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HERMITIANS IN MATRIX ALGEBRAS WITH OPERATOR NORM – II

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Abstract. We continue our investigation of the real space H of Hermitian matrices in $M_n(\mathbb{C})$ with respect to norms on $\mathbb{C}^n$. We complete the commutative case by showing that any proper real subspace of the real diagonal matrices on $\mathbb{C}^n$ can appear as H. For the non-commutative case, we give a complete solution when $n = 3$ and we provide various illustrative examples for $n \geq 4$. We end with a short list of problems.

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1. Introduction. This paper is a continuation of [4]. Standard numerical range notation and basic facts about Hermitians can be found in [2, 3].

Let $X = (\mathbb{C}^n, \|\cdot\|)$ with dual space $X' = (\mathbb{C}^n, \|\cdot\|')$. We denote by H_X, or simply H, the set of matrices in $M_n(\mathbb{C})$ which are Hermitian with respect to the norm $\|\cdot\|$ or, equivalently, with respect to the corresponding operator norm $|\cdot|$ on $M_n(\mathbb{C})$. We write $\dim(H)$ for the real dimension of H. Where several norms are distinguished by subscripts, we shall use the same subscripts for the corresponding sets of Hermitians (and other sets dependent on the norms). Thus, for $\|\cdot\|_1$ and $\|\cdot\|_2$, we write H_1 and H_2, respectively.

A result of Bauer [1, p. 38] is particularly useful. Define an equivalence relation on the set of norms for $\mathbb{C}^n$ as follows. Let norms $\|\cdot\|_1$ and $\|\cdot\|_2$ be *similar* if there exists an (invertible) $L \in M_n(\mathbb{C})$ such that $\|v\|_2 = \|Lv\|_1$ $(v \in \mathbb{C}^n)$. Then, the numerical range of $T \in M_n(\mathbb{C})$ with respect to $\|\cdot\|_1$ is the numerical range of $L^{-1}TL$ with respect to $\|\cdot\|_2$ and, in particular, $T \in H_1$ if and only if $L^{-1}TL \in H_2$. The mapping $T \mapsto L^{-1}TL$ is an isometric isomorphism from $(M_n(\mathbb{C}), |\cdot|_1)$ to $(M_n(\mathbb{C}), |\cdot|_2)$. Since H_1 maps to H_2, $\dim(H_1) = \dim(H_2)$ and $H_1 + iH_1$ is an algebra/C^*-algebra if and only if $H_2 + iH_2$ is an algebra/C^*-algebra. The property of being absolute is not necessarily preserved by similar norms. For example,

$$\|(x, y, z)\|_1 = |x| + |y| + |z| \quad \text{and} \quad \|(x, y, z)\|_2 = |x| + |y + z| + |y - z|$$

define similar norms, but only $\|\cdot\|_1$ is absolute. Examples suggest that in any equivalence class, absolute norms are quite sparse. Indeed, the equivalence class containing the norm in [4, Theorem 3.9] contains no absolute norms.

The paper is divided into five sections. Preliminary results are covered in Section 2, including properties of the similarity relation for norms. In Section 3, we consider the case of H commutative. We answer a question concerning non-absolute norms and prove a non-similarity result on norms which give the same H. Non-commutative H are the subject of Section 4. We give a comprehensive description of the case $n = 3$ and show by examples that for $n = 4$, the situation is much more complicated. For every $n \geq 3$, we show that there is a norm on $\mathbb{C}^n$ giving the minimum dimension of 4. Questions of maximum dimension remain open. We conclude, in Section 5, with a short selection of problems.

NOTES 1.1. We shall make use of the following facts. See [**4**, Lemma 2.1, Theorem 2.3 and Remarks].

(1) A norm on $\mathbb{C}^n$ is absolute if and only if H includes all real diagonal matrices.
(2) If a norm on $\mathbb{C}^n$ is absolute, then $H + iH$ is all diagonal matrices or a direct sum of C^*-algebras.
(3) For any complex Banach space, if H contains A and B with $AB \neq BA$, then I, A, B and $i[A, B] = i(AB - BA)$ are linearly independent (over $\mathbb{C}$ and so also over $\mathbb{R}$). Hence, $\dim(H) \geq 4$.

2. Preliminary results.

Here, we include a number of results most of which will be used in later sections. The first two lemmas are more powerful than [**3**, Lemma 15.2] for inductive arguments.

The following notation is assumed in Lemmas 2.1, 2.3, 2.5 and Corollary 2.4. Let $p = m + n$. For $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ write

$$(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{C}^p \quad \text{and} \quad 0 = (0, \ldots, 0) \text{ (as required).}$$

Let $X = (\mathbb{C}^m, \|\cdot\|_X)$ and $Z = (\mathbb{C}^p, \|\cdot\|_Z)$ with

$$\|x\|_X = \|(x, 0)\|_Z \quad (x \in X).$$

Let $T = [t_{jk}] \in M_p(\mathbb{C})$ have block matrix form $T = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ where $P \in M_m(\mathbb{C})$.

LEMMA 2.1. *Let $\|(x, 0)\|_Z \leq \|(x, y)\|_Z$ for all $(x, y) \in Z$ and let $T \in H_Z$. Then $P \in H_X$.*

Proof. Let $\alpha \in X'$ and let $\gamma = (\alpha, 0) \in Z'$. Then, for $(x, y) \in Z$,

$$|\gamma(x, y)| = |\alpha(x)| \leq \|\alpha\|'_X \|x\|_X = \|\alpha\|'_X \|(x, 0)\|_Z \leq \|\alpha\|'_X \|(x, y)\|_Z$$

so that $\|\gamma\|'_Z \leq \|\alpha\|'_X$. It follows that if $\alpha \in D_X(x)$, then $\gamma \in D_Z(x, 0)$ and

$$\gamma(T(x, 0)) = \begin{bmatrix} \alpha & 0 \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \alpha(Px).$$

So $\alpha(Px) \in \mathbb{R}$ and the result follows. □

NOTE 2.2. The following example illustrates how Lemma 2.1 can be generalised. The same generalisation applies to Lemma 2.3 and Corollary 2.4, below.

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Suppose, in $\mathbb{C}^4$, $\|(w, 0, y, 0)\| \le \|(w, x, y, z)\|$ and $T = [t_{jk}] \in M_4(\mathbb{C})$ is Hermitian. Then, in $\mathbb{C}^2$ with norm defined by $\|(w, y)\| = \|(w, 0, y, 0)\|$, $\begin{bmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{bmatrix}$ is Hermitian.

LEMMA 2.3. *Let $T \in H_Z$ and $R = O$. Then, $P \in H_X$.*

Proof. Let $\alpha \in D_X(x)$. By the Hahn–Banach theorem, α has a norm 1 extension, $\alpha^+ \in Z'$, which must be of the form

$$(\alpha, \beta) = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$$

since $\alpha^+(e_j) = \alpha(e_j) = \alpha_j$ $(j = 1, \ldots, m)$. Then $(\alpha, \beta) \in D_Z(x, 0)$ and

$$(\alpha, \beta)(T(x, 0)) = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} P & Q \\ O & S \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \alpha(Px).$$

So $\alpha(Px) \in \mathbb{R}$ and the result follows. □

It is straightforward to verify that the numerical range of a matrix M with respect to a given norm is equal to the numerical range of its transpose M' with respect to the dual norm. This leads to the following corollary to Lemma 2.3.

COROLLARY 2.4. *Let $T \in H_Z$ and $Q = O$. Then $P \in H_X$.*

Proof. Since $T \in H_Z$, $T' \in H_{Z'}$. So, by Lemma 2.3, $P' \in H_{X'}$ and hence $P \in H_X$. □

Define $\overline{T} = [s_{jk}] \in M_p(\mathbb{C})$ where $s_{jk} = \overline{t_{jk}}$ $(j, k = 1, \ldots, p)$. For $v = (v_1, \ldots, v_p) \in \mathbb{C}^p$ let $\overline{v} = (\overline{v_1}, \ldots, \overline{v_p})$.

