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DYNAMICS OF THE CONSUMPTION–CAPITAL RATIO, THE SAVING RATE, AND THE WEALTH DISTRIBUTION IN THE NEOCLASSICAL GROWTH MODEL

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The paper utilizes a common approach to derive sufficient conditions for strictly monotonic behavior of three ratios along the transition to the steady state in the Ramsey–Cass–Koopmans model: the consumption–capital ratio, the consumption–output ratio (and the saving rate), and the consumption–wage ratio. These conditions are then applied to derive additional results on the transitional dynamics of the distribution of wealth in the model (when individual consumers are differentiated by their initial wealth endowments).

Keywords: Neoclassical, Ramsey–Cass–Koopmans, Growth, Inequality, Wealth Distribution, Saving

1. INTRODUCTION

The Ramsey–Cass–Koopmans (R-C-K) version of the neoclassical model of growth [Ramsey (1928), Cass (1965), Koopmans (1965)] is the most widely used dynamic representation of a perfectly competitive economy that approaches a steady state over time. Romer (1996, pp. 151) describes it as "the natural Walrasian baseline model of the aggregate economy." Yet properties of the model economy in transition to the steady state are not completely understood. In this paper we focus on the transitional dynamics of three variables in the neoclassical growth model: the ratio of aggregate (per capita) consumption to aggregate (per capita) capital stock, the degree of inequality in the distribution of wealth, and the (gross) saving rate.

Our results on the above three variables are derived, in turn, from an analysis of the behavior of three ratios in the R-C-K model: the consumption–capital ratio, the consumption–wage ratio,[1] and the consumption–(gross)output ratio. The consumption–capital ratio features as an important variable in models of economic

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growth, and much of the dynamics of growth models is worked out by analyzing systems where this ratio enters as one of the variables. In this paper, our analysis of the transitional dynamics of the consumption–capital ratio also allows us to draw inferences about the behavior of the consumption–wage ratio. The dynamics of the consumption–wage ratio is significant because it is closely related to the dynamics of wealth distribution in the R-C-K model. Analyzing the behavior of the consumption–output ratio yields results on the transitional dynamics of the (gross) saving rate in the model. Moreover, as in the case of the consumption–capital ratio, such results can be applied to complement existing results on the dynamics of distribution in the model.

Standard expositions of the R-C-K model[2] assume a general neoclassical production technology and a CEMU (constant elasticity of marginal utility) instantaneous utility function. We shall heneceforth refer to the R-C-K model allowing for all such technologies and preference sets as the standard R-C-K model. The transitional dynamics of the consumption–capital ratio, the wealth distribution, and the saving rate in the R-C-K model have, however, been studied assuming more specific forms for production technologies or preferences. Barro and Sala-i-Martin (1995) completely characterize the behavior of the consumption–capital ratio and the saving rate, assuming Cobb–Douglas production technology. Similarly, Caselli and Ventura (2000) graphically analyze the dynamics of the consumption–capital ratio and the wealth distribution assuming logarithmic instantaneous utility. Contributions that simply study the transitional dynamics of the saving rate or the wealth distribution also place restrictions on technologies. Thus, both Smetters (2003), who derives a set of sufficient conditions for monotonic behavior of the saving rate, and Guha (2004), who demonstrates the possibility of a long-run trade-off between growth and wealth inequality in the R-C-K model with technical progress, consider the case of CES production technology. An exception is the paper by Glachant and Vellutini (2002). There, the authors derive necessary and sufficient conditions for wealth inequality to be *locally* increasing and *locally* decreasing around the steady state in the standard R-C-K model. All the above results relate to the continuous-time version of the R-C-K model. Additionally, Obiols-Homs and Urrutia (2005) consider the evolution of wealth inequality in a discrete-time version of the model with log instantaneous utility and Cobb–Douglas production technology.[3]

Despite the restrictions placed on technologies or preferences, the above contributions reveal that the R-C-K model allows a wide variety of transitional behavior in the concerned variables (the consumption–capital ratio, the saving rate, and the degree of wealth inequality). Behavior might be monotonic or nonmonotonic along the transition path, and even when variables behave monotonically, they might be increasing or decreasing in transition. We know, however, that in a sufficiently small neighborhood of the steady state, the ratio of consumption per capita to any twice-differentiable function of the capital stock per capita either remains constant or displays strictly monotonic behavior along the transition to the steady state.[4] Therefore, the consumption–capital ratio, the consumption-output ratio, and the saving rate must be either strictly increasing, strictly decreasing, or constant in some neighborhood of the steady state. Similarly, Glachant and Vellutini (2002) show that in some neighborhood of the steady state, the distribution of wealth in the economy at any given instant either strictly Lorenz-dominates, is strictly Lorenz-dominated by, or has the same Lorenz curve as all subsequent wealth distributions attained along the transition path. Therefore, inequality in the distribution of wealth may also be thought to be either strictly increasing, strictly decreasing, or unchanged in some neighborhood of the steady state.

Two questions follow immediately. First, can we define analytically the size of the neighborhood centered on the steady state in which the above variables are either strictly increasing, strictly decreasing, or constant? Or, rather, can we define conditions (in terms of the initial capital stock per capita and the structural parameters of the model) under which the above variables would be either strictly increasing, strictly decreasing, or constant along the entire transition path to the steady state? Second, can we say anything about when each of these above variables will display a strictly increasing trend, when a strictly decreasing trend, and when a constant trend along the transition path? The literature, as surveyed above, provides only fragmentary answers to these questions, because all results have been derived using specific forms for preferences or technologies. Our objective in this paper is to provide more general answers to these questions by analyzing the standard R-C-K model.

Such questions need not be of purely theoretical interest. Exercises involving numerical simulations of the transitional dynamics in the neoclassical growth model [e.g., King and Rebelo (1993)] suggest limited success in accounting for historical trends in data. However, as in textbooks on economic growth, the neoclassical growth model usually serves as a starting point for empirical research attempting to relate growth theory to experience. Thus, in attempting to explain historical trends through numerical simulations of growth models, investigators often use the predictions from a standard R-C-K model as a benchmark for comparison. Notable examples include Christiano (1989), who considers the behavior of the Japanese saving rate after World War II, and Alvarez-Pelaez and Diaz (2005), who consider the evolution of wealth inequality in the United States in the years 1870–1970. There is, of course, no reason why the saving rate or the degree of wealth or income inequality in an economy should have a strictly monotonic trend over the entire historical period being considered.[5] However, there is often a strictly monotonic trend in the concerned variable during a subperiod before its value ultimately stabilizes at a constant level, or there is a strictly monotonic trend in the concerned variable in the latter part of the period considered.[6] Knowledge of the theoretical conditions under which the concerned variable demonstrates a strictly increasing trend or a strictly decreasing trend in approaching the steady state in the neoclassical growth model can reduce the number of possible specifications of the benchmark model the investigator has to consider.

To get more general results on the transitional dynamics of the consumption–capital ratio, the saving rate, and the distribution of wealth in the R-C-K model, we consider in this paper the standard R-C-K model (in continuous time) with general CEMU preferences and a general neoclassical production technology. A common analytical approach is used to obtain sets of sufficient conditions for both strictly increasing and strictly decreasing trajectories of the consumption–capital ratio, the consumption–wage ratio, and the saving rate (and consumption–output ratio) in the transition to the steady state. These conditions are then used to derive (given heterogeneity in initial wealth endowments of consumers) sets of sufficient conditions for continuously decreasing inequality (convergence) and continuously increasing inequality (divergence) in the distribution of wealth.

Most of the conditions derived in this paper are relatively simple to apply. The results on the consumption–capital ratio and many of those on the wealth distribution involve just one preference parameter, the constant elasticity of marginal utility, and one technological variable, the elasticity of the marginal product of labor with respect to the capital–labor ratio. The results on the saving rate involve, in addition, the output elasticity of capital. Although the emphasis is placed on deriving sufficient conditions for strictly monotonic behavior, our analysis also allows us to derive necessary conditions for strictly monotonic behavior of these variables, as well as necessary and sufficient conditions for there to be no change in the concerned variables in transition.

