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SOME HOMOLOGICAL PROPERTIES OF FOURIER ALGEBRAS ON HOMOGENEOUS SPACES

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Abstract

Let H be a compact subgroup of a locally compact group G. We first investigate some (operator) (co)homological properties of the Fourier algebra $A(G/H)$ of the homogeneous space G/H such as (operator) approximate biprojectivity and pseudo-contractibility. In particular, we show that $A(G/H)$ is operator approximately biprojective if and only if G/H is discrete. We also show that $A(G/H)^{**}$ is boundedly approximately amenable if and only if G is compact and H is open. Finally, we consider the question of existence of weakly compact multipliers on $A(G/H)$.

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1. Introduction

The foundation stone of the (co)homology theory of Banach algebras was laid by Helemskii and Johnson from distinctly different viewpoints. The problems that arise in this subject have been studied by many other mathematicians in the setting of the Fourier algebra $A(G)$ of a locally compact group G. The notion of Fourier algebra on homogeneous spaces of a locally compact group was first introduced and investigated by Forrest [7]. Let G be a locally compact group, H a compact subgroup of G and $A(G/H)$ the Fourier algebra of the homogeneous space G/H, which is the subalgebra consisting of the functions in $A(G)$ that are constant on the left cosets modulo H. It was shown in [8, 17] that many properties of $A(G/H)$ associated with amenability, such as biprojectivity and operator amenability, are closely linked to such properties of $A(G)$. Thus, we are naturally motivated to study approximate (co)homological properties of $A(G/H)$.

Our study of (weakly) compact multipliers of $A(G/H)$ is motivated by the question, if $A(G/H)$ has a non-zero (weakly) compact multiplier, must G/H be discrete? An affirmative answer to this question will provide some characterisations of discreteness of G/H in terms of finite-dimensional ideals of $A(G/H)$.

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The organisation of the paper is as follows. In Section 3 we study operator approximate biprojectivity and pseudo-contractiblity of Fourier algebras on homogeneous spaces. In particular, we show that the Banach algebra $A(G/H)$ is operator approximately biprojective if and only if H is open. We also show that in the presence of approximate identities, discreteness of G/H is a necessary and sufficient condition for $A(G/H)$ to be (operator) pseudo-contractible. The purpose of Section 4 is to investigate the relation between the existence of weakly compact multipliers of $A(G/H)$ and discreteness of G/H. This work generalises the corresponding result of [15].

2. Preliminaries

Let G be a locally compact group with fixed left Haar measure dx. Let $B(G)$ denote the *Fourier–Stieltjes algebra* of G consisting of all coefficient functions arising from all the weakly continuous unitary representations of G. The *Fourier algebra* $A(G)$ is a closed ideal of $B(G)$, consisting of coefficient functions $u(\cdot) = \langle \lambda(\cdot) f, g\rangle$ where λ is the left regular representation of G on the Hilbert space $L^2(G)$ defined by $\lambda(t)f(x) = f(t^{-1}x)$ for all $x, t \in G$ and $f \in L^2(G)$. The dual of $A(G)$ is isometrically isomorphic to $VN(G)$, the *group von Neumann algebra* which is generated by λ in the operator algebra $B(L^2(G))$. See [6, 12] for more information on $B(G)$, $A(G)$ and $VN(G)$.

Let H be a compact subgroup of a locally compact group G and let G/H be the homogeneous space of left cosets of H. Suppose that $p : G \to G/H$ is the canonical quotient map. We write $\tilde{x}$ for the left coset $xH = p(x)$. As in [7], we define the sets

$$\begin{aligned} B(G : H) &= \{u \in B(G) : u \text{ is constant on the left cosets of } H\},\\ A(G : H) &= \overline{\{u \in B(G : H) : p(\operatorname{supp} u) \text{ is compact in } G/H\}}^{\|\cdot\|_{B(G)}}. \end{aligned}$$

