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DIFFERENTIAL GRADED MOTIVES: WEIGHT COMPLEX, WEIGHT FILTRATIONS AND SPECTRAL SEQUENCES FOR REALIZATIONS; VOEVODSKY VERSUS HANAMURA

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Abstract We describe explicitly the Voevodsky's triangulated category of motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (and give a 'differential graded enhancement' of it). This enables us to able to verify that $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ is (anti)isomorphic to Hanamura's $\mathcal{D}(k)$.

We obtain a description of all subcategories (including those of Tate motives) and of all localizations of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. We construct a conservative *weight complex* functor $t : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$; t gives an isomorphism $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \to K_0(\mathrm{Chow}^{\mathrm{eff}})$. A motif is mixed Tate whenever its weight complex is. Over finite fields the Beilinson–Parshin conjecture holds if and only if $t\mathbb{Q}$ is an equivalence.

For a realization D of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ we construct a spectral sequence S (the *spectral sequence of motivic descent*) converging to the cohomology of an arbitrary motif X. S is 'motivically functorial'; it gives a canonical functorial weight filtration on the cohomology of $D(X)$. For the 'standard' realizations this filtration coincides with the usual one (up to a shift of indices). For the motivic cohomology this weight filtration is non-trivial and appears to be quite new.

We define the (rational) *length* of a motif M; modulo certain 'standard' conjectures this length coincides with the maximal length of the weight filtration of the singular cohomology of M.

Keywords: motives; algebraic cycles; realizations and cohomology; weight filtrations and spectral sequences

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Contents

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Introduction

We give an explicit description of the category of effective geometric motives of Voevodsky (see [**36**]). In what follows, DM^{s} is the full triangulated subcategory of the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (defined in [**36**]) generated by motives of smooth varieties (we do not add the kernels of projectors). It is proved that for any motivic complex M (i.e. an object of Voevodsky's $\mathrm{DM}^{\mathrm{eff}}_{-}$ that comes from DM^{s}; in particular, the Suslin complex of an arbitrary variety) there exists a quasi-isomorphic complex M' 'constructed from' the Suslin complexes of smooth projective varieties; M' is unique up to a homotopy. We prove that Hanamura's $\mathcal{D}(k)$ is $\mathcal{D}(k)$ is (anti)equivalent to Voevodsky's $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$.

Our main category $\mathfrak{H}$ is defined as the category of *twisted complexes* over a certain differential graded category whose objects are cubical Suslin complexes; we construct an equivalence $m : \mathfrak{H} \to \mathrm{DM}^{\mathrm{s}}$. In terms of [**6**] our description of DM^{s} gives an *enhancement* of this category. One should think of twisted complexes as of the results of repetitive computation of cones of morphisms in an 'enhanced' triangulated category. One can describe any subcategory of $\mathfrak{H}$ that is generated by a fixed set of objects; this method gives a description of the triangulated category of Tate motives similar to the rational description of [**25**]. Besides, any localization of $\mathfrak{H}$ can be described explicitly using the construction of Drinfeld (see [**10**]).

As an application we consider the problem of constructing exact functors from DM^{s} (i.e. realizations) in terms of cubical Suslin motivic complexes. The most simple and yet quite interesting of functors constructed by our method are the *truncation functors* t_N that correspond to the canonical filtration of the Suslin complex. The target of t_0 is just the category $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ ($\mathrm{Corr}_{\mathrm{rat}}$ is 'almost' the category of effective Chow motives, see § 1.1). t_0 extends to a conservative weight complex functor $t : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$. We prove that t induces an isomorphism $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \to K_0(\mathrm{Chow}^{\mathrm{eff}})$, thus answering the question of [**14**].

We show that if W denotes the weight filtration on $H^i(X)$ for a 'standard' realization H then $W_{l+N}H^i(X)/W_{l-1}H^i(X)$ factorizes through t_N.

