (10). If A is negative then a_1 is positive for arbitrary α and sufficiently large β. Rearranging a_2 into the form

$$a_2 = \beta[\alpha(l_3 A + f_{3Y}(i_3 + s_3)) + \gamma + i_2 l_3] + \alpha i_2(A + f_{1Y}),$$

we see that a_2 is positive for arbitrary α, β and sufficiently large γ. The coefficient a_4 is always positive for arbitrary positive values of parameters α, β, γ. Therefore, we can say that conditions (8) are satisfied for an arbitrary α and sufficiently large β and γ.

As parameter β can be, taking into account its economic sense, very large, let us order Δ_3 as a polynomial with respect to β. Denoting

$$\begin{aligned} a_1 &= a_{11}\beta + a_{12}(\alpha), \\ a_2 &= a_{21}(\alpha, \gamma)\beta + a_{22}(\alpha), \\ a_3 &= a_{31}(\alpha, \gamma)\beta, \\ a_4 &= a_{41}(\alpha, \gamma)\beta, \end{aligned}$$

the relation Δ_3 can be expressed in the form

$$\Delta_3(\alpha, \beta, \gamma) = g_1(\alpha, \gamma)\beta^2 + g_2(\alpha, \gamma)\beta + g_3(\alpha, \gamma), \tag{11}$$

where

$$\begin{aligned} g_1(\alpha, \gamma) &= (a_{21}a_{31} - a_{11}a_{41})a_{11}, \\ g_2(\alpha, \gamma) &= ((a_{12}a_{21} + a_{11}a_{22})a_{31} - 2a_{11}a_{12}a_{41} - a_{31}^2), \\ g_3(\alpha, \gamma) &= a_{12}a_{22}a_{31} - a_{12}^2 a_{41}. \end{aligned}$$

Express the coefficient $g_1(\alpha, \gamma)$ in the form

$$g_1(\alpha, \gamma) = \mathscr{A}\alpha^2 + \mathscr{B}\alpha + \mathscr{C},$$

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where

$$\begin{aligned}\mathscr{A} = \mathscr{A}(\gamma) &= (l_3A+(i_3+s_3)f_{3Y})(\gamma A+i_2((A+f_{1Y})l_3+s_3f_{3Y})-(i_3+s_3)J_Y)l_3,\\ \mathscr{B} = \mathscr{B}(\gamma) &= (A(\gamma+i_2l_3)^2+(\gamma+i_2l_3)i_2s_3f_{3Y}+i_2l_3(i_2l_3f_{1Y}-i_3J_Y)\\ &\quad +\gamma(i_3+s_3)(i_2f_{3Y}-J_Y))l_3,\\ \mathscr{C} = \mathscr{C}(\gamma) &= \gamma i_2(\gamma+i_2l_3)l_3.\end{aligned}$$

The expression A in the coefficient $\mathscr{A}$ can be positive, zero or negative. As Kaldor showed in [3], in practice the value f_{1Y} is considerably bigger than the value f_{2Y} at the equilibrium E^*, which causes the value A to be negative quite often. Thus, suppose $A < 0$. The coefficient $\mathscr{C}$ is always positive, the coefficient $\mathscr{B}$ is negative for sufficiently large γ and the sign of $\mathscr{A}$ depends on the sign of

$$l_3A+(i_3+s_3)f_{3Y}.$$

Let us first analyse the following equation:

$$g_1(\alpha,\gamma) = \mathscr{A}\alpha^2+\mathscr{B}\alpha+\mathscr{C} = 0. \tag{12}$$

There are three cases to consider.

Case 1. $l_3A+(i_3+s_3)f_{3Y} > 0$.

Then $\mathscr{A} < 0$ and equation (12) has two roots of different signs

$$\alpha_1^{(1)} = \frac{-\mathscr{B}+\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}} < 0, \quad \alpha_1^{(2)} = \frac{-\mathscr{B}-\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}} > 0.$$

Case 2. $l_3A+(i_3+s_3)f_{3Y} = 0$.

Then $\mathscr{A} = 0$ and equation (12) has a positive root

$$\alpha_2^{(1)} = -\frac{\mathscr{C}}{\mathscr{B}} > 0.$$

Case 3. $l_3A+(i_3+s_3)f_{3Y} < 0$.

