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ORDERING PROPERTIES OF ORDER STATISTICS FROM HETEROGENEOUS POPULATIONS: A REVIEW WITH AN EMPHASIS ON SOME RECENT DEVELOPMENTS

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In this paper, we review some recent results on the stochastic comparison of order statistics and related statistics from independent and heterogeneous proportional hazard rates models, gamma variables, geometric variables, and negative binomial variables. We highlight the close connections that exist between some classical stochastic orders and majorization-type orders.

1. INTRODUCTION

Order statistics and related statistics have received considerable attention in the literature as they play an important role in many areas including statistical inference, goodness-of-fit tests, reliability theory, economics, and operations research. Let $X_{i:n}$ denote the ith order statistic arising from independent random variables $X_1, \ldots, X_n$ having possibly different probability distributions. A lot of work has been done in the literature on order statistics for the case when the underlying variables are independent and identically distributed (i.i.d.). Due to the complexity of the distribution theory in the case when samples are non-i.i.d., only limited results are available in the literature; see, for example, David and Nagaraja [15], Balakrishnan and Rao [4,5], and Balakrishnan [3] for a comprehensive discussion on order statistics arising from independent and non-identically distributed (i.ni.d.) random variables.

In this review paper, we discuss the existing results placing special emphasis to recent developments on stochastic comparisons of order statistics from various samples. Incidentally, Kochar [27] and Boland, Shaked, and Shanthikumar [11], Boland, Hu, and Shaked [10],

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Khaledi and Kochar [25], and Kochar and Xu [30] have all presented reviews on this topic earlier up to 1998, 2002, and 2007, respectively. In Section 2, we focus on stochastic comparisons of order statistics and sample ranges from proportional hazard rates (PHR) models. Sections 3 and 4 are devoted to stochastic comparisons of order statistics from gamma, geometric, and negative binomial samples.

We first recall some notions of stochastic orders, and majorization and related orders which are most pertinent to the discussions in this paper. Throughout the paper, the term *increasing* is used for *monotone non-decreasing* and similarly *decreasing* is used for *monotone non-increasing.*

1.1. Stochastic Orders

Definition 1.1: *For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be the corresponding survival functions. Then:*

(i) *X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in x;*
(ii) *X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x;*
(iii) *X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$) if $F_Y(x)/F_X(x)$ is increasing in x;*
(iv) *X is said to be smaller than Y in the stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\overline{F}_Y(x) \geq \overline{F}_X(x)$;*
(v) *X is said to be smaller than Y in the increasing convex order (denoted by $X \leq_{\mathrm{icx}} Y$) if $\int_t^\infty \overline{F}_X(x)\mathrm{d}x \leq \int_t^\infty \overline{F}_Y(x)\mathrm{d}x$ for all t;*
(vi) *X is said to be smaller than Y in the mean residual life order (denoted by $X \leq_{\mathrm{mrl}} Y$) if $\mathsf{E}X_t \leq \mathsf{E}Y_t$, where $X_t = (X - t | X > t)$ is the residual life at age $t > 0$ of the random lifetime X.*

Definition 1.2: *The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be larger than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ (denoted by $\mathbf{X} \overset{\mathrm{st}}{\succeq} \mathbf{Y}$) in the multivariate stochastic order if*

$$\mathsf{E}[\phi(\mathbf{X})] \geq \mathsf{E}[\phi(\mathbf{Y})]$$

for all increasing functions ϕ. It is well known that multivariate stochastic order implies component-wise stochastic order. For elaborate details on various stochastic orders, one may refer to Shaked and Shanthikumar [42] and Müller and Stoyan [38].

One of the basic criteria for comparing variability in probability distributions is the *dispersive* order.

Definition 1.3: *A random variable X is said to be less dispersed than another random variable Y (denoted by $X \leq_{\mathrm{disp}} Y$) if*

$$F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$$

for $0 \leq u \leq v \leq 1$, where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G of X and Y, respectively.

A weaker variability order, called the *excess wealth* order, is defined as below.

Definition 1.4: *X is said to have less excess wealth than Y (denoted by $X \leq_{\mathrm{ew}} Y$) if*

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(t)\mathrm{d}t \leq \int_{G^{-1}(p)}^{\infty} \overline{G}(t)\mathrm{d}t, \quad 0 \leq p \leq 1.$$

Then, the following implications are well known:

$$X \leq_{\mathrm{disp}} Y \Longrightarrow X \leq_{\mathrm{ew}} Y \Longrightarrow \mathrm{Var}(X) \leq \mathrm{Var}(Y).$$

Definition 1.5: *X is said to be smaller than Y in the convex transform order (denoted by $X \leq_{\mathrm{c}} Y$) if $G^{-1}F(x)$ is convex in x on the support of X.*

Definition 1.6: *X is said to be smaller than Y in the star order (denoted by $X \leq_{\star} Y$) if $G^{-1}F(x)/x$ is increasing in x on the support of X.*

The convex transform order as well as the star order are scale invariant. The star order is also called the more IFRA (increasing failure rate in average) order in reliability theory. It is known from Marshall and Olkin [36] that

$$X \leq_{\star} Y \Longrightarrow \mathrm{cv}(X) \leq \mathrm{cv}(Y),$$

where $\mathrm{cv}(X) = \sqrt{\mathrm{Var}(X)}/\mathsf{E}(X)$ is the coefficient of variation of X. Detailed discussions on these two orders can be found in Barlow and Proschan [8] and Marshall and Olkin [36].

1.2. Majorization and Related Orders

We will use the notion of majorization in our discussion as it is quite useful to establish inequalities. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 1.7:

(i) *A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n$ is said to majorize another vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re^n$ (written as $\mathbf{x} \overset{\mathrm{m}}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) *A vector $\mathbf{x} \in \Re^n$ is said to weakly supermajorize another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(iii) *A vector $\mathbf{x} \in \Re^n$ is said to weakly submajorize another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \succeq_{\mathrm{w}} \mathbf{y}$) if*

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(iv) *A vector* $\mathbf{x} \in \Re_+^n$ *is said to be p-larger than another vector* $\mathbf{y} \in \Re_+^n$ *(written as* $\mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y}$*) if*

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

(v) *A vector* $\mathbf{x} \in \Re_+^n$ *is said to reciprocal majorize another vector* $\mathbf{y} \in \Re_+^n$ *(written as* $\mathbf{x} \overset{\mathrm{rm}}{\succeq} \mathbf{y}$*) if*

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \ge \sum_{i=1}^{j} \frac{1}{y_{(i)}} \quad \text{for } j = 1, \ldots, n.$$

Those functions that preserve the majorization ordering are said to be Schur-convex. Evidently, $\mathbf{x} \overset{\mathrm{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$, and $\mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y}$ is equivalent to $\log(\mathbf{x}) \overset{\mathrm{w}}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ is the vector of logarithms of the coordinates of $\mathbf{x}$. It is known that

$$\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\mathrm{p}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\mathrm{rm}}{\succeq} \mathbf{y}$$

for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$. For more details on majorization, p-larger and reciprocal majorization orders and their applications, one may refer to Marshall, Olkin, and Arnold [35], Bon and Păltănea [12], and Zhao and Balakrishnan [49].

2. PHR CASE

Independent random variables $X_1, \ldots, X_n$ are said to follow the PHR model if, for $i = 1, \ldots, n$, the survival function of X_i can be expressed as

$$\overline{F}_i(x) = [\overline{F}(x)]^{\lambda_i},$$

where $\overline{F}(x)$ is the survival function of some baseline random variable X. If $r(t)$ denotes the hazard rate function of the baseline distribution F, then the survival function of X_i is given by

$$\overline{F}_i(x) = \exp\{-\lambda_i R(x)\}$$

for $i = 1, \ldots, n$, where $R(x) = \int_0^x r(t)\mathrm{d}t$ is the cumulative hazard rate of X. Many well-known distributions are special cases of the PHR model such as exponential, Weibull, Pareto, and Lomax distributions. A classic example of such a situation is when the components have independent exponential distributions with respective hazard rates $(\lambda_1, \ldots, \lambda_n)$. In reliability engineering and system security, it is of great interest to study the effect on the survival function, the hazard rate function and other characteristics of order statistics when the vector $(\lambda_1, \ldots, \lambda_n)$ gets changed to another vector $(\lambda_1^*, \ldots, \lambda_n^*)$.

2.1. Comparisons Between two Heterogeneous PHR Samples

Pledger and Proschan [40] were the first time to deal with this problem, and they established the following result.

Theorem 2.1: *Let* $(X_1, \ldots, X_n)$ *be a vector of independent random variables with proportional hazard rate vector* $(\lambda_1, \ldots, \lambda_n)$*, and* $(X_1^*, \ldots, X_n^*)$ *be another vector of independent random variables with proportional hazard rate vector* $(\lambda_1^*, \ldots, \lambda_n^*)$*. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{k:n} \ge_{\mathrm{st}} X_{k:n}^*. \quad \textbf{(2.1)}$$

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Proschan and Sethuraman [41] strengthened this result from componentwise stochastic order to multivariate stochastic order, that is, under the same setup as in Theorem 2.1, they proved that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{st}}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*). \tag{2.2}$$

In the case of Weibull distributions with common shape parameter α and scale parameter vectors $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots, \lambda_n^*)$, it follows immediately from (2.2) that

$$(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \overset{\mathrm{m}}{\succeq} ((\lambda_1^*)^\alpha, \ldots, (\lambda_n^*)^\alpha) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{st}}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*). \tag{2.3}$$

Khaledi and Kochar [26] also provided a similar result in the Weibull case as follows.

THEOREM 2.2: *Let $(X_1, \ldots, X_n)$ be a vector of independent Weibull random variables with common shape parameter $\alpha \le 1$ and scale parameter vector $(\lambda_1, \ldots, \lambda_n)$, and $(X_1^*, \ldots, X_n^*)$ be another vector of independent Weibull random variables with common shape parameter α and scale parameter vector $(\lambda_1^*, \ldots, \lambda_n^*)$. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{st}}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*).$$

Khaledi and Kochar [26] also compared the smallest order statistics from heterogeneous Weibull samples and obtained the following stronger results:

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{1:n} \ge_{\mathrm{hr}} X_{1:n}^* \quad \text{for } 0 < \alpha \le 1$$

and

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{1:n} \le_{\mathrm{hr}} X_{1:n}^* \quad \text{for } \alpha \ge 1.$$

For the exponential case, Khaledi and Kochar [23] partially improved the ordering property in (2.1) by weakening the sufficient condition as

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \ge_{\mathrm{st}} X_{n:n}^*, \tag{2.4}$$

while Khaledi and Kochar [25] extended the result in (2.4) from the exponential case to the PHR model, but they also showed there by means of a counterexample that the result in (2.4) may not hold for other order statistics. Moreover, as asserted in Kochar and Xu [30], the result in (2.4) cannot be strengthened to the hazard rate order or the reversed hazard rate order. Dykstra, Kochar, and Rojo [17] proved, in the exponential framework, that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \ge_{\mathrm{rh}} X_{n:n}^*. \tag{2.5}$$

With the help of a counterexample, Boland, EL-Neweihi, and Proschan [9] showed that (2.1) cannot be strengthened from the usual stochastic order to the hazard rate order;

but for the case when $n = 2$, they established, in the exponential framework, that

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{hr}} X_{2:2}^*. \quad \textbf{(2.6)}$$

Dykstra et al. [17] further improved (2.6) from the hazard rate order to the likelihood ratio order as

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{lr}} X_{2:2}^*. \quad \textbf{(2.7)}$$

Joo and Mi [21] gave some sufficient conditions under which the hazard rate order in (2.6) holds. Specifically, they proved, under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, that

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\text{hr}} X_{2:2}^*. \quad \textbf{(2.8)}$$

Zhao and Balakrishnan [50] covered all the results in (2.6)–(2.8) and established the following two equivalent characterizations.

Theorem 2.3: *Let (X_1, X_2) be a vector of independent exponential random variables with respective hazard rates λ_1 and λ_2, and (X_1^*, X_2^*) be another vector of independent exponential random variables with respective hazard rates λ_1^* and λ_2^*. Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Then,*

(i)

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\text{lr}} [\geq_{\text{rh}}]\, X_{2:2}^*; \quad \textbf{(2.9)}$$

(ii)

$$(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longleftrightarrow X_{2:2} \geq_{\text{hr}} [\geq_{\text{st}}]\, X_{2:2}^*. \quad \textbf{(2.10)}$$

As an immediate consequence of Theorem 2.3, we have the following corollary.

Corollary 2.1: *Let (X_1, X_2) be a vector of independent exponential random variables with respective hazard rates λ_1 and λ_2, and (X_1^*, X_2^*) be another vector of independent exponential random variables with common hazard rate λ. Suppose $\lambda \leq \max(\lambda_1, \lambda_2)$. Then,*

(i) $\lambda \geq \frac{\lambda_1 + \lambda_2}{2} \Longleftrightarrow X_{2:2} \geq_{\text{lr}} [\geq_{\text{rh}}] X_{2:2}^*$;
(ii) $\lambda \geq \sqrt{\lambda_1 \lambda_2} \Longleftrightarrow X_{2:2} \geq_{\text{hr}} [\geq_{\text{st}}] X_{2:2}^*$.

Remark 2.1: In fact, the result in Corollary 2.1 is valid without the assumption that $\lambda \leq \max(\lambda_1, \lambda_2)$. Let $Z_\lambda\, [Z_\mu]$ be the second order statistic of a random sample of size 2 from an exponential distribution with common hazard rate $\lambda\, [\mu]$. Assume $\lambda < \mu$. We then have $Z_\lambda \geq_{\text{lr}} Z_\mu$ from Theorem 1.C.33 of Shaked and Shanthikumar [42]. Based on this fact, it can be concluded that the result in Corollary 2.1 is also valid for the case when $\lambda > \max(\lambda_1, \lambda_2)$.

Remark 2.2: Theorem 2.4 of Joo and Mi [21] stated the condition in Part (i) of Theorem 2.3 as a sufficient condition for the hazard rate order, but they left the case $\lambda_1 + \lambda_2 > \lambda_1^* + \lambda_2^*$ as an open question. It can be seen that Part (ii) of Theorem 2.3 provides a complete answer to this problem since it establishes the characterization condition $\lambda_1 \lambda_2 \leq \lambda_1^* \lambda_2^*$ for the hazard rate order which does include the condition $\lambda_1 + \lambda_2 \leq \lambda_1^* + \lambda_2^*$ as a special case. For example, let $(\lambda_1, \lambda_2) = (2, 7)$ and $(\lambda_1^*, \lambda_2^*) = (3, 5)$. We then have $\lambda_1 \lambda_2 \leq \lambda_1^* \lambda_2^*$ and $\lambda_1 + \lambda_2 > \lambda_1^* + \lambda_2^*$, and in this case we see from Figure 1 that $X_{2:2} \geq_{\text{hr}} X_{2:2}^*$.