Some of the results that we can derive for specific technologies or preferences as special cases of the more general propositions in our paper already exist in the literature. However, even for the specific technologies or preferences that have been studied in the literature, it is possible to add to existing knowledge. Consider, for example, the case of CES production technology and assume that the economy approaches the steady state from below. Smetters (2003) provides sufficient conditions for the saving rate to be strictly increasing when the elasticity of factor substitution is greater than one and strictly decreasing when the elasticity is less than one. We can, however, also derive sufficient conditions for the saving rate to be strictly decreasing when the elasticity of factor substitution is greater than one and strictly increasing when the elasticity is less than one. We can, of course, also derive results for specifications not considered in the literature. One of the cases that Caselli and Ventura (2000) consider, in discussing the evolution of wealth inequality, is that of Cobb–Douglas production technology and log instantaneous utility . The results in this paper allow us to completely characterize the dynamics of wealth inequality for Cobb–Douglas production technology and any CEMU instantaneous utility function.

Our results on the dynamics of wealth distribution are significant in a few other respects. We show that they can also be used to obtain sufficient conditions for convergence and divergence in the distribution of income along the transition path. Moreover, they can be used to derive some simple policy conclusions for the R-C-K model. We demonstrate how some of the results on wealth distribution continue to hold when we introduce into the model a redistributive tax–subsidy regime where a constant proportional tax on personal wealth is balanced by equal lump sum subsidies to all individuals. The same results hold with a minor alteration when the tax is on income from wealth or on total income. In both cases, we are able to state sufficient conditions under which a sudden unanticipated increase in the redistributive tax rate leads to convergence or divergence in the distribution of wealth.

A common modification of the standard R-C-K model involves the introduction of a subsistence level of consumption. Christiano (1989) and Alvarez-Pelaez and Diaz (2005) show respectively that numerical simulations of such a modified model can outperform the standard model in explaining historical trends in the Japanese saving rate and in U.S. wealth inequality. Chatterjee (1994), in a modified one-factor R-C-K model, and Obiols-Homs and Urrutia (2005), for the case of Cobb–Douglas production technology, analytically consider the evolution of wealth inequality in the presence of a subsistence level of consumption. We show that our results for the standard R-C-K model have only to be slightly altered to obtain sufficient conditions for convergence or divergence in the distribution of wealth for this modified model.

Finally, a question might be raised about the ethical significance of our results on the distribution of wealth. Chatterjee (1994) correctly points out that provided the normative criterion used depends solely on the consumption streams of the infinitely lived consumers in the R-C-K model, *changes* in the distribution of wealth along the transition path have no normative significance. The consumption stream of any individual consumer in the R-C-K model is determined entirely by the *initial* distribution of wealth and the time path of capital stock per capita in the economy (which is independent of wealth distribution).

However, other normative criteria might conceivably be of interest. Note that the infinitely lived consumers in the R-C-K model may be interpreted as infinitely lived dynasties, constituted at each instant by a specific generation of individual members, who are motivated by intergenerational altruism in deciding their current rates of consumption and saving (bequest). In this case, the wealth of members of a dynasty at any instant can be interpreted as a bequest from preceding generations.

The institution of private inheritance of wealth has often been criticized as violating principles of justice or fairness[7]. Mechanisms designed to modify the working of this institution—taxation of and restrictions on intergenerational transfers of assets—have, in turn, been criticized as violating principles of individual freedom and dampening incentives for effort, enterprise, and thrift. Much of the disagreement between opponents and defenders of such mechanisms can possibly be resolved if the private economy, beginning from an initial distribution of inheritances, can itself be shown to achieve a more equal distribution of inheritances in the long run. The R-C-K model, being the most common example of a Pareto-efficient economy with intergenerational altruism, appears to be a good starting point for addressing that question.

Viewed from this perspective, the conditions for strictly increasing (decreasing) wealth inequality in the R-C-K model can be interpreted as the conditions under which the argument favoring restrictions on the institution of inheritance is relatively strong (weak). This interpretation can be further strengthened if we note that the results of our analysis suggest[8] that a simple redistributive wealth tax may help (hinder) redressal of historical inequalities in wealth precisely in those cases where the wealth distribution in the private economy has historically been subject to divergence (convergence) over time.

This paper has five more sections. Section 2 restates the assumptions for the R-C-K model and introduces the notations used in the paper. Section 3 considers the transitional dynamics of the consumption–capital ratio. In Section 4 we demonstrate how the dynamics of the consumption–wage ratio are related to the dynamics of wealth distribution in the R-C-K model. This relation is then used to analyze the dynamics of distribution in the model. In Section 5 we consider the transitional behavior of the consumption–output ratio and the saving rate in the model. Some concluding comments are presented in Section 6.

2. THE MODEL

Consider a one-sector R-C-K economy without population growth or technical progress.[9] Because we intend to discuss some distributive properties of the R-C-K model, we take a heterogeneous-agent version of the model. The population consists of infinitely-lived individuals numbered $1, 2, \ldots, N$. The individuals are differentiated only by the amount of wealth owned by them at an initial point in time 0. Following Caselli and Ventura (2000), we assume that N is sufficiently large and the initial distribution of wealth sufficiently egalitarian so that in choosing their optimal consumption plans, individuals can neglect the effect of their choices on the time path of factor rental prices. For every $i \in \{1, 2, \ldots, N\}$ and every $\tau \in [0, \infty)$, individual i at time τ chooses a pair (c_i, a_i) that solves

Problem $\boldsymbol{P}(i,\tau)$: $\max \int_{\tau}^{\infty} \frac{1}{1-\theta}[\{c_i(t)\}^{1-\theta} - 1]\, e^{-\rho(t-\tau)}dt$
subject to: $\dot{a}_i(t) = w(t) + r(t)a_i(t) - c_i(t)$, for all $t \geq \tau$;
$\lim_{t\to\infty} a_i(t)\, e^{-\int_{\tau}^{t} r(v)dv} \geq 0$; $c_i(t) \geq 0$, for all $t \geq \tau$; $a_i(\tau)$ given.

The notation used in the paper is standard except for a few additions. $W(t)$ denotes the present discounted value at time t ($t \geq 0$) of the stream of wage earnings of any individual time t onward. That is, $W(t) = \int_{t}^{\infty} w(\tau)\, e^{-\int_{t}^{\tau} r(v)dv} d\tau$. $\omega(k)$ is used to denote the elasticity of the marginal product of labour at any given capital–labor ratio k. Then, $\omega(k) = \{-k^2 f''(k)\}/\{f(k) - kf'(k)\}$ for all $k > 0$. The output elasticity of capital $f'(k)\, k/f(k)$ at any given capital–labor ratio k is denoted by $\alpha(k)$. Also, let k^* and c^* be the respective steady state per capita values of capital stock and consumption defined by $f'(k^*) = \rho + \delta$ and $c^* = f(k^*) - \delta k^*$, where δ is the constant rate of depreciation of capital.

Then, provided that for all $i \in \{1, 2, \ldots, N\}$, $a_i(0) > -W(0)$, it is known that a unique perfect foresight competitive equilibrium growth path exists for this economy, for which

$$\forall t \geq 0 : \dot{c}(t)/c(t) = (1/\theta)\{r(t) - \rho\}, \tag{1.i}$$

$$\forall t \geq 0 : \dot{k}(t) = w(t) + r(t)k(t) - c(t), \tag{1.ii}$$

$$\lim_{t\to\infty} k(t)\, e^{-\int_0^t r(v)dv} = 0, \tag{1.iii}$$

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$$\text{for all } t \geq 0, \dot{c}(t) \lesseqqgtr 0 \text{ and } \dot{k}(t) \lesseqqgtr 0 \text{ according to whether } k(0) \gtreqqless k^*, \quad \textbf{(1.iv)}$$

$$\lim_{t\to\infty} k(t) = k^* \text{ and } \lim_{t\to\infty} c(t) = c^*. \quad \textbf{(1.v)}$$

3. MONOTONIC BEHAVIOR OF THE CONSUMPTION–CAPITAL RATIO

In this section we derive separate sets of sufficient conditions for the consumption–capital ratio c/k to be strictly increasing, strictly decreasing, or constant along the transition path to the steady state.