LEMMA 2.5. *Let $\|v\|_Z = \|\overline{v}\|_Z$ for all $v \in Z$ and let $T \in H_Z$. Then $\overline{T} \in H_Z$ and it follows easily that*

$$\operatorname{Re} T = \tfrac{1}{2}(T + \overline{T}) \in H_Z \quad \textit{and} \quad i \operatorname{Im} T = \tfrac{1}{2}(T - \overline{T}) \in H_Z.$$

Proof. For $\gamma \in Z'$ and $v \in Z$, $|\overline{\gamma}(\overline{v})| = |\gamma(v)|$. Hence, $\|\overline{\gamma}\|' = \|\gamma\|'$ and $\gamma \in D_Z(v)$ if and only if $\overline{\gamma} \in D_Z(\overline{v})$. It is straightforward to verify that $\gamma(\overline{T}v) = \overline{\gamma}(T\overline{v})$ from which the result follows. □

LEMMA 2.6. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be similar norms for $\mathbb{C}^n$. Then, the corresponding dual norms, $\|\cdot\|_1'$ and $\|\cdot\|_2'$, are also similar.*

Proof. Let $\|v\|_2 = \|Lv\|_1$ $(v \in \mathbb{C}^n)$. We regard vectors v and ϕ in the space and its dual space, respectively, as row or column vectors as dictated by their matrix products with $L \in M_n(\mathbb{C})$. We have

$$|\phi(v)| = |(\phi L^{-1})(Lv)| \le \|\phi L^{-1}\|_1' \|Lv\|_1 = \|\phi L^{-1}\|_1' \|v\|_2$$

so that $\|\phi\|_2' \le \|\phi L^{-1}\|_1'$. If $\|v\|_1 = 1$ and $|(\phi L^{-1})v| = \|\phi L^{-1}\|_1'$ then $\|L^{-1}v\|_2 = 1$ and

$$|\phi(L^{-1}v)| = |(\phi L^{-1})(v)| = \|\phi L^{-1}\|_1'.$$

Hence $\|\phi\|_2' = \|K\phi\|_1'$ where K is the transpose of L^{-1}. □

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PROPOSITION 2.7. *Let $\|\cdot\|$ be a norm for $\mathbb{C}^n$. There is a similar norm $\|\cdot\|_0$ such that, for $v \in \mathbb{C}^n$,*

$$\|v\|_\infty \leq \|v\|_0 \leq \|v\|_1$$

where $\|v\|_\infty$ and $\|v\|_1$ are the ℓ_∞ and ℓ_1 norms, respectively.

We first prove two lemmas. Let $X = (\mathbb{C}^n, \|\cdot\|)$ with $n \geq 2$. Let $\mathcal{S}^+$ be the set of matrices $T = [\phi_{jk}] \in M_n(\mathbb{C})$ where the rows $\phi_j = (\phi_{j1}, \ldots, \phi_{jn})$ $(j = 1, \ldots, n)$ are functionals in X' with $\|\phi_j\|' = 1$, and let $\mathcal{S} = \{T \in \mathcal{S}^+ : \det(T) \neq 0\}$. Then, $\mathcal{S} \neq \emptyset$ (consider ϕ_j, $(j = 1, \ldots, n)$ linearly independent).

Let $\mathcal{D} = \{|\det(T)| : T \in \mathcal{S}\}$. There exists $M > 0$ such that, for all $\theta = (\theta_1, \ldots, \theta_n) \in X'$, $\max_j |\theta_j| \leq M\|\theta\|'$. So, for $T \in \mathcal{S}$, the Leibniz formula for $\det(T)$ gives $|\det(T)| \leq n!M$. Hence, $\mathcal{D}$ is bounded above.

LEMMA 2.8. *For some $T \in \mathcal{S}$, $|\det(T)| = \max \mathcal{D}$.*

Proof. With respect to the operator norm, $\mathcal{S}^+$ is compact and $T \mapsto |\det(T)|$ is continuous. Since $\det(T) = 0$ for $T \in \mathcal{S}^+ \setminus \mathcal{S}$, it follows that $|\det(T)| = \max \mathcal{D}$ for some $T \in \mathcal{S}$. □

For $T \in \mathcal{S}$, the columns of $T^{-1} = [x_{jk}]$ are given by

$$x_k = (x_{1k}, \ldots, x_{nk}) = \frac{1}{\Delta}(c_{1k}, \ldots, c_{nk}), \tag{2.1}$$

where $\Delta = \det(T)$ and $[c_{jk}]$ is the cofactor matrix of T. Observe that each $\det(T)x_k$ is independent of $\phi_{1k}, \ldots, \phi_{nk}$.

LEMMA 2.9. *Let $T = [\phi_{jk}] \in \mathcal{S}$ with $|\det(T)| = \max \mathcal{D}$. Then, each column of $T^{-1} = [x_{jk}]$, regarded as a vector in X, has norm 1.*

Proof. We show that $\|x_1\| = 1$. The proofs for $\|x_k\|$ $(k = 2, \ldots, n)$ are similar. Since $\phi_1(x_1) = 1$, $\|x_1\| \geq 1$. Suppose $\|x_1\| = 1/t > 1$. Then, $tx_1 = y_1 = (y_{11}, \ldots, y_{n1})$ has a support functional $\psi_1 = (\psi_{11}, \ldots, \psi_{1n})$ (say), and $\psi_1, \phi_2, \ldots, \phi_n$ are linearly independent since $\psi_1(x_1) = 1/t$ and $\phi_2(x_1) = \cdots = \phi_n(x_1) = 0$. Let V be T with first row, $(\phi_{11}, \ldots, \phi_{1n})$, replaced with $(\psi_{11}, \ldots, \psi_{1n})$, let $V^{-1} = [z_{jk}]$ and let $\Gamma = \det(V)$. Then, considering (2.1) and the observation following, it follows that

$$z_1 = (z_{11}, \ldots, z_{n1}) = \frac{1}{\Gamma}\Delta(x_{11}, \ldots, x_{n1}) = \frac{\Delta}{\Gamma t}(y_{11}, \ldots, y_{n1}) = \frac{\Delta}{\Gamma t}y_1.$$

We have $\psi_1(z_1) = 1$, and since $\psi \in D(y_1)$, $\psi_1(y_1) = 1$. Hence, $\Delta = \Gamma t$ so that $|\Delta| < |\Gamma|$. This contradicts $|\det(T)| = \max \mathcal{D}$. Hence $\|x_1\| = 1$. □

Proof of Proposition 2.7. With respect to $\|\cdot\|$, let $T \in \mathcal{S}$ with $|\det(T)| = \max \mathcal{D}$. For $v, \theta \in \mathbb{C}^n$, define

$$\|v\|_0 = \|T^{-1}v\| \quad \text{so that} \quad \|\theta\|_0' = \|\theta T\|'.$$

Let $T = [\phi_{jk}]$ and $T^{-1} = [x_{jk}]$, and let $\{e_1, \ldots, e_n\}$ be the usual basis for $\mathbb{C}^n$. Then regarding $e_1, \ldots, e_n$ as vectors in X' and X in turn we have, for $m = 1, \ldots, n$,

$$\|e_m\|_0' = \|\phi_m\|' = 1 \quad \text{and} \quad \|e_m\|_0 = \|x_m\| = 1.$$

Hence, for $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$,

$$\max_m |v_m| = \max_m |e_m(v)| \le \|v\|_0 = \|v_1 e_1 + \cdots + v_n e_n\|_0 \le |v_1| + \cdots + |v_n|$$

and the result follows. □

Note that the norm $\|\cdot\|_0$ in Proposition 2.7 has the property that $\|e_k\|_0 = \|\xi_k\|_0' = 1$ for all the usual basis vectors e_k and ξ_k in the space and its dual space, respectively. We call such a norm *doubly normalised.*

We use Proposition 2.7 to establish a further result along the lines of Lemmas 2.1 and 2.3.

LEMMA 2.10. *Let* $X = (\mathbb{C}^m, \|\cdot\|_1)$, $Y = (\mathbb{C}^n, \|\cdot\|_2)$ *and* $Z = (\mathbb{C}^{m+n}, \|\cdot\|_0)$ *where*

$$\|(x, y)\|_0 = \max\{\|x\|_1, \|y\|_2\}.$$

Let T be the block matrix $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, *where P is* $m \times m$ *and S is* $n \times n$. *Then*

$$T \in H_0 \iff P \in H_1,\ S \in H_2 \text{ and } Q, R \text{ are zero-matrices}.$$

Proof. Part (i). Suppose, first, that both $\|\cdot\|_1$ and $\|\cdot\|_2$ are doubly normalised. Let $\Gamma = (\phi, \psi) \in Z'$. Then

$$\Gamma(z) = \Gamma(x, y) = \Gamma(x, 0) + \Gamma(0, y) = \phi(x) + \psi(y) \quad \text{and} \quad \|\Gamma\|_0' = \|\phi\|_1' + \|\psi\|_2'.$$

For $\|(x, y)\|_0 = 1$,

$$D_0(x, y) = \begin{cases} \{(\phi, 0) : \phi \in D_1(x)\} & \text{if } \|x\|_1 = 1 > \|y\|_2 \ (\textit{Case 1}), \\ \{(0, \psi) : \psi \in D_2(y)\} & \text{if } \|x\|_1 < 1 = \|y\|_2 \ (\textit{Case 2}), \end{cases}$$

and when $\|x\|_1 = \|y\|_2 = 1$

$$D_0(x, y) = \{((1-t)\phi, t\psi) : \phi \in D_1(x),\ \psi \in D_2(y),\ 0 \le t \le 1\} \quad (\textit{Case 3}).$$

Let $U = \begin{bmatrix} P & O \\ O & S \end{bmatrix}$. Then considering $\Gamma(Uz)$, where $\Gamma \in D_0(z)$, in the three cases above, leads to

$$U \in H_0 \iff P \in H_1 \text{ and } S \in H_2. \tag{2.2}$$

Let $T \in H_0$. It remains to show that $P \in H_1$, $S \in H_2$ and Q, R are zero matrices.