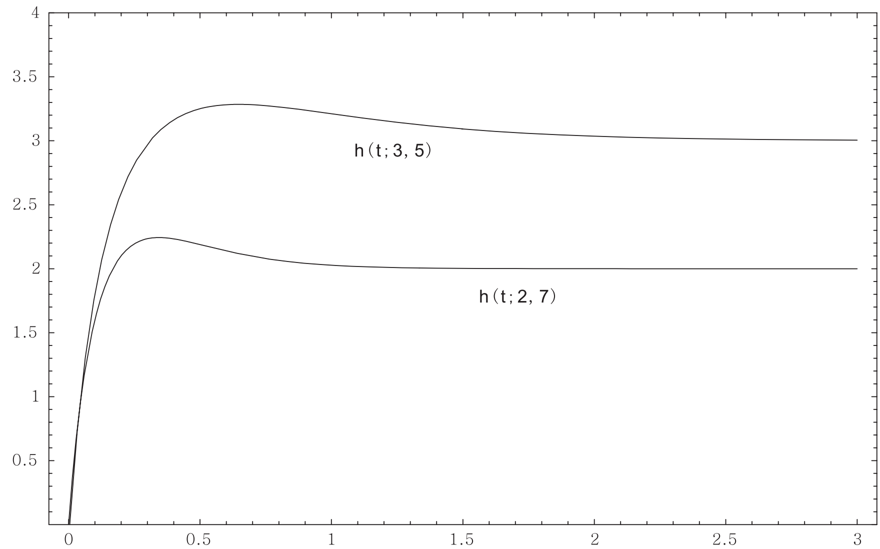

FIGURE 1. Plot of the hazard rate functions of two lifetimes of parallel systems with exponential units, where $h(t;a,b)$ denotes the hazard rate function for the case with exponential parameter vector (a,b).

The results in (2.9) and (2.10) can also be extended from the exponential case to the PHR model as follows.

THEOREM 2.4: *Let* (X_1, X_2) *be a vector of independent random variables with respective survival functions* $\overline{F}^{\lambda_1}$ *and* $\overline{F}^{\lambda_2}$*, and* (X_1^*, X_2^*) *be another vector of independent random variables with respective survival functions* $\overline{F}^{\lambda_1^*}$ *and* $\overline{F}^{\lambda_2^*}$*. Suppose* $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$*. Then,*

(i)

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{lr}} X_{2:2}^*;$$

(ii)

$$(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{hr}} X_{2:2}^*.$$

Zhao and Balakrishnan [51] established the following result for the mean residual life order.

THEOREM 2.5: *Let* (X_1, X_2) *be a vector of independent exponential random variables with respective hazard rates* λ_1 *and* λ_2*, and* (X_1^*, X_2^*) *be another vector of independent exponential random variables with respective hazard rates* λ_1^* *and* λ_2^**. Suppose* $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$*. Then,*

$$(\lambda_1, \lambda_2) \overset{\text{rm}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{mrl}} X_{2:2}^*.$$

Remark 2.3: If $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, Theorem 2.3 states that $\lambda_1 + \lambda_2 \le \lambda_1^* + \lambda_2^*$ and $\lambda_1\lambda_2 \le \lambda_1^*\lambda_2^*$ are necessary and sufficient conditions for $X_{2:2} \ge_{\text{lr}} X_{2:2}^*$ and $X_{2:2} \ge_{\text{hr}} X_{2:2}^*$, respectively. It can be seen that Theorem 2.5 provides a sufficient condition for the mean residual life order which includes the condition $\lambda_1\lambda_2 \le \lambda_1^*\lambda_2^*$ as a special case since

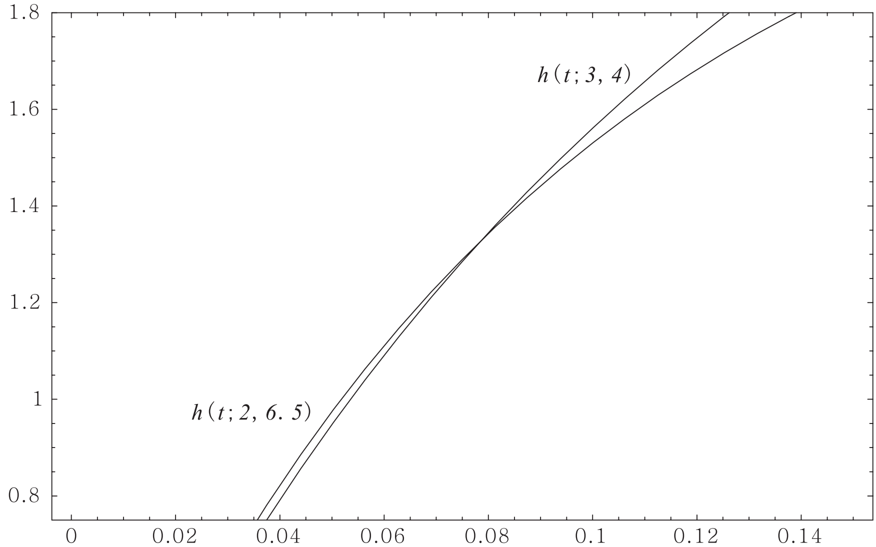

Figure 2. Plot of the hazard rate functions of two lifetimes of parallel systems with exponential units, where $h(t; a, b)$ denotes the hazard rate function for the case with exponential parameter vector (a, b).

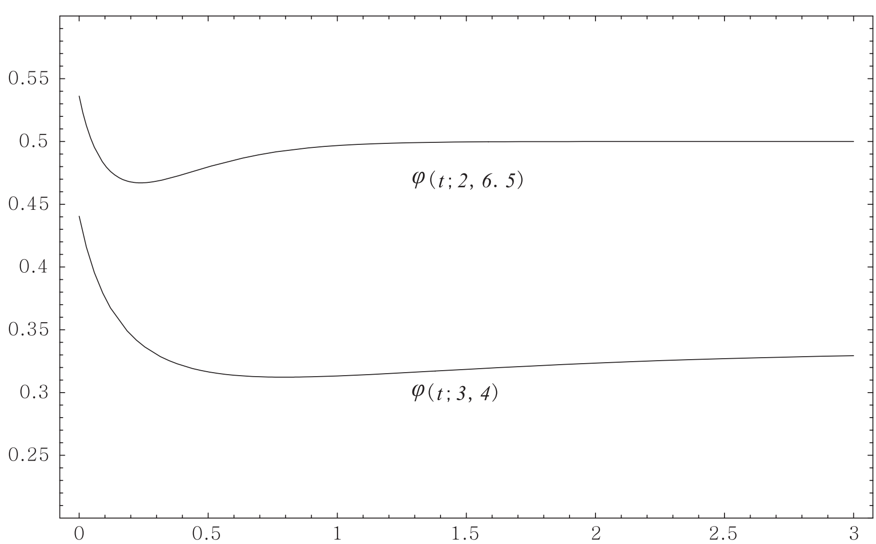

Figure 3. Plot of the mean residual life functions of two lifetimes of parallel systems with exponential units, where $\varphi(t; a, b)$ denotes the mean residual life function for the case with exponential parameter vector (a, b).

$(\lambda_1, \lambda_2) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies $(\lambda_1, \lambda_2) \overset{\text{rm}}{\succeq} (\lambda_1^*, \lambda_2^*)$. For example, let $(\lambda_1, \lambda_2) = (2, 6.5)$ and $(\lambda_1^*, \lambda_2^*) = (3, 4)$. Then, we have $\lambda_1 + \lambda_2 > \lambda_1^* + \lambda_2^*$, $\lambda_1\lambda_2 > \lambda_1^*\lambda_2^*$, and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} > \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}$, and while $X_{2:2} \geq_{\text{hr}} X_{2:2}^*$ does not hold, $X_{2:2} \geq_{\text{mrl}} X_{2:2}^*$ does hold, as can be seen in Figures 2 and 3.

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Also, the following result is an immediate consequence of Theorem 2.5.

COROLLARY 2.2: *Let (X_1, X_2) be a vector of independent exponential random variables with respective hazard rates λ_1 and λ_2, and (X_1^*, X_2^*) be another vector of independent exponential random variables with common hazard rate λ. Then,*

$$\lambda \geq \frac{2}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} \Longleftrightarrow X_{2:2} \geq_{\mathrm{mrl}} X_{2:2}^*.$$

The following result extends Theorem 2.5 to the PHR model.

THEOREM 2.6: *Let (X_1, X_2) be a vector of independent random variables with respective survival functions $\overline{F}^{\lambda_1}$ and $\overline{F}^{\lambda_2}$, and (X_1^*, X_2^*) be another vector of independent random variables with respective survival functions $\overline{F}^{\lambda_1^*}$ and $\overline{F}^{\lambda_2^*}$. Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Then, if the baseline distribution F is decreasing failure rate (DFR), we have*

$$(\lambda_1, \lambda_2) \overset{\mathrm{rm}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{mrl}} X_{2:2}^*.$$

Zhao and Balakrishnan [55] stochastically compared the variability between the maxima in terms of the dispersive order and the excess wealth order, and established the following result.

THEOREM 2.7: *Let (X_1, X_2) be a vector of independent random variables with respective survival functions $\overline{F}^{\lambda_1}$ and $\overline{F}^{\lambda_2}$, and (X_1^*, X_2^*) be another vector of independent random variables with respective survival functions $\overline{F}^{\lambda_1^*}$ and $\overline{F}^{\lambda_2^*}$. Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. If the baseline distribution F is DFR, then*

(i)

$$(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{disp}} X_{2:2}^*;$$

(ii)

$$(\lambda_1, \lambda_2) \overset{\mathrm{rm}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{ew}} X_{2:2}^*.$$

Joo and Mi [21] pointed out that the hazard rate order between two maxima does not necessarily hold in the case when $\lambda_1 \leq \lambda_1^* \leq \lambda_2 \leq \lambda_2^*$. Da, Ding, and Li [14] gave a sufficient condition for the hazard rate order to hold in this case, and established, under the exponential setup and the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2 \leq \lambda_2^*$, that

$$(\lambda_1, \lambda_2^*) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2) \text{ [or equivalently, } \lambda_1 + \lambda_2^* \leq \lambda_1^* + \lambda_2] \Longrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*. \tag{2.11}$$

Yan, Da, and Zhao [46] discussed this problem further and strengthened the result in (2.11) as follows.

THEOREM 2.8: *Let (X_1, X_2) be a vector of independent random variables with respective survival functions $\overline{F}^{\lambda_1}$ and $\overline{F}^{\lambda_2}$, and (X_1^*, X_2^*) be another vector of independent random*

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variables with respective survival functions $\overline{F}^{\lambda_1^*}$ *and* $\overline{F}^{\lambda_2^*}$. *Suppose* $\lambda_1 \leq \lambda_1^* \leq \lambda_2 \leq \lambda_2^*$. *Then,*

(i)
$$(\lambda_1, \lambda_2^*) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2) \ [\textit{or equivalently}, \lambda_1 + \lambda_2^* \leq \lambda_1^* + \lambda_2] \Longrightarrow X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*;$$

(ii)
$$(\lambda_1, \lambda_2^*) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2) \ [\textit{or equivalently}, \frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_2^*}{\lambda_1^*}] \Longrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^*.$$

Presented below are some numerical examples provided by Yan et al. [46] which demonstrate the main results in Theorem 2.8. For ease of descriptions, let $h_{(\lambda_1, \lambda_2)}(t)$ and $f_{(\lambda_1, \lambda_2)}(t)$ denote the hazard rate and density functions of the maximum from two independent exponential random variables with respective hazard rates λ_1 and λ_2.

Example 2.1:

(a) *Set* $\lambda_1 = 2$, $\lambda_2 = 4.5$, $\lambda_1^* = 4$ *and* $\lambda_2^* = 6$. *It may be easily verified that the assumption in Part (i) of Theorem 2.8 is satisfied. So, we have* $X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*$. *This coincides with what is displayed in Figure 4(a);*

(b) *Set* $\lambda_1 = 2$, $\lambda_2 = 4.5$, $\lambda_1^* = 2.05$ *and* $\lambda_2^* = 8$. *The assumption in Part (i) of Theorem 2.8 is violated, and Figure 4 (b) shows that the likelihood ratio function* $f_{(2,4.5)}(t)/f_{(2.05,8)}(t)$ *has a locally decreasing trend, which means* $X_{2:2} \not\geq_{\mathrm{lr}} X_{2:2}^*$ *and* $X_{2:2} \not\leq_{\mathrm{lr}} X_{2:2}^*$;

(c) *Set* $\lambda_1 = 2.05$, $\lambda_2 = 8$, $\lambda_1^* = 4$ *and* $\lambda_2^* = 11$. *Although the assumption in Part (i) of Theorem 2.8 is not satisfied, as seen in Figure 4(c),* $X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*$ *still holds.*

Example 2.2:

(a) *Set* $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_1^* = 2.4$ *and* $\lambda_2^* = 3.4$. *It can be readily seen that the assumption in Part (ii) of Theorem 2.8 is satisfied. So, we have* $h_{(2,3)}(t) \leq h_{(2.4,3.4)}(t)$. *This coincides with what is displayed in Figure 5(a);*

(b) *Set* $\lambda_1 = 2.4$, $\lambda_2 = 3.4$, $\lambda_1^* = 2.405$ *and* $\lambda_2^* = 5$. *The assumption in Part (ii) of Theorem 2.8 is violated, and Figure 5 (b) shows that* $h_{(2.4,3.4)}(t)$ *and* $h_{(2.405,5)}(t)$ *cross each other;*

(c) *Set* $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_1^* = 2.405$ *and* $\lambda_2^* = 5$. *Although the assumption in Part (ii) of Theorem 2.8 is not satisfied, as seen in Figure 5 (c),* $h_{(2.405,5)}(t)$ *is still above* $h_{(2,3)}(t)$ *for all* $t \geq 0$.

Remark 2.4: It can be seen from Examples 2.1 (c) and 2.2 (c) that the conditions of Theorem 2.8 are sufficient but not necessary for the likelihood ratio and hazard rate orders between $X_{2:2}$ and $X_{2:2}^*$, respectively. Also, by extensive empirical check, we observed that these sufficient conditions are somewhat stringent.

Open Problem 1: Are there sharper sufficient conditions than those presented in Theorem 2.8? It will be of interest to find better sufficient conditions or even equivalent characterization conditions for the likelihood ratio and hazard rate ordering results stated in Theorem 2.8 to hold.