We know that, for all $t \geq 0$, $w(t) = f(k(t)) - k(t) f'(k(t))$ and $r(t) = f'(k(t)) - \delta$. From (1.i) we can see that the rate of growth of consumption $\dot{c}/c$ at any instant is a function of the capital stock per capita k and is independent of the rate of consumption per capita c. In contrast, the rate of growth of capital stock $\dot{k}/k$, as can be seen from (1.ii), is a function of both k and c. Moreover, $\dot{k}/k$ is strictly decreasing in the consumption–capital ratio c/k. Then, at any given instant, given the value of k, there will be a unique value that c/k must assume in order to have $\dot{c}/c = \dot{k}/k$. From (1.i) and (1.ii), for any $t \geq 0$, this value is given by $\{f(k(t))/k(t)\} - (1/\theta) f'(k(t)) + \delta\{(1/\theta) - 1\} + (\rho/\theta)$. Because this value is simply a function of $k(t)$ (the value of t not directly entering into its determination), we can say that with every $k > 0$, there is associated a unique value that the consumption–capital ratio c/k must take in order to have $d(c/k)/dt = 0$. This value is defined by $z(k) = \{f(k)/k\} - (1/\theta) f'(k) + \delta\{(1/\theta) - 1\} + (\rho/\theta)$. Moreover, because $\dot{c}/c$ is independent of c and $\dot{k}/k$ is negatively related to c, $d(c/k)/dt \gtreqqless 0$ according to whether $c/k \gtreqqless z(k)$.

Besides being a differentiable function, the only general property that we can ascribe to $z(k)$ is the following: given that k^* and $\bar{k}$ are values of k such that $f'(k^*) = \rho + \delta$ and $f(\bar{k})/\bar{k} = \delta$, $z(k) > 0$ for all k in some interval containing both k^* and $\bar{k}$ in its interior.[10] Moreover, for all $k > 0$, $z'(k) = -f''(k)[(1/\theta) - \{1/\omega(k)\}]$. This implies that for all $k > 0$, $z'(k) \gtreqqless 0$ according to whether $\omega(k) \gtreqqless \theta$.

Note also that as k approaches its steady state value k^*, the consumption–capital ratio in the economy c/k approaches $c^*/k^* = \{f(k^*)/k^*\} - \delta = z(k^*)$.

We can now easily derive some general results relating to the transitional dynamics of the consumption–capital ratio. Consider, for example, conditions necessary for the consumption–capital ratio to be constant when the economy is approaching the steady state from below. We know that to have a constant c/k in the transition to the steady state, $c/k = z(k)$ and $c/k = c^*/k^* = z(k^*)$ for all $k \in [k(0), k^*)$. This implies that $z'(k) = 0$ for all $k \in [k(0), k^*)$, which, in turn, implies that $\omega(k) = \theta$ for all $k \in [k(0), k^*)$. To see that the latter condition is also sufficient for a constant consumption–capital ratio, suppose that at some arbitrary instant $t \geq 0$, $c(t)/k(t) \neq z(k(t)) = z(k^*) = c^*/k^*$. The phase diagram in Figure 1 suggests that from time t onward, c/k must be continuously increasing if $c(t)/k(t) > z(k(t))$ and continuously decreasing if $c(t)/k(t) < z(k(t))$. But

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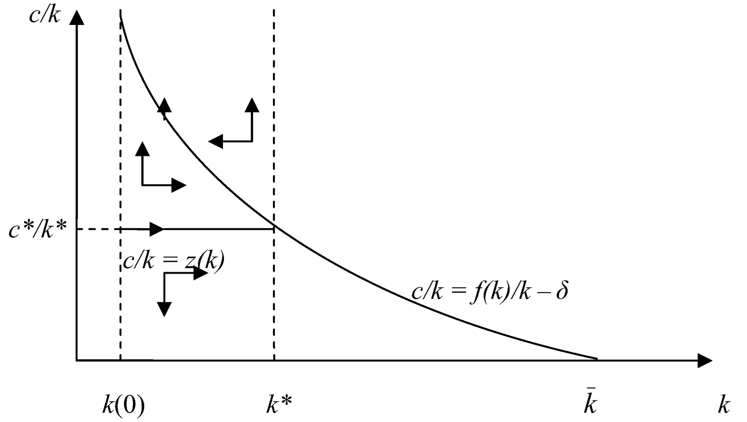

FIGURE 1. Phase diagram when $z'(k) > 0$ for all $k \in [k(0), k^*)$.

this would imply that c/k must be continuously diverging away from c^*/k^* and this is clearly not possible. The following proposition therefore holds.

PROPOSITION 1. *The necessary and sufficient condition for the consumption–capital ratio to remain unchanged in the transition to the steady state is that* $\omega(k) = \theta$ *for all values of k along the transition path.*

This is consistent with the result stated by Barro and Sala-i-Martin (1995, pp. 90) that for a Cobb–Douglas production function with output elasticity of capital α, c/k remains unchanged along the transition path if and only if $\theta = \alpha$. Note that for this production function, $\omega(k) = \alpha$ for all $k > 0$.

We can similarly obtain necessary conditions for the cases of a strictly increasing and a strictly decreasing consumption–capital ratio. Note that, in order for the consumption–capital ratio to be strictly increasing in transition, it must be increasing at $t = 0$. This implies that $c(0)/k(0) > z(k(0))$. However, suppose that $c^*/k^* = z(k^*) \leq z(k(0))$. Then, because $c(0)/k(0) > c^*/k^*$ and $c(t)/k(t) \to c^*/k^*$ as $t \to \infty$, it follows that c/k cannot be strictly increasing along the transition to the steady state. Arguing in a similar manner for the case of a strictly decreasing consumption–capital ratio, the following proposition is easily established.

PROPOSITION 2. *A necessary condition for the consumption–capital ratio to be strictly increasing (decreasing) in the transition to the steady state is that* $z(k(0)) < z(k^*)$ $(z(k(0)) > z(k^*))$.

Finally, we can derive sufficient conditions for a strictly increasing or a strictly decreasing consumption–capital ratio in the transition to the steady state. Consider the case where the economy approaches the steady state from below and $z'(k) > 0$ for all $k \in [k(0), k^*)$.

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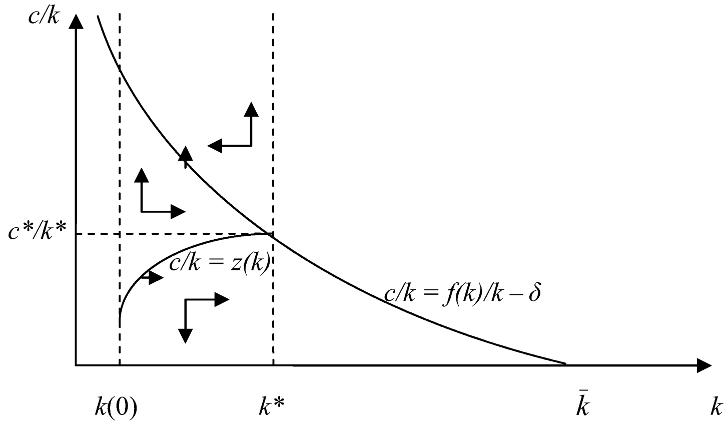

FIGURE 2. Phase diagram when $z'(k) > 0$ for all $k \in [k(0), k^*)$.

From the phase diagram in Figure 2, it is clear that if $c/k \leq z(k)$ for any value of k in the interval $[k(0), k^*)$, then, at all higher values of k in that interval, c/k must be strictly less than $z(k)$ and decreasing. Since $z(k^*) > z(k)$ for all $k \in [k(0), k^*)$, it is then impossible that $c/k \to z(k^*)$ as $k \to k^*$. Hence, for all $k \in [k(0), k^*)$ we must have $c/k > z(k)$. This implies that a sufficient condition for a strictly increasing consumption–capital ratio when the economy approaches the steady state from below is that $z'(k) > 0$, that is, $\omega(k) > \theta$ for all $k \in [k(0), k^*)$. Similar arguments can be used to derive sufficient conditions for the other cases considered in the following proposition.

