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CONDITIONS FOR PERMANENCE AND ERGODICITY OF CERTAIN STOCHASTIC PREDATOR–PREY MODELS

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Abstract

In this paper we derive sufficient conditions for the permanence and ergodicity of a stochastic predator–prey model with a Beddington–DeAngelis functional response. The conditions obtained are in fact very close to the necessary conditions. Both nondegenerate and degenerate diffusions are considered. One of the distinctive features of our results is that they enable the characterization of the support of a unique invariant probability measure. It proves the convergence in total variation norm of the transition probability to the invariant measure. Comparisons to the existing literature and matters related to other stochastic predator–prey models are also given.

Keywords: Ergodicity; extinction; permanence; predator–prey; Beddington–DeAngelis functional response; stationary distribution

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1. Introduction

In this paper we focus on stochastic predator–prey models with a Beddington–DeAngelis functional response. In ecology, a functional response is the intake rate of a consumer as a function of food density. It is associated with the numerical response that is the reproduction rate of a consumer as a function of food density. Holling [6] initiated the study of functional response, where he introduced several types of such responses. The so-called Holling-type II functional response is characterized by a decelerating intake rate following from the assumption that the consumer is limited by its capacity to process food. Similar to Holling-type functional response with an extra term describing mutual interference by predators, Beddington [1] and DeAngelis *et al.* [3] introduced the nowadays well-known Beddington–DeAngelis functional response; see also [25] and the references therein. Such a model represents most of the qualitative features of the ratio-dependent models but avoids the 'low densities problem'.

As the building blocks of the bio- and eco-systems, the basic premise of the predator–prey models is that species compete, evolve, and disperse for the purpose of seeking resources to sustain their struggle and existence. Denote the two population sizes at time t by $x(t)$ and $y(t)$, respectively. Then a general deterministic model called Kolmogorov's predator–prey model

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takes the form

$$\dot{x}(t) = xf(x, y), \qquad \dot{y}(t) = yg(x, y).$$

When $f(x, y) = b - py$ and $g(x, y) = cx - d$, we have the so-called Lotka–Volterra model.

In addition to the study of deterministic models, stochastic predator–prey models have received increasing and resurgent attention. Stochastic models can be considered as the above systems subject to Brownian motion perturbations. Rudnicki [21] provided a detailed analysis for stability in distribution of a stochastic Lotka–Volterra model. Meanwhile, Mao *et al.* [19] and Du and Sam [4] studied general stochastic Lotka–Volterra models using Lyapunov-type functions and exponential martingale inequalities. Recently, Lotka–Volterra models in random environments have also gained much attention [26]. In addition, there is a resurgent interest in treating evolutionary games [5], in which Lotka–Volterra-type equations are one of the central models. Concerning different functional responses, [17] and [18] dealt with the stochastic predator–prey model with Holling functional response of the form

$$\mathrm{d}x(t) = x(t)\bigg(a_1 - b_1 x(t) - \frac{c_1 y(t)}{1 + x(t)}\bigg)\,\mathrm{d}t + \alpha x(t)\,\mathrm{d}B_1(t), \tag{1.1}$$

$$\mathrm{d}y(t) = y(t)\bigg(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{1 + x(t)}\bigg)\,\mathrm{d}t + \beta y(t)\,\mathrm{d}B_2(t), \tag{1.2}$$

where a_i, b_i, c_i, α, and β are appropriate constants, and $B_i(\cdot)$ are standard Brownian motions. Ji *et al.* [10] studied the predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation; see also [9] in which stochastic ratio-dependent predator–prey models were considered. Moreover, several stochastic models with the well-known Beddington–DeAngelis functional response were also studied in [8], [16], and [24]. In ecology models, an important concept is stochastic permanence, which indicates that the species will survive forever. Much effort has been devoted to finding the conditions needed for stochastic permanence. In some of the aforementioned papers, using suitable Lyapunov-type functions, some conditions for extinction or permanence were also provided and ergodicity was investigated; see [8] and [18]. However, as shown later in Section 4 of this paper, their conditions are restrictive and not close to a necessary condition. In other words, there is a considerably large set of parameters satisfying neither their conditions for extinction nor for permanence. Moreover, their results are not applicable to degenerate cases. Thus, although interesting, their work leaves a sizable gap. One of the main goals of this paper is to close this gap. We aim to provide a sufficient and almost necessary condition for permanence (as well as ergodicity) for the following model with a Beddington–DeAngelis functional response:

$$\mathrm{d}x(t) = x(t)\bigg(a_1 - b_1 x(t) - \frac{c_1 y(t)}{m_1 + m_2 x(t) + m_3 y(t)}\bigg)\,\mathrm{d}t + \alpha x(t)\,\mathrm{d}B_1(t), \tag{1.3}$$

$$\mathrm{d}y(t) = y(t)\bigg(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1 + m_2 x(t) + m_3 y(t)}\bigg)\,\mathrm{d}t + \beta y(t)\,\mathrm{d}B_2(t), \tag{1.4}$$

where a_i, b_i, c_i, m_i are positive constants for $i = 1, 2$, $m_3 \geq 0$, $\alpha \neq 0$, $\beta \neq 0$, and $B_1(\cdot)$, $B_2(\cdot)$ are two mutually independent Brownian motions. When $m_3 = 0$, the functional response is said to be of Holling-type II. Moreover, in this paper, we also consider the degenerate case $B_1(\cdot) = B_2(\cdot)$.

The rest of the paper is arranged as follows. In Section 2 we derive a threshold that is used to determine extinction and permanence. To establish the desired result, after considering the dynamics on the boundary, we obtain a threshold λ that enables us to determine the asymptotic behavior of the solution. In particular, it is shown that if $\lambda < 0$, the predator will eventually die out. In the $\lambda > 0$ case, the solution converges to a stationary distribution in total variation norm. Moreover, ergodicity is established. In Section 2 we concentrate on the nondegenerate case, whereas in Section 3 we treat the degenerate case $B_1(\cdot) = B_2(\cdot)$. In the degenerate case, under usual conditions imposed on the Lie algebra generated by the drift and the diffusion coefficients, we investigate the controllability of the associated control systems and use certain results in [15] to prove analogous results to the nondegenerate case; namely, the existence and uniqueness of an invariant probability measure as well as the convergence in total variation of the transition probability. Moreover, the support of the invariant measure is described. Finally, in Section 4 we provide further discussion and insight. Among other things, we point out that the techniques used in this paper can be applied to other stochastic predator–prey models.

2. Threshold between extinction and permanence

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $B_1(t)$ and $B_2(t)$ be two $\mathcal{F}_t$-adapted, mutually independent Brownian motions. It is well known that for any initial value $(x(0), y(0)) \in \mathbb{R}^{2,\circ}_+$ (the interior of $\mathbb{R}^2_+$), there exists a unique global solution to (1.3) and (1.4) that remains in $\mathbb{R}^{2,\circ}_+$ almost surely (a.s.); see [8]. To proceed, we first consider the equation on the boundary:

$$d\varphi(t) = \varphi(t)(a_1 - b_1\varphi(t))\,dt + \alpha\varphi(t)\,dB_1(t). \tag{2.1}$$

By utilizing a comparison theorem, it is easy to check that $x(t) \leq \varphi(t)$ for all $t \geq 0$ a.s. provided that $x(0) = \varphi(0) > 0$ and $y(0) > 0$. If $a_1 \leq \alpha^2/2$, we can easily verify [7, Theorem 3.1(2), p. 447] to show that $\lim_{t\to\infty}\varphi(t) = 0$ a.s. Hence, $\lim_{t\to\infty} x(t) = 0$ a.s., which implies that $\lim_{t\to\infty} y(t) = 0$ a.s. For this reason, we suppose that $a_1 > \alpha^2/2$ throughout the rest of this paper.

Defining $\theta(t) = \ln\varphi(t)$, (2.1) can be written as

$$d\theta(t) = \left(a_1 - \frac{\alpha^2}{2} - b_1\exp(\theta(t))\right)dt + \alpha\,dB_1(t).$$

By solving the Fokker–Planck equation, we show that the process $\theta(t)$ has a unique stationary distribution with density given by $f^*(x) = C\exp(qx - a\exp(x))$, where $q = 2a_1/\alpha^2 - 1 > 0$, $a = 2b_1/\alpha^2 > 0$, and C is the normalizing constant. Since $\theta(t) = \ln\varphi(t)$, it is easily seen that $\varphi(t)$ has a unique stationary distribution $\mu_-(\cdot)$ with density $\phi^*(x) = Cx^{q-1}e^{-ax}$, $x > 0$. It turns out that $C = a^q/\Gamma(q)$ with $\Gamma(\cdot)$ being the gamma function and that $\mu_-(\cdot)$ is the gamma distribution with parameters q and a.