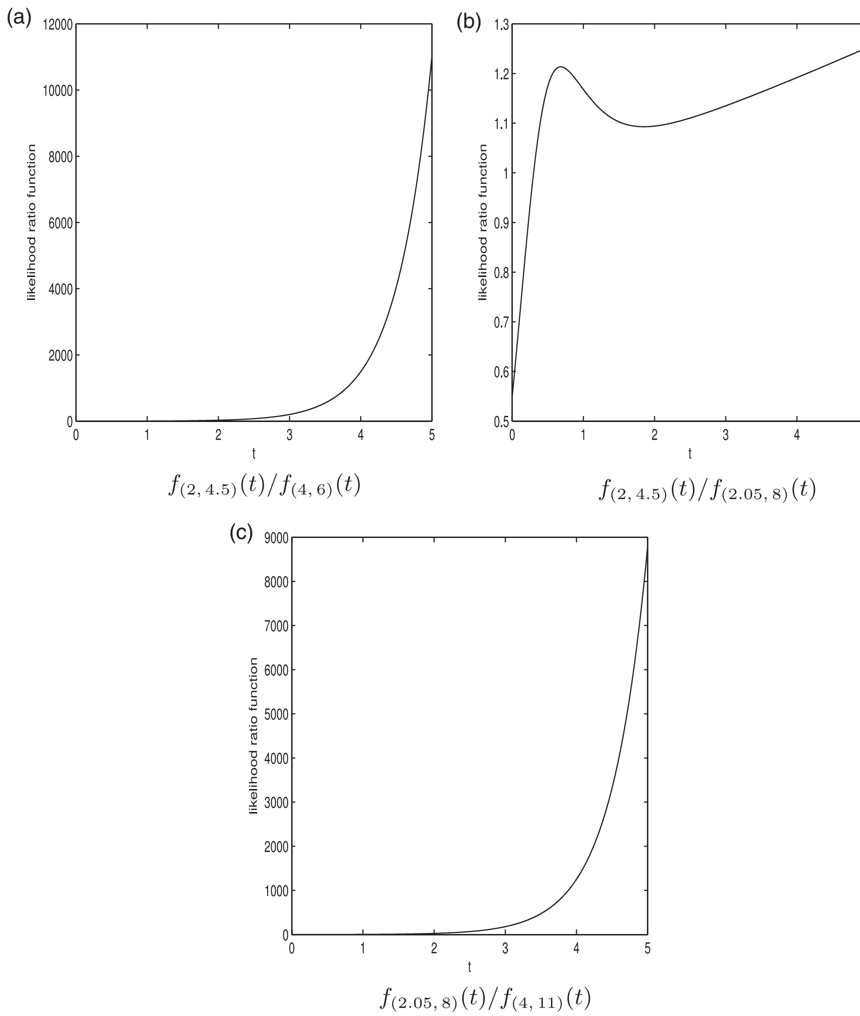

Figure 4. Plots of likelihood ratio functions considered in Example 2.1.

The following counterexample is given to illustrate that the result in Part (ii) of Theorem 2.8 cannot be strengthened from the hazard rate order to the likelihood ratio order.

Example 2.3: *Set* $\lambda_1 = 1$, $\lambda_2 = 80$, $\lambda_1^* = 1.5$, *and* $\lambda_2^* = 115$. *As can be seen in Figure 6, the likelihood ratio function between* $X_{2:2}$ *and* $X_{2:2}^*$ *decreases locally, which means* $X_{2:2} \nleq_{\mathrm{lr}} X_{2:2}^*$, *and* $X_{2:2} \ngeq_{\mathrm{lr}} X_{2:2}^*$.

2.2. Comparisons in Multiple-Outlier PHR Models

Now, let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

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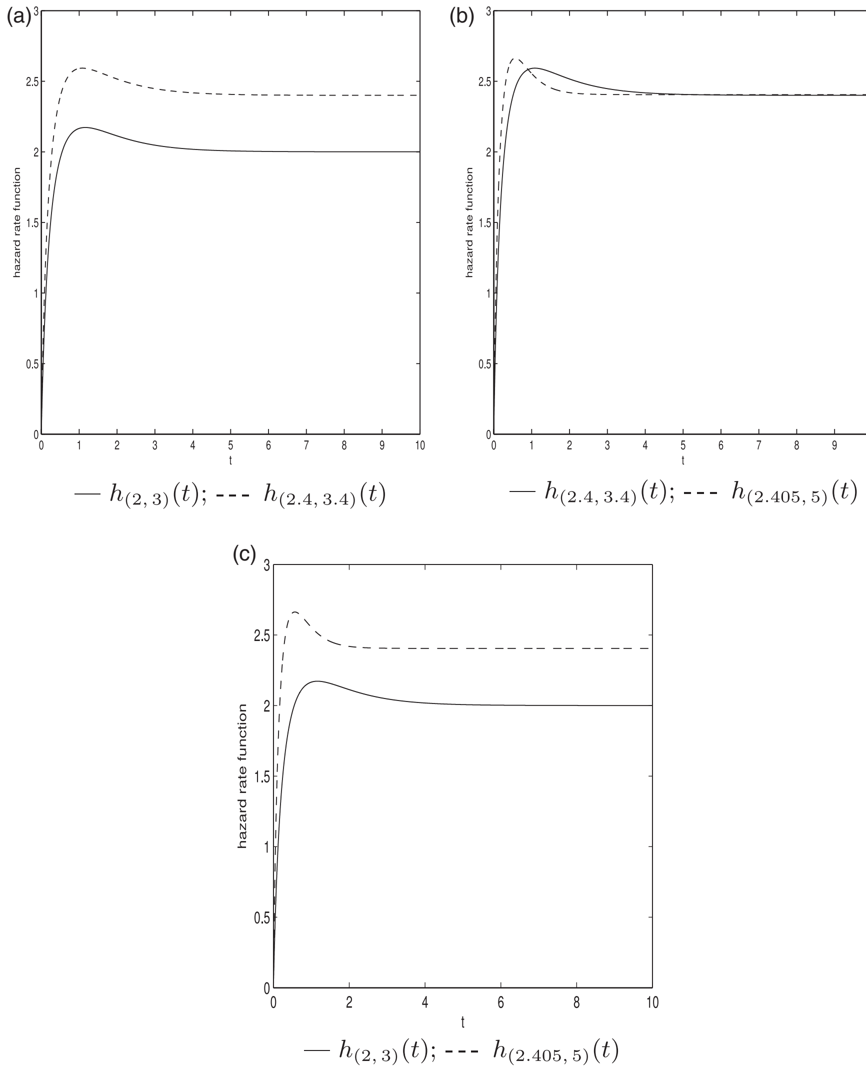

Figure 5. Plots of hazard rate functions considered in Example 2.2.

where $p + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}).$$

Here, by an exponential distribution with parameter λ, we mean the distribution with survival function

$$\overline{F}(t) = \exp(-\lambda t), \quad t > 0, \lambda > 0.$$

Then, Kochar and Xu [32] established the following interesting result for the star ordering.

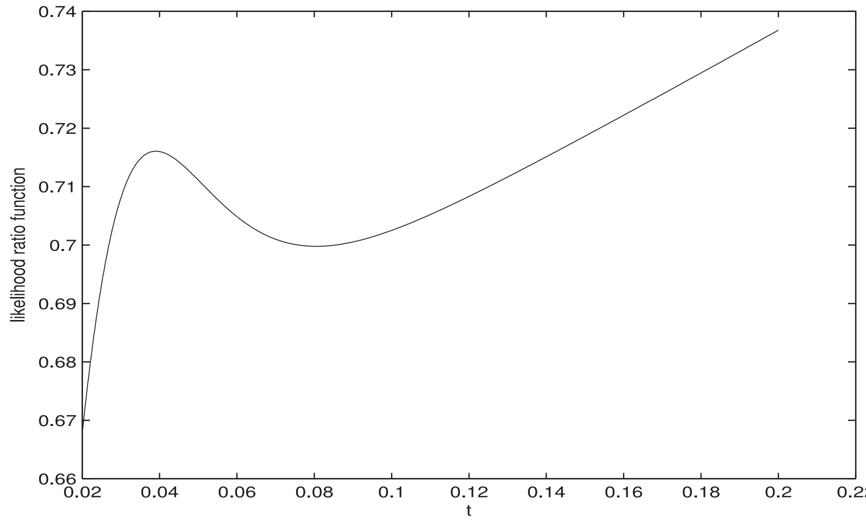

FIGURE 6. Plot of the likelihood ratio function decreasing locally.

THEOREM 2.9: *Under the above multiple-outlier setting, we have*

$$\frac{\lambda_{(2)}}{\lambda_{(1)}} \geq \frac{\lambda^*_{(2)}}{\lambda^*_{(1)}} \Longrightarrow X_{k:n} \geq_\star Y_{k:n}, \quad \textit{for } k = 1, \ldots, n,$$

where $\lambda_{(2)} = \max\{\lambda_1, \lambda_2\}$ *and* $\lambda_{(1)} = \min\{\lambda_1, \lambda_2\}$.

As stated in Kochar and Xu [32], the condition in Theorem 2.9 is very general. For example, it includes any of the following conditions:

(a) $(\lambda_1, \lambda_2) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \lambda_2^*)$;

(b) $(\log(\lambda_1), \log(\lambda_2)) \overset{\mathrm{m}}{\succeq} (\log(\lambda_1^*), \log(\lambda_2^*))$;

(c) $(1/\lambda_1, 1/\lambda_2) \overset{\mathrm{m}}{\succeq} (1/\lambda_1^*, 1/\lambda_2^*)$.

The following example, due to Kochar and Xu [32], provides a counterexample to show that the result in Theorem 2.9 cannot be extended to the general case when the n parameters are all different.

EXAMPLE 2.4: *Let* (X_1, X_2, X_3) *be a vector of independent exponential variables with parameter vector* $(1, 2, 9)$*, and* (Y_1, Y_2, Y_3) *be another vector of independent exponential variables with parameter vector* $(1, 5, 6)$*. Then, it is clear that*

$$(1, 2, 9) \overset{\mathrm{m}}{\succeq} (1, 5, 6).$$

However,

$$\mathrm{cv}(X_{3:3}) = 0.815396 < 0.921265 = \mathrm{cv}(Y_{3:3}),$$

which implies that $X_{3:3} \ngeq_\star Y_{3:3}$.

As a direct consequence of Theorem 2.9, the following corollary shows that order statistics from multiple-outlier exponential model are more skewed than the corresponding statistics from the homogeneous exponential model in the sense of star order.

Corollary 2.3: *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier exponential model with parameters*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p + q = n$, and $Y_1, \ldots, Y_n$ be an i.i.d. random sample from any exponential distribution. Then,

$$X_{k:n} \geq_{\star} Y_{k:n} \quad \text{for } k = 1, \ldots, n.$$

Upon using Corollary 2.3, Kochar and Xu [32] presented the following equivalent characterization results.

Theorem 2.10: *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier exponential model with parameters*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p + q = n$, and $Y_1, \ldots, Y_n$ be a random sample from an exponential distribution with parameter λ. Then:

(i) *if*

$$\lambda \geq \hat{\lambda} = \left[\binom{n}{k}^{-1} \sum_{l \in \mathbb{L}_k} \binom{p}{l} \binom{n-p}{k-l} \lambda_1^l \lambda_2^{k-l} \right]^{1/k},$$

where

$$\mathbb{L}_k = \{l : \max\{k - n + p, 0\} \leq l \leq \min\{p, k\}\},$$

we have,

$$X_{k:n} \geq_{\text{disp}} Y_{k:n} \Longleftrightarrow X_{k:n} \geq_{\text{hr}} Y_{k:n} \Longleftrightarrow X_{k:n} \geq_{\text{st}} Y_{k:n};$$

(ii) *if*

$$\lambda \geq \tilde{\lambda} = \sum_{j=1}^{k} \frac{1}{n-j+1} \left[\sum_{j=n-k+1}^{n} (-1)^{j-n+k-1} \binom{j-1}{n-k} \right.$$
$$\left. \times \sum_{m \in \mathbb{M}_j} \binom{p}{m} \binom{n-p}{j-m} \frac{1}{m\lambda_1 + (j-m)\lambda_2} \right]^{-1},$$

where

$$\mathbb{M}_j = \{m : \max\{j - n + p, 0\} \leq m \leq \min\{p, j\}\},$$

we have,

$$X_{k:n} \geq_{\text{ew}} Y_{k:n} \Longleftrightarrow \mathsf{E}X_{k:n} \geq \mathsf{E}Y_{k:n}.$$

They also discussed the general case of PHR models and presented the following results.

THEOREM 2.11: *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier PHR model with survival functions*

$$(\underbrace{\overline{F}^{\lambda_1}, \ldots, \overline{F}^{\lambda_1}}_{p}, \underbrace{\overline{F}^{\lambda_2}, \ldots, \overline{F}^{\lambda_2}}_{q}),$$

where $p+q=n$, and $Y_1, \ldots, Y_n$ be a random sample from a distribution with survival function $\overline{F}^{\lambda}$. If F is DFR, then

(i)

$$\lambda \geq \hat{\lambda} \Longrightarrow X_{k:n} \geq_{\text{disp}} Y_{k:n};$$

(ii)

$$\lambda \geq \tilde{\lambda} \Longrightarrow X_{k:n} \geq_{\text{ew}} Y_{k:n}.$$

It can be seen that the results in Theorem 2.11 extend the corresponding ones in Theorem 2.10 from exponential case to the PHR models. Along these lines, Zhao and Balakrishnan [54] recently discussed the likelihood ratio order (reversed hazard rate order) and the hazard rate order (usual stochastic order) and obtained the following results.

THEOREM 2.12: *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier exponential model with parameters*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p+q=n$, and $Y_1, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}).$$

Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Then,

(i)

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{\text{w}}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longleftrightarrow X_{n:n} \geq_{\text{lr}} [\geq_{\text{rh}}] Y_{n:n};$$

(ii)

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longleftrightarrow X_{n:n} \geq_{\text{hr}} [\geq_{\text{st}}] Y_{n:n}.$$

The following example, due to Zhao and Balakrishnan [54], illustrates the validity of the results in Theorem 2.12.

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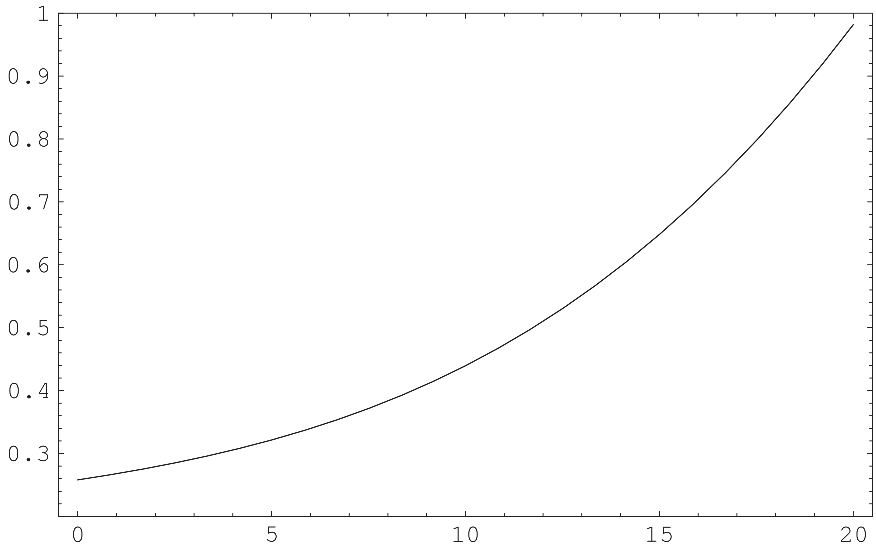

Figure 7. Plot of $f_{X_{5:5}}/f_{Y_{5:5}}$ when $p=2$, $q=3$, $\lambda_1^*=0.05$, $\lambda_1=0.28$, $\lambda_2^*=0.11$, and $\lambda_2=0.26$.

