J. Plasma Physics (2013), vol. 79, part 6, pp. 991–993. © Cambridge University Press 2013
doi:10.1017/S0022377813000846

Exact solution to neutrino-plasma two-flavor dynamics

FERNANDO HAAS[1] and JOSÉ T. MENDONÇA[2]

[1]Departamento de Física, Universidade Federal do Paraná, Curitiba, PR, 81531-990, Brazil
(ferhaas@hotmail.com)
[2]IPFN, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

(Received 5 June 2013; revised 1 July 2013; accepted 1 July 2013; first published online 9 August 2013)

Abstract. It is shown that the two-flavor neutrino oscillation equations admit an exact analytic solution for arbitrarily chosen normalized electron neutrino population, provided the electron plasma density is adjusted in a certain way. The associated formula for the electron plasma density is applied to the cases of exponentially decaying or oscillating electron neutrino populations.

1. Introduction

The energy exchange between neutrino beams and plasma collective modes can be a crucial mechanism e.g. for shocks in type II supernovae (Bingham et al. 2004). The associated neutrino charge coupling (Serbeto et al. 2004) leads to kinetic effects such as neutrino Landau damping (Silva et al. 1999), as well as to the generation of quasi-static magnetic fields (Shukla et al. 1998). The orthodox approach to the neutrino–plasma interaction problem is to assume specific medium properties, and then to solve the dynamical equations, either in approximate or numerical forms. In this respect, one can have sinusoidal variations of the electron density (Schafer and Koonin 1987; Krastev and Smirnov 1989; Koike et al. 2009; Kneller et al. 2013), general time-dependent media (Hollenberg and Päs 2012), stochastic backgrounds (Torrente-Lujan 1999; Benatti and Floreanini 2005) as well as instabilities due to electron density ripples (Shukla 2011). In an inverse way, in the present work, a certain electron density profile is assumed, and then the corresponding medium properties are unveiled. The procedure is restricted to two-flavor neutrino populations. No further approximations are needed.

The work is organized as follows. Section 2 describes the general method, leading to (2.8), the central result of the paper. Section 3 briefly discusses the cases of exponentially decaying or oscillating electron neutrino populations. Section 4 is reserved to final remarks.

2. Exact solution

The equations for neutrino-flavor oscillations in a plasma are well known (Raffelt 1996) and we present them in the form

$$\dot{P}_1 = -\Omega(t)P_2\,, \quad \dot{P}_2 = \Omega(t)P_1 - \Omega_0 P_3\,, \quad \dot{P}_3 = \Omega_0 P_2\,, \tag{2.1}$$

where $\mathbf{P} = (P_1, P_2, P_3)$ is the three-dimensional flavor polarization vector, such that the density matrix can be written as

$$\rho = \frac{N_0}{2}(1 + \mathbf{P}\cdot\boldsymbol{\sigma})\,, \tag{2.2}$$

using the total neutrino number $N_0 = N_e + N_\mu$ and the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, with $N_{e,\mu}$ being the electron (muon) neutrino populations. In (2.1),

$$\Omega(t) = \omega_0(\cos 2\theta_0 - \xi(t))\,, \quad \Omega_0 = \omega_0 \sin 2\theta_0\,, \tag{2.3}$$

where we have introduced the characteristic oscillation frequency $\omega_0 = \Delta m^2/2E$, with $\Delta m^2 = m_2^2 - m_1^2$ being the square mass difference between mass eigenstates and E the energy associated with the neutrino Dirac spinor, while θ_0 is the pertinent mixing angle. Finally, we have $\xi(t) = \sqrt{2}G_F n_e/\omega_0$ being the coupling function between the neutrino and the embedding plasma medium, where G_F is the Fermi constant and n_e is the electron plasma density. In our analysis, it is important to keep in mind that $P_3 = (N_e - N_\mu)/N_0$.

From the first and the last equations in (2.1), we get

$$\Omega = -\frac{\dot{P}_1}{P_2}\,, \quad P_2 = \frac{\dot{P}_3}{\Omega_0}\,. \tag{2.4}$$

Substituting the results shown in (2.4) into the mid equality in (2.1) and integrating, once yields

$$I = \dot{P}_3^2 + \Omega_0^2\left(P_3^2 + P_1^2\right) = \Omega_0^2\,, \tag{2.5}$$

where I is a constant of motion, $dI/dt = 0$. The last equality in (2.5) follows from $\dot{P}_3 = \Omega_0 P_2$ and the normalization condition, $|\mathbf{P}| = 1$. Our central result comes from the fact that (2.5) can be solved up to a sign choice for P_1 in terms of P_3, or

$$P_1 = \pm\frac{(\Omega_0^2 - \dot{P}_3^2 - \Omega_0^2 P_3^2)^{1/2}}{\Omega_0}\,. \tag{2.6}$$

Correspondingly, using (2.4) and (2.6), we find

$$\Omega = \pm\frac{\ddot{P}_3 + \Omega_0^2 P_3}{(\Omega_0^2 - \dot{P}_3^2 - \Omega_0^2 P_3^2)^{1/2}}\,. \tag{2.7}$$

Therefore, we have a very simple recipe to generate exact solutions for the two-flavor neutrino-plasma oscillation equations. Instead of prescribing a given plasma density n_e as usual, one can start choosing P_3, which is interpreted as the normalized difference between neutrino flavor populations. Afterward, (2.6) and the last in (2.4) give resp. the coherences P_1 and P_2. Finally, (2.7) gives the corresponding Ω, which is linked to the plasma medium properties. To have meaningful solutions, at least some requirements should be taken into account, namely $|P_3| \leqslant 1$, otherwise one would eventually get negative flavor populations. In addition, P_3 should be a double-differentiable function of time.