It was shown by Forrest in [7] that $B(G : H)$ is a closed subalgebra of $B(G)$ and is the range of the projection map P_H defined on $B(G)$ by $P_H u(x) = \int_H u(xh)\,dh$ for $x \in G$. When P_H is restricted to $A(G)$, it is a projection onto $A(G : H)$. Since the elements of $B(G : H)$ are constant on cosets of H, it is also identified as an algebra of functions on G/H. We denote by $A(G/H)$ and $B(G/H)$ the corresponding Fourier and Fourier–Stieltjes algebras on the homogeneous space G/H. Here we recall some of the well-known properties of the Fourier algebra $A(G/H)$ that we shall need. The algebra $A(G/H)$ is a commutative, semisimple and regular Banach algebra with $A(G/H) \subseteq C_0(G/H)$, the space of continuous functions on G/H that vanish at infinity. The Gelfand structure space of $A(G/H)$ is G/H. Finally, the dual of $A(G/H)$ is isometrically isomorphic to $VN(G/H)$ and this space is the closure of $\{\lambda(f) : f \in L^1(G),\ f \text{ constant on the left cosets of } H\}$ in $VN(G)$, with respect to the weak* topology. Note that in general, $VN(G/H)$ need not be a von Neumann algebra.

Let $\mathcal{A}$ be a Banach algebra. The first Arens product $\Box$ on $\mathcal{A}^{**}$ is defined by the following three steps. For a, b in $\mathcal{A}$, T in $\mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$, define $T \cdot a$, $m \cdot T \in \mathcal{A}^*$ and $m \Box n \in \mathcal{A}^{**}$ by

$$\langle T \cdot a, b\rangle = \langle T, ab\rangle, \quad \langle m \cdot T, a\rangle = \langle m, T \cdot a\rangle, \quad \langle m \Box n, T\rangle = \langle m, n \cdot T\rangle.$$

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As is well known, this multiplication naturally induces a Banach algebra multiplication on $\mathcal{A}^{**}$ which extends that on $\mathcal{A}$.

By analogy with the case of topological groups, we define

$$UCB(\widehat{G/H}) = \operatorname{span}\{\tilde{u} \cdot T : \tilde{u} \in A(G/H), T \in VN(G/H)\}^{-\|\cdot\|_{VN(G/H)}}$$

The dual of the space $UCB(\widehat{G/H})$ equipped with the multiplication induced by that on $VN(G/H)^*$ is also a Banach algebra. A linear functional $m \in VN(G/H)^*$ is called a *topologically invariant mean* on $VN(G/H)$ if $\|m\| = \langle m, \lambda(\tilde{e})\rangle = 1$ and $\langle m, \tilde{u} \cdot \Phi\rangle = \tilde{u}(\tilde{e})\langle m, \Phi\rangle$ for all $\Phi \in VN(G/H)$ and $\tilde{u} \in A(G/H)$, where $\lambda(\tilde{e})$ is the multiplicative linear functional on $A(G/H)$ defined by $\lambda(\tilde{e})(\tilde{u}) = \tilde{u}(\tilde{e})$. We denote by $TIM(\widehat{G/H})$ the set of all topologically invariant means on $VN(G/H)$. It was shown by Chu and Lau [3] that $VN(G/H)$ always admits a topologically invariant mean. Throughout, H will denote a compact subgroup of a locally compact group G.

3. Some (co)homological properties of $A(G/H)$

A *derivation* from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is a linear map $D : \mathcal{A} \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for all $a, b \in \mathcal{A}$. For each $x \in X$, the map $ad_x : \mathcal{A} \to X$ defined by

$$ad_x(a) = a \cdot x - x \cdot a, \quad a \in \mathcal{A},$$

is a derivation, which is called the *inner derivation* induced by x.

A Banach algebra $\mathcal{A}$ is called *boundedly approximately amenable* if for every Banach $\mathcal{A}$-bimodule X, every bounded derivation D from $\mathcal{A}$ into X^* is bounded approximately inner, that is, there is a bounded net (D_α) of inner derivations such that

$$D(a) = \lim_\alpha D_\alpha(a),$$

for all $a \in \mathcal{A}$.