By the strong law of large numbers type result [22, Theorem 3.16, p. 46], we deduce that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi^p(s)\,ds &= \frac{a^q}{\Gamma(q)}\int_0^\infty x^{p+q-1}e^{-ax}\,dx \\ &= \frac{\Gamma(p+q)}{a^p\Gamma(q)} \\ &:= K_p < \infty \quad \text{a.s. for all } p > 0.\end{aligned} \tag{2.2}$$

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In particular, with $p=1$, $K_1 = q/a = (a_1 - \alpha^2/2)/b_1$. This property implies that

$$\lim_{t\to\infty}\frac{1}{t}\ln\varphi(t) = \lim_{t\to\infty}\left(\frac{1}{t}\int_0^t\left(a_1 - \frac{\alpha^2}{2} - b_1\varphi(s)\right)\mathrm{d}s\right) + \alpha\lim_{t\to\infty}\frac{B_1(t)}{t} = 0.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t) \le 0 \tag{2.3}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x^p(s)\,\mathrm{d}s \le K_p. \tag{2.4}$$

Let $\psi(t)$ be the solution to

$$\mathrm{d}\psi(t) = \psi(t)\left(-a_1 + \frac{c_2}{m_2} - b_2\psi(t)\right)\mathrm{d}t + \beta\psi(t)\,\mathrm{d}B_2(t).$$

Then $y(t) \le \psi(t)$ for all $t \ge 0$ a.s. provided that $y(0) = \psi(0) > 0$. Hence, with probability 1, we have

$$\limsup_{t\to\infty}\frac{1}{t}\ln y(t) \le 0, \tag{2.5}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t y^p(s)\,\mathrm{d}s \le \widehat{K}_p \quad \text{for some constant } \widehat{K}_p > 0. \tag{2.6}$$

Define the threshold

$$\lambda := -a_2 - \frac{\beta^2}{2} + \int_0^\infty \frac{c_2 x}{m_1 + m_2 x}\mu_-(\mathrm{d}x) = -a_2 - \frac{\beta^2}{2} + \frac{a^q}{\Gamma(q)}\int_0^\infty \frac{c_2 x^q \mathrm{e}^{-ax}}{m_1 + m_2 x}\,\mathrm{d}x.$$

Theorem 2.1. *If $\lambda < 0$ then the predator is eventually extinct, that is,* $\lim_{t\to\infty} y(t) = 0$ *a.s. Moreover, as $t \to \infty$ the distribution of $x(t)$ converges weakly to $\mu_-(\cdot)$, that is, the gamma distribution with parameters $q = 2a_1/\alpha^2 - 1$ and $a = 2b_1/\alpha^2$, respectively.*

Proof. Let $\overline{y}(t)$ be the solution to

$$\mathrm{d}\overline{y}(t) = \overline{y}(t)\left(-a_2 - b_2\overline{y}(t) + \frac{c_2\varphi(t)}{m_1 + m_2\varphi(t)}\right)\mathrm{d}t + \beta\overline{y}(t)\,\mathrm{d}B_2(t),$$

where $\varphi(t)$ is the solution to (2.1). By using the comparison theorem, $y(t) \le \overline{y}(t)$ a.s. given that $\varphi(0) = x(0)$ and $\overline{y}(0) = y(0)$. In view of the Itô formula and the ergodicity of $\varphi(t)$, we have

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\ln\overline{y}(t) &= \limsup_{t\to\infty}\left(\frac{1}{t}\int_0^t\left(-a_2 - \frac{\beta^2}{2} - b_2\overline{y}(s) + \frac{c_2\varphi(s)}{m_1 + m_2\varphi(s)}\right)\mathrm{d}s + \beta\frac{B_2(t)}{t}\right)\\
&\le \lim_{t\to\infty}\frac{1}{t}\int_0^t\left(-a_2 - \frac{\beta^2}{2} + \frac{c_2\varphi(s)}{m_1 + m_2\varphi(s)}\right)\mathrm{d}s + \beta\lim_{t\to\infty}\frac{B_2(t)}{t}\\
&= \lambda < 0 \quad \text{a.s.}
\end{aligned}$$

That is, $y(t)$ converges to 0 at an exponential rate a.s. The remaining part of the assertion can be proved by the arguments in [21, Lemma 7]. □

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Theorem 2.2. *If $\lambda > 0$, the process $(x(t), y(t))$ has an invariant probability measure concentrated on $\mathbb{R}_+^{2,\circ}$.*

Proof. For any initial value $(x(0), y(0)) \in \mathbb{R}_+^{2,\circ}$, we have

$$\begin{aligned}\frac{1}{t}\ln y(t) &= -\frac{1}{t}\int_0^t b_2 y(s)\,\mathrm{d}s + \frac{1}{t}\int_0^t \left(-a_2 - \frac{\beta^2}{2} + \frac{c_2\varphi(s)}{m_1 + m_2\varphi(s)}\right)\mathrm{d}s \\ &\quad - \frac{1}{t}\int_0^t \left(\frac{c_2\varphi(s)}{m_1 + m_2\varphi(s)} - \frac{c_2 x(s)}{m_1 + m_2 x(s)}\right)\mathrm{d}s \\ &\quad - \frac{1}{t}\int_0^t \left(\frac{c_2 x(s)}{m_1 + m_2 x(s)} - \frac{c_2 x(s)}{m_1 + m_2 x(s) + m_3 y(s)}\right)\mathrm{d}s + \beta\frac{B_2(t)}{t} \\ &\geq \frac{1}{t}\int_0^t \left(-a_2 - \frac{\beta^2}{2} + \frac{c_2\varphi(s)}{m_1 + m_2\varphi(s)}\right)\mathrm{d}s \\ &\quad - \frac{1}{t}\int_0^t \left(\frac{c_2}{m_1}(\varphi(s) - x(s)) + \left(\frac{c_2 m_3}{m_1 m_2} + b_2\right)y(s)\right)\mathrm{d}s + \beta\frac{B_2(t)}{t}. \end{aligned} \tag{2.7}$$

Letting $t \to \infty$, (2.5) and (2.7) yield that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{c_2}{m_1}(\varphi(s) - x(s)) + \left(\frac{c_2 m_3}{m_1 m_2} + b_2\right)y(s)\right)\mathrm{d}s \geq \lambda \quad \text{a.s.} \tag{2.8}$$

Similarly, we have

$$\begin{aligned}\frac{1}{t}\ln x(t) &= \frac{1}{t}\int_0^t \left(a_1 - \frac{\alpha^2}{2} - b_1\varphi(s)\right)\mathrm{d}s \\ &\quad + \frac{1}{t}\int_0^t \left(b_1(\varphi(s) - x(s)) - \frac{c_1 y(s)}{m_1 + m_2 x(s) + m_3 y(s)}\right)\mathrm{d}s + \alpha\frac{B_1(t)}{t} \\ &\geq \frac{1}{t}\int_0^t \left(a_1 - \frac{\alpha^2}{2} - b_1\varphi(s)\right)\mathrm{d}s + \frac{1}{t}\int_0^t \left(b_1(\varphi(s) - x(s)) - \frac{c_1 y(s)}{m_1}\right)\mathrm{d}s \\ &\quad + \alpha\frac{B_1(t)}{t}. \end{aligned} \tag{2.9}$$

From (2.2), (2.3), and (2.9), it follows that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(-b_1(\varphi(s) - x(s)) + \frac{c_1}{m_1}y(s)\right)\mathrm{d}s \geq 0 \quad \text{a.s.} \tag{2.10}$$

Dividing both sides of (2.8) and (2.10) by c_2/m_1 and b_1, respectively, and adding them side by side, we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,\mathrm{d}s \geq \frac{b_1 m_1^2 m_2 \lambda}{c_1 c_2 m_2 + b_1 c_2 m_1 m_3 + b_1 b_2 m_1^2 m_2} =: \overline{m} > 0 \quad \text{a.s.}$$

