The semilattice J, the inverse semigroups $N(E_\alpha)$ and the mappings ϕ determine N to within isomorphism.

Conversely, any semilattice of inverse semigroups has a structure determined in the above manner.

The set of all sets of semilattices J of inverse semigroups $N(E_\alpha)$ and the corresponding mappings ϕ associated as above with N forms a set of complete invariants for N.

REFERENCES

1. G. B. Preston, Inverse semi-groups, *London Math. Soc.*, **29** (1954), 396–403.
2. A. H. Clifford, Bands of semigroups, *Proc. Amer. Math. Soc.* **5** (1954), 499–504.
3. A. H. Clifford, Semigroups admitting relative inverses, *Ann. of Math.*, **42** (1941), 1037–1049.
4. A. E. Liber, K teorii obobščennyh grupp, *Dokl. Akad. Nauk. SSSR*, **97** (1954), 25–28.
5. R. Croisot, Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples, *Ann. de L'École Norm.*, (3) **70** (1953), 361–379.
6. W. D. Munn and R. Penrose, A note on inverse semigroups, *Proc. Cambridge Phil. Soc.* **51** (1955), 396–399.
7. V. V. Vagner, Obobščennye gruppy, *Dokl. Akad. Nauk. SSSR*, **84** (1952), 1119–1122.
8. A. H. Clifford, Extensions of semigroups, *Trans. Amer. Math. Soc.* 68 (1950), 165–173.

ROYAL MILITARY COLLEGE OF SCIENCE
SHRIVENHAM

ON THE CLOSED GRAPH THEOREM

by ALEX. P. ROBERTSON *and* WENDY ROBERTSON

(Received 8th September, 1956)

The closed graph theorem is one of the deeper results in the theory of Banach spaces and one of the richest in its applications to functional analysis. This note contains an extension of the theorem to certain classes of topological vector spaces. For the most part, we use the terminology and notation of N. Bourbaki [1], contracting " locally convex topological vector space over the real or complex field " to " convex space "; here we confine ourselves to convex spaces.

Suppose that E is a separated (i.e. Hausdorff) convex space and that its dual E' has the weak topology $\sigma(E', E)$. Then E is called *fully complete* if a vector subspace M' of E' is closed whenever $M' \cap U^0$ is closed for every neighbourhood U of the origin in E. A fully complete space is complete; a closed vector subspace of a fully complete space is fully complete and so also is a quotient by a closed vector subspace (H. S. Collins [2]). Any Fréchet space is fully complete (J. Dieudonné and L. Schwartz [3], Théorème 5, Corollaire). There are other fully complete spaces; for example the algebraic dual E^* of any vector space E is fully complete under the topology $\sigma(E^*, E)$ (Collins, [2], Corollary 17.2). It is not difficult to show that the dual E' of a Fréchet space E is fully complete under any topology between the topology of compact convergence and the Mackey topology $\tau(E', E)$; in particular the strong dual of a reflexive Fréchet space is fully complete.

Fully complete spaces have also been studied by V. Pták [5] in connection with the open mapping theorem (homomorphism theorem). He calls a linear mapping t of a convex space E onto a convex space F almost open if, for each neighbourhood U of the origin in E, $\overline{t(U)}$ is a neighbourhood of the origin in F. Then Pták's main result ([5], Theorem 4.5) is that every continuous almost open mapping is open if and only if E is fully complete, so that, in particular, a continuous linear mapping of a fully complete space onto a convex Baire space is open.

The following lemma is implicit in the proof of Pták's theorem but we reproduce the relevant details so that the results here are self-contained.