A (completely contractive) Banach algebra $\mathcal{A}$ is called (*operator*) *approximately biprojective* if there is a net (ρ_α) of (completely) bounded $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\widehat{\otimes}\mathcal{A}$ (respectively, $\mathcal{A}\widehat{\otimes}_{\mathrm{op}}\mathcal{A}$) such that $\pi \circ \rho_\alpha(a) \to a$ for all $a \in \mathcal{A}$, where $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ (respectively, $\pi : \mathcal{A}\widehat{\otimes}_{\mathrm{op}}\mathcal{A} \to \mathcal{A}$) is the product morphism and $\widehat{\otimes}$ ($\widehat{\otimes}_{\mathrm{op}}$) is the (operator) projective tensor product of (completely contractive) Banach algebras. The fact that $A(G)$ is the predual of a von Neumann algebra allows us to equip $A(G)$ with a natural operator space structure. We know that a closed subspace of an operator space is also an operator space, hence $A(G/H)$ and $VN(G/H)$ are operator spaces. A Banach algebra which is also an operator space is said to be completely contractive if the multiplication is a complete contraction (see [5] for more details). In this section we characterise some operator (co)homological properties of $A(G/H)$.

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THEOREM 3.1. *The following conditions are equivalent:*

(i) *H is open;*
(ii) *$A(G/H)$ is operator biprojective;*
(iii) *$A(G/H)$ is operator approximately biprojective.*

PROOF. (i) ⇒(ii) follows from [17, Theorem 4.6] and (ii) ⇒(iii) is obvious.

(iii) ⇒(i). Suppose that $A(G/H)$ is operator approximately biprojective and let

$$\rho_\alpha : A(G/H) \to A(G/H)\widehat{\otimes}_{\mathrm{op}}A(G/H)$$

be a net of completely bounded $A(G/H)$-bimodule morphisms such that $\pi \circ \rho_\alpha(\tilde{u}) \to \tilde{u}$ for all $\tilde{u} \in A(G/H)$. Let $\iota : A(G/H) \to A(G/H)$ be the identity map and $\lambda(\tilde{e})$ the (completely) bounded functional on $A(G/H)$ defined by $\lambda(\tilde{e})(\tilde{u}) = \tilde{u}(\tilde{e})$. Put

$$S = \iota \otimes \lambda(\tilde{e}) : A(G/H)\widehat{\otimes}_{\mathrm{op}}A(G/H) \to A(G/H).$$

By checking with elementary tensors, one can see that S satisfies

$$S(\tilde{u} \cdot \varphi) = \tilde{u}S(\varphi), \quad S(\varphi \cdot \tilde{u}) = \tilde{u}(\tilde{e})S(\varphi) \quad \text{and} \quad S(\varphi)(\tilde{e}) = \pi(\varphi)(\tilde{e})$$

for all $\tilde{u} \in A(G/H)$ and $\varphi \in A(G/H)\widehat{\otimes}_{\mathrm{op}}A(G/H)$. Let $S_\alpha : A(G/H) \to A(G/H)$ be the net of completely bounded left $A(G/H)$-module maps defined by

$$S_\alpha := S \circ \rho_\alpha.$$

It easy to see that

$$S_\alpha(\tilde{u}\tilde{v}) = \tilde{v}(\tilde{e})S_\alpha(\tilde{u})$$

for all $\tilde{u}, \tilde{v} \in A(G/H)$. Given $\tilde{u}_0 \in A(G/H)$ with $u_0(\tilde{e}) = 1$, we can find α_0 such that $\pi \circ \rho_{\alpha_0}(\tilde{u}_0)(\tilde{e}) \neq 0$. Putting $\tilde{v}_0 = S_{\alpha_0}(\tilde{u}_0) \in A(G/H)$, we have

$$\tilde{v}_0(\tilde{e}) = S_{\alpha_0}(\tilde{u}_0)(\tilde{e}) = S(\rho_{\alpha_0}(\tilde{u}_0))(\tilde{e}) = \pi(\rho_{\alpha_0}(\tilde{u}_0))(\tilde{e}) \neq 0.$$

Moreover, for each $\tilde{u} \in A(G/H)$,

$$\tilde{u}\tilde{v}_0 = \tilde{u}S_{\alpha_0}(\tilde{u}_0) = S_{\alpha_0}(\tilde{u}\tilde{u}_0) = S_{\alpha_0}(\tilde{u}_0\tilde{u}) = S_{\alpha_0}(\tilde{u}_0)\tilde{u}(\tilde{e}) = \tilde{u}(\tilde{e})\tilde{v}_0.$$