Then $\mathscr{A} > 0$ and $\mathscr{B}^2-4\mathscr{A}\mathscr{C} > 0$ for sufficiently large γ because $\mathscr{B}^2-4\mathscr{A}\mathscr{C} = A^2\gamma^4+g(\gamma)$, $\lim_{\gamma\to\infty}\frac{g(\gamma)}{\gamma^4} = 0$. Equation (12) has two positive roots,

$$\alpha_3^{(1)} = \frac{-\mathscr{B}-\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}} > 0, \quad \alpha_3^{(2)} = \frac{-\mathscr{B}+\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}} > 0.$$

Now consider an arbitrary γ such that conditions (8) are satisfied, $\mathscr{B} < 0$ and $\mathscr{B}^2 - 4\mathscr{A}\mathscr{C} > 0$, denote it by γ^* and fix it. Take $\delta = \frac{1}{\beta}$ and define the function

$$F(\alpha,\delta,\gamma^*) = \delta^2\Delta_3\left(\alpha,\frac{1}{\delta},\gamma^*\right) = g_1(\alpha,\gamma^*)+g_2(\alpha,\gamma^*)\delta+g_3(\alpha,\gamma^*)\delta^2.$$

Then:

(1) $F(\alpha_i^{(j)},0,\gamma^*) = g_1(\alpha_i^{(j)},\gamma^*) = 0, \quad i = 1,2,3;\ j = 1,2,\ i \neq j,$ where $\alpha_i^{(j)}$ are positive roots of equation (12).

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(2) For $\frac{\partial F(\alpha_i^{(j)},0,\gamma^*)}{\partial \alpha}$ we get

Case 1:

$\frac{\partial F(\alpha_1^{(2)},0,\gamma^*)}{\partial \alpha} = 2\alpha_1^{(2)}\mathscr{A} + \mathscr{B} = 2\frac{-\mathscr{B}-\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}}\mathscr{A} + \mathscr{B} = -\sqrt{\mathscr{B}^2 - 4\mathscr{A}\mathscr{C}} < 0.$

Case 2:

$\frac{\partial F(\alpha_2^{(1)},0,\gamma^*)}{\partial \alpha} = \mathscr{B} < 0.$

Case 3:

$\frac{\partial F(\alpha_i^{(j)},0,\gamma^*)}{\partial \alpha} = 2\alpha_i^{(j)}\mathscr{A} + \mathscr{B}$ if

(a) $\frac{\partial F(\alpha_3^{(1)},0,\gamma^*)}{\partial \alpha} = 2\frac{-\mathscr{B}-\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}}\mathscr{A} + \mathscr{B} = -\sqrt{\mathscr{B}^2 - 4\mathscr{A}\mathscr{C}} < 0,$

(b) $\frac{\partial F(\alpha_3^{(2)},0,\gamma^*)}{\partial \alpha} = 2\frac{-\mathscr{B}+\sqrt{\mathscr{B}^2-4\mathscr{A}\mathscr{C}}}{2\mathscr{A}}\mathscr{A} + \mathscr{B} = +\sqrt{\mathscr{B}^2 - 4\mathscr{A}\mathscr{C}} > 0.$

On the basis of the Implicit Function Theorem we can formulate the following lemma.

Lemma 1 *For an arbitrary sufficiently small positive* δ^* *there exists in all cases 1-3 an* α^* *such that*

$$F(\alpha^*, \delta^*, \gamma^*) = \delta^* \Delta_3\left(\alpha^*, \frac{1}{\delta^*}, \gamma^*\right) = 0,$$

which means that

$$\Delta_3\left(\alpha^*, \frac{1}{\delta^*}, \gamma^*\right) = \Delta_3(\alpha^*, \beta^*, \gamma^*) = 0, \quad \beta^* = \frac{1}{\delta^*}.$$

From these considerations and Lemma 1 we can make a statement about the existence of a critical triple of parameters.

Theorem 1 *If parameters* β *and* γ *are sufficiently large, then there exists a critical triple* $(\alpha_0, \beta_0, \gamma_0)$ *of model (5).*

2.3 Stability of an equilibrium

Now consider a critical triple $(\alpha_0, \beta_0, \gamma_0)$ of model (5) whose existence is guaranteed by Theorem 1. Fix β_0 and γ_0 and investigate model (5) on the interval $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, $\varepsilon > 0$.

The question of stability of the equilibrium $E^*(\gamma)$ of model (4) in a small neighbourhood of the critical value α_0 was solved in [5], and is formulated in the next theorem.

Theorem 2 *The equilibrium* E^* *of model (4) in a small neighbourhood of the critical value* α_0 *is*

Case 1. asymptotically stable for $\alpha < \alpha_0$ *and unstable for* $\alpha > \alpha_0$,

Case 2. asymptotically stable for $\alpha < \alpha_0$ *and unstable for* $\alpha > \alpha_0$,

Case 3.

(a) *asymptotically stable for* $\alpha < \alpha_0$ *and unstable for* $\alpha > \alpha_0$,

(b) *unstable for* $\alpha < \alpha_0$ *and asymptotically stable for* $\alpha > \alpha_0$.