Example 2.5:

(a) *Setting* $p=2$, $q=3$, $\lambda_1^*=0.05$, $\lambda_1=0.28$, $\lambda_2^*=0.11$ *and* $\lambda_2=0.26$ *in Part (i) of Theorem 2.12, we find*

$$(0.05,0.05,0.28,0.28,0.28) \overset{\mathrm{w}}{\succeq} (0.11,0.11,0.26,0.26,0.26).$$

Figure 7 presents the plot of the ratio of the two density functions from which it can be seen that $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ *is increasing in* $t\in\Re_+$, *which is consistent with the result in Part (i) of Theorem 2.12.*

(b) *Setting* $p=2$, $q=3$, $\lambda_1^*=1/8$, $\lambda_1=4$, $\lambda_2^*=1/2$ *and* $\lambda_2=2$ *in Part (ii) of Theorem 2.12, we find*

$$(1/8,1/8,4,4,4) \overset{\mathrm{p}}{\succeq} (1/2,1/2,2,2,2),$$

but the $\overset{\mathrm{w}}{\succeq}$ *order does not hold between these two vectors. Figures 8 and 9 present plots of the ratios of two density and survival functions, respectively, from which it can be seen that* $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ *in Figure 8 is not monotone while* $\overline{F}_{X_{5:5}}(t)/\overline{F}_{Y_{5:5}}(t)$ *in Figure 9 is increasing in* $t\in\Re_+$. *These are consistent with the result in Part (ii) of Theorem 2.12.*

For the PHR models, we have the following analogous results.

Theorem 2.13: *Let* $X_1,\ldots,X_n$ *be independent random variables following a PHR model with survival functions*

$$(\underbrace{[\overline{F}(x)]^{\lambda_1},\ldots,[\overline{F}(x)]^{\lambda_1}}_{p},\underbrace{[\overline{F}(x)]^{\lambda_2},\ldots,[\overline{F}(x)]^{\lambda_2}}_{q}),$$

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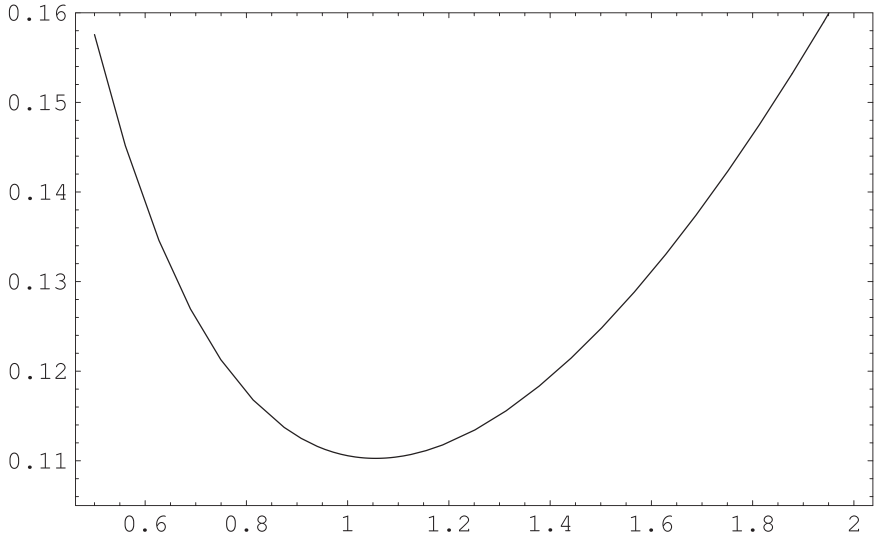

FIGURE 8. Plot of $f_{X_{5:5}}/f_{Y_{5:5}}$ when $p = 2$, $q = 3$, $\lambda_1^* = 1/8$, $\lambda_1 = 4$, $\lambda_2^* = 1/2$, and $\lambda_2 = 2$.

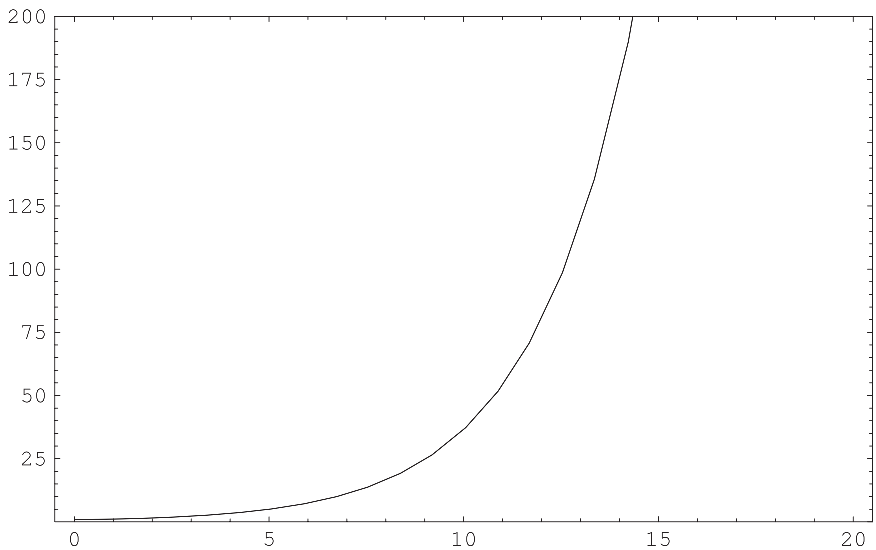

FIGURE 9. Plot of $\overline{F}_{X_{5:5}}/\overline{F}_{Y_{5:5}}$ when $p = 2$, $q = 3$, $\lambda_1^* = 1/8$, $\lambda_1 = 4$, $\lambda_2^* = 1/2$, and $\lambda_2 = 2$.

where $p + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent random variables following a PHR model with survival functions

$$(\underbrace{[\overline{F}(x)]^{\lambda_1^*}, \ldots, [\overline{F}(x)]^{\lambda_1^*}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2^*}, \ldots, [\overline{F}(x)]^{\lambda_2^*}}_{q}).$$

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Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. *Then,*

(i)

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \overset{\text{w}}{\succeq} (\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}) \Longrightarrow X_{n:n} \le_{\text{lr}} Y_{n:n};$$

(ii)

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}) \Longrightarrow X_{n:n} \le_{\text{hr}} Y_{n:n}.$$

Open Problem 2: Similar to likelihood ratio order [reversed hazard rate order] and hazard rate order [usual stochastic order] mentioned above, it would be also of interest to check whether, under the condition $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, the result

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \overset{\text{rm}}{\succeq} (\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}) \Longrightarrow X_{n:n} \ge_{\text{mrl}} Y_{n:n}$$

holds.

Next, we present a result for the dispersive order.

Theorem 2.14: *Let* $X_1,\ldots,X_n$ *be independent random variables following the multiple-outlier exponential model with scale parameters*

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}),$$

where $p+q=n$, *and* $Y_1,\ldots,Y_n$ *be another set of independent random variables following the multiple-outlier exponential model with scale parameters*

$$(\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}).$$

Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. *Then,*

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}) \Longrightarrow X_{n:n} \ge_{\text{disp}} Y_{n:n}.$$

Proof: From Part (ii) of Theorem 2.12, it follows that

$$(\underbrace{\lambda_1,\ldots,\lambda_1}_{p},\underbrace{\lambda_2,\ldots,\lambda_2}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_1^*,\ldots,\lambda_1^*}_{p},\underbrace{\lambda_2^*,\ldots,\lambda_2^*}_{q}) \Longrightarrow X_{n:n} \ge_{\text{st}} Y_{n:n}.$$

Since the assumption satisfies the condition in Theorem 2.9, we also have

$$X_{n:n} \ge_{\star} Y_{n:n}.$$

On the other hand, it is known from Ahmed et al. [1] that, for two continuous random variables X and Y, if $X \le_{\star} Y$, then

$$X \le_{\text{st}} Y \Longrightarrow X \le_{\text{disp}} Y.$$

From these facts, we therefore can conclude that

$$X_{n:n} \geq_{\text{disp}} Y_{n:n}.$$

■

The above result can be readily extended to the general case of PHR models as follows.

THEOREM 2.15: *Let $X_1, \ldots, X_n$ be independent random variables following a PHR model with survival functions*

$$(\underbrace{[\overline{F}(x)]^{\lambda_1}, \ldots, [\overline{F}(x)]^{\lambda_1}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2}, \ldots, [\overline{F}(x)]^{\lambda_2}}_{q}),$$

where $p+q=n$, and $Y_1, \ldots, Y_n$ be another set of independent random variables following a PHR model with survival functions

$$(\underbrace{[\overline{F}(x)]^{\lambda_1^*}, \ldots, [\overline{F}(x)]^{\lambda_1^*}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2^*}, \ldots, [\overline{F}(x)]^{\lambda_2^*}}_{q}).$$

Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. If F is DFR, then

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{\text{p}}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} \geq_{\text{disp}} Y_{n:n}.$$

OPEN PROBLEM 3: Similar to dispersive order mentioned above, it would be of interest to see whether, under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, the result

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{\text{rm}}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} \geq_{\text{ew}} Y_{n:n}$$

holds.

2.3. Comparisons Between Heterogeneous and Homogeneous PHR Samples

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be another random sample of size n from an exponential distribution with hazard rate $\lambda_{\text{am}} = \sum_{i=1}^n \lambda_i/n$, the arithmetic mean of λ_i's, and denote by $Y_{n:n}$ the corresponding largest order statistic. Dykstra et al. [17] then proved that

$$X_{n:n} \geq_{\text{hr}} Y_{n:n} \quad \text{and} \quad X_{n:n} \geq_{\text{disp}} Y_{n:n}, \tag{2.12}$$

which was strengthened by Kochar and Xu [29] as

$$X_{n:n} \geq_{\text{lr}} Y_{n:n}. \tag{2.13}$$

Khaledi and Kochar [23] strengthened the result in (2.12), under a weaker condition, by proving that if $Z_1, \ldots, Z_n$ is a random sample of size n from an exponential distribution with hazard rate $\lambda_{\text{gm}} = (\prod_{i=1}^n \lambda_i)^{1/n}$, the geometric mean of λ_is, then

$$X_{n:n} \geq_{\text{hr}} Z_{n:n} \quad \text{and} \quad X_{n:n} \geq_{\text{disp}} Z_{n:n}. \tag{2.14}$$

Subsequently, Kochar and Xu [29] and Khaledi and Kochar [26] extended the results in (2.13) and (2.14) from the exponential case to the general PHR case as follows.

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Theorem 2.16: *Let $X_1, \ldots, X_n$ be independent random variables having survival function $\overline{F}^{\lambda_i}, i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n[Z_1, \ldots, Z_n]$ be a random sample with common survival function $\overline{F}^{\lambda_{\rm am}}[\overline{F}^{\lambda_{\rm gm}}]$. Then,*

(i) $X_{n:n} \geq_{\rm lr} Y_{n:n}$;
(ii) $X_{n:n} \geq_{\rm hr} Z_{n:n}$;
(iii) $X_{n:n} \geq_{\rm disp} Z_{n:n}$, *if F is DFR.*

The next example, due to Khaledi and Kochar [26], shows the DFR condition in Part (iii) of Theorem 2.16 can not be dispensed with.

Example 2.6: *Let X_1 and X_2 be independent random variables with X_i having survival function*

$$\overline{F}_i(x) = (1-x)^{\lambda_i}, \quad 0 \leq x \leq 1, \quad i = 1, 2.$$

Let Y_1 and Y_2 be independent random variables with common survival function

$$\overline{G}(x) = (1-x)^{(\lambda_1\lambda_2)^{1/2}}, \quad 0 \leq x \leq 1.$$

If $\lambda_1 = 1$ and $\lambda_2 = 4$, then we have

$$\mathrm{Var}(X_{2:2}) = 73/720 < 11/225 = \mathrm{Var}(Y_{2:2}),$$

from which it can be concluded that Part (iii) of Theorem 2.16 may not hold for the case when F, the baseline distribution, is not DFR. Notice that here F, being a uniform distribution on $(0, 1)$, is IFR.

The following corollary can be directly obtained from Theorem 2.16.

Corollary 2.4: *Let $X_1, \ldots, X_n$ be independent Weibull random variables with X_i having shape parameter α and scale parameter λ_i, for $i = 1, \ldots, n$. Let $Z_1, \ldots, Z_n$ be an independent Weibull random sample with common shape parameter α and scale parameter $\lambda_{\rm gm}$. Then,*

(i) $X_{n:n} \geq_{\rm hr} Z_{n:n}$ *for all* $\alpha > 0$;
(ii) $X_{n:n} \geq_{\rm disp} Z_{n:n}$ *if* $0 < \alpha \leq 1$.

In this connection, Fang and Zhang [18] recently considered the case when $\alpha > 1$ for the dispersive order and established the following result.

Theorem 2.17: *Let $X_1, \ldots, X_n$ be independent Weibull random variables with X_i having shape parameter α and scale parameter λ_i, for $i = 1, \ldots, n$. Let $X_1^*, \ldots, X_n^*$ be an independent Weibull random sample with common shape parameter α and scale parameter λ. If $\alpha > 1$, then*

$$\lambda \geq \frac{(\prod_{i=1}^n \lambda_i^\alpha)^{1/n}}{\lambda_{\min}^{\alpha-1}} \Longrightarrow X_{n:n} \geq_{\rm disp} X_{n:n}^*,$$

where $\lambda_{\min} = \min\{\lambda_1, \ldots, \lambda_n\}$.

OPEN PROBLEM 4: Upon noting that

$$\frac{(\prod_{i=1}^{n}\lambda_i^{\alpha})^{1/n}}{\lambda_{\min}^{\alpha-1}} \geq \sqrt[n]{\prod_{i=1}^{n}\lambda_i} = \lambda_{\mathrm{gm}},$$

can we establish, similar to the result for the hazard rate order in Part (i) of Corollary 2.4, under the setup of Theorem 2.17, the result that

$$\lambda \geq \lambda_{\mathrm{gm}} \Longrightarrow X_{n:n} \geq_{\mathrm{disp}} X^*_{n:n}, \quad \text{for all } \alpha > 0?$$

Suppose $\lambda_1^* = \cdots = \lambda_n^* = \lambda$. For the exponential case, Bon and Păltănea [13] provided the necessary and sufficient condition on λ for $X_{k:n} \geq_{\mathrm{st}} X^*_{k:n}$ of the following form:

$$X_{k:n} \geq_{\mathrm{st}} X^*_{k:n} \Longleftrightarrow \lambda \geq \left[\binom{n}{k}^{-1} \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\ldots\lambda_{i_k}\right]^{\frac{1}{k}}.$$

Kochar and Xu [31] proved that the largest order statistic from heterogeneous exponential variables is more skewed in the sense of convex transform order than that from homogeneous exponential variables, which is quite a general result as there is no any restriction on the parameters. They also proved the following two characterization results in this regard:

$$X_{n:n} \geq_{\mathrm{disp}} X^*_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{gm}}$$

and

$$X_{n:n} \geq_{\mathrm{ew}} X^*_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{ew}},$$

where

$$\lambda_{\mathrm{ew}} = \left(\sum_{i=1}^{n}\frac{1}{i}\right)\left[\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq j_1<\cdots<j_k\leq n}\frac{1}{\sum_{i=1}^{k}\lambda_{j_i}}\right]^{-1}.$$

Mao and Hu [34] generalized the above results and established the following characterizations.