Since $A(G/H)$ separates points in G/H, we conclude that $\tilde{v}_0/\tilde{v}_0(\tilde{e}) = \mathbf{1}_{\tilde{e}}$, the characteristic function at $\{\tilde{e}\}$. Hence, H is open. □

It follows from [17, Theorem 3.3] that if G has an abelian subgroup of finite index, then the maximal structure in $A(G/H)$ coincides with the operator space structure. In this case, the operator approximate biprojectivity of $A(G/H)$ is equivalent to its approximate biprojectivity. Therefore, we obtain the following result as a consequence of Theorem 3.1.

COROLLARY 3.2. *Suppose that G has an abelian subgroup of finite index. Then $A(G/H)$ is approximately biprojective if and only if H is open.*

Following [10], a (completely contractive) Banach algebra $\mathcal{A}$ is (*operator*) *pseudo-contractible* if there is a net (φ_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ (respectively, $\mathcal{A}\widehat{\otimes}_{\mathrm{op}}\mathcal{A}$) such that for each $a \in \mathcal{A}$,

$$a \cdot \varphi_\alpha = \varphi_\alpha \cdot a, \quad \pi(\varphi_\alpha)a \to a.$$

THEOREM 3.3. *The following conditions are equivalent:*

(i) *$A(G/H)$ has an approximate identity and H is open;*
(ii) *$A(G/H)$ has an approximate identity and is (operator) approximately biprojective;*
(iii) *$A(G/H)$ is (operator) pseudo-contractible.*

PROOF. We will prove the operator space version of the theorem. The other case follows from similar arguments and [10, Proposition 3.8].

(ii) $\Leftrightarrow$ (i) follows from Theorem 3.1.

(i) $\Rightarrow$ (iii). Let $(\tilde{u}_\alpha)$ be an approximate identity for $A(G/H)$ and let H be open. Since the elements with compact support are dense in $A(G/H)$, we can assume that each $\tilde{u}_\alpha$ has compact, and hence finite, support F_α. For every α, define $\varphi_\alpha \in A(G/H)\widehat{\otimes}_{\mathrm{op}}A(G/H)$ by

$$\varphi_\alpha = \sum_{\tilde{x} \in F_\alpha} \tilde{u}_\alpha(\tilde{x})(\mathbf{1}_{\tilde{x}} \otimes \mathbf{1}_{\tilde{x}}).$$

It is not hard to see that $\pi(\varphi_\alpha) = \tilde{u}_\alpha$ and $\tilde{v} \cdot \varphi_\alpha = \varphi_\alpha \cdot \tilde{v}$ for all $\tilde{v} \in A(G/H)$. Therefore, $A(G/H)$ is operator pseudo-contractible.

(iii) $\Rightarrow$ (i). Let (φ_α) be an operator approximate diagonal for $A(G/H)$ such that $\tilde{v} \cdot \varphi_\alpha = \varphi_\alpha \cdot \tilde{v}$ for all $\tilde{v} \in A(G/H)$. Let $S = \iota \otimes \lambda(\tilde{e})$ be as in the proof of Theorem 3.1. Arguing as in the proof of Theorem 3.1, there is $\tilde{u}_0 = S(\varphi_{\alpha_0})$ such that $\tilde{u}_0/\tilde{u}_0(\tilde{e}) = \mathbf{1}_{\tilde{e}}$. Hence, H is open. □

THEOREM 3.4. *$A(G/H)^{**}$ is boundedly approximately amenable if and only if G is compact and H is open.*

PROOF. Let G be compact and H be open. Then, $A(G/H) = A(G/H)^{**} = \mathbb{C}^n$, for some positive integer n, so the result is clear.