LEMMA 1. *Let F be fully complete under the topology ξ, and let η be a coarser topology under which F is a separated convex space. If, under the topology ξ, F has a base of neighbourhoods of the origin whose η-closures are neighbourhoods under η, then ξ and η are identical.*

Proof. Let F' be the dual of F under ξ. The dual M' of F under η is a vector subspace of F', dense in F' under $\sigma(F', F)$. If U is any ξ-closed absolutely convex (i.e. convex and circled) neighbourhood of the origin in F under ξ, the η-closure $\overline{U}$ of U is an η-neighbourhood, so that U and $\overline{U}$ have the same polar $U^0 \cap M'$ in M'. Hence $U^0 \cap M'$ is $\sigma(M', F)$-compact and so $\sigma(F', F)$-closed. Since F is fully complete under ξ, this implies that M' is $\sigma(F', F)$-closed. Thus $M' = F'$. It follows that U, being absolutely convex, has the same closure in ξ and η, and so $\overline{U} = U$. Thus ξ is identical with η.

THEOREM 1 (*Closed graph theorem*). *Suppose that t is a linear mapping of the barrelled space E into the fully complete space F and that the graph of t in $E \times F$ is closed. Then t is continuous.*

Proof. Since $\overline{t(E)}$ is fully complete and the graph of t is closed in $E \times \overline{t(E)}$, it is sufficient to prove the theorem when $t(E)$ is dense in F.

Let $\mathcal{V}$ be a base of closed absolutely convex neighbourhoods of the origin in F under the given topology ξ. We begin by defining another topology η on F, taking as a base of neighbourhoods of the origin the sets

$$\widetilde{V} = \overline{t\left(\overline{t^{-1}(V)}\right)}, \quad V \in \mathcal{V}.$$

Each V is clearly absolutely convex and since

$$V = \overline{V \cap t(E)} \subseteq \overline{t\left(t^{-1}(V)\right)} \subseteq \widetilde{V},$$

each $\widetilde{V}$ is also absorbent. Thus the sets $\widetilde{V}$ form a base $\widetilde{\mathcal{V}}$ of neighbourhoods of the origin in a topology η on F coarser than ξ.

To show that F is separated under η we use the fact that the graph G of t is closed. For suppose that $y \in \widetilde{V}$ for all $\widetilde{V} \in \widetilde{\mathcal{V}}$, and let (U, V) be a neighbourhood of the origin in $E \times F$. Then $y \in \frac{1}{2}\widetilde{V}$, so that $y + \frac{1}{2}V$ meets $t\left(\overline{t^{-1}(\frac{1}{2}V)}\right)$. There is therefore a point $x_1 \in \overline{t^{-1}(\frac{1}{2}V)}$ with $t(x_1) \in y + \frac{1}{2}V$. Hence $x_1 \in t^{-1}(\frac{1}{2}V) + U$, and so there is a point $x_2 \in U$ with $x_1 - x_2 \in t^{-1}(\frac{1}{2}V)$. Then

$$t(x_2) \in t(x_1) - \tfrac{1}{2}V \subseteq y + \tfrac{1}{2}V + \tfrac{1}{2}V = y + V,$$

and so $(x_2, t(x_2)) \in G \cap (U, y + V)$. Thus $(0, y) \in \overline{G} = G$, and this implies that $y = 0$.

Next we use the fact that E is barrelled to prove that, for each $V \in \mathcal{V}$, $\widetilde{V}$ is contained in the closure of V under η. Let $y \in \widetilde{V}$; then, for each $W \in \mathcal{V}$, $y + \frac{1}{2}W$ meets $t\left(\overline{t^{-1}(V)}\right)$, so that there is a point $x_1 \in \overline{t^{-1}(V)}$ with $t(x_1) \in y + \frac{1}{2}W$. Now $\overline{t^{-1}(\frac{1}{2}W)}$ is a barrel, and therefore a neighbourhood of the origin in E, so that $x_1 \in t^{-1}(V) + \overline{t^{-1}(\frac{1}{2}W)}$. Thus there is a point $x_2 \in t^{-1}(V)$ with $x_1 - x_2 \in \overline{t^{-1}(\frac{1}{2}W)}$ and then

$$t(x_2) \in t(x_1) - t\left(\overline{t^{-1}(\tfrac{1}{2}W)}\right) \subseteq y + \tfrac{1}{2}W + \tfrac{1}{2}\widetilde{W} \subseteq y + \widetilde{W}.$$

Thus $y + \widetilde{W}$ meets V, and hence y belongs to the η-closure of V.