For $0 < \hbar < \overline{m} < H < \infty$, Hölder's inequality yields that

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{y(s)\geq\hbar\}}\, y(s)\,\mathrm{d}s \leq \left(\frac{1}{t}\int_0^t \mathbf{1}_{\{y(s)\geq\hbar\}}\,\mathrm{d}s\right)^{1/2}\left(\frac{1}{t}\int_0^t y^2(s)\,\mathrm{d}s\right)^{1/2},$$

which implies that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{y(s)\geq\hbar\}}\,\mathrm{d}s &\geq \bigg(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{y(s)\geq\hbar\}}\,y(s)\,\mathrm{d}s\bigg)^2\bigg(\limsup_{t\to\infty}\frac{1}{t}\int_0^t y^2(s)\,\mathrm{d}s\bigg)^{-1}\\ &\geq \bigg(\liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s-\hbar\bigg)^2\bigg(\limsup_{t\to\infty}\frac{1}{t}\int_0^t y^2(s)\,\mathrm{d}s\bigg)^{-1}\\ &\geq \frac{(\overline{m}-\hbar)^2}{\widehat{K}_2}\quad \text{a.s.}\end{aligned} \tag{2.11}$$

In addition, (2.4) and (2.6) imply that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{y(s)\geq H\}}\,\mathrm{d}s \leq \frac{1}{H}\limsup_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s\leq\frac{\widehat{K}_1}{H}\quad\text{a.s.,} \tag{2.12}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{x(s)\geq H\}}\,\mathrm{d}s \leq \frac{1}{H}\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s\leq\frac{K_1}{H}\quad\text{a.s.} \tag{2.13}$$

It follows from (2.11)–(2.13) that for $\hbar<\overline{m}/2$, $H>8(K_1+\widehat{K}_1)\widehat{K}_2/\overline{m}^2$,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(x(s),y(s))\in A\}}\,\mathrm{d}s\geq\frac{(\overline{m}-\hbar)^2}{\widehat{K}_2}-\frac{K_1+\widehat{K}_1}{H}>\frac{\overline{m}^2}{8\widehat{K}_2}\quad\text{a.s.,} \tag{2.14}$$

where $A=\{(x,y)\colon 0<x\leq H,\hbar\leq y\leq H\}$. By virtue of Fatou's lemma, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t P(s,(x,y),A)\,\mathrm{d}s\geq\frac{\overline{m}^2}{8\widehat{K}_2}\quad\text{for all } (x,y)\in\mathbb{R}^{2,\circ}_+, \tag{2.15}$$

where $P(t,(x,y),\cdot)$ is the transition probability of $(x(t),y(t))$. By the invariance of $\mathcal{M}=\{x\geq 0,\ y>0\}$ under (1.3) and (1.4), we can consider the Markov process $(x(t),y(t))$ on the state space $\mathcal{M}$. It is easy to show that $(x(t),y(t))$ has the Feller property. Thus, (2.15) implies that there is an invariant probability measure μ^* on $\mathcal{M}$; see [20]. Since $y(t)\to 0$ provided that $x(0)=0$, $\lim_{t\to\infty}P(t,(0,y),K)=0$ for all compact set $K\subset\mathcal{M}$. Thus, we must have $\mu^*(\{x=0,y>0\})=0$ (equivalently $\mu^*(\mathbb{R}^{2,\circ}_+)=1$). Furthermore, by the invariance of $\mathbb{R}^{2,\circ}_+$, μ^* is an invariant probability measure of $(x(t),y(t))$ on $\mathbb{R}^{2,\circ}_+$. □

Since $B_1(\cdot)$ and $B_2(\cdot)$ are independent, the diffusion is nondegenerate. It is well known that the existence of an invariant probability measure is equivalent to a positive recurrence. Hence, the invariant probability is unique and the strong law of large numbers holds; see [14, Theorems 3.1 and 3.3]. We have the following result.

Theorem 2.3. *If* $\lambda>0$ *then (1.3) and (1.4) have a unique invariant probability measure* μ^* *with support* $\mathbb{R}^{2,\circ}_+$. *Moreover,*

(i) *for any* μ^**-integrable* $f(x,y)\colon\mathbb{R}^{2,\circ}_+\to\mathbb{R}$, *we have*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x(s),y(s))\,\mathrm{d}s=\int f(x,y)\mu^*(\mathrm{d}x,\,\mathrm{d}y)\quad\textit{a.s. for all } (x(0),y(0))\in\mathbb{R}^{2,\circ}_+,$$

(ii) *and*

$$\lim_{t\to\infty}\|P(t,(x,y),\cdot)-\mu^*(\cdot)\|=0\quad\textit{for all } (x,y)\in\mathbb{R}^{2,\circ}_+,$$

where $\|\cdot\|$ *is the total variation norm.*

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Proof. Theorem 2.3(i) was proved in [14, Theorem 3.3]; we refer the reader to [12, Proposition 5.1] (correction [13]) or [2] for the proof of Theorem 2.3(ii). □

As a direct corollary of Theorem 2.3, if $\lambda > 0$, systems (1.3) and (1.4) are stochastically permanent in the sense that for any $\varepsilon > 0$, there is some $\delta \in (0, 1)$ such that $\liminf_{t\to\infty} P(t, x, y, [\delta, \delta^{-1}]^2) > 1-\varepsilon$. Moreover, from (2.4) and (2.6), it follows that we have the following limits:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x^p(s)\,\mathrm{d}s = \int x^p \mu^*(\mathrm{d}x,\ \mathrm{d}y) \quad \text{a.s. for all } (x(0), y(0)) \in \mathbb{R}_+^{2,\circ},\ p > 0,$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t y^p(s)\,\mathrm{d}s = \int y^p \mu^*(\mathrm{d}x,\ \mathrm{d}y) \quad \text{a.s. for all } (x(0), y(0)) \in \mathbb{R}_+^{2,\circ},\ p > 0.$$

3. Degenerate case

Suppose that $B_1(\cdot) = B_2(\cdot) = W(\cdot)$. We consider the following system:

$$\mathrm{d}x(t) = x(t)\bigg(a_1 - b_1 x(t) - \frac{c_1 y(t)}{m_1 + m_2 x(t) + m_3 y(t)}\bigg)\mathrm{d}t + \alpha x(t)\,\mathrm{d}W(t), \tag{3.1}$$

$$\mathrm{d}y(t) = y(t)\bigg(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1 + m_2 x(t) + m_3 y(t)}\bigg)\mathrm{d}t + \beta y(t)\,\mathrm{d}W(t). \tag{3.2}$$

Owing to the symmetry of the Brownian motion, we can suppose that $\alpha > 0$. Since the estimates in the previous section still hold for this case, we have $\lim_{t\to\infty} y(t) = 0$ when $\lambda < 0$ while $x(t)$ converges weakly to the stationary distribution μ_- of $\varphi(t)$. In what follows, we suppose that $\lambda > 0$ for which the process has an invariant probability measure μ^* on $\mathbb{R}_+^{2,\circ}$. Setting $\xi(t) = \ln x(t)$ and $\eta(t) = \ln y(t)$, (3.1) and (3.2) can be expressed as

$$\mathrm{d}\xi(t) = \bigg(a_1 - \frac{\alpha^2}{2} - b_1 \mathrm{e}^{\xi(t)} - \frac{c_1 \mathrm{e}^{\eta(t)}}{m_1 + m_2 \mathrm{e}^{\xi(t)} + m_3 \mathrm{e}^{\eta(t)}}\bigg)\mathrm{d}t + \alpha\,\mathrm{d}W(t), \tag{3.3}$$

$$\mathrm{d}\eta(t) = \bigg(-a_2 - \frac{\beta^2}{2} - b_2 \mathrm{e}^{\eta(t)} + \frac{c_2 \mathrm{e}^{\xi(t)}}{m_1 + m_2 \mathrm{e}^{\xi(t)} + m_3 \mathrm{e}^{\eta(t)}}\bigg)\mathrm{d}t + \beta\,\mathrm{d}W(t). \tag{3.4}$$