Conversely, assume that $A(G/H)^{**}$ is boundedly approximately amenable. Let $m \in TIM(\widehat{G/H})$. Then it is easy to see that

$$n \Box m = \langle n, \lambda(\tilde{e})\rangle m$$

for all $n \in A(G/H)^{**}$. Therefore, $I := m \Box A(G/H)^{**}$ is a closed two-sided ideal in $A(G/H)^{**}$. The mapping $n \mapsto m \Box n$ from $A(G/H)^{**}$ onto I is a continuous projection, which implies that I is complemented. By [9, Corollary 2.4], I admits a right approximate identity, say (e_α). Since for each $n \in TIM(\widehat{G/H})$, we have $n \Box e_\alpha = e_\alpha$ and $n = m \Box n \in I$, we conclude that

$$n = \lim_\alpha n \Box e_\alpha = \lim_\alpha e_\alpha.$$

This implies that $TIM(\widehat{G/H})$ is a singleton and so H must be open by [13, Corollary 1.9]. Thus, $A(G/H)$ is an ideal in its bidual by Corollary 4.2 and, from [2, Lemma 5.2],

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$A(G/H)^{**}$ is unital. Therefore, $A(G/H)$ has a bounded approximate identity. Following [7], we will identify $A(G/H)$ with $A(G:H)$. Thus, $A(G/H)$ can be viewed as a closed subspace of $A(G)$. Hence, by [4, page 816], $A(G/H)$ is a weakly sequentially complete Banach algebra, whence $A(G/H)$ is Arens regular by [1, Theorem 1.6]. Therefore, by [4, Theorem 2.9.39], $A(G/H)$ has an identity and so H is of finite index. Therefore, G must be compact. □

4. Weakly compact multipliers of $A(G/H)$

A bounded linear operator on a Banach algebra $\mathcal{A}$ is called a right (respectively, left) multiplier if it satisfies $R(ab) = aR(b)$ (respectively, $L(ab) = L(a)b$) for all $a, b \in \mathcal{A}$. In particular, for each $a \in \mathcal{A}$ the multiplication operators $\rho_a : \mathcal{A} \to \mathcal{A}$ and $\ell_a : \mathcal{A} \to \mathcal{A}$ defined by $\rho_a(b) = ba$ and $\ell_a(b) = ab$ are a right multiplier and a left multiplier of $\mathcal{A}$, respectively. For the general theory of multipliers we refer to Larsen [14].

In the group setting Lau in [15] proved that a locally compact group G is discrete if and only if its Fourier algebra $A(G)$ has a nonzero compact or even weakly compact multiplier. In this section, we partially extend this result to the setting of homogeneous spaces. Let H be a compact subgroup of G and let

$$S(G/H) = \{\tilde{v} \in A(G/H) : \|\tilde{v}\| = \tilde{v}(\tilde{e}) = 1\}.$$

Then $S(G/H)$ is a commutative semigroup with pointwise multiplication.

THEOREM 4.1. *The following conditions are equivalent:*

(i) *H is open;*
(ii) *$\rho_{\tilde{u}}$ is compact for every $\tilde{u} \in A(G/H)$;*
(iii) *there exists $\tilde{u} \in A(G/H)$ such that $\rho_{\tilde{u}}$ is weakly compact and $\tilde{u}(\tilde{e}) \neq 0$.*

PROOF. (i) ⇒ (ii). Suppose that H is open and let $\tilde{a} \in G/H$. Putting $\tilde{u} = \mathbf{1}_{\tilde{a}}$, we have $\rho_{\tilde{u}}(A(G/H)) = \{\lambda \mathbf{1}_{\tilde{a}} : \lambda \in \mathbb{C}\}$. This implies that $\rho_{\tilde{u}}$ is compact. Therefore, $\rho_{\tilde{u}}$ is compact for every $\tilde{u} \in A(G/H)$ with finite support. Since the set of all $\tilde{u} \in A(G/H)$ such that $\tilde{u}$ has finite support is dense in $A(G/H)$, a simple approximation argument shows that $\rho_{\tilde{u}}$ is compact for all $\tilde{u} \in A(G/H)$.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i). Let K be the norm closure of the set $\{\tilde{u}\tilde{v} : \tilde{v} \in S(G/H)\}$. Then K is a weakly compact convex subset of $A(G/H)$. For each $\tilde{v} \in S(G/H)$ we can define an affine map $r_{\tilde{v}} : K \to K$ by $r_{\tilde{v}}(\varphi) = \tilde{v}\varphi$ for all $\varphi \in K$. By the Markov–Kakutani fixed point theorem, the commuting family $\{r_{\tilde{v}} : \tilde{v} \in S(G/H)\}$ has a fixed point $\varphi_0 \in K$, that is, $\tilde{v}\varphi_0 = \varphi_0$ for all $\tilde{v} \in S(G/H)$. Moreover, $\varphi_0(\tilde{e}) \neq 0$. Now we show that $S(G/H)$ generates $A(G/H)$. Indeed, let $P : A(G) \to A(G/H)$ be the contractive projection