Case 1. For $\phi \in D_1(x)$, $(\phi, 0) \in D_0(x, 0)$ and $(\phi, 0)(T(x, 0)) = \phi(Px)$. So $\phi(Px) \in \mathbb{R}$ and hence $P \in H_1$.

Case 2. Similarly, $S \in H_2$.

Case 3. From (2.2), $U \in H_0$ and hence $V = T - U = \begin{bmatrix} O & Q \\ R & O \end{bmatrix} \in H_0$. Taking $t = 0$ gives

$$(\phi, 0) \in D_0(x, y) \quad \text{and hence} \quad (\phi, 0)(V(x, y)) = \phi(Qy) \in \mathbb{R}$$

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for all $\|x\|_1 = \|y\|_2 = 1$ and $\phi \in D_1(x)$. Let $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_m\}$ be the usual bases for Y and X', respectively. Let q_{jk} be the (j, k)-th entry in Q and choose x so that $\xi_j \in D_1(x)$. Then, for any $|\lambda| = 1$,

$$(\xi_j, 0)(V(x, \lambda e_k)) = \lambda\xi_j(Qe_k) = \lambda q_{jk} \in \mathbb{R}.$$

So $q_{jk} = 0$ and hence $Q = O$. Taking $t = 1$, a similar argument gives $R = O$.

Part (ii). Now remove the restriction that $\|\cdot\|_1$ and $\|\cdot\|_2$ are doubly normalised. In view of Proposition 2.7, there are doubly normalised norms $\|\cdot\|_{01}$ and $\|\cdot\|_{02}$ similar to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let $\|x\|_{01} = \|Ax\|_1$ $(x \in \mathbb{C}^m)$ and $\|y\|_{02} = \|By\|_2$ $(y \in \mathbb{C}^n)$. Define $\|(x, y)\|_{00} = \max\{\|x\|_{01}, \|y\|_{02}\}$. Then

$$\|(x, y)\|_{00} = \|C(x, y)\|_0 \quad \text{where} \quad C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

and

$$\begin{aligned} T \in H_0 \quad &\Longleftrightarrow \quad C^{-1}TC \in H_{00} \Longleftrightarrow \begin{bmatrix} A^{-1} & O \\ O & B^{-1} \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in H_{00} \\ &\Longleftrightarrow \quad \begin{bmatrix} A^{-1}PA & A^{-1}QB \\ B^{-1}RA & B^{-1}SB \end{bmatrix} \in H_{00} \\ &\Longleftrightarrow \quad A^{-1}PA \in H_{01},\ B^{-1}SB \in H_{02},\ A^{-1}QB = O,\ B^{-1}RA = O \text{ (by Part (i))} \\ &\Longleftrightarrow \quad P \in H_1,\ S \in H_2,\ Q = O,\ R = O \end{aligned}$$

as required. □

EXAMPLE 2.11. Let $\mathbb{C}^p = \mathbb{C}^{m+n}$ have norm defined for $(z, w) = (z_1, \ldots, z_m, w_1, \ldots, w_n)$ by

$$\|(z, w)\| = \max\{|z_r|, |z_s + z_t|, |w_u| : r, s, t = 1, \ldots, m,\ s < t,\ u = 1, \ldots, n\}.$$

Then, H consists of all real diagonal matrices of the form $\begin{bmatrix} rI & O \\ O & \Delta \end{bmatrix}$ where I is $m \times m$ and Δ is $n \times n$, and H has real dimension $n + 1$. See [4, Theorem 3.2].

3. Commutative *H*.

In [4, Section 3], we noted that when H is commutative, we may suppose, via a similarity change of norm on $\mathbb{C}^n$, that H is a real subspace of the real diagonal matrices on $\mathbb{C}^n$. We asked if the converse is true.

THEOREM 3.1. *Let K be any proper real subspace of the real diagonal matrices on $\mathbb{C}^n$ with $I \in K$. Then, there exists a non-absolute norm on $\mathbb{C}^n$ for which $H = K$.*

Proof. Suppose that $\dim(K) = m$ (thus $m < n$). Choose any basis for K which contains I. Perform the usual row operations on these diagonals to form a triangular basis. If necessary, exchange the order of the usual basis in $\mathbb{R}^n$ to arrive at a basis for H given by I and the diagonal matrices D_j with entries:

$$(0, 1, a_{23}, a_{24}, \ldots, a_{2n}), (0, 0, 1, a_{34}, \ldots, a_{3n}), \ldots, (0, 0, \ldots, 0, 1, a_{mm+1}, \ldots, a_{mn}) \tag{3.1}$$

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with all $a_{jk} \in \mathbb{R}$. Let $w = (w_1, \dots, w_n)$ with each $w_j \in \mathbb{T}$, and let Ω_n denote the set of nth roots of unity. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define

$$\|z\| = \max\{|Lz| : w \in \mathbb{T}^n, \gamma_j \in \Omega_n\}, \tag{3.2}$$

where $L = L_1 + L_2$ and

$$L_1 z = w_1\gamma_1 z_1 + w_1 w_2 \gamma_2 z_2 + w_1 w_2^{a_{23}} w_3 \gamma_3 z_3 + \cdots + w_1 w_2^{a_{2m}} w w_3^{a_{3m}} \dots w_m \gamma_m z_m,$$

$$L_2 z = w_1 w_2^{a_{2\,m+1}} w_3^{a_{3\,m+1}} \dots w_m^{a_{m\,m+1}} \gamma_{m+1} z_{m+1} + \cdots + w_1 w_2^{a_{2n}} w_3^{a_{3n}} \dots w_m^{a_{mn}} \gamma_n z_n.$$

It is clear that $\|\cdot\|$ is a seminorm on $\mathbb{C}_n$. We complete the proof with four lemmas. □

LEMMA 3.2. *The seminorm* $\|\cdot\|$ *given by* (3.2) *is a norm.*

Proof. Suppose $\|z\| \neq 0$. Taking a scalar multiple, we may assume that some $z_k = 1$ and that $|z_j| \leq 1$ $(j \neq k)$. Now take all $w_j = 1$ and $\gamma_k = 1$. Choose the remaining γ_j so that $\mathrm{Re}(\gamma_j z_j) \geq 1$. Then $\mathrm{Re}(Lz) \geq 1$, and hence $\|z\| \neq 0$. □

LEMMA 3.3. *Each diagonal matrix* D_j *is Hermitian.*

Proof. For $t \in \mathbb{R}$, we have

$$\exp(itD_j) = \mathrm{diag}(1, 1, \dots, 1, \exp(it), \exp(ia_{j\,j+1}t), \dots, \exp(ia_{jn}t)).$$

In the formula for $\|\exp(itD_j)z\|$, each exponential in the entries for $\exp(itD_j)$ is absorbed into $w_{j+k}^{a_{j\,j+k}}$ and hence the norm is unchanged as $\|z\|$ and D_j is Hermitian. □

LEMMA 3.4. *Let* T *be a Hermitian matrix with respect to* $\|\cdot\|$. *Then,* T *is a real diagonal matrix.*

Proof. For $\gamma \in \Omega_n$, let ϕ be the functional given by

$$\phi_\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n).$$

Clearly $\|\phi_\gamma\| \leq 1$, and for each j, we have $\gamma_j^{-1}\phi_\gamma(e_j) = 1$. Hence $\|\gamma_j^{-1}\phi_\gamma\| = 1$ and so $\gamma_j^{-1}\phi_\gamma(Te_j) = 1 \in \mathbb{R}$. Thus

$$\gamma_j^{-1}\gamma_1 t_{1j} + \gamma_j^{-1}\gamma_2 t_{2j} + t_{jj} + \gamma_j^{-1}\gamma_n t_{nj} \in \mathbb{R} \tag{3.3}$$

for all choices of γ. We may choose values of γ_1 so that, for $j \neq 1$, $\gamma_j^{-1}\gamma_1$ takes the values $1, \omega, \dots, \omega^{n-1}$ where $\omega = \exp(2\pi i/n)$, and similarly for each other term in (3.3). Add to give $nt_{jj} \in \mathbb{R}$. Now choose values of γ_1 so that, for $j \neq 1$, every value of $\gamma_j^{-1}\gamma_1$ is the fixed value ω^ν and all the other values are as above. Add again to give $\omega^\nu t_{1j} \in \mathbb{R}$ for all ν, and so $t_{1j} = 0$. Similarly for all the other terms. Hence, T is a real diagonal matrix. □

LEMMA 3.5. *The only real diagonal Hermitians with respect to* $\|\cdot\|$ *are determined by the basis* (3.1).