2.4 Existence and stability of limit cycles

We now perform a bifurcation analysis. Before transforming model (5) to its partial normal form on the centre manifold, we perform Taylor expansion of the functions on the right-hand side of the model, then translate the critical value α_0 onto the origin and find the Jordan form of matrix **A**. After the Taylor expansion of model (5) at E_1^*, we obtain

$$\begin{aligned}
\dot{Y}_1 &= \alpha(f_{1Y} - f_{2Y} + J_Y - t_1)Y_1 - \alpha i_2 K_1 - \alpha(i_3 + s_3)R_1 \\
&\quad + \frac{1}{2}\alpha(f_{1Y}^{(2)} - f_{2Y}^{(2)} + J_Y^{(2)})Y_1^2 + \frac{1}{6}\alpha(f_{1Y}^{(3)} - f_{2Y}^{(3)} + J_Y^{(3)})Y_1^3 \\
&\quad + \frac{1}{24}\alpha(f_{1Y}^{(4)} - f_{2Y}^{(4)} + J_Y^{(4)})Y_1^4 + \mathcal{O}(|Y_1|^5), \\
\dot{K}_1 &= f_{1Y}Y_1 - i_2K_1 - i_3R_1 + \frac{1}{2}f_{1Y}^{(2)}Y_1^2 + \frac{1}{6}f_{1Y}^{(3)}Y_1^3 \\
&\quad + \frac{1}{24}f_{1Y}^{(4)}Y_1^4 + \mathcal{O}(|Y_1|^5), \\
\dot{R}_1 &= \beta f_{3Y}Y_1 - \beta l_3R_1 - \beta M_1 + \frac{1}{2}\beta f_{3Y}^{(2)}Y_1^2 + \frac{1}{6}\beta f_{3Y}^{(3)}Y_1^3 \\
&\quad + \frac{1}{24}\beta f_{3Y}^{(4)}Y_1^4 + \mathcal{O}(|Y_1|^5), \\
\dot{M}_1 &= J_YY_1 + \gamma R_1 + \frac{1}{2}J_Y^{(2)}Y_1^2 + \frac{1}{6}J_Y^{(3)}Y_1^3 + \frac{1}{24}J_Y^{(4)}Y_1^4 + \mathcal{O}(|Y_1|^5),
\end{aligned} \tag{13}$$

where $f_{1Y}^{(i)} = \frac{d^if_1(E^*)}{dY^i}$, $f_{2Y}^{(i)} = (1-t_1)^i\,\frac{d^if_2(E^{*D})}{d(Y^D)^i}$, $f_{3Y}^{(i)} = \frac{d^if_3(E^*)}{dY^i}$, $J_Y^{(i)} = \frac{d^iJ(E^*)}{dY^i}$, $i = 1,2,3,4$.

Let us write $\alpha_1 = \alpha - \alpha_0$. After this transformation for the fixed parameters $\beta = \beta_0$, $\gamma = \gamma_0$, we have

$$\begin{aligned}
\dot{Y}_1 &= \alpha_0\left[(f_{1Y} - f_{2Y} + J_Y - t_1)Y_1 - i_2K_1 - (i_3 + s_3)R_1\right] \\
&\quad + \left[(f_{1Y} - f_{2Y} + J_Y - t_1)Y_1 - i_2K_1 - (i_3 + s_3)R_1\right]\alpha_1 \\
&\quad + \sum_{k=2}^{4}\alpha_0\frac{1}{k!}(f_{1Y}^{(k)} - f_{2Y}^{(k)} + J_Y^{(k)})Y_1^k \\
&\quad + \sum_{k=2}^{4}\frac{1}{k!}(f_{1Y}^{(k)} - f_{2Y}^{(k)} + J_Y^{(k)})Y_1^k\alpha_1 + \mathcal{O}(|Y_1|^5), \\
\dot{K}_1 &= f_{1Y}Y_1 - i_2K_1 - i_3R_1 + \sum_{k=2}^{4}\frac{1}{k!}f_{1Y}^{(k)}Y_1^k + \mathcal{O}(|Y_1|^5), \\
\dot{R}_1 &= \beta_0\left[f_{3Y}Y_1 - l_3R_1 - M_1\right] + \sum_{k=2}^{4}\beta_0\frac{1}{k!}f_{3Y}^{(k)}Y_1^k + \mathcal{O}(|Y_1|^5), \\
\dot{M}_1 &= J_YY_1 + \gamma_0R_1 + \sum_{k=2}^{4}\frac{1}{k!}J_Y^{(k)}Y_1^k + \mathcal{O}(|Y_1|^5).
\end{aligned} \tag{14}$$

Consider matrix **M** which changes matrix $\mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ into its Jordan form **J**. Then the transformation $x = \mathbf{M}y$, $x = (Y_1, K_1, R_1, M_1)^T$, $y = (Y_2, K_2, R_2, M_2)^T$ takes model (14) into the model

$$\begin{aligned}\dot{Y}_2 &= i\omega Y_2 + F_1(Y_2, K_2, R_2, M_2, \alpha_1),\\ \dot{K}_2 &= -i\omega K_2 + F_2(Y_2, K_2, R_2, M_2, \alpha_1),\\ \dot{R}_2 &= \lambda_3 R_2 + F_3(Y_2, K_2, R_2, M_2, \alpha_1),\\ \dot{M}_2 &= \lambda_4 M_2 + F_4(Y_2, K_2, R_2, M_2, \alpha_1),\end{aligned} \tag{15}$$

where $Re\lambda_3 < 0$, $Re\lambda_4 < 0$, $K_2 = \overline{Y}_2$, $F_2 = \overline{F}_1$. (Notation '$\overline{}$' means complex conjugation.)