The conditions of Lemma 1 are now all satisfied, and it follows that ξ and η are identical. Therefore, for each $V \in \mathcal{V}$, $\widetilde{V} = V$ and $\overline{t^{-1}(V)} \subseteq t^{-1}(\widetilde{V}) = t^{-1}(V)$. But $\overline{t^{-1}(V)}$ is a neighbourhood of the origin in E, so that t is continuous, and the theorem is proved.

Suppose that (E_α) is a family of convex spaces, and that, for each α, there is linear mapping u_α of E_α into a vector space E, with $E = \bigcup_\alpha u_\alpha(E_\alpha)$. Then there is a finest convex space topology on E in which each u_α is continuous, and E with this topology is called the inductive limit of the family (E_α) by the linear mappings (u_α). (See, e.g., A. Grothendieck [4], Introduction, IV.) If t is a linear mapping of E into another convex space F, then t is continuous if and only if each mapping $t \circ u_\alpha$ of E_α into F is continuous.

The next theorem extends the result of Grothendieck ([4] Introduction, Théorème B). First we need a lemma (cf. [1], Chapitre III, §1, Proposition 1).

Lemma 2. *If E is a convex space and H a non-meagre vector subspace of E then, under the induced topology, H is barrelled.*

Proof. Let B be a barrel in H. Then $H \subseteq \bigcup_{n=1}^{\infty} n\overline{B}$, so that $\overline{B}$ is a neighbourhood of the origin in E. Hence $\overline{B} \cap H$ is a neighbourhood of the origin in the induced topology on H. But $B = \overline{B} \cap H$, since B is closed in H, and so H is barrelled.

Theorem 2. *Suppose that E is an inductive limit of convex Baire spaces and that F is a separated inductive limit of a sequence of fully complete spaces. If t is a linear mapping of E into F and if the graph of t in $E \times F$ is closed, then t is continuous.*

Proof. Suppose first that E is a convex Baire space and that F is the inductive limit of the sequence (F_n) of fully complete spaces by the linear mappings (v_n). Then

$$E = t^{-1}(F) = t^{-1}\left(\bigcup_{n=1}^{\infty} v_n(F_n)\right) = \bigcup_{n=1}^{\infty} t^{-1}(v_n(F_n)),$$

so that there is some n for which $H = t^{-1}(v_n(F_n))$ is non-meagre and $\overline{H} = E$. For this n write $K = F_n / v_n^{-1}(0)$, and let w be the (1-1) linear mapping of K into F associated with v_n.

Then, since the graph G of t is closed, the graph L of the mapping $w^{-1} \circ t$ of H into K is also closed. For L is the inverse image of G by the continuous mapping $(x, y) \to (x, w(y))$ of $H \times K$ into $E \times F$.

Since H is barrelled by Lemma 2 and K, the quotient of a fully complete space by a closed vector subspace, is fully complete, $w^{-1} \circ t$ is continuous on H by Theorem 1. Now K is complete and H dense in E, so that $w^{-1} \circ t$ has a continuous linear extension s mapping E into K. On H, $w \circ s = t$; we show now that this holds on E.

Suppose not; then, for some $x_1 \in E$, $(x_1, w(s(x_1))) \notin G$. Since G is closed, there is a neighbourhood (U, V) of the origin in $E \times F$ with $(x_1 + U, w(s(x_1)) + V)$ not meeting G. Since $w \circ s$ is continuous, there is a neighbourhood $U_1 \subseteq U$ of the origin in E with $w(s(U_1)) \subseteq V$. Also H is dense in E, and so there is a point $x_2 \in H \cap (x_1 + U_1)$. Then

$$(x_2, t(x_2)) = (x_2, w(s(x_2))) \in (x_1 + U, w(s(x_1)) + V),$$

and this last set does not meet G. This is a contradiction, so that $w \circ s = t$ on E, and therefore t is continuous.