THEOREM 2.18: *Let $X_1,\ldots,X_n$ be independent exponential random variables with respective hazard rates $\lambda_1,\ldots,\lambda_n$, and $X_1^*,\ldots,X_n^*$ be i.i.d. exponential random variables with a common hazard rate λ. Then,*

(i) $X_{n:n} \geq_{\mathrm{lr}} X^*_{n:n} \Longleftrightarrow X_{n:n} \geq_{\mathrm{rh}} X^*_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{am}}$;

(ii) $X_{n:n} \geq_{\mathrm{hr}} X^*_{n:n} \Longleftrightarrow X_{n:n} \geq_{\mathrm{st}} X^*_{n:n} \Longleftrightarrow X_{n:n} \geq_{\mathrm{disp}} X^*_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{gm}}$;

(iii) $X_{n:n} \geq_{\mathrm{mrl}} X^*_{n:n} \Longleftrightarrow X_{n:n} \geq_{\mathrm{icx}} X^*_{n:n} \Longleftrightarrow X_{n:n} \geq_{\mathrm{ew}} X^*_{n:n} \Longleftrightarrow \lambda \geq \lambda_{\mathrm{ew}}$;

(iv) $X_{n:n} \leq_{\mathrm{order}} X^*_{n:n} \Longleftrightarrow \lambda \leq \min_{1\leq i\leq n}\lambda_i$, *where $\leq_{\mathrm{order}}$ denotes any one of the orders $\leq_{\mathrm{lr}}, \leq_{\mathrm{hr}}, \leq_{\mathrm{rh}}, \leq_{\mathrm{st}}$, and $\leq_{\mathrm{disp}}$.*

Recently, some attention has been paid to ordering results concerning the second order statistic of exponentials, viz., the lifetimes of the $(n-1)$-out-of-n systems, which are commonly referred to as *fail-safe systems*; see Barlow and Proschan [7]. Păltănea [39] proved that

$$X_{2:n} \geq_{\rm hr} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm hr} = \sqrt{\frac{\sum_{1\leq i<j\leq n} \lambda_i\lambda_j}{\binom{n}{2}}} \tag{2.15}$$

and

$$X_{2:n} \leq_{\rm hr} X^*_{2:n} \Longleftrightarrow \lambda \leq \lambda_{\rm u} = \frac{\min_{1\leq i\leq n} \Lambda_i}{n-1}, \tag{2.16}$$

where $\Lambda_i = \sum_{j=1}^n \lambda_j - \lambda_i$. Zhao, Li, and Balakrishnan [57] obtained the corresponding characterization on the likelihood ratio order as follows:

$$X_{2:n} \geq_{\rm lr} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm lr} = \frac{1}{2n-1}\left[2\Lambda(1) + \frac{\Lambda(3)-\Lambda(1)\Lambda(2)}{\Lambda^2(1)-\Lambda(2)}\right], \tag{2.17}$$

where $\Lambda(k) = \sum_{i=1}^n \lambda_i^k, k = 1, 2, 3$, and

$$X_{2:n} \leq_{\rm lr} X^*_{2:n} \Longleftrightarrow \lambda \leq \lambda_{\rm u}. \tag{2.18}$$

As an immediate consequence of (2.17) and (2.18), the following result compares the corresponding second order statistics in terms of the likelihood ratio order for the case when both exponential samples are heterogeneous. If $X_1, \ldots, X_n$ are independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and $X_1^*, \ldots, X_n^*$ are another set of independent exponential random variables with respective hazard rates $\mu_1, \ldots, \mu_n$, then

$$\lambda_{\rm lr} \leq \mu_u = \frac{\sum_{i=1}^n \mu_i - \max_{1\leq i\leq n} \mu_i}{n-1} \Longrightarrow X_{2:n} \geq_{\rm lr} X^*_{2:n}. \tag{2.19}$$

Zhao and Balakrishnan [48] presented the following characterization for the mean residual life order as

$$X_{2:n} \geq_{\rm mrl} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm mrl} = \frac{(2n-1)}{n(n-1)\left(\sum_{n=1}^n \frac{1}{\Lambda_i} - \frac{n-1}{\Lambda}\right)}, \tag{2.20}$$

where $\Lambda = \sum_{i=1}^n \lambda_i$, and moveover

$$X_{2:n} \leq_{\rm mrl} X^*_{2:n} \Longleftrightarrow \lambda \leq \lambda_{\rm u}. \tag{2.21}$$

As a consequence of Theorems 2.20 and 2.21, the following result, similar to (2.19), provides a comparison of the second order statistics in terms of the mean residual life order for the case when both exponential samples are heterogenous:

$$\lambda_{\rm mrl} \leq \mu_u \Longrightarrow X_{2:n} \geq_{\rm mrl} X^*_{2:n}.$$

Remark 2.5: Note that the characterization results in (2.15), (2.17), and (2.20) in terms of the hazard rate order, likelihood ratio order and mean residual life order, respectively, are all under the same setup. Therefore, based on these three characterizations and the fact

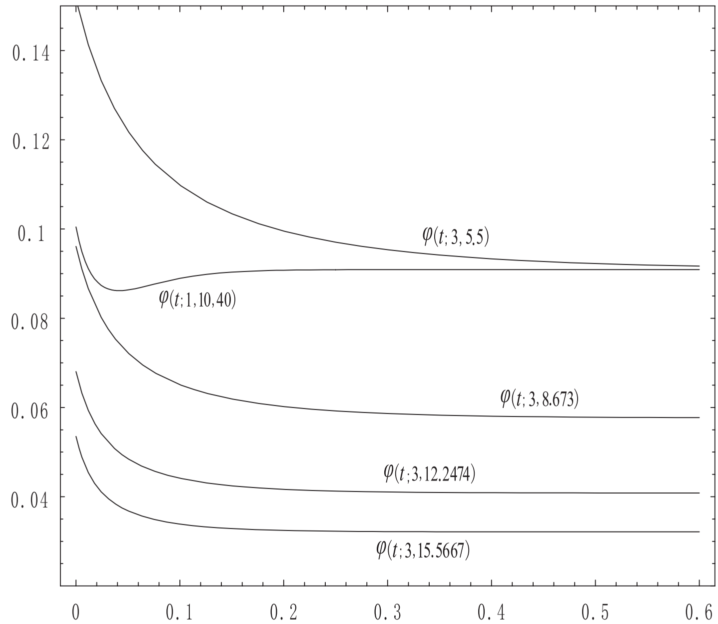

Figure 10. Plots of mean residual life functions of $X_{2:3}$ from three exponentials with hazards $(1, 10, 40)$ and $Y_{2:3}$ from i.i.d. exponentials with parameter $\lambda_{\mathrm{u}} = 5.5$, $\lambda_{\mathrm{mrl}} = 8.673$, $\lambda_{\mathrm{hr}} = 12.2474$, and $\lambda_{\mathrm{lr}} = 15.5667$.

that the likelihood ratio order implies the hazard rate order which in turn implies the mean residual life order, the following interesting inequalities can be obtained between different means:

$$\lambda_{\mathrm{mrl}} \le \lambda_{\mathrm{hr}} \le \lambda_{\mathrm{lr}} \le \lambda_{\mathrm{am}}. \tag{2.22}$$

For example, for the non-negative vector $(\lambda_1, \lambda_2, \lambda_3) = (1, 10, 40)$, we have

$$\lambda_{\mathrm{mrl}} \approx 8.673, \quad \lambda_{\mathrm{hr}} \approx 12.2474, \quad \lambda_{\mathrm{lr}} \approx 15.5667, \quad \lambda_{\mathrm{am}} = 17,$$

which support the order in (2.22).

Example 2.7: *Let (X_1, X_2, X_3) be a vector of independent exponential random variables with hazard rate vector $(1, 10, 40)$. Denote by $\varphi(t; 1, 10, 40)$ and $r(t; 1, 10, 40)$ the corresponding mean residual life and hazard rate functions of the second order statistic $X_{2:3}$. Let (Y_1, Y_2, Y_3) be another vector of i.i.d. exponential random variables with common hazard rate λ, and denote by $\varphi(t; 3, \lambda)$ and $r(t; 3, \lambda)$ the corresponding mean residual life and hazard rate functions of $Y_{2:3}$. Figure 10 presents the mean residual life functions of $X_{2:3}$ and $Y_{2:3}$ for λ taking $\lambda_{\mathrm{u}} = 5.5$, $\lambda_{\mathrm{mrl}} = 8.673$, $\lambda_{\mathrm{hr}} = 12.2474$, and $\lambda_{\mathrm{lr}} = 15.5667$. It can be seen that the best bounds for $\varphi(t; 1, 10, 40)$ are $\varphi(t; 3, \lambda_{\mathrm{mrl}})$ and $\varphi(t; 3, \lambda_{\mathrm{u}})$, with the former being the best approximation near the origin and the latter having the same limit as $\varphi(t; 1, 10, 40)$. In Figure 11, we have presented the corresponding hazard rate functions. Clearly, the best bounds for $r(t; 1, 10, 40)$ are $r(t; 3, \lambda_{\mathrm{hr}})$ and $r(t; 3, \lambda_{\mathrm{u}})$, but if $\lambda = \lambda_{\mathrm{mrl}}$, the hazard rates $r(t; 1, 10, 40)$ and $r(t; 3, \lambda_{\mathrm{mrl}})$ are not comparable.*

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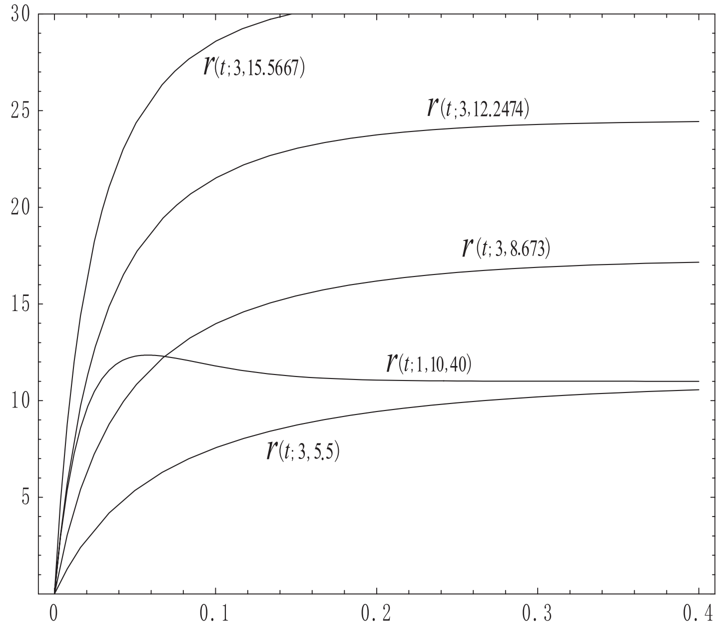

Figure 11. Plots of hazard rate functions of $X_{2:3}$ from three exponentials with hazards $(1, 10, 40)$ and $Y_{2:3}$ from i.i.d. exponentials with parameter $\lambda_{\rm u} = 5.5$, $\lambda_{\rm mrl} = 8.673$, $\lambda_{\rm hr} = 12.2474$, and $\lambda_{\rm lr} = 15.5667$.

Zhao and Balakrishnan [52] and Zhao, Li, and Da [58] discussed the dispersive order and excess wealth order, respectively, and established the characterizations

$$X_{2:n} \geq_{\rm disp} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm hr}, \tag{2.23}$$

$$X_{2:n} \geq_{\rm ew} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm mrl} X^*_{2:n} \Longleftrightarrow \mathsf{E}X_{2:n} \geq \mathsf{E}X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm mrl}, \tag{2.24}$$

and

$$X_{2:n} \leq_{\rm ew} X^*_{2:n} \Longleftrightarrow X_{2:n} \leq_{\rm disp} X^*_{2:n} \Longleftrightarrow \lambda \leq \lambda_{\rm u}. \tag{2.25}$$

In fact, we can establish the following general result for the exponential case.

Theorem 2.19: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and $X^*_1, \ldots, X^*_n$ be i.i.d. exponential random variables with a common hazard rate λ. Then,*

(i) $X_{2:n} \geq_{\rm lr} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm rh} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm lr}$;
(ii) $X_{2:n} \geq_{\rm hr} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm st} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm disp} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm hr}$;
(iii) $X_{2:n} \geq_{\rm mrl} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm icx} X^*_{2:n} \Longleftrightarrow X_{2:n} \geq_{\rm ew} X^*_{2:n} \Longleftrightarrow \lambda \geq \lambda_{\rm mrl}$;
(iv) $X_{2:n} \leq_{\rm order} X^*_{2:n} \Longleftrightarrow \lambda \leq \lambda_{\rm u}$, *where* $\leq_{\rm order}$ *denotes any one of the orders* $\leq_{\rm lr}, \leq_{\rm hr}, \leq_{\rm rh}, \leq_{\rm mrl}, \leq_{\rm disp}$, *and* $\leq_{\rm ew}$.

Proof: (i) We only need to prove that $X_{2:n} \geq_{\rm rh} X^*_{2:n} \Longrightarrow \lambda \geq \lambda_{\rm lr}$. Since $X_{2:n}$ has its distribution function as

$$F_{X_{2:n}}(t) = 1 - \sum_{i=1}^{n} e^{-\Lambda_i t} + (n-1)e^{-\Lambda t}, \quad t \geq 0,$$

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and its density function as

$$f_{X_{2:n}}(t) = \sum_{i=1}^{n} \Lambda_i e^{-\Lambda_i t} - (n-1)\Lambda e^{-\Lambda t}, \quad t \geq 0,$$

by applying Taylor's expansion at the origin, we get

$$f_{X_{2:n}}(t) = \left[(n-1)\Lambda^2 - \sum_{i=1}^{n} \Lambda_i^2\right] t - \frac{1}{2}\left[(n-1)\Lambda^3 - \sum_{i=1}^{n} \Lambda_i^3\right] t^2 + o(t^2),$$

and

$$F_{X_{2:n}}(t) = \frac{1}{2}\left[(n-1)\Lambda^2 - \sum_{i=1}^{n} \Lambda_i^2\right] t^2 + o(t^2).$$

Thus,

$$r_{X_{2:n}}(t) = \frac{f_{X_{2:n}}(t)}{F_{X_{2:n}}(t)} = \frac{2}{t} - \frac{(n-1)\Lambda^3 - \sum_{i=1}^{n} \Lambda_i^3}{(n-1)\Lambda^2 - \sum_{i=1}^{n} \Lambda_i^2} + o(1),$$

and likewise,

$$r_{X^*_{2:n}}(t) = \frac{f_{X^*_{2:n}}(t)}{F_{X^*_{2:n}}(t)} = \frac{2}{t} - (2n-1)\lambda + o(1).$$

Since $X_{2:n} \geq_{\rm rh} X^*_{2:n}$ implies $r_{X_{2:n}}(t) \geq r_{X^*_{2:n}}(t)$ for all $t \geq 0$, we have

$$\lambda \geq \frac{(n-1)\Lambda^3 - \sum_{i=1}^{n} \Lambda_i^3}{(2n-1)\left[(n-1)\Lambda^2 - \sum_{i=1}^{n} \Lambda_i^2\right]} = \frac{1}{2n-1}\left[2\Lambda(1) + \frac{\Lambda(3) - \Lambda(1)\Lambda(2)}{\Lambda^2(1) - \Lambda(2)}\right] = \lambda_{\rm lr}.$$

The results in (ii), (iii) and (iv) can all be readily obtained from (2.15)–(2.25). ■

The following theorem, due to Zhao and Balakrishnan [52] and Zhao et al. [58], presents the analogous results for the general case of PHR models.