Denote by $(\xi^{u,v}(t), \eta^{u,v}(t))$ the solution with initial value (u, v) to (3.3) and (3.4) and let $\widehat{P}(t, (u, v), \cdot)$ be its transition probability. Put

$$A(u, v) = \begin{pmatrix} a_1 - \dfrac{\alpha^2}{2} - b_1 \mathrm{e}^u - \dfrac{c_1 e^v}{m_1 + m_2 \mathrm{e}^u + m_3 \mathrm{e}^v} \\ -a_2 - \dfrac{\beta^2}{2} - b_2 \mathrm{e}^v + \dfrac{c_2 \mathrm{e}^u}{m_1 + m_2 \mathrm{e}^u + m_3 \mathrm{e}^v} \end{pmatrix}, \qquad B(u, v) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

To proceed, we first recall the notion of the Lie bracket. If $X(x) = (X_1, X_2)^\top$ and $Y(x) = (Y_1, Y_2)^\top$ are vector fields on $\mathbb{R}^2$ then the Lie bracket $[X, Y]$ is a vector field given by

$$[X, Y]_i(x) = \bigg(X_1 \frac{\partial Y_i}{\partial x_1}(x) - Y_1 \frac{\partial X_i}{\partial x_1}(x)\bigg) + \bigg(X_2 \frac{\partial Y_i}{\partial x_2}(x) - Y_2 \frac{\partial X_i}{\partial x_2}(x)\bigg), \qquad i = 1, 2.$$

We impose the following condition.

Assumption 3.1. *The Lie algebra* $\mathfrak{g}(u, v)$ *generated by* $A(u, v)$, $B(u, v)$ *satisfies the following:* $\dim \mathfrak{g}(u, v) = 2$ *at every* $(u, v) \in \mathbb{R}^2$. *In other words, we say that the set of vectors* $A, B, [A, B], [A, [A, B]], [B, [A, B]], \ldots$ *spans* $\mathbb{R}^2$.

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This assumption appears to be satisfied for most practical situations. It seems to be satisfied for any $a_i, b_i, c_i, m_1, m_2, m_3, \alpha > 0, i = 1, 2, \beta \neq 0$, and $a_1 - \alpha^2/2 > 0$, although verifying this assumption for our model in general involves cumbersome calculations. For specific parameters, the assumption can be verified by direct calculations. Note that the set of (u, v) at which vectors $A, B, [A, B], [A, [A, B]], [B, [A, B]], \dots$ do not span $\mathbb{R}^2$ are the roots of a system $\det(A, B) = 0, \det(A, [A, B]) = 0, \dots$ each of which is a polynomial equation of unknowns e^u, e^v. Thus, we can show that there is no (u, v) satisfying the above system of equations after taking into account a sufficient number of these equations.

To describe the support of the invariant measure μ^* and to prove the ergodicity of (3.3) and (3.4), we need to investigate the following control system:

$$\dot{u}_\phi(t) = \alpha\phi(t) + a_1 - \frac{\alpha^2}{2} - b_1 e^{u_\phi(t)} - \frac{c_1 e^{v_\phi(t)}}{m_1 + m_2 e^{u_\phi(t)} + m_3 e^{v_\phi(t)}}, \tag{3.5}$$

$$\dot{v}_\phi(t) = \beta\phi(t) - a_2 - \frac{\beta^2}{2} - b_2 e^{v_\phi(t)} + \frac{c_2 e^{u_\phi(t)}}{m_1 + m_2 e^{u_\phi(t)} + m_3 e^{v_\phi(t)}}, \tag{3.6}$$

where ϕ is taken from the set of piecewise continuous real-valued functions defined on $\mathbb{R}_+$. Let $(u_\phi(t, u, v), v_\phi(t, u, v))$ be the solution to (3.5) and (3.6) with control ϕ and initial value (u, v). Denote by $\mathcal{O}_1^+(u, v)$ the reachable set from (u, v), that is, the set of $(u', v') \in \mathbb{R}^2$ such that there exists a $t \geq 0$ and a control $\phi(\cdot)$ satisfying $u_\phi(t, u, v) = u', v_\phi(t, u, v) = v'$. It should be noted that Assumption 3.1 guarantees the accessibility of (3.5) and (3.6), i.e. $\mathcal{O}_1^+(u, v)$ has a quality, nonempty interior for every $(u, v) \in \mathbb{R}^2$; see [11]. We first recall some concepts introduced in [15]. Let U be a subset of $\mathbb{R}^2$ satisfying the property that for any $w_1, w_2 \in U$, we have $w_2 \in \overline{\mathcal{O}_1^+(w_1)}$. Then there is a unique maximal set $V \supset U$ such that this property still holds for V. Such a V is called a control set. A control set C is said to be invariant if $\overline{\mathcal{O}_1^+(w)} \subset \overline{C}$ for all $w \in C$.

Putting $z_\phi = v_\phi - (\beta/\alpha)u_\phi$, we have an equivalent system

$$\dot{u}_\phi(t) = \alpha\phi(t) + g(u_\phi(t), z_\phi(t)), \qquad \dot{z}_\phi(t) = h(u_\phi(t), z_\phi(t)), \tag{3.7}$$

where

$$g(u, z) = a_1 - \frac{\alpha^2}{2} - b_1 e^u - \frac{c_1 e^{z+(\beta/\alpha)u}}{m_1 + m_2 e^u + m_3 e^{z+(\beta/\alpha)u}},$$

and

$$h(u, z) = -\left(a_2 + \frac{\beta^2}{2} + \frac{\beta}{\alpha}\left(a_1 - \frac{\alpha^2}{2}\right)\right) - b_2 e^{z+(\beta/\alpha)u} + \frac{\beta}{\alpha} b_1 e^u + \frac{c_2 e^u + (\beta/\alpha)c_1 e^{z+(\beta/\alpha)u}}{m_1 + m_2 e^u + m_3 e^{z+(\beta/\alpha)u}}.$$

Denote by $\mathcal{O}_2^+(u, z)$ the set of $(u', z') \in \mathbb{R}^2$ such that there is a $t > 0$ and a control $\phi(\cdot)$ such that $u_\phi(t, u, z) = u', z_\phi(t, u, v) = z'$.

Claim 3.1. For any $u_0, u_1, z_0 \in \mathbb{R}$, and $\varepsilon > 0$, there exists a control ϕ and some $T > 0$ such that $u_\phi(T, u_0, z_0) = u_1, |z_\phi(T, u_0, z_0) - z_0| < \varepsilon$.

For the proof, suppose that $u_0 < u_1$ and let $\rho_1 = \sup\{|g(u, z)|, |h(u, z)| : u_0 \leq u \leq u_1, |z - z_0| \leq \varepsilon\}$. We choose $\phi(t) \equiv \rho_2$ with $(\alpha\rho_2\rho_1^{-1} - 1)\varepsilon \geq u_1 - u_0$. It is easy to check that with this control, there is a $T \in [0, \varepsilon\rho_1^{-1}]$ such that $u_\phi(T, u_0, z_0) = u_1, |z_\phi(T, u_0, z_0) - z_0| < \varepsilon$. If $u_0 > u_1$, we can construct $\phi(t)$ similarly.

Claim 3.2. For any $z_0 > z_1$, there is a $u_0 \in \mathbb{R}$, a control ϕ, and some $T > 0$ such that $z_\phi(T, u_0, z_0) = z_1$ and that $u_\phi(t, u_0, z_0) = u_0$ for all $0 \le t \le T$.

Indeed, if $\beta > 0$ and $-u_0$ is sufficiently large, then there is a $\rho_3 > 0$ such that $h(u_0, z) < -\rho_3$ for all $z_1 \le z \le z_0$. This property, combined with (3.7), implies the existence of a control ϕ and a $T > 0$ satisfying the desired claim. In the $\beta < 0$ case, choosing u_0 to be sufficiently large, we have the same result.

Claim 3.3. If $0 < \beta < \alpha$, for any $z_0 < z_1$, if u_0 is sufficiently large, $\inf_{z\in[z_0,z_1]} h(u_0, z) > 0$, which implies that there is a control ϕ and a $T > 0$ satisfying $z_\phi(T, u_0, z_0) = z_1$ and $u_\phi(t, u_0, z_0) = u_0$ for all $0 \le t \le T$.