Theorem 3 *There exists a polynomial transformation*

$$\begin{aligned}Y_2 &= Y_3 + h_1(Y_3, K_3, \alpha_1),\\ K_2 &= K_3 + h_2(Y_3, K_3, \alpha_1),\\ R_2 &= R_3 + h_3(Y_3, K_3, \alpha_1),\\ M_2 &= M_3 + h_4(Y_3, K_3, \alpha_1),\end{aligned} \tag{16}$$

where $h_j(Y_3, K_3, \alpha_1)$ are non-linear polynomials with constant coefficients of the kind

$$h_j(Y_3, K_3, \alpha_1) = \sum_{m_1+m_2+m \geqslant 2, m \in \{0,1\}}^{4-2m} h_j^{(m_1,m_2,m)} {Y_3}^{m_1} {K_3}^{m_2} {\alpha_1}^{m},\ j = 1, 2, 3, 4,$$

which transforms model (15) to its partial normal form on the centre manifold

$$\begin{aligned}\dot{Y}_3 &= i\omega Y_3 + \delta_1 Y_3 \alpha_1 + \delta_2 {Y_3}^2 K_3\\ &\quad + U^0(Y_3, K_3, R_3, M_3, \alpha_1) + U^*(Y_3, K_3, R_3, M_3, \alpha_1),\\ \dot{K}_3 &= -i\omega K_3 + \overline{\delta}_1 K_3 \alpha_1 + \overline{\delta}_2 Y_3 {K_3}^2 + \overline{U^0} + \overline{U^*},\\ \dot{R}_3 &= \lambda_3 R_3 + V^0(Y_3, K_3, R_3, M_3, \alpha_1) + V^*(Y_3, K_3, R_3, M_3, \alpha_1),\\ \dot{M}_3 &= \lambda_4 M_3 + W^0(Y_3, K_3, R_3, M_3, \alpha_1) + W^*(Y_3, K_3, R_3, M_3, \alpha_1),\end{aligned} \tag{17}$$

where $U^0(Y_3, K_3, 0, 0, \alpha_1) = V^0(Y_3, K_3, 0, 0, \alpha_1) = W^0(Y_3, K_3, 0, 0, \alpha_1) = 0$ *and*

$$\begin{aligned}&U^*(\sqrt{|\alpha_1|}Y_3, \sqrt{|\alpha_1|}K_3, \sqrt{|\alpha_1|}R_3, \sqrt{|\alpha_1|}M_3, \alpha_1)\\ &= V^*(\sqrt{|\alpha_1|}Y_3, \sqrt{|\alpha_1|}K_3, \sqrt{|\alpha_1|}R_3, \sqrt{|\alpha_1|}M_3, \alpha_1)\\ &= W^*(\sqrt{|\alpha_1|}Y_3, \sqrt{|\alpha_1|}K_3, \sqrt{|\alpha_1|}R_3, \sqrt{|\alpha_1|}M_3, \alpha_1) = O((\sqrt{|\alpha_1|})^5).\end{aligned}$$

The resonant coefficients δ_1 and δ_2 in model (17) are determined by the formulae

$$\delta_1 = \frac{\partial^2 F_1}{\partial Y_2 \partial \alpha_1},$$

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$$\delta_2 = -\frac{1}{2i\omega}\frac{\partial^2 F_1}{\partial {Y_2}^2}\frac{\partial^2 F_1}{\partial Y_2 \partial K_2} + \frac{1}{i\omega}\frac{\partial^2 F_1}{\partial Y_2 \partial K_2}\frac{\partial^2 F_2}{\partial Y_2 \partial K_2} + \frac{1}{6i\omega}\frac{\partial^2 F_1}{\partial {K_2}^2}\frac{\partial^2 F_2}{\partial {Y_2}^2}$$
$$- \frac{1}{\lambda_3}\frac{\partial^2 F_1}{\partial Y_2 \partial R_2}\frac{\partial^2 F_3}{\partial Y_2 \partial K_2} + \frac{1}{4i\omega - 2\lambda_3}\frac{\partial^2 F_1}{\partial K_2 \partial R_2}\frac{\partial^2 F_3}{\partial {Y_2}^2} \tag{18}$$
$$- \frac{1}{\lambda_4}\frac{\partial^2 F_1}{\partial Y_2 \partial M_2}\frac{\partial^2 F_4}{\partial Y_2 \partial K_2} + \frac{1}{4i\omega - 2\lambda_4}\frac{\partial^2 F_1}{\partial K_2 \partial M_2}\frac{\partial^2 F_4}{\partial {Y_2}^2} + \frac{1}{2}\frac{\partial^3 F_1}{\partial {Y_2}^2 \partial K_2},$$

while all derivatives are calculated at the values $Y_2 = 0$, $K_2 = 0$, $R_2 = 0$, $M_2 = 0$, $\alpha_1 = 0$.