Proof. Suppose, towards a contradiction, that there is a further real diagonal Hermitian Δ with respect to $\|\cdot\|$. By row operations as before, we may suppose that Δ has zeros in the first m positions and entry 1 in some position N with $m < N \leq n$. Let X be the subspace of $\mathbb{C}^n$ spanned by $e_1, \dots, e_m, e_N$. By [3, Lemma 15.2] (or by Lemma 2.2), $\Delta|_X$ is Hermitian and also for the restriction of every other basis element in (3.1). For X with the induced norm, the space of diagonal Hermitians has full dimension and so by [4, Lemma 2.1] the

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norm on X is absolute. We show that this is false. Let all entries of $\xi \in X$ be 1, so that $\|\xi\| = m + 1$, and let all entries of η be 1 except for the last one which is $\exp(i\theta)$. The norm formula for η has only $m + 1$ terms in it. We may choose $w_2, \ldots, w_m$ successively so that the first m terms each line up as 1, but for infinitely many choices of θ, the last value fails to line up, and hence the norm is not absolute. □

Note in passing that the norm given by (3.2) is an extension of the ℓ_1 norm on $\mathbb{C}^m$.

The next proposition shows that there are uncountably many different similarity equivalence classes on $\mathbb{C}^2$ with H equal to all real diagonal matrices.

PROPOSITION 3.6. *Let $p, q \in [1, \infty]$ and let $\|\cdot\|_p$ and $\|\cdot\|_q$ denote, respectively, the ℓ_p and ℓ_q norm on $\mathbb{C}^2$. If $p \neq q$, then $\|\cdot\|_p$ and $\|\cdot\|_q$ are not similar.*

Proof. Write $p \sim q$ when $\|\cdot\|_p$ and $\|\cdot\|_q$ are similar. We show that $p \sim q$ implies $p = q$. Suppose first that $p, q \in [1, \infty)$ with $q \neq 2$, and $p \sim q$. Then, there exists (invertible) $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$ such that, for $(x, y) \in \mathbb{C}^2$, $\|(x, y)\|_p = \|T(x, y)\|_q$. Thus

$$(|ax + by|^q + |cx + dy|^q)^{1/q} = (|x|^p + |y|^p)^{1/p}. \tag{3.4}$$

Putting $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$ gives

$$|a|^q + |c|^q = |b|^q + |d|^q = 1 \tag{3.5}$$

and putting $(x, y) = (1, \lambda) = (1, e^{it})$ gives

$$F(t) = |a + \lambda b|^q + |c + \lambda d|^q = 2^{q/p}. \tag{3.6}$$

Differentiating (3.6) with respect to t gives

$$F'(t) = -q[|a + \lambda b|^{q-2} \operatorname{Im}(\lambda \bar{a} b) + |c + \lambda d|^{q-2} \operatorname{Im}(\lambda \bar{c} d)] = 0 \tag{3.7}$$

for all $|\lambda| = 1$ with $|a + \lambda b|$ and $|c + \lambda d|$ non-zero.

Case 1. Suppose $\bar{a}b$ and $\bar{c}d$ are non-zero. We can choose μ with $|\mu| = 1$ such that $|a \pm \mu b|$, $|c \pm \mu d|$, $\operatorname{Re}(\mu \bar{a} b)$, $\operatorname{Im}(\mu \bar{c} d)$ are all non-zero. Putting $\lambda = \pm\mu$ in (3.7) gives

$$|a \pm \mu b|^{q-2} \frac{\operatorname{Im}(\mu \bar{a} b)}{\operatorname{Im}(\mu \bar{c} d)} + |c \pm \mu d|^{q-2} = 0 \quad \text{so that} \quad |c \pm \mu d| = R|a \pm \mu b| \quad \text{(say)}.$$

Then, by (3.6), $(1 + R^q)|a \pm \mu b|^q = 2^{q/p}$. Hence, $|a + \mu b| = |a - \mu b|$ so that $\operatorname{Re}(\mu \bar{a} b) = 0$ contradicting the choice of μ.

Case 2. Suppose $\bar{a}b = 0$ and $\bar{c}d \neq 0$. [$\bar{a}b \neq 0$ and $\bar{c}d = 0$ is similar.] Then, by (3.7), $|c + \lambda d| = 0$ for almost all $|\lambda| = 1$. Hence, $c = d = 0$ contradicting $\bar{c}d \neq 0$.

Case 3. Suppose $\bar{a}b = 0$ and $\bar{c}d = 0$. By (3.5), we cannot have $a = c = 0$ or $b = d = 0$. If $a = d = 0$, then (3.5) gives $|b| = |c| = 1$ and (3.4) then gives $\|(x, y)\|_p = \|(x, y)\|_q$. So $p = q$. Similarly if $b = d = 0$.

The above arguments show that $p \not\sim q$ when $p, q \in [1, \infty)$ with $q \neq 2$ and $p \neq q$. For such p and q, the ordered pair (p, q) is indicated on the table, below, by •. By Lemma 2.6, if $p \not\sim q$, then $p' \not\sim q'$ where $pp' = p + p'$ and $qq' = q + q'$. Each such further

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(p', q') is indicated by $\circ$. And if $p \not\sim q$, then $q \not\sim p$. Such further (q, p) are each indicated by $*$.

$q \backslash p =$	1	u	2	v	∞
1		$\bullet$	$\bullet$	$\bullet$	
u	$\bullet$		$\bullet$	$\bullet$	$\circ$
2	$*$	$*$		$*$	$*$
v	$\bullet$	$\bullet$	$\bullet$		$\circ$
∞		$\circ$	$\circ$	$\circ$	

$(1 < u < 2 < v < \infty)$

To complete the proof, we show that $\infty \not\sim 1$. Let $p = \infty$ and $q = 1$. Then, (3.4), (3.5), (3.6) and (3.7) remain valid provided we substitute $\max\{|x|, |y|\}$ for $(|x|^p + |y|^p)^{1/p}$ in (3.4) and 1 for $2^{q/p}$ in (3.6). The arguments in *Case 1* (with 1 for $2^{q/p}$) and *Case 2* give the same contradictions as before. *Case 3* leads to the contradiction $2 = 1$. □

4. Non-Commutative *H*.

In this section, we investigate the problem of finding non-absolute norms on $\mathbb{C}^n$ for which H is non-commutative. It is clear from the Remarks at the end of [4] that there are no such norms on $\mathbb{C}^2$. In [4, Theorem 3.9], we gave such a norm on $\mathbb{C}^3$ and we prove below that all H for such norms on $\mathbb{C}^3$ are isometrically isomorphic as real Lie algebras. But first, we generalise that example to $\mathbb{C}^n$ for $n \geq 4$.

For $j, k = 1, \ldots, n$, let E_{jk} be the usual elementary matrices in $M_n(\mathbb{C})$, and for $j < k$ define $G_{jk} = i(E_{jk} - E_{kj})$. This notation is used at several points in this section for different values of n.

THEOREM 4.1. *Let $\mathbb{C}^n$ have norm defined by*

$$\|(z_1, \ldots, z_n)\|_0 = \max\left\{\left|\sum_{j=1}^n \alpha_j z_j\right| : \alpha_j \in \mathbb{R}, \sum_{j=1}^n \alpha_j^2 = 1\right\}.$$

Then, H has a basis consisting of I and every G_{jk}. Hence, $\dim(H) = 1 + \frac{1}{2}n(n-1)$.