Alternatively, we can use $P_3 = 2N_e/N_0 - 1$ to express the results in terms of the electron neutrino population. From (2.4), (2.6) and (2.7), we get

$$P_1 = \pm 2\left(\bar{N}_e - \bar{N}_e^2 - \frac{\dot{\bar{N}}_e^2}{\Omega_0^2}\right)^{1/2}, \quad P_2 = \frac{2\dot{\bar{N}}_e}{\Omega_0},$$

$$\Omega = \pm\frac{\ddot{\bar{N}}_e + \Omega_0^2(\bar{N}_e - 1/2)}{\Omega_0\left(\bar{N}_e - \bar{N}_e^2 - \dot{\bar{N}}_e^2/\Omega_0^2\right)^{1/2}}, \tag{2.8}$$

where $\bar{N}_e \equiv N_e/N_0$. The results in (2.8) compactly represent the basic findings of this work.

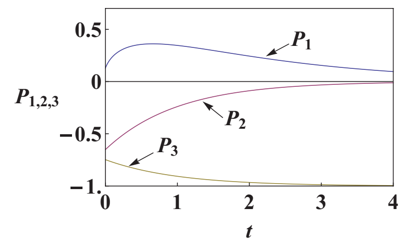

Figure 1. (Colour online) Polarization vector components for an exponentially decaying electron neutrino population, according to (2.8) and (3.1). Parameters, $\omega_0 = 1, r_0 = 1$, $\sin^2 2\theta_0 = 0.15$, $\bar{N}_e(t_0)\exp(t_0/r_0) = 0.13$.

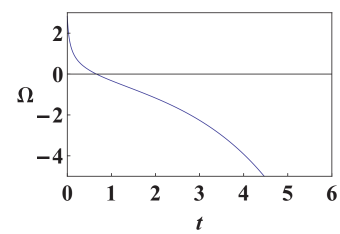

Figure 2. (Colour online) Function $\Omega(t)$ for an exponentially decaying electron neutrino population, according to (2.8) and (3.1) and the same parameters as in Fig. 1.

3. Applications

3.1. *Exponentially decaying electron neutrino population*

As a first example, consider the case of an exponentially decaying electron neutrino population,

$$\bar{N}_e = \bar{N}_e(t_0)\exp\left(-\frac{t-t_0}{r_0}\right), \tag{3.1}$$

which models the change of the electron number density along the path of the solar neutrinos moving radially from the central region to the surface of the Sun (Petkov 1988, 1997). In this context, r_0 is the scale height and $t - t_0$ is the distance traveled by the neutrinos. To have meaningful solutions from (2.8) (or, real $P_{1,2,3}$), one should have $\bar{N}_e(t_0)\exp(t_0/r_0) > (1 + 1/\Omega_0^2 r_0^2)^{-1}$, as can be readily verified. We use stretched time and space variables so that $\omega_0 = 1, r_0 = 1$. Moreover, the mixing angle satisfies $\sin^2 2\theta_0 = 0.15$, so that $\Omega_0 = 0.39$. Finally, we chose $\bar{N}_e(t_0)\exp(t_0/r_0) = 0.13$, which assures the produced solutions to be non-complex. The resulting polarization vector components are shown in Fig. 1, while $\Omega(t)$ is shown in Fig. 2, with the plus sign chosen in (2.8). It can be shown that in this case, one has the asymptotic dependence $\Omega \propto -\Omega_0\exp(t/2r_0)$ when $t \to \infty$. Evidently, an infinite class of profiles can be generated via the same procedure. One can, e.g., consider the case of an oscillating electron neutrino population, discussed in the following.

3.2. *Periodic electron density*

Now, consider an initially unpolarized electron neutrino beam,

$$\bar{N}_e = \frac{1}{2} + \frac{\varepsilon}{2}\sin\tilde{\Omega}t, \tag{3.2}$$

including an amplitude parameter $\varepsilon \geqslant 0$ and an arbitrary frequency $\tilde{\Omega}$. A simple analysis shows that $\varepsilon < \mathrm{Inf}(1, \Omega_0/\tilde{\Omega})$ is the condition to avoid singularities. The corresponding polarization vector components and $\Omega(t)$ function are shown resp. in Figs. 3 and 4, for $\Omega_0 = 0.39$ as before and for $\varepsilon = 0.39, \tilde{\Omega} = 1.0$.