Proof The unknown terms $h_j^{m_1,m_2,m}$, $j = 1,2,3,4$, and the resonant terms δ_1 and δ_2 can be found by the standard procedure, which is described, for example, in [7]. As the whole process of finding them is rather elaborate, we do not present it here. □

Suppose now that the real parts of both resonant coefficients δ_1 and δ_2 are different from zero. In polar coordinates

$$Y_3 = re^{i\varphi},\ K_3 = re^{-i\varphi},\ R_3 = u_1,\ M_3 = u_2$$

and model (17) has the form

$$\begin{aligned}
\dot r &= r(\rho r^2 + \tau\alpha_1) + R^0(r,\varphi,u_1,u_2,\alpha_1) + R^*(r,\varphi,u_1,u_2,\alpha_1),\\
\dot\varphi &= \omega + \mu\alpha_1 + \upsilon r^2 + \frac{1}{r}\big(\Phi^0(r,\varphi,u_1,u_2,\alpha_1) + \Phi^*(r,\varphi,u_1,u_2,\alpha_1)\big),\\
\dot u_1 &= \lambda_3 u_1 + U_1^0(r,\varphi,u_1,u_2,\alpha_1) + U_1^*(r,\varphi,u_1,u_2,\alpha_1),\\
\dot u_2 &= \lambda_4 u_2 + U_2^0(r,\varphi,u_1,u_2,\alpha_1) + U_2^*(r,\varphi,u_1,u_2,\alpha_1),
\end{aligned} \tag{19}$$

where $\rho = \mathrm{Re}\,\delta_2$, $\tau = \mathrm{Re}\,\delta_1$, $\mu = \mathrm{Im}\,\delta_1$, $\upsilon = \mathrm{Im}\,\delta_2$, and superscripts '$0, *$' have the same meaning as in Theorem 3. The following equation

$$\rho r^2 + \tau\alpha_1 = 0$$

is the bifurcation equation of model (17). It determines the behaviour of solutions in a neighbourhood of the equilibrium of model (4). Using the results from the bifurcation theory (see e.g. [2]), we can formulate the following theorem.

Theorem 4 *For the coefficients* ρ, τ *in the bifurcation equation, suppose that the following conditions hold:*

(1) *If* $\rho < 0$, *then there exists a stable limit cycle for every small enough* $\alpha_1 > 0$ *if* τ *is positive, and for every small enough* $\alpha_1 < 0$ *if* τ *is negative.*
(2) *If* $\rho > 0$, *then there exists an unstable limit cycle for every small enough* $\alpha_1 < 0$ *if* τ *is positive, and for every small enough* $\alpha_1 > 0$ *if* τ *is negative.*

3 Numerical example

Based on macroeconomic indicators of the Slovak Republic in the year 2008 (Internet sources: http://portal.statistics.sk/showdoc.do?docid=39668; http://www.nbs.sk/sk/statisticke-udaje/udajove-kategorie-sdds/ urokove-sadzby/urokove-sadzby-nbs/zakladna-urokova-sadzba-nbs- limitna-urokova-sadzba-pre-dvojtyzdnove-repo; http://www.nbs.sk/sk/statisticke-udaje/menova-a-bankova-statistika/ menova-statistika-penaznych-financnych-instituciiM3-PFI), and the interest rate of the United Kingdom as the foreign country in 2008 (http://www.bankofengland.co.uk/monetarypolicy/Pages/decisions.aspx), we specify the functions I, S, T, L, J in model (4) as follows:

$$\begin{aligned} I &= \tfrac{1}{10}\sqrt{Y^3} - \tfrac{8}{25}K - 2R + \tfrac{1283}{50},\\ S &= \tfrac{1}{100}\left(\tfrac{2}{25}Y + \tfrac{1}{5}\right)^2 + R + \tfrac{4003}{31250},\\ T &= \tfrac{19}{100}Y - \tfrac{1}{5},\\ L &= \tfrac{1}{5}\sqrt{Y} - \tfrac{1}{2}R + \tfrac{643}{200},\\ J &= -Y + \tfrac{801}{50}, \end{aligned}$$

taking the government expenditures $G = 3$ (in 10^{10} EUR), and the interest rate of the United Kingdom $R_f = \frac{5}{100}$.