Proof. Since $\exp(itG_{jk}) = I + i(\sin t)G_{jk} + (\cos t - 1)P_{jk}$ where $P_{jk} = E_{jj} + E_{kk}$, it follows that each $G_{jk} \in H$. For the subsequent arguments in [4], we simply expand each 3-vector to an n-vector by inserting zeros arbitrarily. Finally, we note the linear independence of I and the G_{jk} since the matrices have no non-zero overlapping entries. □

Next, we consider norms on $\mathbb{C}^3$ and investigate the dimension and algebraic structure of H when H is non-commutative and, in particular, when $H + iH$ is not C^*.

THEOREM 4.2. *Let $\mathbb{C}^3$ have a norm for which H is non-commutative.*

(1) *If the norm on $\mathbb{C}^3$ is similar to an absolute norm, then* $\dim(H) \geq 5$.
(2) *If the norm on $\mathbb{C}^3$ is not similar to any absolute norm, then* $\dim(H) = 4$ *and all such H are isometrically isomorphic as real Lie algebras.*

Note that [4, Theorem 3.9] shows that there does exist a norm on $\mathbb{C}^3$ such that H is non-commutative and $\dim(H) = 4$.

Proof. (1) Let the norm on $\mathbb{C}^3$ be similar to an absolute norm. Then, $H + iH$ is a C^*-algebra and since H is non-commutative, it is the direct sum of full matrix algebras. Since

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$I \in H$ the isomorphic split into a direct sum cannot be $M_2(\mathbb{C}) \oplus \{0\}$. Hence, $\dim(H) \geq 5$.

(2) Let A and B be non-commuting Hermitians. So, the set $\mathcal{S} = \{I, A, B, i[A, B]\}$ is, or can be extended to, a basis for H. Then, A has eigenvalues $\alpha_1 \geq \alpha_2 \geq \alpha_3$ with $\alpha_1 > \alpha_3$ (otherwise $AB = BA$). Replacing A in the basis with $(\alpha_1 - \alpha_3)^{-1}(A - \alpha_3 I)$, we may assume that A has eigenvalues $1 \geq t \geq 0$. Then, we may replace the given norm with a similar norm under which $P^{-1}\mathcal{S}P = \{I, D, E, i[D, E]\}$ where D is the diagonal matrix diag$(1, t, 0)$.

The remainder of the proof takes the form of a sequence of lemmas. Here, we set out the steps involved.

We show first, in Lemma 4.3, that if $t \in (0, 1)$ and $t \neq \frac{1}{2}$, then the norm is absolute. Then, in Lemma 4.4, we show that when $t = \frac{1}{2}$ and E has a particular form, again the norm is absolute. This is then used in Lemma 4.5 to show that if $t = 0$ or $t = 1$, the norm is absolute. Thus, we reach a position where the norm can only be non-absolute if $t = \frac{1}{2}$. Under these restrictions, Lemma 4.6 gives $\dim(H) = 4$. Finally, Lemma 4.7 shows (by comparing with H_0 in [4, Theorem 3.9]) that all H with $\dim(H) = 4$ are isometrically isomorphic as real Lie algebras. □

We shall assume, in Lemmas 4.3–4.7, that D and E are as defined in (2), above, with $t \in [0, 1]$. Note that since A and B do not commute neither do D and E.

LEMMA 4.3. *Let $t \in (0, 1)$, $t \neq \frac{1}{2}$. Then, the norm on $\mathbb{C}^3$ is absolute.*

Proof. Let

$$E = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$$

be Hermitian. Let $E_1 = i[D, E]$ and $E_n = i[D, E_{n-1}]$ $(n = 2, 3, \ldots)$. Then, writing $s = 1 - t$, we have, for $n = 4m$,

$$E_n = \begin{bmatrix} 0 & bs^n & c \\ ds^n & 0 & gt^n \\ h & jt^n & 0 \end{bmatrix} \quad \to \quad U = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ h & 0 & 0 \end{bmatrix} \in H \quad \text{as } n \to \infty$$

and, for $n = 4m + 1$,

$$E_n = i\begin{bmatrix} 0 & bs^n & c \\ -ds^n & 0 & gt^n \\ -h & -jt^n & 0 \end{bmatrix} \quad \to \quad V = \begin{bmatrix} 0 & 0 & ic \\ 0 & 0 & 0 \\ -ih & 0 & 0 \end{bmatrix} \in H \quad \text{as } n \to \infty.$$

Hence

$$i[U, V] = 2ch\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 2chW \quad \text{(say)}$$

is Hermitian. If $ch \neq 0$ then, since $t \neq \frac{1}{2}$, I, D and W are linearly independent diagonal matrices. So, all diagonal matrices are in H, and hence the norm on $\mathbb{C}^3$ must be absolute.

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Suppose now that $ch = 0$. Then, $c = h = 0$ since otherwise U would be a non-zero nilpotent contradicting $U \in H$. If $s < t$ then, writing $u = s/t$, we have, for $n = 4m$,

$$(1/t^n)E_n = \begin{bmatrix} 0 & bu^n & 0 \\ du^n & 0 & g \\ 0 & j & 0 \end{bmatrix} \rightarrow U' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & j & 0 \end{bmatrix} \in H \quad \text{as } n \rightarrow \infty$$

and, for $n = 4m + 1$,

$$(1/t^n)E_n = i\begin{bmatrix} 0 & bu^n & 0 \\ -du^n & 0 & g \\ 0 & -j & 0 \end{bmatrix} \rightarrow V' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & ig \\ 0 & -ij & 0 \end{bmatrix} \in H \quad \text{as } n \rightarrow \infty.$$

Hence

$$i[U', V'] = 2gj\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 2gjW' \quad \text{(say)}$$

is Hermitian. If $dg \neq 0$ then I, D and W' are linearly independent, since $t \neq 2$, and again the norm must be absolute.

If $t < s$, a similar argument forces an absolute norm if $bd \neq 0$. That leaves us with the case where $bd = ch = gj = 0$. Using the above nilpotent argument, all of b, d, c, h, g, j must be 0 so that $E = \text{diag}(a, f, k)$ which commutes with D, a contradiction. □

The values of $t \in [0, 1]$ not covered by Lemma 4.3 are 0, $\frac{1}{2}$ and 1. Of these $t = 0$ and $t = 1$ are equivalent. The next lemma shows that $t = \frac{1}{2}$ may also imply that the norm is absolute.

LEMMA 4.4. *Let $t = \frac{1}{2}$ and let*

$$E = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ h & 0 & 0 \end{bmatrix} \quad \textit{with } ch > 0.$$

Then, the norm on $\mathbb{C}^3$ is absolute.

Proof. It is enough to show that every real diagonal matrix is Hermitian. Define Hermitian matrices

$$D' = 2D - I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad F = i[D, E] = \begin{bmatrix} 0 & 0 & ic \\ 0 & 0 & 0 \\ -ih & 0 & 0 \end{bmatrix}.$$

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Since $ch > 0$, $\overline{c}/|c| = h/|h| = \lambda$ (say) and hence

$$(\mathrm{Re}\,\lambda)E + (\mathrm{Im}\,\lambda)F = \begin{bmatrix} 0 & 0 & \lambda c \\ 0 & 0 & 0 \\ \overline{\lambda} h & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & |c| \\ 0 & 0 & 0 \\ |h| & 0 & 0 \end{bmatrix} \in H.$$

So, without loss, we shall assume that $c = 1$ and $h > 0$. Let $\phi = (\alpha, \beta, \gamma)$ be a support functional for $v = (x, y, z)$. Then, $\phi(v) = 1$ and $\phi(D'v)$, $\phi(Ev)$, $\phi(Fv) \in \mathbb{R}$ so that

$$\alpha x + \beta y + \gamma z = 1, \tag{4.1}$$
$$\alpha x - \gamma z \in \mathbb{R}, \tag{4.2}$$
$$\alpha z + \gamma hx \in \mathbb{R}, \tag{4.3}$$
$$\alpha z - \gamma hx \in i\mathbb{R}. \tag{4.4}$$

From (4.3) and (4.4), we have

$$\alpha z = \overline{\gamma hx} = p \text{ (say) and } \gamma hx = \overline{p}. \tag{4.5}$$

Case 1. Let αz or γx be non-zero. It follows from (4.5) that both are non-zero and hence that α, z, γ, x and p are non-zero. Then from (4.5), $\alpha = p/z$ and $\gamma = \overline{p}/(hx)$. Substituting in (4.2) gives $px/z - \overline{p}z/(hx) \in \mathbb{R}$ and hence

$$(px/z) - r(\overline{px/z}) \in \mathbb{R} \quad \text{where} \quad r = (1/h)|z/x|^2 > 0.$$

It follows that $px/z \in \mathbb{R}$. Then, (4.5), (4.2) and (4.1) give, in turn, αx, γz, $\beta y \in \mathbb{R}$. Hence, for any real $T = \mathrm{diag}(a, f, k)$, we have

$$\phi(Tv) = \alpha ax + \beta fy + \gamma kz \in \mathbb{R} \tag{4.6}$$

so that T is Hermitian.