PROPOSITION 3.

(i) *A sufficient condition for the consumption–capital ratio to be strictly increasing (decreasing) as the economy approaches the steady state from below (above) is that $\omega(k) > \theta$ for all values of k along the transition path.*

(ii) *A sufficient condition for the consumption–capital ratio to be strictly decreasing (increasing) as the economy approaches the steady state from below (above) is that $\omega(k) < \theta$ for all values of k along the transition path.*

Proof. Because $k(0) > 0$ and $\lim_{k\to 0} f'(k) > \rho + \delta$, from (1.iv) and (1.v) it follows that $k(t) > 0$, for all $t \geq 0$.

From (1.i) and (1.ii), substituting for $w(t)$ and $r(t)$, it follows that for all $t \geq 0$,

$$\{\dot{c}(t)/c(t)\} - \{\dot{k}(t)/k(t)\} = \{c(t)/k(t)\} - [\{f(k(t)) - \delta k(t)\}/k(t)] + (1/\theta)\{f'(k(t)) - \rho - \delta\}.$$

Thus, for all $t \geq 0$, it can be shown that

$$(d/dt)[\{\dot{c}(t)/c(t)\} - \{\dot{k}(t)/k(t)\}] = \{c(t)/k(t)\}[\{\dot{c}(t)/c(t)\} - \{\dot{k}(t)/k(t)\}] + [w(t)\dot{k}(t)/\{k(t)\}^2][1 - (1/\theta)\omega(k(t))]. \quad \textbf{(2)}$$

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Note from (1.i), (1.ii), and (1.v) that

$$\lim_{t\to\infty}\{\dot{c}\,(t)\,/c\,(t)\} = \lim_{t\to\infty}\{\dot{k}\,(t)\,/k\,(t)\} = 0. \quad \textbf{(3)}$$

It therefore follows from (2) and (3) that for all $t \geq 0$,

$$[\forall\tau \geq t : \dot{k}\,(\tau)\,[1-(1/\theta)\,\omega\,(k\,(\tau))] > 0] \rightarrow [\{\dot{c}\,(t)\,/c\,(t)\} - \{\dot{k}\,(t)\,/k\,(t)\}] < 0 \quad \textbf{(4)}$$

$$[\forall\tau \geq t : \dot{k}\,(\tau)\,[1-(1/\theta)\,\omega\,(k\,(\tau))] < 0] \rightarrow [\{\dot{c}\,(t)\,/c\,(t)\} - \{\dot{k}\,(t)\,/k\,(t)\}] > 0. \quad \textbf{(5)}$$

Given (1.iv) and (1.v), Proposition 3 follows from (4) and (5). ■

Remark 3.1. We can illustrate the applicability of Proposition 3 with a few examples. Note that for the case of a Cobb–Douglas production function with output elasticity of capital α, $\omega(k) = \alpha$ for all $k > 0$. For the case of Cobb–Douglas technology, therefore, the above proposition reduces to the result stated by Barro and Sala-i-Martin (1995, pp. 90). Similarly, for the production function $f(k) = k(A - B\ln k)$, $A > 0$, $B > 0$, used by Cass and Yaari (1967), $\omega(k) = 1$ and Proposition 3 is straightforward to apply, given $k(0) < e^{(A/B)-1}$ so that $f'(k(0)) > 0$. Assume instead CES production technology with $f(k) = A[\alpha k^{-\beta} + (1-\alpha)]^{-(1/\beta)}$, $A > 0, \alpha \in (0, 1)$. If elasticity of substitution $\sigma = 1/(1+\beta) > 1$, then $\omega(k) = (1+\beta)\{f'(k)/(\alpha^{-1/\beta}A)\}^{\beta/(1+\beta)}$ is strictly increasing in k. It follows from Proposition 3 that, given $\lim_{k\to\infty} f'(k) = \alpha^{-(1/\beta)}A < \delta$, c/k is strictly increasing (decreasing) as the economy approaches the steady state from below if $\omega(k(0)) > \theta$ $[\omega(k^*) < \theta]$. If, instead, $\sigma < 1$ then $\omega(k)$ is strictly decreasing in k and, given $\lim_{k\to 0} f'(k) = \alpha^{-(1/\beta)}A > \rho + \delta$, c/k is strictly increasing (decreasing) if $\omega(k^*) > \theta$ $[\omega(k(0)) < \theta]$. Similarly, sufficient conditions can be derived for when the economy is approaching the steady state from above.

Remark 3.2. Caselli and Ventura (2000) graphically analyze the dynamics of c/k for CES technology with $\sigma < 1$, assuming a logarithmic instantaneous utility function ($\theta = 1$). They demonstrate that when the economy is approaching the steady state from below, for sufficiently small values of ρ, c/k is strictly decreasing in the neighborhood of the steady state but may display nonmonotonic behavior for sufficiently small $k\,(0)$. The results stated in the previous paragraph for CES technology with $\sigma < 1$ show that, given technology and the values of ρ and δ, for sufficiently small values of θ, c/k is going to be strictly increasing in transition, so that nonmonotonic behavior can be ruled out. Similarly, given $k\,(0)$, for sufficiently large values of θ, c/k is going to be strictly decreasing in transition, so that the possibility of nonmonotonic behavior can again be ruled out. Thus, nonmonotonic behavior of the consumption–capital ratio, illustrated by Caselli and Ventura, is possible, for a given technology and given values of δ, ρ, and $k\,(0)$, only in an intermediate range of values of θ. Caselli and Ventura's example relates to a case where $\theta (= 1)$ lies in that intermediate range.

4. CONVERGENCE AND DIVERGENCE IN THE DISTRIBUTION OF WEALTH

4.1. Direction of Change in Wealth Inequality

We shall say that wealth inequality in an economy is decreasing (increasing) at any point in time t, if for every pair of individuals with unequal wealth holdings, the difference between the share of aggregate wealth held by the richer individual and the share of aggregate wealth held by the poorer individual is decreasing (increasing) at time t; that is, if for all $i, j \in \{1, 2, \ldots N\}$ such that $a_i(t) > a_j(t)$, it is true that

$$d[\{a_i(t)/Nk(t)\} - \{a_j(t)/Nk(t)\}]/dt < 0 (> 0).$$

From the equations of motion for individual wealth and capital stock per capita, we get

$$d[\{a_i(t)/Nk(t)\} - \{a_j(t)/Nk(t)\}]/dt = \{1/Nk(t)\}[\{a_j(t)/k(t)\} - \{a_i(t)/k(t)\}]\left[w(t) - \left[\int_t^\infty e^{\int_t^\tau [\{(1-\theta)/\theta\}r(v)-(\rho/\theta)]dv} d\tau\right]^{-1} W(t)\right].$$

Therefore, there is convergence (divergence) in the distribution of wealth along the transition to the steady state if the quantity

$$X\,(t) = w\,(t) - \left[\int_t^\infty e^{\int_t^\tau [\{(1-\theta)/\theta\}r(v)-(\rho/\theta)]dv} d\tau\right]^{-1} W\,(t)$$

is positive (negative) for all $t \geq 0$.[11]

Why is the sign of $X(t)$ crucial in determining the direction of change in wealth inequality? In the R-C-K model, the rate of consumption of individual i ($i \in \{1, 2, \ldots, N\}$) at any time $t \geq 0$ is given by $c_i(t) = \{W(t)/\mu(t)\} + \{a_i\,(t)\,/\mu\,(t)\}$, where $\mu\,(t) = \int_t^\infty e^{\int_t^\tau [\{(1-\theta)/\theta\}r(v)-(\rho/\theta)]dv} d\tau > 0$. The first term on the R.H.S. can be thought of as individual i's consumption out of his or her human wealth and the second term as individual i's consumption out of his or her (nonhuman) wealth. Consumption out of human wealth is the same for every individual, whereas consumption out of (nonhuman) wealth is proportional to the size of (nonhuman) wealth, which varies across individuals. The rate of change of individual i's (nonhuman) wealth, equal to that individual's rate of saving, is then given by $\dot{a}_i(t) = X(t) + [r(t) - \{1/\mu(t)\}]a_i(t)$, which also has two components. The second term representing saving out of income from (nonhuman) wealth, is proportional to the magnitude of (nonhuman) wealth of the individual, while the first term, representing saving out of income from human wealth, is the same for every individual in the economy. It follows that there is convergence, no change in inequality, or divergence in the distribution of wealth in the economy at $t \geq 0$ according to whether $X(t) \gtreqless 0$.