Lemma 3.1. *Suppose that $\beta < 0$ or $\beta \ge \alpha$. Let $c^* := \sup\{\bar{z} \colon \overline{\sup_{u\in R}}\{h(u, z)\} > 0$ for all $z \le \bar{z}$.$\}$. Then $c^* > -\infty$, (c^* may be ∞) and for any $(u, z) \in \mathbb{R}^2$, $\overline{\mathcal{O}_2^+(u, z)} \supset \{(u', z') \colon z' \le c^*\}$.*

Proof. Note that

$$\lambda = -a_2 - \frac{\beta^2}{2} + \int_0^\infty \frac{c_2 x}{m_1 + m_2 x} \mu_-(\mathrm{d}x) > 0.$$

In view of Jensen's inequality,

$$\int_0^\infty \frac{c_2 x}{m_1 + m_2 x} \mu_-(\mathrm{d}x) \le \frac{c_2 \int_0^\infty x\mu_-(\mathrm{d}x)}{m_1 + m_2 \int_0^\infty x\mu_-(\mathrm{d}x)} = \frac{c_2(a_1 - \alpha^2/2)b_1^{-1}}{m_1 + m_2(a_1 - \alpha^2/2)b_1^{-1}}.$$

If $\mathrm{e}^{\bar{u}} = (a_1 - \alpha^2/2)b_1^{-1}$, we have

$$\begin{aligned} h(\bar{u}, z) = {} & \frac{c_1(a_1 - \alpha^2/2)b_1^{-1}}{m_1 + m_2(a_1 - \alpha^2/2)b_1^{-1} + m_3\mathrm{e}^{z+(\beta/\alpha)\bar{u}}} - \left(a_2 + \frac{\beta^2}{2}\right) + b_2\mathrm{e}^{z+(\beta/\alpha)\bar{u}} \\ & + \frac{(\beta/\alpha)c_1\mathrm{e}^{z+(\beta/\alpha)\bar{u}}}{m_1 + m_2\mathrm{e}^{\bar{u}} + m_3\mathrm{e}^{z+(\beta/\alpha)\bar{u}}}. \end{aligned}$$

Since

$$\frac{c_1(a_1 - \alpha^2/2)b_1^{-1}}{m_1 + m_2(a_1 - \alpha^2/2)b_1^{-1}} - \left(a_2 + \frac{\beta^2}{2}\right) > 0,$$

$h(\bar{u}, z) > 0$ when e^z is sufficiently small. Now we move to the second assertion. Note that it follows directly from the continuous dependence of solutions on initial values that if $\overline{\mathcal{O}_2^+(w_2)} \subset \overline{\mathcal{O}_2^+(w_1)}$ provided $w_2 \in \overline{\mathcal{O}_2^+(w_1)}(\omega_1, \omega_2 \in \mathbb{R}^2)$. For $(u, z) \in \mathbb{R}^2$, define $\mathfrak{z}_{u,z} = \sup\{z_1 \colon$ there exists u_1 such that $(u_1, z_1) \in \overline{\mathcal{O}_2^+(u, z)}\}$. For any $(u_1, z_1) \in \mathbb{R}^2$, it is easy to derive from Claims 3.1 and 3.2 that $\overline{\mathcal{O}_2^+(u_1, z_1)} \supset \{(u', z') \colon z' \le z_1\}$. Hence, $\overline{\mathcal{O}_2^+(u, z)} \supset \{(u_1, z_1) \colon z_1 \le \mathfrak{z}_{u,z}\}$. If $\mathfrak{z}_{u,z} < c^*$ there is some $\widehat{u} \in \mathbb{R}$ such that $h(\widehat{u}, \mathfrak{z}_{u,z}) > 0$. Since $h(\cdot)$ is continuous, there is an $\widehat{z} > \mathfrak{z}_{u,z}$ such that $\inf\{h(\widehat{u}, z) \colon z \in [\mathfrak{z}_{u,z}, \widehat{z}]\} > 0$. As a result, there is a control ϕ and a $T > 0$ such that $z_\phi(T, \widehat{u}, \mathfrak{z}_{u,z}) = \widehat{z}$ and $u_\phi(t, \widehat{u}, \mathfrak{z}_{u,z}) = \widehat{u}$ for all $t \in [0, T]$. That is, $(\widehat{u}, \widehat{z}) \in \mathcal{O}_2^+(\widehat{u}, \mathfrak{z}_{u,z}) \subset \overline{\mathcal{O}_2^+(u, z)}$, which contradicts the definition of $\mathfrak{z}_{u,z}$. The proof is complete. □

Proposition 3.1. *The control system in (3.5) and (3.6) has only one invariant control set C. If $0 < \beta < \alpha$, $C = \mathbb{R}^2$. If $\beta < 0$ or $\beta \ge \alpha$, $C = \{(u, v) \colon v - (\beta/\alpha)u \le c^*\}$.*

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Proof. If $0 < \beta < \alpha$, it follows from Claims 3.1, 3.2, and 3.3 that for any $(u_1, z_1), (u_2, z_2) \in \mathbb{R}^2$, $(u_2, z_2) \in \overline{\mathcal{O}_2^+(u_1, z_1)}$. Hence, for any $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have $(u_2, v_2) \in \overline{\mathcal{O}_1^+(u_1, v_1)}$. This implies that $\mathbb{R}^2$ is a unique invariant control set. Now, consider the case $\beta < 0$ or $\beta \geq \alpha$ for which the conclusion of this proposition is a direct corollary of Lemma 3.1 if $c^* = \infty$. If $c^* < \infty$, it is seen from the definition of c^* that $h(u, c^*) \leq 0$ for all $u \in \mathbb{R}$. Consequently, for all control ϕ, we have $z_\phi(t, u, z) \leq c^*$ for all $t \geq 0$ provided that $z \leq c^*$. In other words, $\overline{\mathcal{O}_2^+(u, z)} \subset \{(u', z') \colon z' \leq c^*\}$. This claim combined with Lemma 3.1 implies that $\overline{\mathcal{O}_2^+(u, z)} = \{(u', z') \colon z' \leq c^*\}$ for all $u \in \mathbb{R}, z \leq c^*$. As a result, $\{(u', z') \colon z' \leq c^*\}$ is an invariant control set for (3.7). The uniqueness of this invariant control set is obtained with the property that $\{(u', z') \colon z' \leq c^*\} \subset \overline{\mathcal{O}_2^+(u, z)}$ for every $(u, z) \in \mathbb{R}^2$. Equivalently, $C := \{(u, v) \colon v - (\beta/\alpha)u \leq c^*\}$ is a unique invariant control set for (3.5) and (3.6). □

Note that if $\lambda > 0$, there is an invariant probability measure π^* of (3.3) and (3.4) that is associated with μ^* of (3.1) and (3.2). Since there is only one invariant control set C, it follows from Assumption 3.1 that π^* is the unique invariant probability measure with support C. Moreover, for all $(u, v) \in C$ and a π^*-integrable function f, we have

$$\mathbb{P}\biggl\{\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\xi^{u,v}(s), \eta^{u,v}(s))\,\mathrm{d}s = \int_{\mathbb{R}^2} f(u', v')\pi^*(\mathrm{d}u', \mathrm{d}v')\biggr\} = 1. \tag{3.8}$$

These results are proved in [15]. Moreover, from [12, Proposition 5.1], it follows that

$$\lim_{t\to\infty} \|\widehat{P}(t, (u, v), \cdot) - \pi^*(\cdot)\| \to 0 \quad \text{for all } (u, v) \in C, \tag{3.9}$$

where $\|\cdot\|$ is the total variation norm, if we can verify the following Hörmander condition.

Assumption 3.2. *The ideal $\mathfrak{g}_0$ in $\mathfrak{g}$ generated by B satisfies* $\dim \mathfrak{g}_0(u, v) = 2$ *at every* $(u, v) \in C$. *In other words, the set of vectors* $B, [A, B], [B, [A, B]], [B, [B, A, B]], \dots$ *spans* $\mathbb{R}^2$.

We aim to prove that (3.8) (under Assumption 3.1) and (3.9) (under Assumption 3.2) hold for all $(u, v) \in \mathbb{R}^2$. We need only consider the case $\beta < 0$ or $\beta \geq \alpha$ since $C = \mathbb{R}^2$ in the $0 < \beta < \alpha$ case.

Proposition 3.2. *Suppose that* $\beta \geq \alpha, \lambda > 0$. *Then, for each initial value* $(u, v) \in \mathbb{R}^2$, *we have* $\tau_{C^\circ}^{u,v}$ *a.s. with* $\tau_{C^\circ}^{u,v} = \inf\{t > 0 \colon (\xi^{u,v}(t), \eta^{u,v}(t)) \in C^\circ\}$.