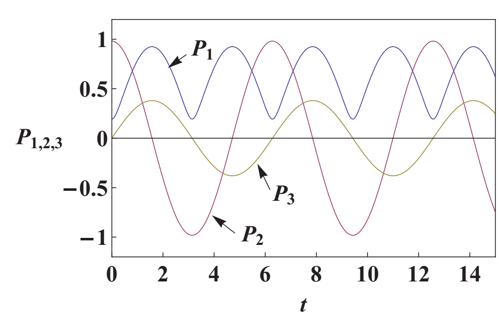

Figure 3. (Colour online) Polarization vector components for an oscillating electron neutrino population, according to (2.8) and (3.2). Parameters, $\Omega_0 = 0.39, \varepsilon = 0.39, \tilde{\Omega} = 1.0$.

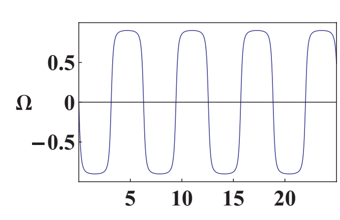

Figure 4. (Colour online) Function $\Omega(t)$ for an oscillating electron neutrino population, according to (2.8) and (3.2) and the same parameters as in Fig. 3.

4. Conclusion

In this work, the usual route for solving the two-flavor neutrino-plasma oscillation equations has been subverted. Namely, instead of setting a certain electron plasma density and then looking for the polarization vector components, here the third component $P_3(t)$ and equivalently the electron neutrino population $N_e(t)$ are chosen *ab initio*. Consequently, simple formulas for the coherences $P_{1,2}(t)$ are readily found. The necessary condition for the recipe to work is to adjust the function $\Omega(t)$ and hence the electron plasma density $n_e(t)$ so that (2.7) holds. The results can be expressed in terms of the electron neutrino population only, see (2.8). In a sense, our exact neutrino flavor solution has similarities with the celebrated Bernstein–Greene–Kruskal equilibria for the Vlasov–Poisson system (Bernstein et al. 1957), where arbitrarily chosen electrostatic potentials can be constructed provided specific trapped electron distributions are set.

Acknowledgements

Fernando Haas acknowledges CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) for financial support. This work is dedicated to the memory of Prof. Padma Kant Shukla.

References

Benatti, F. and Floreanini, R. 2005 Dissipative neutrino oscillations in randomly fluctuating matter. *Phys. Rev.* D **71**, 013003.

Bernstein, I. B., Greene, J. M. and Kruskal, M. D. 1957 Exact nonlinear plasma oscillations. *Phys. Rev.* **1087**, 546–550.

Bingham, R., Silva, L. O., Mendonça, J. T., Shukla, P. K., Mori, W. B. and Serbeto, A. 2004 Neutrino plasma coupling in dense astrophysical plasmas. *Plasma Phys. Control. Fusion* **46**, B327–B334.

Hollenberg, S. and Päs, H. 2012 Adiabatic and nonadiabatic perturbation theory for coherence vector description of neutrino oscillations. *Phys. Rev.* D **85**, 013013.

Kneller, J. P., McLaughlin, G. C. and Patton, K. M. 2013 Stimulated neutrino transformation with sinusoidal density profiles. *J. Phys. G.: Nucl. Part. Phys.* **40**, 055002.

Koike, M., Ota, T., Saito, M. and Satoa, J. 2009 Fourier analysis of the parametric resonance in neutrino oscillations. *Phys. Lett.* B **675**, 69–72.

Krastev, P. I. and Smirnov, A. Y. 1989 Parametric effects in neutrino oscillations. *Phys. Lett.* B **226**, 341–346.

Petkov, S. T. 1988 Exact analytic description of two-neutrino oscillations in matter with exponentially varying density. *Phys. Lett.* B **200**, 373–379.

Petkov, S. T. 1997 Describing analytically the matter-enhanced two-neutrino transitions in a medium. *Phys. Lett.* B **406**, 355–365.

Raffelt, G. G. 1996 *Stars as Laboratories for Fundamental Physics*. University of Chicago Press.

Schafer, A .and Koonin, S. E. 1987 Influence of density fluctuations on solar neutrino conversion. *Phys. Lett.* B **185**, 417–420.

Serbeto, A., Rios, L. A., Mendonça, J. T. and Shukla, P. K. 2004 Neutrino (antineutrino) effective charge in a magnetized electron-positron plasma. *Phys. Plasmas* **11**, 1352–1357.

Shukla, P. K. 2011 Amplification of neutrino oscillations by a density ripple in dense plasmas. *J. Plasma Phys.* **77**, 289–291.

Shukla, P. K., Bingham, R., Mendonça, J. T. and Stenflo, L. 1998 Neutrinos generating inhomogeneities and magnetic fields in the early universe. *Phys. Plasmas* **5**, 2815–2817.

Silva, L. O., Bingham, R., Dawson, J. M., Mendonça, J. T. and Shukla, P. K. 1999 Neutrino driven streaming instabilities in a dense plasma. *Phys. Rev. Lett.* **83**, 2703–2706.

Torrente-Lujan, E. 1999 Finite dimensional systems with random external fields and neutrino propagation in fluctuating media. *Phys. Rev.* D **59**, 073001.