THEOREM 2.20: *Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}$ for $i = 1, \ldots, n$. Let $X^*_1, \ldots, X^*_n$ be a random sample with common survival function $\overline{F}^{\lambda}$. If F is DFR, then*

(i) $\lambda \geq \lambda_{\rm hr} \Longrightarrow X_{2:n} \geq_{\rm disp} X^*_{2:n}$;
(ii) $\lambda \geq \lambda_{\rm mrl} \Longrightarrow X_{2:n} \geq_{\rm ew} X^*_{2:n}$;
(iii) $\lambda \leq \lambda_{\rm u} \Longrightarrow X_{2:n} \leq_{\rm disp} X^*_{2:n}$.

Finally, we turn our attention to the sample ranges. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i for $i = 1, \ldots, n$, $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with hazard rate $\lambda_{\rm am}$, and

$Z_1, \ldots, Z_n$ be another random sample of size n from an exponential distribution with hazard rate $\lambda_{\rm gm}$. Then, Kochar and Rojo [28] proved that

$$X_{n:n} - X_{1:n} \geq_{\rm st} Y_{n:n} - Y_{1:n}. \tag{2.26}$$

Subsequently, Khaledi and Kochar [24] improved this result as

$$X_{n:n} - X_{1:n} \geq_{\rm st} Z_{n:n} - Z_{1:n}.$$

Kochar and Xu [29] strengthened the result in (2.26) from the usual stochastic order to the reversed hazard rate order as

$$X_{n:n} - X_{1:n} \geq_{\rm rh} Y_{n:n} - Y_{1:n}.$$

Genest, Kochar, and Xu [19] further proved that

$$X_{n:n} - X_{1:n} \geq_{\rm lr} Y_{n:n} - Y_{1:n} \quad \text{and} \quad X_{n:n} - X_{1:n} \geq_{\rm disp} Y_{n:n} - Y_{1:n}.$$

The following theorem, due to Mao and Hu [34], presents two characterizations.

THEOREM 2.21: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and $X_1^*, \ldots, X_n^*$ be i.i.d. exponential random variables with a common hazard rate λ. Then,*

(i)

$$X_{n:n} - X_{1:n} \geq_{\rm lr} X^*_{n:n} - X^*_{1:n} \Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\rm rh} X^*_{n:n} - X^*_{1:n} \Longleftrightarrow \lambda \geq \lambda_{\rm am};$$

(ii)

$$X_{n:n} - X_{1:n} \leq_{\rm order} X^*_{n:n} - X^*_{1:n} \Longleftrightarrow \lambda \leq \min_{1\leq i\leq n} \lambda_i,$$

where $\leq_{\rm order}$ is any one of the orders $\leq_{\rm lr}, \leq_{\rm hr}, \leq_{\rm rh}$, and $\leq_{\rm st}$.

Under the setup of Theorem 2.21, Zhao and Li [56] presented the following equivalent characterization:

$$X_{n:n} - X_{1:n} \geq_{\rm st} X^*_{n:n} - X^*_{1:n} \Longleftrightarrow \lambda \geq \lambda_{\rm range-st}, \tag{2.27}$$

where

$$\lambda_{\rm range-st} = \left(\frac{\prod_{i=1}^n \lambda_i}{\lambda_{\rm am}}\right)^{1/(n-1)}.$$

As an immediate consequence of (2.27), we can get a simple upper bound for the distribution function as

$$\mathsf{P}(X_{n:n} - X_{1:n} \leq x) \leq \left(1 - e^{-\lambda_{\rm range-st} x}\right)^{n-1}, \quad x \geq 0.$$

The counterexample below, due to Zhao and Li [56], demonstrates that the result in (2.27) cannot be strengthened to the reversed hazard rate order.

EXAMPLE 2.8: *Consider (X_1, X_2, X_3), an independent exponential random vector with hazard rate vector $(\lambda_1, \lambda_2, \lambda_3) = (5.5, 5.5, 40)$, and (Y_1, Y_2, Y_3), i.i.d. exponential random*

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variables with the hazard rate vector $\lambda_{\text{range-st}} = \sqrt{\frac{1210}{17}}$. *Denote by* $F_{R_3(X)}$ *and* $F_{R_3(Y)}$ *the distribution functions of* $R_3(X) = X_{3:3} - X_{1:3}$ *and* $R_3(Y) = Y_{3:3} - Y_{1:3}$, *respectively. Then, we have*

$$\frac{F_{R_3(X)}(0.05)}{F_{R_3(Y)}(0.05)} \approx 0.761353 > 0.742095 \approx \frac{F_{R_3(X)}(0.06)}{F_{R_3(Y)}(0.06)}.$$

Thus, the ratio $\frac{F_{R_3(X)}(x)}{F_{R_3(Y)}(x)}$ *is not increasing with respect to* $x \geq 0$, *which implies* $X_{3:3} - X_{1:3} \ngeq_{rh} Y_{3:3} - Y_{1:3}$.

Recently, Xu and Balakrishnan [44] proved that the sample range from heterogeneous exponential variables is stochastically larger than that from a homogeneous exponential sample in the sense of the star order, that is, under the setup of Theorem 2.21, we have

$$X_{n:n} - X_{1:n} \geq_{\star} X^*_{n:n} - X^*_{1:n}. \tag{2.28}$$

As a direct consequence of (2.28), the following result provides a bound for the coefficient of variation for the range of heterogeneous exponential samples:

$$\text{cv}(X_{n:n} - X_{1:n}) \geq \frac{\sqrt{\sum_{k=1}^{n-1} \frac{1}{k^2}}}{\sum_{k=1}^{n-1} \frac{1}{k}}.$$

With the help of (2.28), Xu and Balakrishnan [44] also presented the following characterizations.

THEOREM 2.22: *Under the setup of Theorem 2.21, we have*

(i)

$$\begin{aligned} X_{n:n} - X_{1:n} \geq_{\text{hr}} X^*_{n:n} - X^*_{1:n} &\Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\text{st}} X^*_{n:n} - X^*_{1:n} \\ &\Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\text{disp}} X^*_{n:n} - X^*_{1:n} \\ &\Longleftrightarrow \lambda \geq \lambda_{\text{range-st}}; \end{aligned}$$

(ii)

$$\begin{aligned} X_{n:n} - X_{1:n} \geq_{\text{ew}} X^*_{n:n} - X^*_{1:n} &\Longleftrightarrow \mathsf{E}(X_{n:n} - X_{1:n}) \geq \mathsf{E}(X^*_{n:n} - X^*_{1:n}) \\ &\Longleftrightarrow \lambda \geq \lambda_{\text{range-ew}}, \end{aligned}$$

where

$$\lambda_{\text{range-ew}} = \sum_{k=1}^{n-1} \frac{1}{k} \left[\sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} \frac{1}{\sum_{i=1}^{k} \lambda_{j_i}} - \frac{1}{\sum_{k=1}^{n} \lambda_k} \right]^{-1}.$$

OPEN PROBLEM 5: In the case of general spacings, Xu and Balakrishnan [44] conjectured that

$$X_{k:n} - X_{1:n} \geq_{\star} X^*_{k:n} - X^*_{1:n}.$$

As mentioned by them, the key step will be to prove that

$$X_{k:n} \geq_{\star} X^*_{k:n},$$

which has been shown to be true for the multiple-outlier exponential models by Kochar and Xu [32], but the general result remains open.

For the PHR case, Kochar and Xu [30] established the following result.

Theorem 2.23: *Let* $X_1, \ldots, X_n$ *be independent random variables with* X_i *having survival function* $\overline{F}^{\lambda_i}$ *for* $i = 1, \ldots, n$. *Let* $X_1^*, \ldots, X_n^*$ *be a random sample with common survival function* $\overline{F}^{\lambda_{\mathrm{am}}}$. *Then,*

$$X_{n:n} - X_{1:n} \geq_{\mathrm{st}} X_{n:n}^* - X_{1:n}^*.$$

We also have the following result for the PHR models which compares the sample range and the largest order statistic.

Theorem 2.24: *Let* $X_1, \ldots, X_n$ *be independent random variables with* X_i *having survival function* $\overline{F}^{\lambda_i}$ *for* $i = 1, \ldots, n$. *Let* $X_1^*, \ldots, X_{n-1}^*$ *be a random sample of size* $n-1$ *with common survival function* $\overline{F}^{\lambda_{\mathrm{range-st}}}$. *If* F *is NWU, then*

$$X_{n:n} - X_{1:n} \geq_{\mathrm{st}} X_{n-1:n-1}^*.$$

Proof: Let $H(x) = -\log \overline{F}(x)$ be the cumulative hazard of F. The NWU property of F implies that $\overline{F}(x+y) \geq \overline{F}(x)\overline{F}(y)$ for $x, y \geq 0$, which is actually equivalent to $H(x+y) \leq H(x) + H(y)$ for $x, y \geq 0$. From David and Nagaraja [15], the distribution of $R(X) = X_{n:n} - X_{1:n}$ is given by

$$\begin{aligned} F_{R(X)}(x) &= \sum_{i=1}^{n} \int_0^\infty \lambda_i [\overline{F}(u)]^{\lambda_i - 1} f(u) \prod_{j \neq i}^{n} ([\overline{F}(u)]^{\lambda_j - 1} - [\overline{F}(u+x)]^{\lambda_j - 1}) \mathrm{d}u \\ &= \sum_{i=1}^{n} \int_0^\infty \lambda_i e^{-\lambda_i H(u)} r(u) \prod_{j \neq i}^{n} (e^{-\lambda_j H(u)} - e^{-\lambda_j H(u+x)}) \mathrm{d}u \\ &\leq \sum_{i=1}^{n} \lambda_i \prod_{j \neq i}^{n} [1 - e^{-\lambda_j H(x)}] \int_0^\infty r(u) e^{-\sum_{i=1}^n \lambda_i H(u)} \mathrm{d}u \\ &= \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \prod_{j \neq i}^{n} [1 - e^{-\lambda_j H(x)}] \end{aligned}$$

for $x > 0$, where $r(u)$ is the hazard rate of F. For convenience, we us the simpler notation $\hat{\lambda} = \lambda_{\mathrm{range-st}}$. From (2.27), it is known that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \prod_{j \neq i}^{n} [1 - e^{-\lambda_j x}] \leq (1 - e^{-\hat{\lambda} x})^{n-1}$$

for $x > 0$. Replacing x with $H(x)$ in the above inequality, we get

$$\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \prod_{j \neq i}^{n} [1 - e^{-\lambda_j H(x)}] \leq (1 - e^{-\hat{\lambda} H(x)})^{n-1} \qquad \mathbf{(2.29)}$$

for $x > 0$. It can be readily observed that the right hand side of (2.29) is the distribution function of $X_{n-1:n-1}^*$, which yields the desired result. ■

3. GAMMA CASE

Gamma distribution is one of the most commonly used distributions in statistics, reliability and life-testing. It has also been widely applied in actuarial science since many total insurance claim distributions have similar shape to that of gamma distributions: non-negatively supported, skewed to the right and unimodal. Let X be a gamma random variable with shape parameter r and scale parameter λ. Then, X has its pdf as

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \exp(-\lambda x), \quad x > 0.$$

It is a flexible family of distributions with decreasing, constant, and increasing failure rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. Now, let $X_1, \ldots, X_n$ be independent gamma random variables with X_i having shape parameter r and scale parameter λ_i, $i = 1, \ldots, n$, and $X_1^*, \ldots, X_n^*$ be another set of independent gamma random variables with X_i^* having shape parameter r and scale parameter λ_i^*. Then, in this section, we describe some ordering results between order statistics from these two sets of random variables.

3.1. Comparisons Between two Heterogeneous Gamma Samples

Hu [20] proved under the scale model framework that, if $0 < r \leq 1$, then

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{st}}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*). \tag{3.1}$$

It should be mentioned here that the result in (3.1) was also proved independently by Sun and Zhang [43], who also established that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} X_{n:n}^* \tag{3.2}$$

and

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{1:n} \leq_{\mathrm{st}} X_{1:n}^* \quad \text{for } r > 1.$$

The result in (3.2) was further strengthened by Khaledi, Farsinezhad, and Kochar [22] as

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*. \tag{3.3}$$

Obviously, the results in (3.1) and (3.3) extend the corresponding results in (2.2) and (2.4) from the exponential case to the gamma case. Recently, Misra and Misra [37] obtained the following interesting result for the reversed hazard rate order.

THEOREM 3.1: *Let $(X_1, \ldots, X_n)$ be a vector of independent gamma random variables with common shape parameter r and scale parameter vector $(\lambda_1, \ldots, \lambda_n)$, and $(X_1^*, \ldots, X_n^*)$ be another vector of independent gamma random variables with common shape parameter r and scale parameter vector $(\lambda_1^*, \ldots, \lambda_n^*)$. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{rh}} X_{n:n}^*.$$

It can be seen that the result in Theorem 3.1 extends the corresponding result in (2.5), established earlier by Dykstra et al. [17], from the exponential case to the gamma case.

For the two-dimensional case, Zhao [47] established the following two results for the likelihood ratio and hazard rate orders.

Theorem 3.2: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter r and scale parameters λ_1 and λ_2, and (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape parameter r and scale parameters λ_1^* and λ_2^*, respectively. Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. Then,*

(i)

$$(\lambda_1, \lambda_2) \stackrel{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{lr}} X_{2:2}^*;$$

(ii)

$$(\lambda_1, \lambda_2) \stackrel{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \ge_{\text{hr}} X_{2:2}^*, \quad \text{if } r \le 1.$$

As an immediate consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.1: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $r > 0$ and scale parameters λ_1 and λ_2, and (X_1^*, X_2^*) be another vector of independent gamma random variables with common shape and scale parameters r and λ, respectively. Then,*

(i)

$$\lambda \ge \frac{\lambda_1 + \lambda_2}{2} \Longrightarrow X_{2:2} \ge_{\text{lr}} X_{2:2}^*;$$

(ii)

$$\lambda \ge \sqrt{\lambda_1 \lambda_2} \Longrightarrow X_{2:2} \ge_{\text{hr}} X_{2:2}^*, \quad \text{if } r \le 1.$$

In order to illustrate the result in Theorem 3.2, we provide the following two numerical examples taken from Zhao [47].