The proof of this proposition is divided into several lemmas. We consider only the $c^* < \infty$ case since the assertion is trivial if $c^* = \infty$. Let us first explain the idea of the proof. Denote $d_1 = \ln H, d_2 = \ln \hbar$, where $\hbar, H$ are defined as in the proof of Theorem 2.2. Since the process is recurrent relative to $\widehat{A} := \{(u, v) \colon u \leq d_1, d_2 \leq v \leq d_1\}$, in order to show $\tau_{C^\circ}^{u,v} < \infty$, we need to estimate (uniformly) the probability of entering C° from $\widehat{A}$. The difficulty is that $\widehat{A}$ is not compact. Therefore, we divide $\widehat{A}$ into $\widehat{A}_1 = \{(u, v) \colon u < d_5, d_2 \leq v \leq d_1\}$ and $\widehat{A}_2 = \widehat{A} \setminus \widehat{A}_1$, where $-d_5$ is sufficiently large. Noting that $\widehat{A}_2$ is compact and using the support theorem and the Feller property, we can obtain a positive lower bound for the probability of entering C from $\widehat{A}_2$. To obtain a similar result for $\widehat{A}_1$, we will analyze the property of the drift when $-u$ is sufficiently large and then estimate using the exponential martingale inequality.

Fix $0 < \delta < \min\{a_1 - \alpha^2/2, a_2 + \beta^2/2\}$. Thus, there is a $d_3 < d_2$ such that for all $u \leq \alpha\beta^{-1}(d_3 - c^*), v \leq d_3$, we have

$$a_1 - \frac{\alpha^2}{2} - b_1 \mathrm{e}^u - \frac{c_1 \mathrm{e}^v}{m_1 + m_2 \mathrm{e}^u + m_3 \mathrm{e}^v} \geq \delta \text{ and } -a_2 - \frac{\beta^2}{2} - b_2 \mathrm{e}^v + \frac{c_2 \mathrm{e}^u}{m_1 + m_2 \mathrm{e}^u + m_3 \mathrm{e}^v} \leq -\delta.$$

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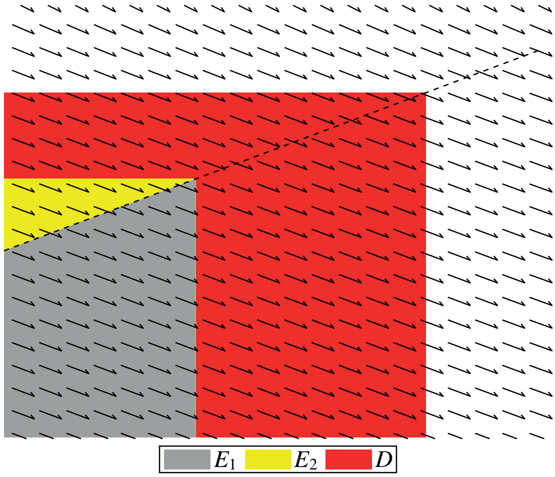

FIGURE 1: Illustration of E_1, E_2, and D. The dashed line is the boundary $\{v = c^* + (\beta/\alpha)u\}$ of C. The arrows are the vector field of the drift.

Let $d_4 \leq \min\{(\alpha/\beta)(d_3 - c^*), d_3\} - \ell$, where $\ell > 0$ is chosen such that $2\exp(-\delta\ell/(\alpha + \beta)^2) < 1$. Construct open sets $D = \{(u, v) \in \mathbb{R}^2 \colon u < (\alpha/\beta)(d_3 - c^*), v < d_3\}$ and $E = \{(u, v) \in \mathbb{R}^2, u, v \leq d_4\}$. Then put $E_1 = E^\circ \cap C^\circ$, $E_2 = E \setminus E_1$; see Figure 1.

Lemma 3.2. *Suppose that $\beta \geq \alpha$. There is a $\tilde{p} > 0$ such that*

$$\mathbb{P}\bigg\{\xi^{u,v}(\sigma_D^{u,v}) = \frac{\alpha}{\beta}(d_3 - c^*), \eta^{u,v}(\sigma_D^{u,v}) < d_3\bigg\} \geq \tilde{p}_1 \quad \text{for all } (u, v) \in E,$$

where $\sigma_D^{u,v}$ is the first time $(\xi^{u,v}(t), \eta^{u,v}(t))$ exits D.

Proof. Define

$$\widehat{T}_{u,v} = \frac{2}{\delta}\bigg(\frac{\alpha}{\beta}(d_3 - c^*) - u + \ell\bigg).$$

By the well-known exponential martingale inequality, we have

$$\mathbb{P}(\Omega_1) > \tilde{p}_1 := 1 - 2\exp(-\delta\ell/(\alpha + \beta)^2),$$

where

$$\Omega_1 := \bigg\{\omega \colon \sup_{0 \leq t \leq \widehat{T}_{u,v}} \bigg\{|W(t)| - \frac{\delta}{2(\alpha + \beta)}t\bigg\} < \frac{\ell}{\alpha + \beta}\bigg\}.$$

For $\omega \in \Omega_1$ and $u, v \leq d_4$, from the properties of Ω_1, (3.3), and (3.4), it follows that

$$\begin{aligned}\xi^{u,v}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) &\geq u + \delta(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) - \frac{\alpha\delta}{2(\alpha + \beta)}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) - \frac{\alpha\ell}{\alpha + \beta} \\ &\geq u - \ell + \frac{\delta}{2}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}),\end{aligned} \tag{3.10}$$

and that

$$\eta^{u,v}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) \leq d_4 - \delta(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) + \frac{\beta\delta}{2(\alpha + \beta)}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) + \ell < d_3. \tag{3.11}$$

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If $\sigma_D^{u,v} > T_{u,v}$, from (3.10), it follows that $\xi^{u,v}(\widehat{T}_{u,v}) \ge u - \ell + (\delta/2)\widehat{T}_{u,v} \ge (\alpha/\beta)(d_3 - c^*)$ which is a contradiction. Hence, $\sigma_D^{u,v} \le T_{u,v}$ for all $\omega \in \Omega_1$. Furthermore, (3.11) implies that for $\omega \in \Omega_1$, $\eta^{u,v}(\sigma_D^{u,v}) = \eta^{u,v}(\sigma_D^{u,v} \wedge \widehat{T}_{u,v}) < d_3$ and consequently $\xi^{u,v}(\sigma_D^{u,v}) = (\alpha/\beta)(d_3 - c^*)$. As a result,

$$\mathbb{P}\bigg\{\xi^{u,v}(\sigma_D^{u,v}) = \frac{\alpha}{\beta}(d_3 - c^*), \eta^{u,v}(\sigma_D^{u,v}) < d_3\bigg\} \ge \mathbb{P}(\Omega_1) \ge \tilde{p}_1 \quad \text{for all } (u, v) \in E.$$

The lemma is proved. □

Lemma 3.3. *Suppose that $\beta \ge \alpha$. There are $d_5 \in \mathbb{R}$, $\tilde{p}_2 > 0$, and $\overline{T} > 0$ such that*

$$\mathbb{P}\{\tau_E^{u,v} \le \overline{T}\} \ge \tilde{p}_2 \quad \text{for all } u \le d_5,\ d_2 \le v \le d_1,$$

where $\tau_E^{u,v}$ is the first time $(\xi^{u,v}(t), \eta^{u,v}(t))$ enters E.