We prove that a motif X belongs to a triangulated category $M \subset \mathfrak{H}$ generated by motives of a given set of smooth projective varieties P_i whenever the same is true for $t(X)$ (as a complex of Chow motives). In particular, the motif of a smooth variety is a mixed Tate one if and only if its weight complex (as defined by Gillet and Soulé) is.

For any realization D of motives that belongs to a wide class of 'enhanceable' realizations (i.e. of realizations that admit a differential graded 'enhancement') we construct a family of 'truncated realizations'. In particular, this could be applied to 'standard' realizations and motivic cohomology; an interesting new family of realizations is obtained. This yields a canonical spectral sequence S converging to the cohomology of $D(X)$ of an arbitrary motif X. S could be called the *spectral sequence of motivic descent*. The E_1-terms of S are expressed in cohomology of smooth projective varieties, the E_n-terms of S have a nice description in terms of $t_{2n-2}(X)$, $n \geqslant 1$. S is canonical (starting from E_1) and 'motivically functorial', it is also functorial with respect to transformations of functors. S gives a canonical non-trivial weight filtration for 'differential graded' realiza-

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tions of motives; for the 'standard' realizations this filtration coincides with the usual one (up to a shift of indices).

The simplest case of S for motivic cohomology is the Bloch's long exact localization sequence for higher Chow groups (see [**4**]). Our 'weight' filtration on motivic cohomology is non-trivial and was not mentioned in the literature; it gives a new filtration on the K-theory of a smooth variety X.

We also study motives with compact support ($M^{\mathrm{c}}_{\mathrm{gm}}$ in the notation of Voevodsky). We give an explicit description of $M^{\mathrm{c}}_{\mathrm{gm}}$ for a variety Z and prove that the weight complex of Gillet and Soulé can be described as $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(Z)))$ (with arrows reversed); the functor h of Guillen and Navarro Aznar is (essentially) $t_0(m^{-1}(M_{\mathrm{gm}}(Z)))$.

We define the 'length' of a motif (*stupid*, *fine* or *rational*); this is a natural motivic analogue of the length of weight filtration for a mixed Hodge structure. For a smooth variety X the length of $M_{\mathrm{gm}}(X)$ lies between the length of the weight filtration of the singular cohomology of X and the dimension of X. If certain 'standard' conjectures are valid then the rational length of a motif coincides with the (appropriately defined) length of the weight filtration of its singular realization.

We note that in the current paper we apply several results of [**36**] that use resolution of singularities; yet applying de Jong's alterations one can easily extend (most of) our results (at least) to motives with rational coefficients over an arbitrary perfect k. For a finite k the Beilinson–Parshin conjecture (that the only non-zero $H^i(X,\mathbb{Q}(n))$ for smooth projective X is H^{2n}) holds if and only if $t\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q})$ is an equivalence (note that here $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$ and $t\mathbb{Q}$ denote the appropriate idempotent completions). Much can also be proved with integral coefficients.

The author would like to note that several interesting results of this paper (in particular, the properties of t_0 and t) follow just from the fact that $\mathfrak{H} = \mathrm{Tr}(J)$ for J being a negative differential graded category (see § 2.4). Hence these results are also valid for any other example of this situation. Moreover, in [**7**] a set of axioms of so-called *weight structures* for a triangulated category C is introduced (see Remark 7.4.4). This (abstract) approach allows to extend (most of) the results of the current paper to a wide class of triangulated categories and realizations that do not necessarily have a differential graded 'enhancement'; in particular, it can be applied to the stable homotopy category. The definition of the weight filtration is quite easy in this context. Yet in this abstract setting it is difficult to define truncation functors (especially the 'higher' ones).

Besides, recently Levine (in his very interesting paper [**29**]) extended some of the results of the current paper to relative motives (i.e. motives over a regular base S, which is essentially of finite type over k). This paper also contains a nice exposition of the yoga of cubical objects (which is essential for the current paper) and of related tensor products.