Example 3.1: *Let (X_1, X_2) be a vector of independent heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(1, 4)$. Denote by $h(t; 1, 4)$ the corresponding hazard rate function of the maximum order statistic $X_{2:2}$. Let (Y_1, Y_2) be a vector of independent heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 3.5)$, and denote by $h(t; 2, 3.5)$ the corresponding hazard rate function of $Y_{2:2}$. We then have $1 \le 2 \le 3.5 \le 4$ and $(1, 4) \stackrel{\text{w}}{\succeq} (2, 3.5)$. It can be seen from Figure 12 that $h(t; 1, 4) \le h(t; 2, 3.5)$ which is in accordance with the result of Part (i) Theorem 3.2. Let (Z_1, Z_2) be a vector of another set of heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 2.5)$. Denote by $h(t; 2, 2.5)$ the corresponding hazard rate function of $Z_{2:2}$. Note that the condition in Part (i) of Theorem 3.2 does not hold even though we have $(2, 2.5) \stackrel{\text{w}}{\succeq} (2, 3.5)$, and in this case we cannot compare the hazard rate functions as seen in Figure 12.*

Example 3.2: *Let (X_1, X_2) be a vector of independent gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(2, 8)$. Denote by $h(t; 2, 8)$ the corresponding hazard rate function of the maximum order statistic $X_{2:2}$. Let (Y_1, Y_2) be a vector of two independent gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(4, 4)$, and denote by $h(t; 4, 4)$ the corresponding hazard rate function of $Y_{2:2}$. Figure 13 presents the hazard rate functions of $X_{2:2}$ and $Y_{2:2}$. It can be seen that $h(t; 2, 8) \le h(t; 4, 4)$ which is in accordance with the result in Part (ii) of Theorem 3.2. Let (Z_1, Z_2) be a vector of two independent gamma random variables with common shape*

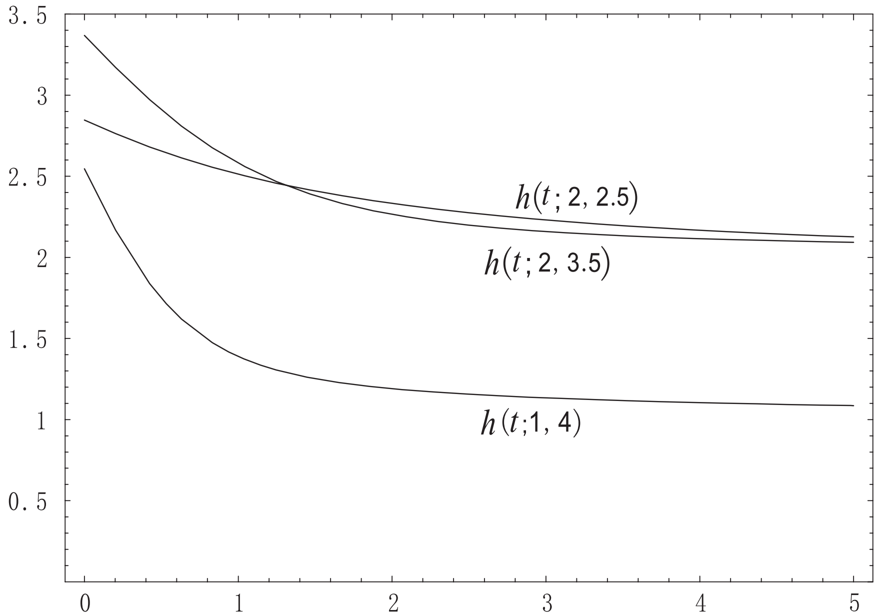

Figure 12. Plots of hazard rate functions of $X_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(1, 4)$, $Y_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(2, 3.5)$, and $Z_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(2, 2.5)$.

parameter $r = 0.5$ *and scale parameter vector* $(3.5, 3.5)$*, and denote by* $h(t; 3.5, 3.5)$ *the corresponding hazard rate function of* $Z_{2:2}$*. It is clear that the hazard rate order does not hold between* $X_{2:2}$ *and* $Z_{2:2}$ *as seen in Figure 13.*

Zhao and Balakrishnan [53] compared stochastically the maxima in terms of the dispersive and star orders.

Theorem 3.3: *Under the same setup as in Theorem 3.2, we have*

(i)

$$(\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{disp}} [\geq_\star] \, X_{2:2}^*;$$

(ii)

$$(1/\lambda_1, 1/\lambda_2) \overset{\mathrm{m}}{\succeq} (1/\lambda_1^*, 1/\lambda_2^*) \Longrightarrow X_{2:2} \geq_\star X_{2:2}^*.$$

The following example, due to Zhao and Balakrishnan [53], shows the dispersive order in Part (i) of Theorem 3.3 cannot be extended to the general case when $n > 2$.

Example 3.3: *Let* (X_1, X_2, X_3) *be an independent exponential random vector with parameter vector* $(0.1, 2, 9)$*, and* (Y_1, Y_2, Y_3) *be another independent exponential random vector with*

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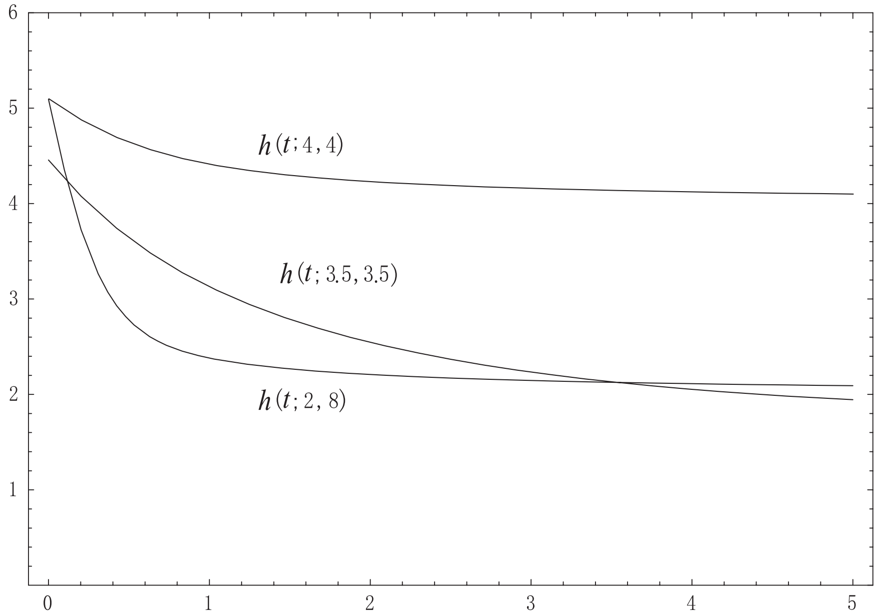

FIGURE 13. Plots of hazard rate functions of $X_{2:2}$ from two independent gamma distributions with common shape parameter 0.5 and scale parameter vector $(2, 8)$, $Y_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(4, 4)$, and $Z_{2:2}$ from two gamma distributions with common shape parameter 0.5 and scale parameter vector $(3.5, 3.5)$.

parameter vector $(0.1, 4, 5)$. *It is clear that*

$$(0.1, 2, 9) \overset{\text{p}}{\succeq} (0.1, 4, 5).$$

Now, when $X_i \sim Exp(\theta_i), i = 1, 2, 3,$ *are independent random variables, it can be readily shown that (see Arnold, Balakrishnan, and Nagaraja [2])*

$$\mathsf{E}(X_{3:3}) = \theta_1 + \theta_2 + \theta_3 - \frac{\theta_1\theta_2}{\theta_1 + \theta_2} - \frac{\theta_1\theta_3}{\theta_1 + \theta_3} - \frac{\theta_2\theta_3}{\theta_2 + \theta_3} + \frac{\theta_1\theta_2\theta_3}{\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3}$$

and

$$\mathsf{E}(X_{3:3}^2) = 2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - \frac{2}{(\frac{1}{\theta_1} + \frac{1}{\theta_2})^2} - \frac{2}{(\frac{1}{\theta_1} + \frac{1}{\theta_3})^2} - \frac{2}{(\frac{1}{\theta_2} + \frac{1}{\theta_3})^2} + \frac{2}{(\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3})^2}.$$

By using these expressions, in this case, we find the variances of $X_{3:3}$ *and* $Y_{3:3}$ *to be*

$$\text{Var}[X_{3:3}] = 99.5619 \leq 99.8326 = \text{Var}[Y_{3:3}],$$

which implies that $X_{3:3} \not\geq_{\text{disp}} Y_{3:3}$.

Remark 3.1: As in the case of dispersive order, one may also wonder whether the result in Theorem 3.3 can be extended to the general case when $n > 2$, say for the star order. In this regard, Example 2.4 in the preceding section can serve as a counterexample to give a negative answer.

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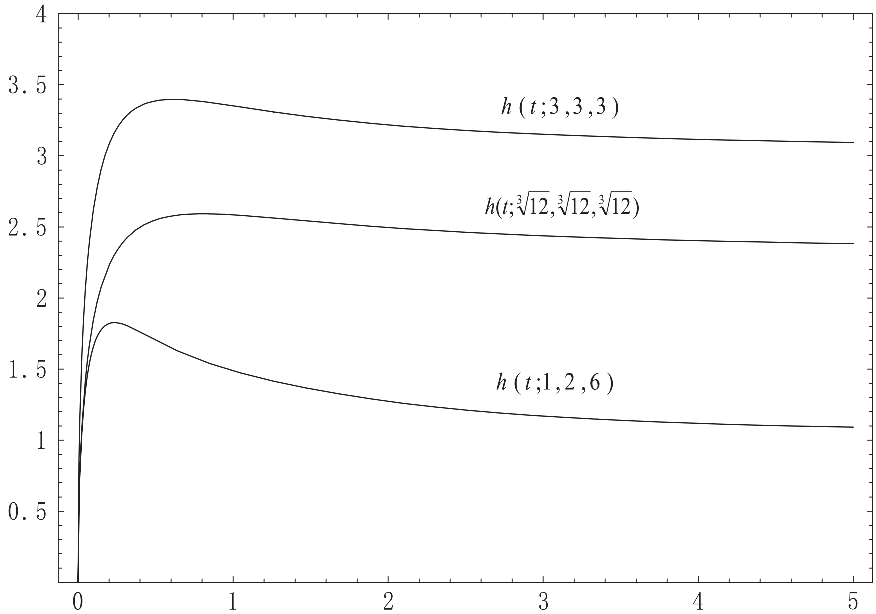

FIGURE 14. Plots of the hazard rate functions when $r = 0.5$.

3.2. Comparisons Between Heterogeneous and Homogeneous Gamma Samples

Recently, Balakrishnan and Zhao [6] established the following result for the gamma case.

THEOREM 3.4: *Let $X_1, \ldots, X_n$ be independent gamma random variables with X_i having shape parameter r and scale parameter λ_i for $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be a random sample of size n from a gamma distribution with shape parameter r and a common scale parameter $\lambda \geq \lambda_{\mathrm{gm}}$. If $r \leq 1$, then*

$$X_{n:n} \geq_{\mathrm{hr}} Y_{n:n}.$$

As a direct consequence of Theorem 3.4, we can obtain an upper bound on the hazard rate function of $X_{n:n}$ from heterogeneous gamma variables in terms of the hazard rate function of $Y_{n:n}$ from an i.i.d. gamma sample. The following numerical example, due to Balakrishnan and Zhao [6], can be used to illustrate this fact.

EXAMPLE 3.4: *Let (X_1, X_2, X_3) be a vector of independent heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 6)$, and $h(t; 1, 2, 6)$ denote the hazard rate function of $X_{3:3}$. Let (Y_1, Y_2, Y_3) be an i.i.d. gamma random sample with common shape parameter 0.5 and scale parameter 3 (the arithmetic mean of $(1, 2, 6)$), and let $h(t; 3, 3, 3)$ denote the hazard rate function of $Y_{3:3}$. Let (Z_1, Z_2, Z_3) be an i.i.d. gamma random sample with common shape parameter 0.5 and scale parameter $\sqrt[3]{12}$ (the geometric mean of $(1, 2, 6)$), and let $h(t; \sqrt[3]{12}, \sqrt[3]{12}, \sqrt[3]{12})$ denote the hazard rate function of $Z_{3:3}$. Figure 14 presents a plot of the hazard rate functions of these three largest order statistics, which can be seen to be in accordance with the result of Theorem 3.4. It can also be seen that the upper bound given by $h(t; \sqrt[3]{12}, \sqrt[3]{12}, \sqrt[3]{12})$ is better than that offered by $h(t; 3, 3, 3)$.*

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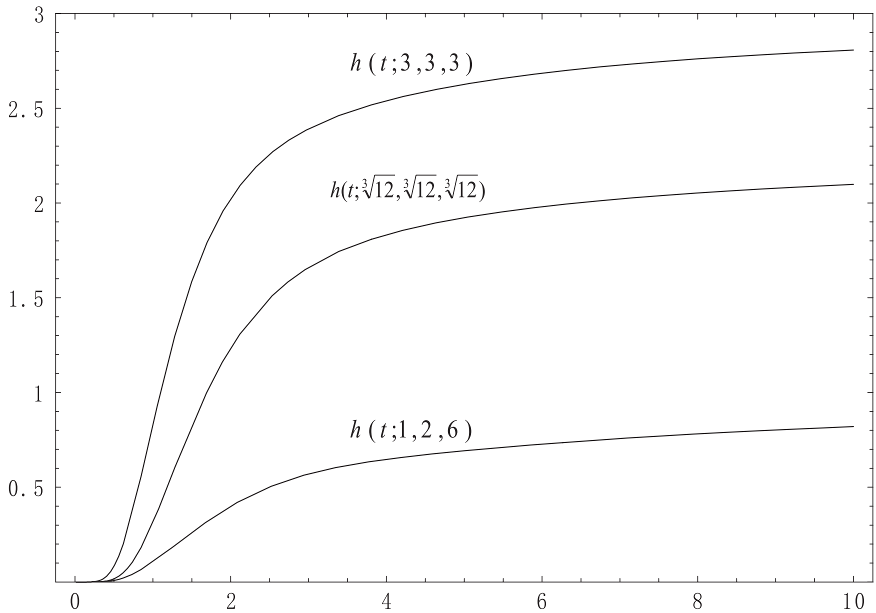

FIGURE **15.** Plots of the hazard rate functions when $r = 3$.

OPEN PROBLEM 6: A natural question that arises is whether the result in Theorem 3.4 also holds for the case when the shape parameter is larger than 1. It is possible that this may be true as can be seen in Figure 15 (the hazard rate plots under the same setup as in Figure 14, but the shape parameter is now 3), but it remains as an open problem.

OPEN PROBLEM 7: Moreover, it would be interesting to see whether the result in Theorem 3.4 can be established for the likelihood ratio order. For the exponential case, such results have been derived by Khaledi and Kochar [23] and Kochar and Xu [29].