Proof. It is readily seen that there are $\sigma_1 < d_4$, $G_1 > 0$, and $\delta_1 > 0$ such that

$$\sup_{u \le \sigma_1, v \in \mathbb{R}} \bigg\{a_1 - \frac{\alpha^2}{2} - b_1 e^u - \frac{c_1 e^v}{m_1 + m_2 e^u + m_3 e^v}\bigg\} \le G_1,$$

and that

$$\sup_{u \le \sigma_1, v \in \mathbb{R}} \bigg\{-a_2 - \frac{\beta^2}{2} - b_2 e^v + \frac{c_2 e^u}{m_1 + m_2 e^u + m_3 e^v}\bigg\} < -\delta_1.$$

Fix $\delta_2 > 0$. Define $\overline{T} = 2((d_1 - d_4 + \delta_2)/\delta_1)$ and $d_5 = \sigma_1 - \delta_2 - (G_1 + \delta_1/2)\overline{T}$ and the stopping time

$$\zeta^{u,v} = \inf\{t > 0 \colon \xi^{u,v}(t) \ge \sigma_1 \text{ or } \eta^{u,v}(t) \le d_4\}.$$

By the exponential martingale inequality, we have $\mathbb{P}\{\Omega_2\} > \tilde{p}_2 := 1 - \exp(-\delta_1\delta_2/(\alpha+\beta)^2) > 0$, where

$$\Omega_2 := \bigg\{\omega \colon \sup_{0 \le t \le \overline{T}} \bigg\{W(t) - \frac{\delta_1}{2(\alpha+\beta)} t\bigg\} < \frac{\delta_2}{\alpha+\beta}\bigg\}.$$

For $\omega \in \Omega_2$ and $u < d_5$, $d_2 \le v \le d_1$, from the properties of Ω_2 and (3.3) and (3.4), it follows that

$$\begin{aligned}
\xi^{u,v}(\zeta^{u,v} \wedge \overline{T}) &< u + G_1(\zeta^{u,v} \wedge \overline{T}) + \frac{\alpha\delta_1}{2(\alpha+\beta)}(\zeta^{u,v} \wedge \overline{T}) + \frac{\alpha\delta_2}{\alpha+\beta} \\
&\le d_5 + \delta_2 + \bigg(G_1 + \frac{\delta_1}{2}\bigg)\overline{T} \\
&= \sigma_1,
\end{aligned} \tag{3.12}$$

and that

$$\begin{aligned}
\eta^{u,v}(\zeta^{u,v} \wedge \overline{T}) &< d_1 - \delta_1(\zeta^{u,v} \wedge \overline{T}) + \frac{\beta\delta_1}{2(\alpha+\beta)}(\zeta^{u,v} \wedge \overline{T}) + \frac{\beta\delta_2}{\alpha+\beta} \\
&\le d_1 + \delta_2 - \frac{\delta_1}{2}(\zeta^{u,v} \wedge \overline{T}).
\end{aligned} \tag{3.13}$$

If $\zeta^{u,v} > \overline{T}$, from (3.13), we deduce that $\eta^{u,v}(\overline{T}) < d_1 + \delta_2 - (\delta_1/2)(\overline{T}) = d_4$, which contradicts the definition of $\zeta^{u,v}$. Hence, for $\omega \in \Omega_2$, we have $\zeta^{u,v} \le \overline{T}$. Moreover, (3.12) implies that $\xi^{u,v}(\zeta^{u,v}) < \sigma_1$. In view of the definition of $\zeta^{u,v}$, we have $\eta^{u,v}(\zeta^{u,v}) = d_4$ in Ω_2, consequently $\tau_E^{u,v} \le \overline{T}$ in Ω_2. As a result, for any $u \le d_5, d_2 \le v \le d_1$, $\mathbb{P}\{\tau_E^{u,v} \le \overline{T}\} \ge \mathbb{P}(\Omega_2) \ge \tilde{p}_2$. □

Lemma 3.4. *Suppose that $\beta \geq \alpha, \lambda > 0$. For any $(u, v) \in \mathbb{R}^2$, the process $(\xi^{u,v}(t), \eta^{u,v}(t))$ is recurrent relative to E, that is, there is a sequence of random variables $\{t_n(\omega)\}$ such that $t_n(\omega) \uparrow \infty$ as $n \to \infty$ and that $(\xi^{u,v}(t_n), \eta^{u,v}(t_n)) \in E$ for all $n \in \mathbb{N}$ for almost all ω.*

Proof. Since $E_1 \subset \overline{\mathcal{O}_1^+(u, v)}$ for all $(u, v) \in \mathbb{R}^2$, it follows from the support theorem (see [7, Theorem 8.1, p. 518] or [23]) for diffusion processes, that there is a $T_{u,v} > 0$ such that $\mathbb{P}\{(\xi^{u,v}(T_{u,v}), \eta^{u,v}(T_{u,v})) \in E_1\} > 2p^{u,v} > 0$. Since the process $(\xi(t), \eta(t))$ is Feller and E_1 is an open set, there is a neighborhood $V_{u,v}$ of (u, v) such that for $\mathbb{P}\{(\xi^{u',v'}(T_{u,v}), \eta^{u',v'}(T_{u,v})) \in E_1\} > p_{u,v}$ for all $(u', v') \in V_{u,v}$. Let d_5 be as in Lemma 3.3, we consider the compact set $K = \{(u, v) : d_5 \leq u \leq d_1, d_2 \leq v \leq d_1\}$. By the Heine–Borel theorem, there is a finite number of $V_{u_i,v_i}, i = 1, \ldots, n$ such that $K \subset \bigcup_{i=1}^n V_{u_i,v_i}$. Letting $\overline{T}_K = \max\{T_{u_i,v_i}, i = 1, n\}$ and $\overline{p}_K = \min\{p_{u_i,v_i}, i = 1, n\}$, we claim that for any $(u, v) \in K$, $\mathbb{P}\{\tau_E^{u,v} \leq T_K\} \geq \mathbb{P}\{\tau_{E_1}^{u,v} \leq T_K\} \geq p_K > 0$. Combining this result with the conclusion of Lemma 3.3, we derive that there are $\widehat{T} > 0$, $\widehat{p} > 0$ such that

$$\mathbb{P}(\tau_E^{u,v} < \widehat{T}) \geq \widehat{p} \quad \text{for all } (u, v) \in \widehat{A} := \{(u, v) : u \leq d_1, d_2 \leq v \leq d_1\}. \tag{3.14}$$

Since (2.14) is equivalent to

$$\frac{1}{t} \int_0^t \mathbf{1}_{\{(\xi^{u,v}(s), \eta^{u,v}(s)) \in \widehat{A}\}} \, \mathrm{d}s > 0 \quad \text{a.s. for all } (u, v) \in \mathbb{R}^2,$$

the process $(\xi^{u,v}(t), \eta^{u,v}(t))$ is recurrent relative to $\widehat{A}$. Using this property, the strong Markov property, and (3.14), we can conclude the recurrence relative to E of $(\xi^{u,v}(t), \eta^{u,v}(t))$. □

Proof of Proposition 3.2. Since $(\xi^{u,v}(t), \eta^{u,v}(t))$ is recurrent relative to $\widehat{A}$ and E, we can define the following sequences of stopping times:

$$\begin{aligned} \varsigma_1 &= \inf\{t > 0 : \xi^{u,v}(t), \eta^{u,v}(t)) \in E\}, \\ \upsilon_n &= \inf\{t > \varsigma_n : \xi^{u,v}(t), \eta^{u,v}(t)) \in \widehat{A}\}, \\ \varsigma_{n+1} &= \inf\{t > \upsilon_n : \xi^{u,v}(t), \eta^{u,v}(t)) \in E\}, \end{aligned}$$

which are finite a.s.

We also define $\iota_n = \inf\{t > \varsigma_n : \xi^{u,v}(t), \eta^{u,v}(t)) \notin D\}$. Since $E \subsetneq D \subsetneq \widehat{A}^c$, it is easy to see that $\varsigma_n < \iota_n < \upsilon_n$. Consider a sequence of events $O_n := \{\xi^{u,v}(\iota_n) = (\alpha/\beta)(d_3 - c^*), \eta^{u,v}(\iota_n) < d_3\}$. If we are in the time ς_n then O_n is the future information, while we already know whether O_{n-1} has happened. Moreover, it follows from Lemma 3.2 that $\mathbb{P}(O_n^c \mid \xi^{u,v}(\varsigma_n) = u', \eta^{u,v}(\varsigma_n) = v') \leq 1 - \tilde{p}_1$ for all $(u', v') \in E$. Hence, using the strong Markovian property of $(\xi^{u,v}(t), \eta^{u,v}(t))$, we can prove that

$$\mathbb{P}\biggl(\bigcap_{k=1}^n O_k^c\biggr) \leq (1 - \tilde{p}_1)^n \to 0 \quad \text{as } n \to \infty.$$

This means that almost surely, O_n must occur for some $n = n(\omega)$. Whenever O_n occurs, we have $(\xi^{u,v}(\iota_n), \eta^{u,v}(\iota_n)) \in C^\circ$. The proof is complete. □

For the $\beta < 0$ case, we have a similar result.