$$(Pu)(x) = \int_H u(xh)\, dh$$

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as defined in [7, Theorem 3.3]. Then it is easy to see that $P(A(G) \cap P^1(G)) \subseteq S(G/H)$, where $P^1(G)$ is the set of all positive definite functions on G having value 1 at e. Since $A(G) \cap P^1(G)$ generates $A(G)$, it follows that $S(G/H)$ generates $A(G/H)$. Moreover, $A(G/H)$ separates points in G/H. Therefore, $\varphi_0/\varphi_0(\tilde{e}) = \mathbf{1}_{\tilde{e}}$, whence H is open. □

It is known that a Banach algebra $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$ if and only if multiplication operators in $\mathcal{A}$ are weakly compact (see [4, page 248]). The next result generalises [16, Theorem 3.7].

Corollary 4.2. *H is open if and only if $A(G/H)$ is an ideal in $VN(G/H)^*$.*

Before we give the next result, recall that $\lambda(\tilde{e})$ is a multiplicative linear functional on $A(G/H)$ defined by $\lambda(\tilde{e})(\tilde{u}) = \tilde{u}(\tilde{e})$.

Corollary 4.3. *The following conditions are equivalent:*

(i) *H is open;*
(ii) *$A(G/H)$ has a one-dimensional ideal I such that $\lambda(\tilde{e})|_I \neq 0$;*
(iii) *$A(G/H)$ has a finite-dimensional ideal I such that $\lambda(\tilde{e})|_I \neq 0$.*

Proof. (i) ⇒ (ii). Suppose that H is open. Then for each $\tilde{a} \in G/H$ the set $\mathbb{C}\mathbf{1}_{\tilde{a}}$ is a nonzero one-dimensional ideal in $A(G/H)$.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i). Let I be a finite-dimensional ideal in $A(G/H)$ with $\lambda(\tilde{e})|_I \neq 0$. Then there is $\tilde{u} \in I$ such that $\tilde{u}(\tilde{e}) \neq \{0\}$ and $\rho_{\tilde{u}}$ has finite rank. Therefore, Theorem 4.1 implies that H is open. □

Corollary 4.4. *H is open if and only if there exists $\tilde{u} \in B(G/H)$ such that $\rho_{\tilde{u}}$ is weakly compact on $B(G/H)$ and $\tilde{u}(\tilde{e}) \neq 0$.*

Proof. Let $\tilde{v} \in A(G/H)$ be such that $\tilde{v}(\tilde{e}) \neq 0$. Then $\rho_{\tilde{v}\tilde{u}}$ is weakly compact on $A(G/H)$. Hence, H is open by Theorem 4.1. For the converse, choose $\tilde{u} = \mathbf{1}_{\tilde{e}}$. □

Proposition 4.5. *H is open if and only if there is a weakly compact right multiplier $R : VN(G/H)^* \to VN(G/H)^*$ and $m \in VN(G/H)^*$ such that $R(m) \in A(G/H)$ and $\langle R(m), \lambda(\tilde{e})\rangle \neq 0$.*

Proof. Suppose that H is open and consider $\tilde{u} = \mathbf{1}_{\tilde{e}} \in A(G/H)$. Then the map $\Lambda_{\tilde{u}} : VN(G/H)^* \to VN(G/H)^*$ defined by $\Lambda_{\tilde{u}}(m) = m\Box\tilde{u}$, is a weakly compact right multiplier of $VN(G/H)^*$ with the desired properties.