4. DISCRETE CASE

4.1. Geometric Case

The geometric distribution is the discrete counterpart of the exponential distribution since they both possess lack of memory property and constant hazard rates. For a geometric random variable X with parameter $p \in (0, 1)$, the probability mass function is given by

$$\mathsf{P}(X = k) = p(1-p)^k, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}.$$

Mao and Hu [34] proved the following result for this geometric case.

THEOREM 4.1: *Let $X_1, \ldots, X_n$ be independent geometric random variables with parameters $p_1, \ldots, p_n$, and $X_1^*, \ldots, X_n^*$ be another set of independent geometric random variables with parameters $p_1^*, \ldots, p_n^*$, respectively. Then,*

$$(p_1, \ldots, p_n) \overset{\mathrm{p}}{\succeq} (p_1^*, \ldots, p_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*.$$

The result in Theorem 4.1 is an analogue of (2.4). Mao and Hu [34] further showed that Theorem 4.1 might not hold for other order statistics by using the following counterexample.

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Example 4.1: *Let (X_1, X_2, X_3) be a vector of independent geometric variables with parameter vector $(p_1, p_2, p_3) = (0.01, 0.1, 0.9)$, and (X_1^*, X_2^*, X_3^*) be another vector of independent geometric variables with parameter vector $(p_1^*, p_2^*, p_3^*) = (0.1, 0.2, 0.5)$. Then, we have*

$$(p_1, p_2, p_3) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*, p_3^*).$$

But, observe that

$$\mathsf{P}(X_{1:3} \geq 1) = 0.081 < 0.360 = \mathsf{P}(X_{1:3}^* \geq 1),$$

which implies that $X_{1:3} \not\succeq_{\mathrm{st}} X_{1:3}^*$.

Moreover, they also used the following counterexample to show an analogue of (2.5) does not hold for the reversed hazard rate order under the geometric framework.

Example 4.2: *Let (X_1, X_2, X_3) be a vector of independent geometric variables with parameter vector $(p_1, p_2, p_3) = (0.2, 0.4, 0.8)$, and (X_1^*, X_2^*, X_3^*) be another vector of independent geometric variables with parameter vector $(p_1^*, p_2^*, p_3^*) = (0.4, 0.4, 0.6)$. Then, we have*

$$(p_1, p_2, p_3) \overset{\mathrm{m}}{\succeq} (p_1^*, p_2^*, p_3^*).$$

However,

$$\frac{F_{X_{1:3}}(1)}{F_{X_{1:3}}(0)} = 3.456 < 3.584 = \frac{F_{X_{1:3}^*}(1)}{F_{X_{1:3}^*}(0)}$$

and

$$\frac{F_{X_{1:3}}(3)}{F_{X_{1:3}}(2)} = 1.3518 > 1.2831 = \frac{F_{X_{1:3}^*}(3)}{F_{X_{1:3}^*}(2)},$$

which imply that $X_{1:3} \not\succeq_{\mathrm{rh}} X_{1:3}^*$.

Recently, Xu and Hu [45] further proved the following multivariate stochastic order result.

Theorem 4.2: *Under the same setup as in Theorem 4.1, we have*

$$(\log(1-p_1), \ldots, \log(1-p_n)) \succeq_{\mathrm{w}} (\log(1-p_1^*), \ldots, \log(1-p_n^*))$$
$$\Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \succeq_{\mathrm{st}} (X_{1:n}^*, \ldots, X_{n:n}^*).$$

The following corollary is a direct consequence of Theorem 4.2.

Corollary 4.1: *Let $X_1, \ldots, X_n$ be independent geometric random variables with parameters $p_1, \ldots, p_n$, respectively, and $Y_1, \ldots, Y_n$ be i.i.d. geometric random variables with a common parameter p. Then,*

$$p \geq p_{\mathrm{cg}} \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \succeq_{\mathrm{st}} (Y_{1:n}, \ldots, Y_{n:n}),$$

where

$$p_{\mathrm{cg}} = 1 - \left\{ \prod_{i=1}^{n} (1-p_i) \right\}^{1/n}.$$

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The following result, due to Mao and Hu [34], compares the largest order statistics from heterogeneous and homogeneous geometric samples in terms of the likelihood ratio order.

THEOREM 4.3: *Let $X_1, \ldots, X_n$ be independent geometric random variables with parameters $p_1, \ldots, p_n$, respectively, and $Y_1, \ldots, Y_n$ be i.i.d. geometric random variables with common parameter p. Then,*

$$p \geq p_{\mathrm{cg}} \Longrightarrow X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}.$$

They also pointed out that the reversed hazard rate order (and hence the likelihood ratio order) does not hold between $X_{n:n}$ and $Y_{n:n}$ under the condition $p \geq p_{\mathrm{am}} = \frac{1}{n}\sum_{i=1}^{n} p_i$ even though it does hold for the corresponding exponential case; see Kochar and Xu [29]. Moreover, they left the question whether the hazard rate order holds between $X_{n:n}$ and $Y_{n:n}$ under the condition $p \geq p_{\mathrm{am}}$ as an open problem. Du, Zhao, and Balakrishnan [16] recently answered this problem partially for the case when $n = 2$ by proving the following result.

THEOREM 4.4: *Let X_1, X_2 be independent geometric variables with parameters p_1, p_2, and X_1^*, X_2^* be another set of independent geometric variables with parameters p_1^*, p_2^*, respectively. Suppose $p_1 \leq p_1^* \leq p_2^* \leq p_2$. Then,*

$$(p_1, p_2) \overset{\mathrm{p}}{\succeq} (p_1^*, p_2^*) \Longleftrightarrow X_{2:2} \geq_{\mathrm{hr}} X_{2:2}^* \Longleftrightarrow X_{2:2} \geq_{\mathrm{st}} X_{2:2}^*.$$

Next, we present a numerical example to illustrate the results established in Theorem 4.4.

EXAMPLE 4.3: *Let (X_1, X_2) be a vector of independent geometric variables with parameter vector $(p_1, p_2) = (1/6, 1/2)$, and $h(k; 1/6, 1/2)$ be the corresponding hazard rate function of $X_{2:2}$. Let (X_1^*, X_2^*) be another vector of independent geometric variables with parameter vector $(p_1^*, p_2^*) = (1/4, 1/5)$, and $h(k; 1/4, 2/5)$ be the corresponding hazard rate function of $X_{2:2}^*$. It can be readily seen that $(p_1^*, p_1) \overset{\mathrm{p}}{\succeq} (p_2^*, p_2)$. Figure 16 presents plots of the hazard rate functions of these two maxima, which are in accordance with the result in Theorem 4.4.*

Du et al. [16] also examined the likelihood ratio order of the maxima in two multiple-outlier geometric samples.

THEOREM 4.5: *Let $X_1, \ldots, X_n$ be independent geometric variables with parameters*

$$(\underbrace{p_1, \ldots, p_1}_{r}, \underbrace{p_2, \ldots, p_2}_{q}),$$

where $r + q = n$, and $Y_1, \ldots, Y_n$ be another set of independent geometric variables with parameters

$$(\underbrace{p_1^*, \ldots, p_1^*}_{r}, \underbrace{p_2^*, \ldots, p_2^*}_{q}),$$

respectively. Then, if $p_1 \leq p_1^* \leq p_2^* \leq p_2$ and

$$\begin{aligned}&(\underbrace{-\log(1-p_1), \ldots, -\log(1-p_1)}_{r}, \underbrace{-\log(1-p_2), \ldots, -\log(1-p_2)}_{q})\\ &\overset{\mathrm{w}}{\succeq} (-\underbrace{\log(1-p_1^*), \ldots, -\log(1-p_1^*)}_{r}, \underbrace{-\log(1-p_2^*), \ldots, -\log(1-p_2^*)}_{q}), \qquad \textbf{(4.1)}\end{aligned}$$

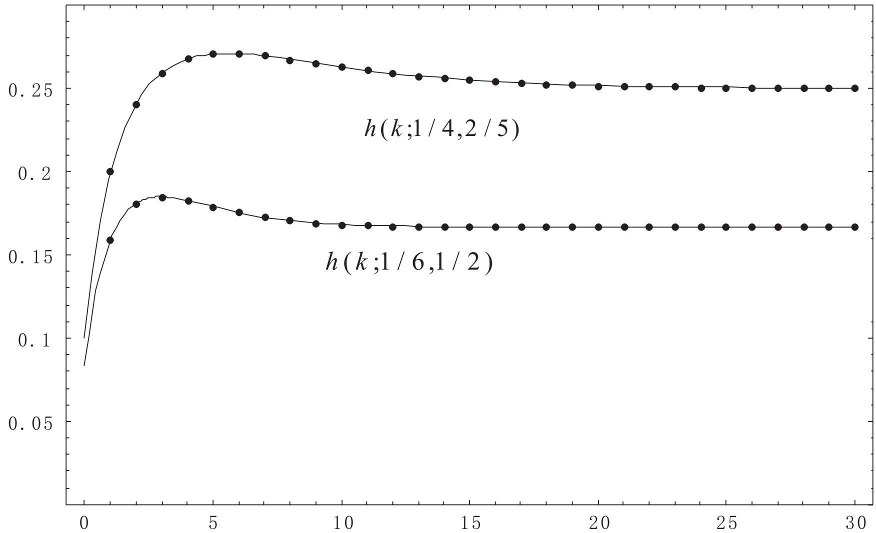

Figure 16. Plots of the hazard rate functions of the maxima of geometric variables with parameter vector $(1/6, 1/2)$ and $(1/4, 2/5)$.

we have

$$X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}.$$

Since the likelihood ratio order implies the hazard rate order, the result in Theorem 4.5 can be used to compare the hazard rate functions of the maxima from two multiple-outlier geometric samples. The following example, from Du et al. [16], illustrates this point.

Example 4.4: *Let* (X_1, X_2, X_3) *be a vector of independent geometric random variables with parameter vector* $(1/6, 1/6, 2/7)$*, and* $h\,(k; 1/6, 1/6, 2/7)$ *be the corresponding hazard rate function of* $X_{3:3}$*. Let* (Y_1, Y_2, Y_3) *be another vector of independent geometric random variables with parameter vector* $(1/5, 1/5, 1/4)$*, and* $h\,(k; 1/5, 1/5, 1/4)$ *be the corresponding hazard rate function of* $Y_{3:3}$*. It can be readily verified that condition (4.1) in Theorem 4.5 is satisfied in this case. Figure 17 presents plots of the hazard rate functions of these two maxima which are readily seen to be in accordance with the result of Theorem 4.5.*

4.2. Negative Binomial Case

The negative binomial distribution is one of the important distributions in statistics, and has wide applications in reliability theory, engineering, game theory, quality control, and communication theory. For a negative binomial random variable X with parameter $(r, p) \in (0, +\infty) \times (0, 1)$, the probability mass function is given by

$$\mathsf{P}(X = k) = \binom{r + k - 1}{k} p^r (1 - p)^k, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Let $X_1, \ldots, X_n$ be a set of independent negative binomial random variables with parameters $(k_1, p_1), \ldots, (k_n, p_n)$, respectively, and let $X_1^*, \ldots, X_n^*$ be another set of independent negative binomial random variables with parameters $(k_1^*, p_1^*), \ldots, (k_n^*, p_n^*)$, respectively, Xu and Hu [45] then obtained the following two results.

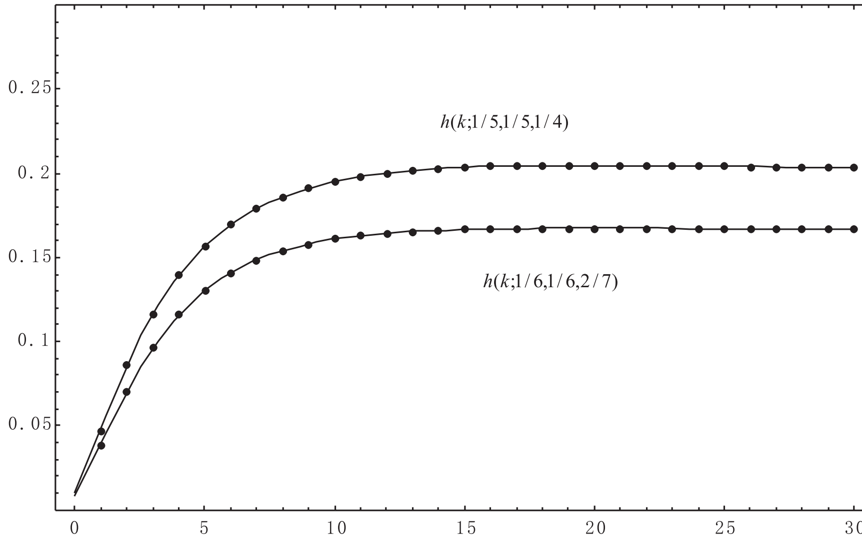

Figure 17. Plots of the hazard rate functions of maxima from two geometric samples with parameter vectors as $(1/6, 1/6, 2/7)$ and $(1/5, 1/5, 1/4)$.

Theorem 4.6: *If $k_i = k_i^* = k$ for $i = 1, \ldots, n$, then*

(i)

$$(p_1, \ldots, p_n) \overset{\text{p}}{\succeq} (p_1^*, \ldots, p_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} X_{n:n}^*;$$

(ii)

$$(1-p_1, \ldots, 1-p_n) \overset{\text{p}}{\succeq} (1-p_1^*, \ldots, 1-p_n^*) \Longrightarrow X_{1:n} \leq_{\text{st}} X_{1:n}^*, \quad \textit{for } k \geq 1;$$

(iii)

$$(\log(1-p_1), \ldots, \log(1-p_n)) \succeq_{\text{w}} (\log(1-p_1^*), \ldots, \log(1-p_n^*)) \\ \Longrightarrow X_{1:n} \geq_{\text{st}} X_{1:n}^*, \quad \textit{for } 0 < k \leq 1.$$

Theorem 4.7: *If $p_i = p_i^* = p$ for $i = 1, \ldots, n$, then*

$$(k_1, \ldots, k_n) \succeq_{\text{w}} [\overset{\text{w}}{\succeq}](k_1^*, \ldots, k_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} X_{n:n}^* [X_{1:n} \leq_{\text{st}} X_{1:n}^*].$$

It is apparent that the result in Part (i) of Theorem 4.6 extends the corresponding one in Theorem 4.1 from the geometric case to the negative binomial case. Also, Xu and Hu [45] gave the following result which can be readily derived by using Part (iii) of Theorem 4.6 and Theorem 2 in Ma [33].

Corollary 4.2: *If $k_i = k_i^* = k \in (0, 1]$ for $i = 1, \ldots, n$ and $p_1^* = \cdots = p_n^* = p$, then*

$$p \geq p_{\text{cg}} \Longrightarrow X_{r:n} \geq_{\text{st}} X_{r:n}^*, \quad r = 1, \ldots, n.$$

Xu and Hu [45] also displayed, with the help of a counterexample, that the result in Corollary 4.2 does not hold for the case when $k_i > 1$ for $i = 1, \ldots, n$.

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