Proposition 3.3. *Suppose that $\beta < 0$, $\lambda > 0$. Then, for each initial data $(u, v) \in \mathbb{R}^2$, $\tau_{C^\circ}^{u,v} < \infty$ a.s.*

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Proof. We only consider the $c^* < \infty$ case for which $C = \{(u, v): v \le c^* - ru\}$ with $r = -\beta/\alpha > 0$. Let $\widehat{A}$ be as in the proof of Lemma 3.4. Divide $\widehat{A}$ into $\widehat{A}_1$ and $\widehat{A}_2$ defined by $\widehat{A}_1 = \widehat{A} \cap C^\circ$ and $\widehat{A}_2 = \widehat{A} \setminus \widehat{A}_1$. It is easy to see that $\widehat{A}_2$ is compact. Using the same arguments as in the proof of Lemma 3.4, we can find $\overline{T}_{\widehat{A}_2} > 0$ such that $\inf_{(u',v')\in\widehat{A}_2} \mathbb{P}(\tau_{C^\circ}^{u',v'} < \overline{T}_{\widehat{A}_2}) > 0$. Since $\widehat{A}_1 \subset C^\circ$, we have

$$\inf_{(u',v')\in\widehat{A}} \mathbb{P}(\tau_{C^\circ}^{u',v'} < \overline{T}_{\widehat{A}_2}) = \inf_{(u',v')\in\widehat{A}_2} \mathbb{P}(\tau_{C^\circ}^{u',v'} < \overline{T}_{A_2}) > 0.$$

Moreover, since $(\xi^{u,v}(t), \eta^{u,v}(t))$ is recurrent relative to $\widehat{A}$, we can use the strong Markov property to obtain the desired conclusion. □

We complete this section by presenting the following theorem.

Theorem 3.1. *Suppose that $\alpha, \beta \neq 0$, $\lambda > 0$, and Assumption 3.1 holds. Then, (3.3) and (3.4) have a unique invariant probability measure π^* satisfying that for any π^*-integrable function f,*

$$\mathbb{P}\bigg\{\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\xi^{u,v}(s), \eta^{u,v}(s))\,\mathrm{d}s = \int_{\mathbb{R}^2} f(u', v')\pi^*(\mathrm{d}u', \mathrm{d}v')\bigg\} = 1 \quad \textit{for all } (u, v) \in \mathbb{R}^2.$$

Moreover, if Assumption 3.2 is satisfied, the transition probability $\widehat{P}(t, (u, v), \cdot)$ converges to $\pi^*(\cdot)$ in total variation as $t \to \infty$.

Proof. The assertions can be proved using (3.8), (3.9), and Propositions 3.2 and 3.3. □

4. Discussion

We compare our results with some of the recent results in the literature. In [8, Theorem 4.1], under the conditions $c_2/m_2 < a_2 + \beta^2/2$ and $a_1 > \alpha^2/2$, it was proved that the predator will eventually die out while the distribution of $x(t)$ converges weakly to the stationary distribution of $u(t)$. In contrast, using Theorem 2.1 of this paper, we obtain the same conclusion provided that $a_1 > \alpha^2/2$ and $\lambda < 0$. Note that $\lambda < 0$ is equivalent to

$$\tilde{\lambda} := \int_0^\infty \frac{c_2 x}{m_1 + m_2 x}\mu_-(\mathrm{d}x) < a_2 + \frac{\beta^2}{2}.$$

It is easy to verify that $\tilde{\lambda} < c_2/m_2$, which indicates that our result on the extinction of the predator is sharper. Furthermore, a suitable Lyapunov function was used in [8] to obtain the ergodicity of system (1.3) and (1.4) for the nondegenerate case as follows; see [8, Theorem 3.1].

Theorem 4.1. *Assume that $(c_2 - a_2 m_2)a_1/b_1 > a_2 m_1$, $b_1 > a_1 m_2/(m_1 + m_2 x^*)$, and $\alpha > 0, \beta > 0$ such that $\delta < \min\{c_2(b_1 - m_2(a_1 - b_1 x^*)/m_1)(m_1 + m_3 y^*)(x^*)^2, b_2 c_1(m_1 + m_2 x^*)(y^*)^2\}$, where $\delta = c_2 x^* \alpha^2/2 + c_1 y^* \beta^2/2$ and (x^*, y^*) is the equilibrium of the deterministic system*

$$\dot{x}(t) = x(t)(a_1 - b_1 x(t)) - \frac{c_1 y(t)}{m_1 + m_2 x(t) + m_3 y(t)})\,\mathrm{d}t, \tag{4.1}$$

$$\dot{y}(t) = \bigg(-a_2 - b_2 y(t) + \frac{c_2 x(t)}{m_1 + m_2 x(t) + m_3 y(t)}\bigg)\,\mathrm{d}t. \tag{4.2}$$

Then there is a stationary distribution $\pi(\cdot)$ for the system in (1.3) and (1.4) and it has ergodic property.

To show that the assumption in [8] is more restrictive than our assumption of ergodicity, let G be the space of the positive parameters $(a_i, b_i, c_i, m_j, \alpha, \beta)$, $i = 1, 2$, $j = 1, 2, 3$, $a_1 > \alpha^2/2$, and

$$G^+ = \{(a_i, b_i, c_i, m_j, \alpha, \beta) \colon \lambda > 0\}, \qquad G^- = \{(a_i, b_i, c_i, m_j, \alpha, \beta) \colon \lambda < 0\}.$$

It is easy to check that λ is a continuous function of parameters. Hence, G^+ and G^- are open. Moreover, the closure $\mathrm{cl}(G^-) = \{\lambda \le 0\} = (G^+)^c$, which is a necessary condition for the extinction of the predator. Let J be the set of parameters satisfying the assumption of Theorem 4.1, we must have $G^- \cup J = \varnothing$. Since J is open, $\mathrm{cl}(G^-) \cup J = \varnothing$ or, equivalently, $J \subset G^+$.

We will show that J is a proper subset of G^+. Choose $a_1, b_1, c_1, a_2, c_2, m_i, i = 1, 3, \alpha, \beta$ such that $\lambda > 0$. This choice can be performed by taking a_1 sufficiently large. Now fix these parameters. Since λ does not depend on b_2, we claim the ergodicity holds for all $b_2 > 0$. It can be proved that there exists $M > 0$ independent of b_2 such that $x^*, y^* < M$, where (x^*, y^*) is the positive equilibrium of (4.1) and (4.2) (if it exists). Thus, for sufficiently small b_2 such that $\delta > b_2 c_1 (m_1 + m_2 x^*)(y^*)^2$, the assumption of Theorem 4.1 does not hold while $\lambda > 0$.

Next we look at the $m_1 = 1, m_2 = 1$, and $m_3 = 0$ cases for which the functional response is said to be Holling-type-II (see (1.1) and (1.2)). We will make a comparison with the findings in [18] in which the authors proved that if $a_1 - \alpha^2/2 > 0$ and $c_2 + a_2 - \beta^2/2 < 0$, the predator will be extinct while $x(t)$ converges weakly to the stationary distribution of $\phi(t)$. Moreover, it was shown that the system is persistent in time-average if

$$a_1 - \frac{\alpha^2}{2} > 0, \qquad a_2 - \frac{\beta^2}{2} > 0, \qquad \frac{a_1 - \alpha^2/2}{c_1} > \frac{c_2 + a_2 - \beta^2/2}{b_2}.$$

In the same manner as in the previous part, we can show that our conditions for extinction or permanence and ergodicity are weaker than those in [18].

We have investigated (1.3) and (1.4) and (3.1) and (3.2) when $\lambda \ne 0$. Note that the set $\{\lambda = 0\}$ has Lebesgue measure 0 in the space of parameters G. Although the set $\{\lambda = 0\}$ is negligible with respect to the Lebesgue measure, it is still interesting to explore the asymptotic behavior of the solution in this critical case. The question of asymptotic behavior corresponding to $\lambda = 0$ remains open. To treat this case, new techniques are needed. Moreover, it seems that our methods are applicable to stochastic predator–prey models with different types of functional responses as well as different diffusion coefficients. Furthermore, our method can be applied to stochastic models with Markovian switching.

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