Conversely, first note that for each $\tilde{v} \in A(G/H)$,

$$\rho_{R(m)}(\tilde{v}) = \tilde{v}R(m) = \tilde{v}\Box R(m) = R(\tilde{v}\Box m) = R \circ \rho_m(\tilde{v}).$$

Using this and the fact that the restriction of R on $A(G/H)$ is weakly compact, we conclude that $\rho_{R(m)} = R \circ \rho_m$ is weakly compact on $A(G/H)$. Since $\langle R(m), \lambda(\tilde{e})\rangle \neq 0$, H must be discrete by Theorem 4.1. □

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REMARK 4.6. The condition $R(m) \in A(G/H)$ cannot be removed in Proposition 4.5. In fact, let m be a topologically invariant mean on $VN(G/H)$. Then, the map $R : VN(G/H)^* \to VN(G/H)^*$ defined by $R(n) = n\Box m = \langle n, \lambda(\tilde{e})\rangle m$ is a rank-one right multiplier of $VN(G/H)^*$ and hence is weakly compact.

Let $B(VN(G/H))$ denote the space of all bounded linear operators on $VN(G/H)$. Let $B_{A(G/H)}(VN(G/H))$ denote the subspace of $B(VN(G/H))$ consisting of all $\Lambda \in B(VN(G/H))$ such that $\Lambda(\tilde{u} \cdot T) = \tilde{u} \cdot \Lambda(T)$ for all $\tilde{u} \in A(G/H)$ and $T \in VN(G/H)$. It is easily verified that the map

$$\tau : UCB(\widehat{G/H})^* \to B_{A(G/H)}(VN(G/H)), \quad m \mapsto m_L$$

induces a contractive and injective algebra homomorphism, where m_L is given by $\langle m_L(T), \tilde{u}\rangle = \langle m, \tilde{u} \cdot T\rangle$ for all $\tilde{u} \in A(G/H)$ and $T \in VN(G/H)$. Our interest in the following property of Fourier algebras $A(G/H)$ stemmed from our study of [15], where it was proved by Lau that if G is an amenable group, then there is an isometric algebra isomorphism between the space $B_{A(G)}(VN(G))$ and the space $UCB(\widehat{G})^*$. As pointed out in [11], the result remains valid if G is replaced by G/H. A natural question is whether the converse holds. The next theorem shows that this question has an affirmative answer.

THEOREM 4.7. *G is amenable if and only if the map*

$$\tau : UCB(\widehat{G/H})^* \to B_{A(G/H)}(VN(G/H)), \quad m \mapsto m_L$$

is surjective.

PROOF. Suppose that G is amenable. By [11, Theorem 20], the map τ is a linear isometry and a surjective algebra homomorphism.

For the converse, let τ be surjective. Then since $id_{VN(G/H)} \in B_{A(G/H)}(VN(G/H))$, there exists $E \in UCB(\widehat{G/H})^*$ such that $\tau(E) = id_{VN(G/H)}$. We extend E to a functional m on $VN(G/H)$ with the same norm. By Goldestine's theorem, there is a net $(\tilde{u}_\alpha)$ in $A(G/H)$ such that $\tilde{u}_\alpha \xrightarrow{w^*} m$ and $\|\tilde{u}_\alpha\| \le \|m\|$ for all α. Therefore, for each $T \in VN(G/H)$ and $\tilde{u} \in A(G/H)$,

$$\begin{aligned}\lim_\alpha \langle \tilde{u}_\alpha \tilde{u}, T\rangle &= \lim_\alpha \langle \tilde{u}_\alpha, \tilde{u} \cdot T\rangle = \langle m, \tilde{u} \cdot T\rangle = \langle E, \tilde{u} \cdot T\rangle \\ &= \langle \tau(E)(T), \tilde{u}\rangle = \langle id_{VN(G/H)}(T), \tilde{u}\rangle = \langle \tilde{u}, T\rangle.\end{aligned}$$

This shows that $(\tilde{u}_\alpha)$ is a bounded weak approximate identity for $A(G/H)$. Applying Mazur's theorem, we obtain a bounded approximate identity for $A(G/H)$. Hence, by [7, Theorem 4.1], G is amenable. □

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