After these specifications, model (4) becomes

$$\begin{aligned} \dot{Y} &= \alpha\left(-\frac{1}{15625}Y^2 + \frac{1}{10}\sqrt{Y^3} - \frac{14879}{12500}Y - \frac{8}{25}K - 3R + \frac{2796969}{62500}\right),\\ \dot{K} &= \frac{1}{10}\sqrt{Y^3} - \frac{8}{25}K - 2R + \frac{1283}{50},\\ \dot{R} &= \beta\left(\frac{1}{5}\sqrt{Y} - \frac{1}{2}R - M + \frac{643}{200}\right),\\ \dot{M} &= -Y + \gamma\left(R - \frac{1}{20}\right) + \frac{801}{50}. \end{aligned} \tag{20}$$

The equilibrium of (20) depends on parameter γ. Put $\gamma = \gamma_0 = 1$. Then the equilibrium is

$$E^* = \left(Y^* = 16,\ K^* = 100,\ R^* = \tfrac{3}{100},\ M^* = 4\right),$$

while the values of Y, K, M are taken in 10^{10} EUR.

The matrix of linear approximation of (20) at γ_0 is given by

$$\mathbf{A}(\alpha, \beta, \gamma_0) = \begin{pmatrix} -\frac{37023}{62500}\alpha & -\frac{8}{25}\alpha & -3\alpha & 0 \\ \frac{3}{5} & -\frac{8}{25} & -2 & 0 \\ \frac{1}{40}\beta & 0 & -\frac{1}{2}\beta & -\beta \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

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Now we determine the critical triple of parameters $(\alpha_0, \beta_0, \gamma_0 = 1)$. Take $\alpha_0 = 1$. Then β_0 is uniquely defined as

$$\frac{67224132268342141 + 6423667\sqrt{100424992444869816929}}{36280172389562500} \approx 3.62725.$$

The eigenvalues of matrix $\mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ are

$$\begin{aligned} \lambda_1 &\approx +2.33888\, i, \\ \lambda_2 &\approx -2.33888\, i, \\ \lambda_3 &\approx -0.182922, \\ \lambda_4 &\approx -2.54307. \end{aligned}$$

The formulae for the calculation of resonant coefficients δ_1 and δ_2, which are introduced in Theorem 3, give the following values:

$$\delta_1 \approx 0.257061 + 0.62249\, i, \quad \delta_2 \approx -0.00032824 + 0.000114456\, i.$$

The bifurcation coefficients are

$$\rho = \operatorname{Re} \delta_2 \approx -0.00032824, \quad \tau = \operatorname{Re} \delta_1 \approx 0.257061.$$

The partial normal form of model (20) on the centre manifold in polar coordinates is

$$\begin{aligned} \dot{r} &= r\,(-0.00032824\, r^2 + 0.257061\, \alpha_1) + R^0(r, \varphi, u_1, u_2, \alpha_1) \\ &\quad + R^*(r, \varphi, u_1, u_2, \alpha_1), \\ \dot{\varphi} &= 2.33888 + 0.62249\, \alpha_1 - 0.000114456 r^2 + \frac{1}{r}(\Phi^0(r, \varphi, u_1, u_2, \alpha_1) \\ &\quad + \Phi^*(r, \varphi, u_1, u_2, \alpha_1)), \\ \dot{u}_1 &= -0.182922\, u_1 + U_1^0(r, \varphi, u_1, u_2, \alpha_1) + U_1^*(r, \varphi, u_1, u_2, \alpha_1), \\ \dot{u}_2 &= -2.54307\, u_2 + U_2^0(r, \varphi, u_1, u_2, \alpha_1) + U_2^*(r, \varphi, u_1, u_2, \alpha_1). \end{aligned} \tag{21}$$

The bifurcation equation

$$\rho\, r^2 + \tau\, \alpha_1 = -0.00032824\, r^2 + 0.257061\, \alpha_1 = 0 \tag{22}$$

in model (21) gives the clear picture on the behaviour of solutions of model (20) around the equilibrium $E^* = (Y^* = 16, K^* = 100, R^* = \frac{3}{100}, M^* = 4)$. As $\rho < 0$ and $\tau > 0$, the limit cycles appear for sufficiently small $\alpha_1 > 0$, or with respect to the original parameter $\alpha = \alpha_1 + 1$ for $\alpha > 1$. In Figures 1–3 there is depicted the projection of the development of Solution 1 with initial values $(Y_0 = 23, K_0 = 107, R_0 = 0.03, M_0 = 4)$ and $\alpha = 1.1$ to the Y–K plane. It can be seen that this solution converges to the stable limit cycle which corresponds to the value $\alpha = 1.1$. It follows from the bifurcation equation (22) that the radii of limit cycles go to zero as α_1 goes to zero from the right, or as α goes to 1 from the right. This fact is illustrated in Figures 4 and 5 where the projection of the development

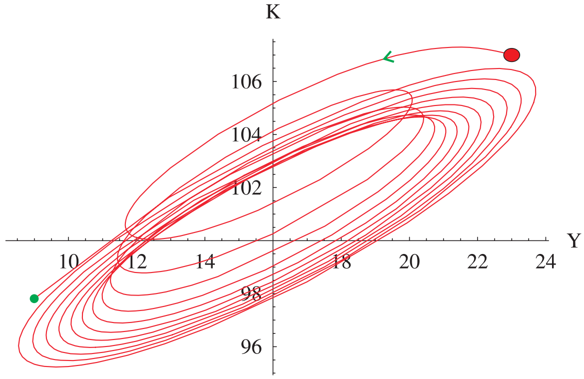

FIGURE 1. (Colour online) Solution 1: the projection of the solution of model (20) with the initial values $Y_0 = 23, K_0 = 107, R_0 = 0.03, M_0 = 4$ and $\alpha = 1.1$ to the Y–K plane for time steps from $t = 0$ to $t = 30$. The equilibrium point is represented by the intersection of the axes. The projection continues in Figures 2 and 3.