Case 2. Let $\alpha z = \gamma x = 0$. If $\alpha = \gamma = 0$ or $x = z = 0$, then (4.1) gives $by \in \mathbb{R}$. If $\alpha = x = 0$, then (4.2) gives $\gamma z \in \mathbb{R}$ and (4.1) gives $\beta y \in \mathbb{R}$. Similarly, if $\gamma = z = 0$, then αx, $\beta y \in \mathbb{R}$. So again (4.6) holds for any real $T = \mathrm{diag}(a, f, k)$, and the proof is complete. □

LEMMA 4.5. *Let $t = 0$ or $t = 1$. Then, there is a similar absolute norm on $\mathbb{C}^3$.*

Proof. We prove the case $t = 1$. Let

$$E = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}.$$

Then

$$F = i[D, E] = \begin{bmatrix} 0 & 0 & ic \\ 0 & 0 & ig \\ -ih & -ij & 0 \end{bmatrix} \quad \text{and} \quad G = -i[D, F] = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & g \\ h & j & 0 \end{bmatrix} \in H.$$

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The eigenvalues of G are $0, \pm\sqrt{ch+gj}$ so that $ch+gj \geq 0$. If $ch+gj=0$ then, since G is Hermitian, $G=O$, and

$$E = \begin{bmatrix} a & b & 0 \\ d & f & 0 \\ 0 & 0 & k \end{bmatrix}$$

commutes with D contradicting the given hypotheses. So, without loss, suppose that $ch+gj=1$. With

$$P = \begin{bmatrix} c & -j & c \\ g & h & g \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} h/2 & j/2 & 1/2 \\ -g & c & 0 \\ h/2 & j/2 & -1/2 \end{bmatrix}$$

we have

$$U = P^{-1}GP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad V = P^{-1}FP = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}.$$

So there is a similar norm such that $\frac{1}{2}(I+U) = \mathrm{diag}(1, \frac{1}{2}, 0)$ and V satisfy the conditions of Lemma 4.4. □

LEMMA 4.6. *Let the norm on* $\mathbb{C}^3$ *be non-absolute, and let* $t = \frac{1}{2}$. *Then* $\dim(H) = 4$.

Proof. Let

$$E = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}.$$

Under the given hypotheses, at least one of b, c, d, g, h, j is non-zero. We show that H is spanned by I, D, F, G where F and G are defined below. Let $E_1 = i[D, E]$ and $E_n = i[D, E_{n-1}]$ $(n = 2, 3, \ldots)$. Then, for $n = 4m$,

$$E_n = \begin{bmatrix} 0 & b/2^n & c \\ d/2^n & 0 & g/2^n \\ h & j/2^n & 0 \end{bmatrix} \quad \to \quad U = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ h & 0 & 0 \end{bmatrix} \in H \quad \text{as } n \to \infty$$

and, for $n = 4m+1$,

$$E_n = i\begin{bmatrix} 0 & b/2^n & c \\ -d/2^n & 0 & g/2^n \\ -h & -j/2^n & 0 \end{bmatrix} \quad \to \quad V = \begin{bmatrix} 0 & 0 & ic \\ 0 & 0 & 0 \\ -ih & 0 & 0 \end{bmatrix} \in H \quad \text{as } n \to \infty.$$

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From $E_4 - U$ and $E_5 - V$, we deduce that

$$F = \begin{bmatrix} 0 & b & 0 \\ d & 0 & g \\ 0 & j & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & ib & 0 \\ -id & 0 & ig \\ 0 & -ij & 0 \end{bmatrix}$$

are Hermitian. Hence

$$E_0 = E - U - F = \begin{bmatrix} a & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & k \end{bmatrix} \quad \text{and} \quad i[F, G] = 2 \begin{bmatrix} bd & 0 & 0 \\ 0 & gj - bd & 0 \\ 0 & 0 & -gj \end{bmatrix}$$

are Hermitian. Considering eigenvalues of E_0, U and $i[F, G]$, it follows that

$$a, f, k, bd, gj \in \mathbb{R} \quad \text{and} \quad ch \geq 0. \tag{4.7}$$

Since the norm is non-absolute, diagonal Hermitians must be linear combinations of I and D so that

$$a + k = 2f \quad \text{and} \quad bd = gj. \tag{4.8}$$

Since D and U are in H and the norm is not absolute, Lemma 4.4 implies that $ch = 0$. If $c = 0$ and $h \neq 0$ (or vice versa), then U is a non-zero nilpotent. So

$$c = h = 0. \tag{4.9}$$

This implies that $F \neq O$ since otherwise E is diagonal and commutes with D. Note that (4.7), (4.8) and (4.9) hold for any E satisfying the given hypotheses. Let

$$E' = \begin{bmatrix} a' & b' & c' \\ d' & f' & g' \\ h' & j' & k' \end{bmatrix} \in H \quad \text{and} \quad F' = \begin{bmatrix} 0 & b' & 0 \\ d' & 0 & g' \\ 0 & j' & 0 \end{bmatrix}$$

with E' not a linear combination of I and D. Then, D and E' do not commute, otherwise all real diagonal matrices would be Hermitian and the norm would be absolute. Hence

$$K = i[F, F'] = i \begin{bmatrix} bd' - db' & 0 & bg' - gb' \\ 0 & db' - bd' + gj' - jg' & 0 \\ jd' - dj' & 0 & jg' - gj' \end{bmatrix} \in H.$$

If $DK \neq KD$, then from (4.9)

$$bg' = gb' \quad \text{and} \quad jd' = dj' \tag{4.10}$$

and if $DK = KD$, then (4.10) remains true since

$$DK - KD = i \begin{bmatrix} 0 & 0 & bg' - gb' \\ 0 & 0 & 0 \\ dj' - jd' & 0 & 0 \end{bmatrix}.$$

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Taking real linear combinations of F and G, it follows that, for all $\lambda \in \mathbb{C}$,

$$C_\lambda = \begin{bmatrix} 0 & \lambda b & 0 \\ \bar{\lambda} d & 0 & \lambda g \\ 0 & \bar{\lambda} j & 0 \end{bmatrix}$$

is Hermitian. At least one of b and g is non-zero, otherwise F is a non-zero nilpotent. Suppose $g \neq 0$ (the other case is similar). Let $\lambda = g'/g$. Then

$$C_\lambda - F' = \begin{bmatrix} 0 & 0 & 0 \\ \bar{\lambda} d - d' & 0 & 0 \\ 0 & \bar{\lambda} j - j' & 0 \end{bmatrix}$$

which is a non-zero nilpotent unless $\bar{\lambda} d = d'$ and $\bar{\lambda} j = j'$. So $F' = C_\lambda$ and H is spanned by I, D, F, G. □

LEMMA 4.7. *Let the norm on $\mathbb{C}^3$ be non-absolute, let $t = \frac{1}{2}$ and let H_0 be the space of Hermitians with respect to $\|\cdot\|_0$ in* [4, Theorem 3.9]. *Then, H and H_0 are isometrically isomorphic as real Lie algebras.*

Proof. From the proof of Lemma 4.6, H has a basis $\{I, D, F, i[D, F]\}$ where

$$F = \begin{bmatrix} 0 & b & 0 \\ d & 0 & g \\ 0 & j & 0 \end{bmatrix}$$

with $bd = jg = r$ (say). Considering eigenvalues, $r > 0$. So, in the basis, we can replace D with $D' = 2D - I$ and

$$F = \begin{bmatrix} 0 & b & 0 \\ r/b & 0 & r/j \\ 0 & j & 0 \end{bmatrix} \quad \text{with} \quad F' = \frac{-1}{\sqrt{2r}} F.$$

Let

$$L = \begin{bmatrix} ib/2j & -b/2j & 0 \\ 0 & 0 & (\sqrt{r/2})/j \\ i/2 & 1/2 & 0 \end{bmatrix} \quad \text{so that} \quad L^{-1} = \begin{bmatrix} -ij/b & 0 & -i \\ -j/b & 0 & 1 \\ 0 & (\sqrt{2/r})j & 0 \end{bmatrix}.$$

Then

$$L^{-1} D' L = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X, \qquad L^{-1} F' L = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} = Y$$

and $\{I, X, Y, i[X, Y]\}$ is a basis for H_0. It follows that H and H_0 are isomorphic as Lie algebras under the mapping $T \mapsto L^{-1} T L$. Since the mapping preserves eigenvalues, the operator norms of $T \in H$ and $L^{-1} T L \in H_0$ are equal. □

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This completes the proof of Theorem 4.2. We give one further result for norms satisfying conditions (2) of the theorem.