Note that for all $t \geq 0$, using (1.i) and the definition of $W(t)$,

$$X(t) = \{\mu(t)\}^{-1}\left[\int_t^{\infty} w(t)\, e^{-\int_t^{\tau} r(v)dv}\left[e^{\int_t^{\tau}\{\dot{c}(v)/c(v)\}dv} - e^{\int_t^{\tau}\{\dot{w}(v)/w(v)\}dv}\right]d\tau\right].$$

It follows that the equilibrium growth path is associated with convergence (divergence) in the distribution of wealth if for all $t \geq 0$, it is true that the rate of growth of per capita consumption $\dot{c}(t)/c(t)$ is greater than (less than) the rate of growth of the wage rate $\dot{w}(t)/w(t)$.

4.2. Main Propositions

Note that convergence (divergence) in the distribution of wealth is related to a strictly increasing (decreasing) consumption–wage ratio c/w in the transition to the steady state. The question naturally arises as to whether one can use the same approach as in the case of the consumption–capital ratio to derive sufficient conditions for a strictly increasing or a strictly decreasing consumption–wage ratio. For the consumption–wage ratio, the counterpart to function $z(k)$ (see Section 3) is given by the function $\varpi(k) = [\{f(k) - \delta k\}/\{f(k) - kf'(k)\}] - [-\{f'(k) - \delta - \rho\}/\{\theta k f''(k)\}]$. At any $k > 0$, c/w is increasing, decreasing, or constant according to whether $c/w \gtreqless \varpi(k)$.

Note that to derive sufficient conditions for a strictly increasing or a strictly decreasing consumption–wage ratio using the same approach as in the case of the consumption–capital ratio, the function $\varpi(k)$ must be differentiable. However, this requires that the function $f(k)$ be thrice differentiable. For production technologies satisfying this condition, the counterparts to Propositions 1 and 3 can be stated.[12]

PROPOSITION 4. *A sufficient condition for the degree of inequality in the distribution of wealth to remain unchanged along the transition to the steady state is that $\varpi'(k) = 0$ for all values of k on the transition path.*

PROPOSITION 5.

(i) *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that $\varpi'(k) > 0$ for all values of k on the transition path.*

(ii) *A sufficient condition for divergence (convergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that $\varpi'(k) < 0$ for all values of k on the transition path.*

Remark 5.1. In the case of Cobb–Douglas production technology, Propositions 4 and 5 allow us to completely characterize the dynamics of wealth inequality. They imply that there is convergence (divergence), no change in inequality, or divergence (convergence) in the wealth distribution as the economy approaches the steady state from below (above) according to whether $\delta + \rho \gtreqless \alpha\delta\theta$. Note that if $\theta = 1$, the case discussed by Caselli and Ventura (2000, p. 918), it is always true that $\delta + \rho > \alpha\delta\theta$. The conditions for convergence, no change in inequality, and divergence in the wealth distribution are respectively the same as the conditions for a strictly decreasing, constant, and strictly increasing saving rate in the R-C-K model with Cobb–Douglas technology [Barro and Sala-i-Martin (1995), Appendix 2B]. This is not surprising, because for a Cobb–Douglas technology, the consumption–wage ratio is monotonically related to the consumption–output ratio. Glachant and Vellutini (2002) provide necessary and sufficient conditions for convergence and divergence in wealth distribution in a local neighborhood of the steady state. Because for Cobb–Douglas technology, the necessary and sufficient conditions for convergence and divergence in wealth distribution along the transition path do not depend on the value of $k(0)$, these conditions should be the same as those obtained by Glachant and Vellutini. In fact, it can be verified that for zero rates of population growth and technical progress, the conditions given by Glachant and Vellutini reduce to the above conditions in the case of a Cobb–Douglas technology.

Proposition 5 does not provide sufficient conditions for a general neoclassical production technology. However, note that for all $t \geq 0$, $\dot{w}(t)/w(t) = \omega(k(t))\{\dot{k}(t)/k(t)\}$; that is, the rate of growth of the wage rate is equal to the product of the elasticity of the wage rate with respect to the capital–labor ratio and the rate of growth of the capital stock per capita.

Therefore, the following proposition is established.

PROPOSITION 6.

(i) *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below is that c/k is strictly increasing (decreasing) along the transition path and $\omega(k) \leq 1$ [$\omega(k) \geq 1$] for all values of k on the transition path.*

(ii) *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from above is that c/k is strictly increasing (decreasing) along the transition path and $\omega(k) \geq 1$ [$\omega(k) \leq 1$] for all values of k on the transition path.*

We can now combine Propositions 3 and 6 to obtain sufficient conditions for convergence and divergence in the distribution of wealth in terms of restrictions on technology and preference parameters.

PROPOSITION 7. ***(i)*** *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that $\theta < \omega(k) \leq 1$ for all values of k on the transition path.*

(ii) *A sufficient condition for divergence (convergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that $1 \leq \omega(k) < \theta$ for all values of k on the transition path.*

Remark 7.1. For the Cass–Yaari (1967) production technology (see Remark 3.1), $\omega(k) = 1$, and Proposition 7 implies that if $\theta < 1$ ($\theta > 1$), then there is convergence (divergence) in the distribution of wealth when the economy approaches the steady state from below and divergence (convergence) in the distribution of wealth when it approaches the steady state from above. Proposition 7 can also be applied easily to the case of CES production technology (see Remark 3.1) with $\sigma > 1$ once we note that for all $k > 0$, $\omega(k) < 1/\sigma < 1$. For CES technology with $\sigma > 1$, there is convergence (divergence) in the distribution of wealth as the economy approaches the steady state from below (above) if $\theta < \omega(k(0))$ $[\theta < \omega(k^*)]$. Deriving sufficient conditions for the case $\sigma < 1$ is slightly more complicated. However, Proposition 7 can be applied once we note that for sufficiently large (small) values of δ and ρ and sufficiently small (large) $k(0)$, for all k along the transition path, $\omega(k) \geq 1$ (≤ 1). Hence, for sufficiently large values of δ and ρ and sufficiently small $k(0)$, there is divergence (convergence) in the distribution of wealth if the economy approaches the steady state from below (above) and $\theta > \omega(k(0))$ $[\theta > \omega(k^*)]$. Similarly, for sufficiently small values of δ and ρ and sufficiently large $k(0)$, there is convergence (divergence) in the distribution of wealth if the economy approaches the steady state from below (above) and $\theta < \omega(k^*)$ $[\theta < \omega(k(0))]$.

4.3. Stone–Geary Utility

If there is a subsistence level of consumption $\bar{c} > 0$ so that instantaneous utility from a rate of consumption C is given by $\{(C - \bar{c})^{1-\theta} - 1\}/(1 - \theta)$, individual i's ($i = 1, 2, \ldots, N$) optimization problem at $\tau \geq 0$ is the problem $P(i, \tau)$ with $c_i(t) - \bar{c}$ replacing $c_i(t)$ and $w(t) - \bar{c}$ replacing $w(t)$ for all $t \geq \tau$. With corresponding restrictions on the initial distribution of wealth it can be shown that a unique equilibrium growth path exists and (1-i)–(1-v) continue to hold provided $c(t)$ is replaced by $c(t) - \bar{c}$ and $w(t)$ is replaced by $w(t) - \bar{c}$. Proposition 7 on the dynamics of the wealth distribution can then be modified easily to yield the following result.