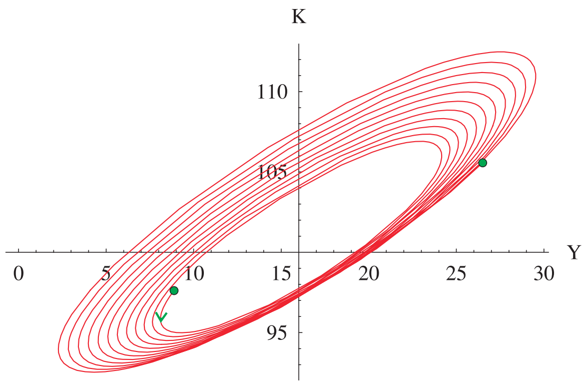

FIGURE 2. (Colour online) Solution 1: time steps from $t = 30$ to $t = 60$.

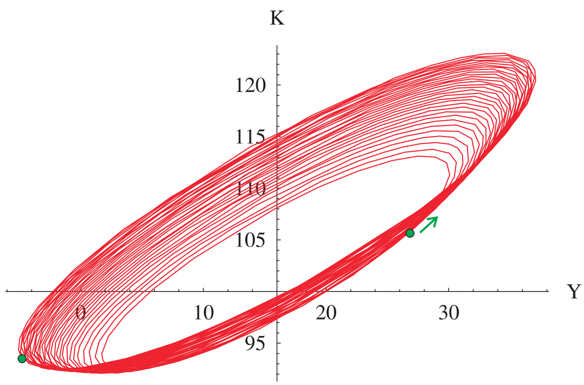

FIGURE 3. (Colour online) Solution 1: time steps from $t = 60$ to $t = 140$.

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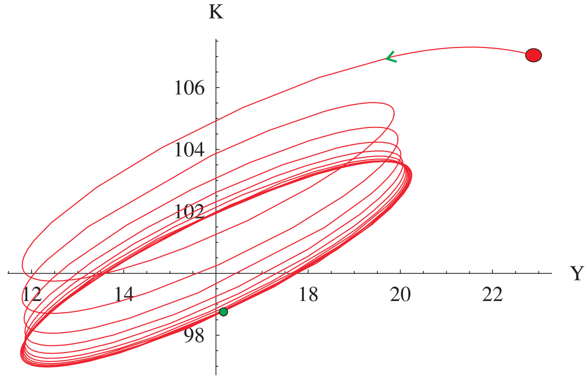

FIGURE 4. (Colour online) Solution 2: the projection of the solution of model (20) with the initial values $Y_0 = 23, K_0 = 107, R_0 = 0.03, M_0 = 4$ and $\alpha = 1.01$ to the Y–K plane for time steps from $t = 0$ to $t = 30$. The equilibrium point is represented by the intersection of the axes. The projection continues in Figure 5.

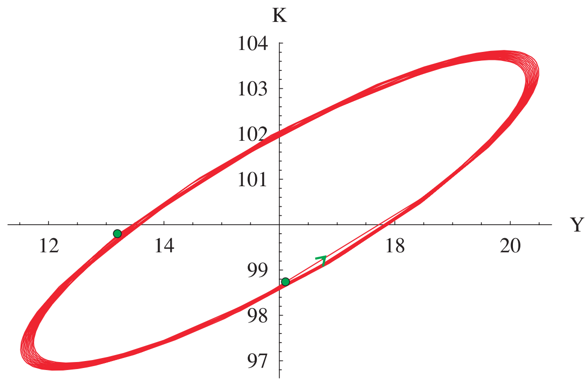

FIGURE 5. (Colour online) Solution 2: time steps from $t = 30$ to $t = 60$.

of Solution 2 with the same initial value, but with the smaller value of the parameter $\alpha = 1.01$, is depicted. We also see from the bifurcation equation (22) that for the values $\alpha_1 \leqslant 0$, i.e. for the values $\alpha \leqslant 1$, the equilibrium E^* is asymptotically stable. In Figure 6, Solution 3 with $\alpha_1 = -0.1$, i.e. $\alpha = 0.9$, illustrates this property. We can see that this solution converges to the equilibrium E^*.