Now consider the general case in which E is the inductive limit of the Baire spaces E_α by the linear mappings u_α. If $t_\alpha = t \circ u_\alpha$, the graph of t_α is the inverse image of the graph of t by the continuous mapping $(x, y) \to (u_\alpha(x), y)$ of $E_\alpha \times F$ into $E \times F$. Thus the graph of t_α is closed, and so, by what has just been proved, t_α is continuous. Hence t is continuous, and this completes the proof of the theorem.

As in Grothendieck's Théorème B, it is clear that replacing the topology of F by a coarser topology does not affect the validity of the results in Theorems 1 and 2.

THEOREM 3 (*Open mapping theorem*). *Suppose that E and F are separated, that either*

(i) *E is fully complete and F barrelled, or*

(ii) *E is an inductive limit of a sequence of fully complete spaces and F is an inductive limit of convex Baire spaces,*

and that t is a continuous linear mapping of E onto F. Then T is open.

Proof. Since t is continuous, $t^{-1}(0)$ is closed. Let $H = E/t^{-1}(0)$. In case (i), H is fully complete. In case (ii), if E is the inductive limit of the sequence (E_n) of fully complete spaces by the linear mappings (u_n), then H is an inductive limit of the sequence of fully complete spaces $H_n = E_n/u_n^{-1}(t^{-1}(0))$.

Then we can write $t = s \circ \phi$, where ϕ is the canonical mapping of E onto H and s is a (1–1) continuous linear mapping of H onto F. The graph of s^{-1}, which is the same as the graph of s, is therefore closed in $F \times H$, and so, by Theorem 1 or 2, s^{-1} is continuous. Hence t is open.

Finally we observe that it follows from the open mapping theorem that fully complete spaces enjoy another property of complete metric spaces :

COROLLARY. *If E is fully complete, or the inductive limit of a sequence of fully complete spaces, and if t is a continuous linear mapping of E into any separated convex space F, then either $t(E)$ is meagre in F or $t(E) = F$.*

Proof. Suppose first that E is fully complete. If $t(E)$ is non-meagre in F, then, by Lemma 2, $t(E)$ is barrelled, and so t is open, by Theorem 3. Hence $t(E)$ is isomorphic with the fully complete space $E/t^{-1}(0)$ and so is fully complete. But $\overline{t(E)} = F$ so that $t(E) = F$.

If E is the inductive limit of the sequence (E_n) by the mappings (u_n), then

$$t(E) = \bigcup_{n=1}^{\infty} t(u_n(E_n)).$$

Hence if $t(E)$ is non-meagre in F, there is some n with $t(u_n(E_n))$ non-meagre in F. By the same argument as before, applied to the mapping $t \circ u_n$, $t(u_n(E_n)) = F$, and so $t(E) = F$.

REFERENCES

1. Bourbaki, N., *Espaces vectoriels topologiques*, Actualités Scientifiques et Industrielles no. 1189, Paris, Hermann, 1953 and no. 1229, Paris, Hermann, 1955.

2. Collins, H. S., Completeness and compactness in linear topological spaces, *Trans. Amer. Math. Soc.*, **79** (1955), 256–280.

3. Dieudonné, J. et Schwartz, L., La dualité dans les espaces ($\mathscr{F}$) et ($\mathscr{L}\mathscr{F}$), *Ann. Inst. Fourier Grenoble* **1** (1950), 61–101.

4. Grothendieck, A., Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16**, 1955.

5. Pták, V., On complete topological linear spaces, *Čehoslovack Mat. Ž.* **3** (78) (1953), 301–364.

THE UNIVERSITY
GLASGOW