PROPOSITION 8. *If* $\bar{c} < \min[f(k(0)) - k(0) f'(k(0)),\ f(k^*) - k^* f'(k^*)]$ *then*

- ***(i)*** *a sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that* $\theta < \omega(k)\frac{f(k) - kf'(k)}{f(k) - kf'(k) - \bar{c}} \leq 1$ *for all values of k on the transition path.*
- ***(ii)*** *a sufficient condition for divergence (convergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that* $1 \leq \omega(k)\frac{f(k) - kf'(k)}{f(k) - kf'(k) - \bar{c}} < \theta$ *for all values of k on the transition path.*

4.4. Redistributive Tax–Subsidy Schemes

Consider an economy where at every instant t the government imposes a proportional tax on individual wealth holdings $a_i(t)$, $i \in \{1, 2 \ldots, N\}$, at the rate

γ and uses the entire tax revenue to give every individual a lump sum subsidy $\gamma k(t)$.[13] Problem P(i, τ) remains unchanged except that for all $t \geq 0$, the rate of interest $r(t)$ is now equal to $f'(k(t)) - \delta - \gamma$. Redefine $W(t)$ as the discounted value at time t of the stream of wages and lump sum subsidies received by any individual from time t onward ($t \geq 0$). Provided that for all $i \in \{1, 2, \ldots, N\}$, $a_i(0) > -W(0)$, a unique equilibrium growth path exists for the economy. This is characterized by (1-i)–(1-v), where now $f'(k^*) = \delta + \rho + \gamma$, $w(t)$ is replaced with $w(t) + \gamma k(t)$ and $r(t) = f'(k(t)) - \delta - \gamma$, for all $t \geq 0$. However, one can check that there is no change in the $z'(k)$ function. Therefore, Proposition 3 would continue to hold.

Now, at any given time t, for any pair of individuals i and j with $a_i(t) > a_j(t)$,

$$d[\{a_i(t)/Nk(t)\} - \{a_j(t)/Nk(t)\}]/dt = \{1/Nk(t)\}[\{a_j(t)/k(t)\}$$

$$- \{a_i(t)/k(t)\}]\left[w(t) + \gamma k(t) - \left[\int_t^\infty e^{\int_t^\tau [\{(1-\theta)/\theta\}r(v) - (\rho/\theta)]dv} d\tau\right]^{-1} W(t)\right].$$

We can proceed as in the no-tax case to derive a modified version of Proposition 6 where the elasticity of the sum of the wage rate and the lump sum subsidy w.r.t. the capital–labor ratio $d\ln(w + \gamma k)/d\ln k$ replaces the elasticity of the wage rate w.r.t. the capital–labor ratio $\omega(k)$ in the statement of Proposition 6. It is, however, easy to check that the former elasticity is greater than, equal to, or less than unity according to whether the latter elasticity is greater than, equal to, or less than unity. Therefore, Proposition 6 continues to hold. Because both Propositions 3 and 6 continue to hold, Proposition 7 also remains true. This has the following implication.

Suppose the economy at a particular point in time is not in steady state and has a value of k that is less than the steady state value of k. Note that the steady state value of k in the economy is smaller the higher the rate of taxation γ. This means that the higher is γ, the smaller is the range of values of k along the transition path. Therefore, the higher the rate of redistributive wealth tax, the less stringent is the set of sufficient conditions for strictly increasing inequality and strictly decreasing inequality in transition to the steady state from below. This would appear to suggest that although a redistributive wealth tax can be an effective weapon in promoting convergence in the distribution of wealth, it needs to be used with caution. With correct information about the economy, the tax can serve as an effective policy instrument when applied under appropriate conditions. However, if information about the economy is incorrect, then the higher the applied rate of taxation, the greater is the possibility that the government could actually end up worsening the distribution of wealth.

Because Proposition 7 continues to hold for an economy with a redistributive wealth tax, we can also state the following corollary.

COROLLARY 1. *Suppose the economy is initially in steady state or is approaching the steady state from below and there is a sudden unanticipated rise in* *the constant proportional rate of the redistributive wealth tax, so that the economy now lies above (below) its new steady state. Then*

- ***(i)*** *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition to the new steady state is that $1 \leq \omega(k) < \theta$ for all values of k on the transition path.*
- ***(ii)*** *A sufficient condition for divergence (convergence) in the distribution of wealth along the transition to the new steady state is that $\theta < \omega(k) \leq 1$ for all values of k on the transition path.*

The above corollary has an interesting implication. Suppose the economy at a given point in time has no taxes and is in steady state or is approaching the steady state from below. Suppose the government initiates a redistributive tax–subsidy scheme to reduce wealth inequalities in the economy. A comparison of Proposition 7 and Corollary 1 suggests that if the tax rate is so high[14] that the economy now lies above the new steady state, the scheme will be effective (counterproductive) precisely in those cases where the private economy has historically witnessed increasing (decreasing) inequality in approaching the steady state from below. However, smaller tax rates (for which the economy lies below the new post-tax steady state) will not be able to reverse the trend of increasing wealth inequality in these cases.

Suppose, instead of a proportional tax on wealth, the government taxes net personal income from wealth at a constant proportional rate γ.[15] It can then be shown that a modified version of Proposition 3 holds, where $\omega(k)$ in the statement of Proposition 3 is replaced by $(1-\gamma)\,\omega(k)$. Also, a modified version of Proposition 6 is true, where $\omega(k)$ in the statement of Proposition 6 is replaced by the elasticity of the sum of the wage rate and the lump sum subsidy w.r.t. the capital–labor ratio $d\ln(w+\gamma rk)/d\ln k$. One can easily verify that $d\ln(w+\gamma rk)/d\ln k \gtreqless 1$ according to whether $(1-\gamma)\,\omega(k) \gtreqless 1$. Therefore, a modified version of Proposition 6 holds, where $\omega(k)$ in the statement of Proposition 6 is replaced by $(1-\gamma)\,\omega(k)$. Given the nature of modifications to Propositions 3 and 6, it follows that Proposition 7 must be modified, replacing $\omega(k)$ by $(1-\gamma)\,\omega(k)$, and we can then obtain the following corollary to the modified proposition.

PROPOSITION 9. *Suppose the economy is initially in steady state or is approaching steady state from below, and there is a sudden unanticipated rise in the constant proportional rate of the redistributive tax on net income from wealth, so that the economy now lies above (below) its new steady state. Then, if γ denotes the new tax rate,*

- ***(i)*** *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition to the new steady state is that $1 \leq (1-\gamma)\,\omega(k) < \theta$ for all values of k on the transition path.*
- ***(ii)*** *A sufficient condition for divergence (convergence) in the distribution of wealth along the transition to the new steady state is that $\theta < (1-\gamma)\,\omega(k) \leq 1$ for all values of k on the transition path.*

4.5. Income Inequality

Extending our notion of changes in wealth inequality to changes in inequality in the distribution of (gross) income, we can say that income inequality in an economy is decreasing (increasing) at any point in time t if for all $i, j \in \{1, 2, \ldots, N\}$ such that $a_i(t) > a_j(t)$, it is true that

$$d[[[w(t) + \{r(t) + \delta\}a_i(t)]/Nf(k(t))]$$
$$-[[w(t) + \{r(t) + \delta\}a_j(t)]/Nf(k(t))]]]/dt < 0(> 0),$$
$$\text{i.e., } d[\alpha(k(t))[\{a_i(t)/Nk(t)\} - \{a_j(t)/Nk(t)\}]]/dt < 0(> 0).$$

We can therefore infer that there is convergence (divergence) in the distribution of income along the equilibrium growth path if there is convergence (divergence) in the distribution of wealth along the equilibrium growth path and the share of capital in (gross) income $\alpha(k(t))$ is nonincreasing (nondecreasing) for all t. Now, suppose the economy is approaching the steady state from below. The share of capital in income would be nonincreasing (nondecreasing) along the equilibrium growth path if for all values of k on that path the elasticity of the wage rate w.r.t. the capital–labor ratio $\omega(k)$ is not less (not greater) than the output elasticity of capital $\alpha(k)$. In the case of the economy approaching the steady state from above, the share of capital in income would be nonincreasing (nondecreasing) along the equilibrium growth path if, for all values of k on that path, $\omega(k)$ is not greater (not less) than $\alpha(k)$. Given Proposition 7, the following set of sufficient conditions can then be stated for convergence (divergence) in the distribution of income.