4 Conclusion

The main aim of governments in economic domain is to guarantee sustainable economic growth. In spite of their efforts, undesirable qualitative changes in the development of economies appear and business crises arise, causing big economic damages. Theoretical reasoning for the occurrence of business crises was given, for example, by Minsky [6]. According to his financial instability hypothesis, a capitalist economy cannot lead to a sustained full employment equilibrium, and serious business crises are unavoidable,

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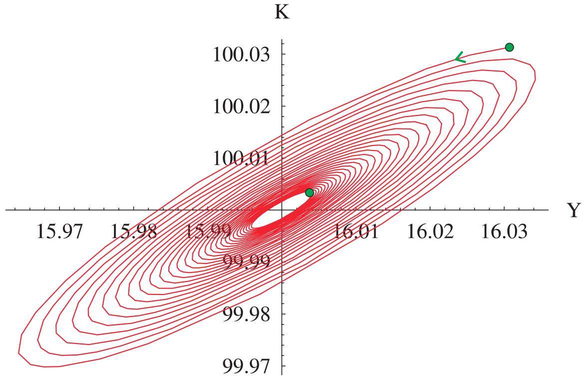

FIGURE 6. (Colour online) Solution 3: the projection of the solution of model (20) with the initial values $Y_0 = 15.7, K_0 = 101, R_0 = 0.03, M_0 = 4$ and $\alpha = 0.9$ to the Y–K plane for time steps from $t = 60$ to $t = 140$. The equilibrium point is represented by the intersection of the axes.

especially due to the unstable nature of the interaction between investment and finance. In dynamic models describing the development of macroeconomic processes business crises are expressed by limit cycles. To eliminate or at least to reduce the consequences of such damaging crises, it is important to know how these phenomena arise and how we can control them.

The aim of this paper was to analyse the arise of limit cycles and their properties in a four-dimensional model of a small open economy. The model describes the development of income Y, physical capital stock K, interest rate R and money stock M. Sufficient conditions for the existence of an equilibrium of the model and its stability were found in [5]. The main result from [5] is recalled in Section 2 by Theorem 2. Limit cycles can appear if the linear approximation matrix of model (4) has at least one pair of purely imaginary eigenvalues. In this paper the case of one pair of purely imaginary eigenvalues with the rest having negative real parts is investigated. Theorem 1 in Section 2 gives sufficient conditions on parameters α, β, γ for the existence of such a situation. It says that at sufficiently large positive parameters β and γ, the case of one pair of purely imaginary eigenvalues with the others having negative real parts is guaranteed.

The character of limit cycles is determined by the signs of the coefficients in the bifurcation equation $\rho r^2 + \tau\alpha_1 = 0$ of model (19). For the practical use of our theoretical results on limit cycles, it is necessary to have a possibility of calculating coefficients ρ and τ. They are the real parts of the first resonant coefficients δ_1 and δ_2 in the partial normal form of the model on the centre manifold. Theorem 3 in Section 3 describes the transformation of the model to its partial normal form on the centre manifold and gives the exact formulae for the calculation of the first resonant coefficients δ_1 and δ_2. The following Theorem 4 gives complete information on the occurrence of limit cycles and their character. Having model (4) with concrete values of functions, we are able to find by solving equation (9) the corresponding critical triple $(\alpha_0, \beta_0, \gamma_0)$ of parameters α, β, γ. Then using the formulae for the calculation of the resonant coefficients δ_1 and δ_2, we can immediately find for which values of a free parameter α in the neighbourhood of its critical value α_0 limit cycles arise and determine their character.

These results can be applied by monetary authorities in small open economies as prognoses for the development of basic macroeconomic variables and to aid their regulations. The functions involved in the model can be constructed on the basis of recent years' data and planned structural macroeconomic modifications. The monetary authorities can simulate possible macroeconomic developments, and, in the case of its unsuitable evolution, they can make necessary changes. Equilibria in models of economic processes express the desirable balance of analysed variables. Their asymptotic stability guaranteee suitable development. If an equilibrium is unstable, what happens in economic reality quite often is that a stable limit cycle around it can stop variables from gaining undesirable values. If it comes out that a business cycle is unavoidable, then monetary authorities should set up such financial regulations to guarantee a stable cycle. Theorems 3 and 4 give them suitable tools for simulating the effects of intended regulations.

Examples of small open economies in the Eurozone which could be considered include states such as Austria, Slovakia, Slovenia and Ireland. In the numerical example in Section 4 we simulated possible development of the Slovak economy in the domain of its business relations with the United Kingdom. Although the functions in the model could not be exactly determined on the base of the data from recent years as the Slovak economy underwent continuous structural changes during last two decades, they approximately express its development. The values of the equilibrium in the example correspond to those of the Slovak economy in 2008, and the interest rate R_f of the foreign country corresponds to that in the United Kingdom in the same year. We have found that, fixing the critical values β_0 and γ_0, cycles arise at $\alpha > \alpha_0$, and they are stable.

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