PROPOSITION 4.8. *Let $\mathbb{C}^3$ have a norm which is not similar to any absolute norm, and let H be non-commutative.*

(1) $\mathrm{alg}(H)$*, the algebra generated by H, is $M_3(\mathbb{C})$.*
(2) *Let $G \in H$. Then $G^2 \in H$ if and only if G is a real multiple of the identity.*

Proof. It is enough to prove the results for $H = H_0$, the Hermitians corresponding to $\|\cdot\|_0$ of [4, Theorem 3.9].

(1) For $\{j, k, m\} = \{1, 2, 3\}$ with $j < k$, we have $G_{jk} \in H_0$ and

$$G_{jk}^2 = I - E_{mm}, \quad E_{jj}G_{jk} = iE_{jk}, \quad G_{jk}E_{jj} = -iE_{kj}.$$

It follows that, for all $j, k = 1, 2, 3$, $E_{jk} \in \mathrm{alg}(H_0)$ and hence $\mathrm{alg}(H_0) = M_3(\mathbb{C})$.

(2) ($\Rightarrow$) Let $G \in H_0$. Since $\{I, G_{12}, G_{13}, G_{23}\}$ is a basis for H_0 (the basis given in [4, Theorem 3.9]), $G = \delta I + K$ where $\delta \in \mathbb{R}$ and

$$K = \alpha G_{12} + \beta G_{13} + \gamma G_{23} \ \text{ with } \alpha, \beta, \gamma \in \mathbb{R}.$$

Then

$$K^2 = \begin{bmatrix} \alpha^2 + \beta^2 & \beta\gamma & -\alpha\gamma \\ \beta\gamma & \alpha^2 + \gamma^2 & \alpha\beta \\ -\alpha\gamma & \alpha\beta & \beta^2 + \gamma^2 \end{bmatrix} \in H_0.$$

Comparing K^2 with a real linear combination of I, G_{12}, G_{13} and G_{23} gives

$$\alpha\beta = \alpha\gamma = \beta\gamma = 0, \tag{4.11}$$

$$\alpha^2 + \beta^2 = \alpha^2 + \gamma^2 = \beta^2 + \gamma^2. \tag{4.12}$$

From (4.11), at least two of α, β, γ are 0 and then, from (4.12), $\alpha = \beta = \gamma = 0$. Hence $K = O$ and $G = \delta I$.

(2) ($\Leftarrow$) Clear. □

As was mentioned in Note 1.1 (3), for any complex Banach space where H is non-commutative, $\dim(H) \geq 4$. The example in [4, Theorem 3.9] shows that $\dim(H) = 4$ can be achieved by a norm on $\mathbb{C}^n$ with $n = 3$. We show next that modified versions of this norm achieve this minimum dimension for any $n \geq 4$.

LEMMA 4.9. *Define norms $\|\cdot\|_1$ and $\|\cdot\|_2$ for $\mathbb{C}^2$ by*

$$\|(x, y)\|_1 = \max\{|x|, \tfrac{1}{2}|x + y|, \tfrac{1}{2}|x - y|\} \quad \textit{and} \quad \|(x, y)\|_2 = \max\{|x|, |y|, |x + y|\}.$$

Then $H_1 = H_2 = \{rI : r \in \mathbb{R}\}$.

Proof. The norms are similar since $\|(x, y)\|_1 = \|P(x, y)\|_2$ where $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. And $H_2 = \{rI : r \in \mathbb{R}\}$ by [4, Lemma 3.1]. □

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PROPOSITION 4.10. *Let $\mathbb{C}^4$ have norm defined by*

$$\|(x, y, z, w)\| = \max\{|ax + by + cz|, \tfrac{1}{2}|ax + by + cz + w| : (a, b, c) \in S\}$$

where $S = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$. Then, $\{I, G_{12}, G_{13}, G_{23}\}$ is a basis for H.

Proof. We make use of $\|\cdot\|_1$ from Lemma 4.9. The matrices I, G_{12}, G_{13} and G_{23} are linearly independent. They are also in H. For example, with $t \in \mathbb{R}$, $\mathcal{C} = \cos t$ and $\mathcal{S} = \sin t$,

$$\|\exp(itG_{12})(x, y, z, w)\| = \|(\mathcal{C}x - \mathcal{S}y, \mathcal{C}y + \mathcal{S}x, z, w)\| = \|(x, y, z, w)\|$$

since, for $(a, b, c) \in S$,

$$a(\mathcal{C}x - \mathcal{S}y) + b(\mathcal{C}y + \mathcal{S}x) + cz + dw = (a\mathcal{C} + b\mathcal{S})x + (b\mathcal{C} - a\mathcal{S})y + cz + dw$$

and $(a\mathcal{C} + b\mathcal{S})^2 + (b\mathcal{C} - a\mathcal{S})^2 + c^2 = a^2 + b^2 + c^2$. We have

$$\begin{aligned}\|(x, 0, 0, w)\| &= \max\{|x|, \tfrac{1}{2}|ax + w| : a \in \mathbb{R}, |a| \le 1\} \\ &= \max\{|x|, \tfrac{1}{2}|x + w|, \tfrac{1}{2}|x - w|\} = \|(x, w)\|_1 \\ &\le \|(x, y, z, w)\|.\end{aligned}$$

For the inequality, use $(\pm 1, 0, 0) \in S$ in the definition of $\|(x, y, z, w)\|$. Similar arguments apply to $\|(0, y, 0, w)\|$ and $\|(0, 0, z, w)\|$. It follows from Lemma 2.1 and Note 2.2 that any $T \in H$ must be of the form

$$\begin{bmatrix} r & * & * & 0 \\ * & r & * & 0 \\ * & * & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \text{where } r \in \mathbb{R}.$$

We also have $\|(x, y, z, 0)\| = \|(x, y, z)\|_0$ where $\|\cdot\|_0$ is the norm for $\mathbb{C}^3$ in [**4**, Theorem 3.9]. So, it follows from Lemma 2.3 that the top-left 3×3 submatrix of T must be a real linear combination of the basis matrices in [**4**, Theorem 3.9], and the proof is complete. □

COROLLARY 4.11. *Let $\mathbb{C}^n$ $(n \ge 5)$ have norm defined, for $v = (x, y, z, w_4, \dots, w_n)$, by*

$$\|v\| = \max\{|ax + by + cz|, \tfrac{1}{2}|ax + by + cz + w_j| : (a, b, c) \in S, j = 4, \dots, n\}.$$

Then, $\{I, G_{12}, G_{13}, G_{23}\}$ is a basis for H.

Proof. Let $T \in H$. Arguing as in the proof of Proposition 4.10, it follows from Lemma 2.1 that T must be of block matrix form $\begin{bmatrix} U & O \\ O & rI \end{bmatrix}$ where U is 3×3 with diagonal (r, r, r), $r \in \mathbb{R}$. Then, Lemma 2.3 completes the proof. □

Our final example is a norm $\|\cdot\|$ on $\mathbb{C}^4$ for which H is non-commutative with $\dim(H) = 5$. The first formulation allows for easy verification that $\|\cdot\|$ is a norm. For $v = (w, x, y, z) \in \mathbb{C}^4$, define

$$\|v\| = \max\{|\lambda\mu w + \nu x + \mu y + \lambda\nu z| : |\lambda| = 1, |\mu|^2 + |\nu|^2 = 1\}. \tag{4.13}$$

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Writing

$$\lambda\mu w + \nu x + \mu y + \lambda\nu z = \alpha\mu + \beta\nu \quad \text{(say)}$$

and applying Cauchy's inequality followed by setting

$$\mu = \frac{\overline{\alpha}}{\sqrt{|\alpha|^2 + |\beta|^2}} \quad \text{and} \quad \nu = \frac{\overline{\beta}}{\sqrt{|\alpha|^2 + |\beta|^2}} \quad (\alpha, \beta \text{ not both } 0)$$

yields

$$\|v\| = \max_{|\lambda|=1} \sqrt{|\lambda w + y|^2 + |x + \lambda z|^2} \tag{4.14}$$

and this translates easily to

$$\|v\| = \sqrt{|w|^2 + |x|^2 + |y|^2 + |z|^2 + 2|w\bar{y} + \bar{x}z|} \tag{4.15}$$

which allows easier computation of particular norm values.