PROPOSITION 10.

(i) *A sufficient condition for convergence (divergence) in the distribution of income along the transition path as the economy approaches the steady state from below (above) is that $\theta < \omega(k) \leq 1$ and $\omega(k) \geq \alpha(k)$ for all values of k on the transition path.*

(ii) *A sufficient condition for divergence (convergence) in the distribution of income along the transition path as the economy approaches the steady state from below (above) is that $1 \leq \omega(k) < \theta$ and $\omega(k) \leq \alpha(k)$ for all values of k on the transition path.*

5. MONOTONIC BEHAVIOR OF THE SAVING RATE

The (gross) saving rate in the economy at any instant $t \geq 0$ is given by $1 - \{c(t)/f(k(t))\}$. Observations about the behavior of the saving rate can therefore be made by analyzing the dynamics of the consumption–output ratio $c/f(k)$. We proceed in our analysis of the dynamics of $c/f(k)$ in the same manner as that used to investigate the behavior of the consumption–capital ratio in Section 3.

From (1.i) and (1.ii), for any $k > 0$, $\zeta(k) = 1 - \{\delta k/f(k)\} - (1/\theta)[1 - \{(\delta + \rho)/f'(k)\}]$ is the value that $c/f(k)$ must take for the rates of growth of consumption and output to be equal in the economy. According to whether $c/f(k) \gtreqqless \zeta(k)$, $c/f(k)$ is increasing, constant, or decreasing in the economy. Like the function $z(k)$, $\zeta(k) > 0$ in some interval containing k^* and $\bar{k}$ in its interior.

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Moreover, for all $k > 0$, it can be shown that $\zeta'(k) = [\{(\delta+\rho)/\theta\}\{f(k) - kf'(k)\}/\{f(k)\}^2][[\omega(k)/\{\alpha(k)\}^2] - \{\delta\theta/(\delta+\rho)\}]$. Therefore, for all $k > 0$, $\zeta'(k) \gtreqless 0$ according to whether $\omega(k)/\{\alpha(k)\}^2 \gtreqless \delta\theta/(\delta+\rho)$. Given that the saving rate in the economy is increasing, constant, or decreasing according to whether the consumption–output ratio is decreasing, constant, or increasing, we can derive the following counterparts to Propositions 1,2, and 3.

PROPOSITION 11. *The necessary and sufficient condition for the saving rate to remain unchanged in the transition to the steady state is that* $\omega(k)/\{\alpha(k)\}^2 = \delta\theta/(\delta+\rho)$ *for all values of k on the transition path.*

PROPOSITION 12. *A necessary condition for the saving rate to be strictly increasing (decreasing) in the transition to the steady state is that* $\zeta(k(0)) > \zeta(k^*)$ $\{\zeta[k(0)] < \zeta(k^*)\}$.

PROPOSITION 13.

(i) *A sufficient condition for the saving rate to be strictly increasing (decreasing) as the economy approaches the steady state from below (above) is that* $\omega(k)/\{\alpha(k)\}^2 < \delta\theta/(\delta+\rho)$ *for all values of k on the transition path.*

(ii) *A sufficient condition for the saving rate to be strictly decreasing (increasing) as the economy approaches the steady state from below (above) is that* $\omega(k)/\{\alpha(k)\}^2 > \delta\theta/(\delta+\rho)$ *for all values of k on the transition path.*

Remark 13.1. For the case of Cobb–Douglas production technology with $\alpha(k) = \omega(k) = \alpha$ for all $k > 0$, Propositions 11 and 13 reduce to the result proved by Barro and Sala-i-Martin (1995, Appendix 2B). For CES production technology (see Remark 3.1) with elasticity of substitution σ, $\omega(k) = (1/\sigma)\alpha(k)$ and $\omega(k)/\{\alpha(k)\}^2 = 1/\{\sigma^2\omega(k)\}$ for all $k > 0$. Proposition 13 can then be applied, using the property of the CES technology that for all $k > 0$, $\omega'(k) \gtrless 0$ according as $\sigma \gtrless 1$. Smetters (2003) provides sufficient conditions for the saving rate to be strictly increasing (decreasing) when $\sigma > 1$ $(\sigma < 1)$ and the economy approaches the steady state from below.[16] These conditions depend on the initial value of capital stock per capita in the economy $k(0)$, and Smetters illustrates in these cases the possibility of nonmonotonic behavior. However, from Proposition 13, we can also derive the following additional result: a sufficient condition for the saving rate to be strictly decreasing (increasing) when $\sigma > 1$ $(\sigma < 1)$ and the economy approaches steady state from below is that $1/\omega(k^*) > \delta\theta\sigma^2/(\delta+\rho)$ $[1/\omega(k^*) < \delta\theta\sigma^2/(\delta+\rho)]$. These latter conditions are independent of $k(0)$, implying thereby that for particular confugurations of the model parameters satisfying these conditions, irrespective of the value of $k(0)$, the saving rate will always display a strictly monotonic trend in transition.

For the Cass–Yaari (1967) production technology, $\omega(k) = 1$ and $\alpha'(k) < 0$ for all $k > 0$. Therefore, $\omega(k)/\{\alpha(k)\}^2$ is a strictly increasing function of k and Proposition 13 implies the following result: a sufficient condition for the saving rate to be strictly decreasing (increasing) when the economy approaches the steady state from below is that $\{\alpha(k(0))\}^2 < (\delta+\rho)/\delta\theta$ $(\{\alpha(k^*)\}^2 > (\delta+\rho)/\delta\theta)$.

Remark 13.2. Proposition 7, containing sufficient conditions for convergence (divergence) in wealth distribution, is derived using sufficient conditions for a strictly increasing (decreasing) consumption–wage ratio. The latter conditions were obtained by combining Proposition 3, which provides sufficient conditions for the consumption–capital ratio to be strictly increasing (decreasing), with sufficient conditions for the wage–capital ratio to be nonincreasing (nondecreasing). However, we can obtain an alternative set of sufficient conditions for strictly monotonic behavior of the consumption–wage ratio by combining Proposition 13, which provides sufficient conditions for the consumption–output ratio to be strictly increasing (decreasing), with sufficient conditions for the share of wages in output to be nonincreasing (nondecreasing). The following result therefore holds.

PROPOSITION 14.

(i) *A sufficient condition for convergence (divergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that* $\{\delta\theta/(\delta+\rho)\}\{\alpha(k)\}^2 < \omega(k) \leq \alpha(k)$ *for all values of k on the transition path.*

(ii) *A sufficient condition for divergence (convergence) in the distribution of wealth along the transition path as the economy approaches the steady state from below (above) is that* $\alpha(k) \leq \omega(k) < \{\delta\theta/(\delta+\rho)\}\{\alpha(k)\}^2$ *for all values of k on the transition path.*

Remark 14.1. Note that for CES production technology, $\alpha(k) = \sigma\omega(k)$, and for all $k > 0$, $\omega'(k) \gtreqless 0$ according as $\sigma \gtreqless 1$. Therefore, Proposition 14(i) implies that for $\sigma > 1$, there is convergence (divergence) in the wealth distribution as the economy approaches steady state from below (above) provided $\theta < (\delta+\rho)/\delta\sigma^2\omega(k^*)$ $[\theta < (\delta+\rho)/\delta\sigma^2\omega(k(0))]$. This is consistent with the conditions for the same kind of transitional behavior derived from Proposition 7 (see Remark 7.1) for CES technology with $\sigma > 1$. Similarly, Proposition 14(ii) implies that for $\sigma < 1$, there is divergence (convergence) in the wealth distribution as the economy approaches steady state from below (above) provided $\theta > (\delta+\rho)/\delta\sigma^2\omega(k^*)$ $[\theta > (\delta+\rho)/\delta\sigma^2\omega(k(0))]$. Comparing with the conditions for the same kind of transitional behavior derived from Proposition 7 (see Remark 7.1) for CES technology with $\sigma < 1$, we find that the two sets of conditions are consistent.

Remark 14.2. Caselli and Ventura (2000) discuss the evolution of wealth inequality in the R-C-K model for the specific cases of Cobb–Douglas technology and CES technology with $\sigma < 1$, assuming log instantaneous utility ($\theta = 1$). Our results for the Cobb–Douglas production technology are given in Remark 5.1 and those for CES production technology in Remarks 7.1 and 14.1. A comparison with the results obtained by Caselli and Ventura (2000) is instructive. For Cobb–Douglas technology, the transitional behavior of wealth inequality for $\theta = 1$ is completely reversed for $\theta > (\delta+\rho)/\alpha\delta$ but is the same for $\theta < (\delta+\rho)/\alpha\delta$. For CES technology with $\sigma < 1$, Caselli and Ventura consider two cases: large values of ρ and small values of ρ. The behavior of wealth inequality around the steady state obtained for the case of large ρ when $\theta = 1$ is found to hold also for sufficiently large values of θ greater than unity. Similarly, the behavior of wealth inequality around the steady state obtained for the case of small ρ when $\theta = 1$ is found to hold also for sufficiently small values of θ less than unity. Remark 14.1, however, also makes it clear that, given technology and given values of δ, ρ, and $k\,(0)$, for sufficiently large values of θ, nonmonotonic behavior in the degree of wealth inequality (of the kind illustrated by Caselli and Ventura) can be ruled out in the economy.

In sum, comparison with Caselli and Ventura's results seems to indicate that when considering transition from below, cases of divergence in wealth distribution obtained for $\theta = 1$ are broadly replicated for sufficiently large values of θ and cases of convergence obtained for $\theta = 1$ are broadly replicated for sufficiently small values of θ. This is not surprising, given that convergence (divergence) in wealth distribution is associated with high (low) rates of consumption growth in the economy, and θ indicates how strong the incentive for consumption smoothing is in the economy. However, as discussion of the case of Cobb–Douglas technology shows, allowing a general CEMU instantaneous utility function with no restrictions on θ might reveal behavior not present when one assumes log instantaneous utility with $\theta = 1$.

6. CONCLUSION

The Ramsey–Cass–Koopmans model of growth describes the behavior of a perfectly competitive economy with a neoclassical production technology that allows the existence of a globally stable steady state. Like the perfectly competitive model in price theory, this neoclassical growth model serves as a benchmark for comparison in growth theory. The aim of this paper has been to further characterize the behavior of a Ramsey–Cass–Koopmans economy in transition to the steady state. The paper concentrates on three aspects of the economy: the consumption–capital ratio, inequality in wealth distribution, and the (gross) saving rate. The decision to consider these three aspects simultaneously is, as the paper demonstrates, due to the common approach that can be brought into analyzing the transitional dynamics of three ratios in the model: the consumption–capital ratio, the consumption–wage ratio, and the consumption–output ratio (which is nothing but the saving rate deducted from unity). The evolution of wealth inequality is related, as is demonstrated in Section 4, to the transitional behavior of the consumption–wage ratio.

We know that in the R-C-K model, the consumption–capital ratio, the saving rate, and the degree of wealth inequality must be either strictly increasing, strictly decreasing, or constant in some neighborhood of the steady state. The specific aim of this paper was to discover the conditions under which such behavior could be extended to the entire transition path. Other contributions in the literature that consider transitional dynamics of the consumption–capital ratio, wealth inequality, or the saving rate in the Ramsey–Cass–Koopmans model do so using specific forms for technologies or preferences. In contrast, our analysis allows us to state sufficient conditions for a strictly increasing, a strictly decreasing, or a constant trend in the concerned variables in terms of preference and technology parameters relating to general CEMU preferences and neoclassical production technologies. This also allows us to derive many of the results in the literature as special cases of our results in this paper. Moreover, our discussion of the dynamics of wealth distribution indicates that the approach used in the paper can also be used to discuss the evolution of wealth inequality in a modified version of the neoclassical growth model with Stone–Geary utility or to consider the effects of redistributive tax–subsidy schemes on wealth inequality in the usual neoclassical model.

Finally, the analysis presented in the paper for the R-C-K model appears extensible to similar one-sector representative-consumer growth models with transitional dynamics. For example, many of the results in the paper regarding transition to the steady state from below are clearly applicable to the one-sector Jones–Manuelli model of endogenous growth discussed in Barro and Sala-i-Martin (1995).

NOTES

1. That is, the ratio of aggregate (per capita) consumption to aggregate (per capita) wage income.
2. See, for example, Barro and Sala-i-Martin (1995) and Romer (1996).
3. See also Chatterjee (1994), which analyzes the evolution of wealth inequality in a "neoclassical growth model" in which capital is the only factor of production and source of income, but equilibrium paths of aggregate output and capital stock are the same as in a R-C-K model with equivalent population size.
4. In mathematical terms, this means that the standard R-C-K model can be shown to have the following property: Consider any system of differential equations in two variables, one of which is the capital stock per capita and the other is a ratio of the consumption per capita to a twice-differentiable function of the capital stock per capita. The (steady state) equilibrium of this system is a saddlepoint.
5. Caselli and Ventura (2000), analytically, and Obiols-Homs and Urrutia (2005), through numerical simulations, have shown that wealth inequality in the R-C-K model can display nonmonotonic behavior. Smetters (2003) demonstrates nonmonotonic behavior of the saving rate in the R-C-K model.
6. These are, for example, illustrated respectively by the cases of the inverted-J curves and the inverted-U curves for inequality cited by Chatterjee (1994, p. 114). Christiano (1989) discusses the hump-shaped trajectory of the saving rate in Japan after World War II. Barro and Sala-i-Martin (1995) consider long-term data given by Maddison (1992) on the gross national saving ratio and the domestic investment ratio for eight countries. They conclude that with the exception of the United States, "the data for the other seven countries show a clear increase in these ratios over time" (p. 9).
7. See, for example, Atkinson (1972, Chapter 5) and Buchanan (1986, Chapter 12).
8. See Section 4.4.
9. For a detailed description see Barro and Sala-i-Martin (1995). A constant rate of exogenous labor-augmenting technical progress or a constant positive rate of growth of population can be introduced without changing the basic character of the results in the paper.
10. The above property can be derived easily from the properties of $f(k)$. Note, in particular, that $k^* < \bar{k}$.
11. This is also the condition obtained by Caselli and Ventura (2000), who assume that the equilibrium growth path is associated with convergence (divergence) in the distribution of wealth if for all $t \geq 0$ and for all $i \in \{1, 2, \ldots, N\}$ it is true that $|\{a_i(t)/k(t)\} - 1|$ is decreasing (increasing).

12. We assume that there is no change in the degree of wealth inequality at any given instant, if and only if the share of every individual in the total wealth of the economy tends to remain unchanged at that instant; i.e., for all $t \geq 0$, there is no change in wealth inequality at t if and only if $X(t) = 0$.

13. If $a_i(t) < 0$ then the individual gets a proportional subsidy $-\gamma a_i(t)$ together with a lump sum subsidy $\gamma k(t)$.

14. If the economy is initially in steady state any positive tax rate will do.

15. Alternatively, this policy regime can be viewed as one with a proportional tax on net income at the rate γ combined with a lump sum subsidy $\gamma(f(k(t)) - \delta k(t))$].

16. These conditions subsume the set of sufficient conditions that can be derived from Proposition 13 for these cases. The claim in Proposition 1(A) by Smetters (2003) that there will always exist a value of $k(0) < k^*$ such that $\theta > (\rho + \delta) / \{\delta\sigma\omega(k(0))\}$ is, however, invalid, because $(\rho + \delta)/\delta$ provides a lower bound to the the RHS expression in this inequality. The proof by Smetters (2003, p. 705) is based on the incorrect assumption that $\lim_{k\to 0} f'(k) = \infty$ for CES production with $\sigma < 1$.

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