Let P, Q, R, S be, respectively, the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$

Clearly I, P, Q, R, S are linearly independent. We show that they form a basis for H. Note that for the Lie algebra structure, we have

$$i[P, Q] = i[P, R] = i[P, S] = 0, \quad \text{and} \quad i[Q, R] = 2S,\ i[R, S] = 2Q,\ i[S, Q] = 2R.$$

LEMMA 4.12. *P, Q, R, S are Hermitian with respect to $\|\cdot\|$.*

Proof. Let T be any of P, Q, R, S. Then, $T^2 = I$ so that $\exp(itT) = \cos t\, I + i \sin t\, T$. We have $\exp(itP) = \text{diag}(e^{it}, e^{-it}, e^{-it}, e^{it})$ and hence

$$\|\exp(itP)v\|^2 = |w|^2 + |x|^2 + |y|^2 + |z|^2 + 2|e^{it}we^{it}\bar{y} + e^{it}\bar{x}e^{it}z| = \|v\|^2.$$

Thus $P \in H$, and similarly $Q \in H$. Next, we have

$$\exp(itR) = \begin{bmatrix} \cos t & 0 & 0 & i\sin t \\ 0 & \cos t & i\sin t & 0 \\ 0 & i\sin t & \cos t & 0 \\ i\sin t & 0 & 0 & \cos t \end{bmatrix}.$$

So, writing $\mathcal{C} = \cos t$ and $\mathcal{S} = \sin t$,

$$\begin{aligned} \|\exp(itR)v\|^2 &= |\mathcal{C}w + i\mathcal{S}z|^2 + |\mathcal{C}x + i\mathcal{S}y|^2 + |\mathcal{C}y + i\mathcal{S}x|^2 + |\mathcal{C}z + i\mathcal{S}w|^2 \\ &\quad + 2|(\mathcal{C}w + i\mathcal{S}z)(\mathcal{C}\bar{y} - i\mathcal{S}\bar{x}) + (\mathcal{C}\bar{x} - i\mathcal{S}\bar{y})(\mathcal{C}z + i\mathcal{S}w)| \\ &= |w|^2 + |x|^2 + |y|^2 + |z|^2 + 2|(\mathcal{C}^2 + \mathcal{S}^2)(w\bar{y} + \bar{x}z)| = \|v\|^2. \end{aligned}$$

Thus $R \in H$, and similarly $S \in H$. □

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Proposition 4.13. *The matrices I, P, Q, R, S form a basis for H.*

Proof. It remains to prove that I, P, Q, R, S span H. Let $T = [t_{jk}] \in H$. Since

$$\|(w, x, 0, 0)\| = \sqrt{|w|^2 + |x|^2} \leq \|(w, x, y, z)\|$$

it follows from Lemma 2.1 that

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \quad \text{is } \ell_2 \text{ Hermitian}$$

so that $t_{11}, t_{22} \in \mathbb{R}$ and $t_{21} = \overline{t_{12}}$. In view of Note 2.2, we can similarly compare $\|(0, 0, y, z)\|$, $\|(w, 0, 0, z)\|$ and $\|(0, x, y, 0)\|$ with $\|(w, x, y, z)\|$ to show that $t_{33}, t_{44} \in \mathbb{R}$ and $t_{43} = \overline{t_{34}}$, $t_{41} = \overline{t_{14}}$, $t_{32} = \overline{t_{23}}$. We also have

$$\|(w, 0, y, 0)\| = |w| + |y| \leq \|(w, x, y, z)\|$$

so that

$$\begin{bmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{bmatrix} \quad \text{is } \ell_1 \text{ Hermitian}$$

giving $t_{13} = t_{31} = 0$. And comparing $\|(0, x, 0, z)\|$ with $\|(w, x, y, z)\|$ similarly gives $t_{24} = t_{42} = 0$. Thus, we can write

$$T = \begin{bmatrix} a & b & 0 & d \\ \overline{b} & f & g & 0 \\ 0 & \overline{g} & m & n \\ \overline{d} & 0 & \overline{n} & s \end{bmatrix} \quad \text{(say) where } a, f, m, s \in \mathbb{R}. \tag{4.16}$$

It is straightforward to verify that $\phi = (\alpha, \beta, \gamma, \delta)$ is a support functional for $u = (w, x, y, z)$ if and only if $\psi = (\alpha, -\beta, \gamma, -\delta)$ is a support functional for $v = (w, -x, y, -z)$. Since $\phi(Tu) = \psi(Uv)$ where

$$U = \begin{bmatrix} a & -b & 0 & -d \\ -\overline{b} & f & -g & 0 \\ 0 & -\overline{g} & m & -n \\ -\overline{d} & 0 & -\overline{n} & s \end{bmatrix},$$

it follows that $U \in H$. Let $V = \frac{1}{2}(T + U)$ and $W = \frac{1}{2}(T - U)$. Then

$$V = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & b & 0 & d \\ \overline{b} & 0 & g & 0 \\ 0 & \overline{g} & 0 & n \\ \overline{d} & 0 & \overline{n} & 0 \end{bmatrix} \quad \text{are in } H.$$

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Since the norm is not absolute, it follows from Note 1.1 (1) that V must be a real linear combination of I, P, Q. It then follows that

$$a+f=m+s \tag{4.17}$$

and this property must hold for the diagonal of any Hermitian matrix. The Hermitian matrices $i[W, R]$ and $i[W, S]$ are, respectively,

$$i\begin{bmatrix} d-\overline{d} & 0 & b-\overline{n} & 0 \\ 0 & g-\overline{g} & 0 & \overline{b}-n \\ n-\overline{b} & 0 & \overline{g}-g & 0 \\ 0 & \overline{n}-b & 0 & \overline{d}-d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d+\overline{d} & 0 & b+\overline{n} & 0 \\ 0 & -g-\overline{g} & 0 & -\overline{b}-n \\ n+\overline{b} & 0 & \overline{g}+g & 0 \\ 0 & -\overline{n}-b & 0 & -\overline{d}-d \end{bmatrix}.$$

Applying (4.17) gives

$$(d-\overline{d})+(g-\overline{g})=(\overline{g}-g)+(\overline{d}-d) \quad \text{and} \quad (d+\overline{d})+(-g-\overline{g})=(\overline{g}+g)+(-\overline{d}-d).$$

Hence $g=\overline{d}$. And comparing with the form of matrix in (4.16), $b-\overline{n}=b+\overline{n}=0$ which gives $b=n=0$. Hence $W=(\operatorname{Re} d)R+(\operatorname{Im} d)S$ and since $T=V+W$, it follows that T is a real linear combination of I, P, Q, R and S, as required. □

For the above norm on $\mathbb{C}^4$, it is straightforward to check that $\operatorname{alg}(H)$ comprises all diagonal and anti-diagonal matrices in $M_4(\mathbb{C})$. It follows that $\operatorname{alg}(H)$ is isomorphic to a direct sum of two copies of $M_2(\mathbb{C})$.

5. Problems. Where it appears, H is the real space of Hermitians in $M_n(\mathbb{C})$ for some norm $\|\cdot\|$ on $\mathbb{C}^n$.

1. Let A be the complex subalgebra of $M_n(\mathbb{C})$ generated by H. Is A always semisimple (and hence isomorphic to a direct sum of full matrix algebras)?
2. Does Proposition 3.6 generalise to $\ell_p(\mathbb{C}^n)$ with $n \geq 3$?
3. The norm $\|\cdot\|_0$ defined in [4, Theorem 3.9] and the dual norm $\|\cdot\|_0'$ have the same set of Hermitians. Are the two norms similar?
4. Find a short proof of Theorem 4.2.
5. For given n, what is the maximal dimension of H when H is not commutative and $\|\cdot\|$ is not similar to any absolute norm?
6. For $n \geq 4$ and H not commutative, are there only finitely many equivalence classes of similar norms?
7. Let K be any real Lie subalgebra of $M_n(\mathbb{C})$ with Lie product given by $i(AB-BA)$, let $I \in K$, and let all matrices in K be diagonable with real eigenvalues. Is there always a norm on $\mathbb{C}^n$ with $H=K$? [The commutative case is given by Theorem 3.1.]

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