Now we list the contents of the paper. More details can be found at the start of each section.

In the first section we recall some basic notation of [**36**] (with minor modifications). Next we describe cubical Suslin complexes and their properties. Most of the proofs are postponed till § 5 since they are not important for the understanding of main results.

We start § 2 by recalling the formalism of differential graded categories (§ 2.1) and twisted complexes (§§ 2.2 and 2.3). Next we use this formalism to construct our main objects of study: a triangulated category $\mathfrak{H}$ and a functor h from $\mathfrak{H}$ into the homotopy category of complexes of Nisnevich sheaves with transfers (§ 2.4). $\mathfrak{H}$ as a triangulated category is generated by motives of smooth projective varieties. We also describe $\mathfrak{H}$ and h more explicitly (in § 2.5). We define 'stupid' filtration for objects of $\mathfrak{H}$ that is similar to the 'stupid' filtration for complexes over an additive category (§ 2.6). Note that (similarly to the case of complexes) the filtration for a motif also depends on the choice of its 'lift' to a certain differential graded category (as the stupid truncation object of $K(A)$ depends on its lift to $C(A)$); yet this filtration is 'functorial enough' for our purposes. In § 2.7 we introduce more differential graded definitions; they will be used in § 7.

In § 3 we prove (Theorem 3.1.1) that h composed with the natural functor from the homotopy category of sheaves with transfers to the derived category gives an equivalence $m : \mathfrak{H} \to \mathrm{DM}^{\mathrm{s}}$.

In § 4 we prove that Voevodsky's $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ is (anti)isomorphic to Hanamura's $\mathcal{D}(k)$.

In § 5 we verify the properties of cubical Suslin complexes. The reader not interested in the proofs of auxiliary results of § 1 may skip this section.

In § 6 using the canonical filtration of the (cubical) Suslin complex we define the 'truncation' functors $t_N : \mathfrak{H} \to \mathfrak{H}_N$ (in § 6.1). $\mathfrak{H}_0$ (i.e. the target of t_0) is just $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. These functors are new though their certain restrictions to varieties were (essentially) considered in [**14**] and [**16**] (and were shown to be quite important). We prove that t_0 extends to $t : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ (in § 6.3). All t_N (see Theorem 6.2.1) and t (Proposition 6.3.1) are conservative. t induces an isomorphism $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \cong K_0(\mathrm{Chow}^{\mathrm{eff}})$ (Theorem 6.4.2). Certainly, this extends to an isomorphism $K_0(\mathrm{DM}_{\mathrm{gm}}) \cong K_0(\mathrm{Chow})$ (Corollary 6.4.3).

We define the *length* of a motif (three types); the *stupid length* (see § 6.2) is not less than the *fine* one (see § 6.3), which is not less than the *rational length*. We prove that motives of smooth varieties of dimension N have stupid length less than or equal to N (parts (1) and (2) of Theorem 6.2.1); besides $t_N(X)$ contains all information on motives of stupid length less than or equal to N (see part (3) of Theorem 6.2.1 and § 7.3).

At the end of the section we calculate $m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))$ for a smooth X explicitly (in § 6.5). Using this result as well as cdh-descent we prove that the weight complex of Gillet and Soulé for X/k could be described as $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X)))$ (with arrows reversed; see § 6.6). Besides, $t_0(m^{-1}(M_{\mathrm{gm}}(X)))$ essentially coincides with the functor h described in Theorem 5.10 of [**16**].

In § 7 we study *realizations* of the category of motives and their connections with (certain) weight filtrations. The differential graded categories formalism yields a general recipe of constructing realizations (see § 7.1). It is quite easy to determine which of those *enhanceable* realizations can be factorized through t_N.

We verify that the étale and motivic cohomology are enhanceable realizations (see §§ 7.2 and 7.5, and part (1) of Remark 7.3.1). Very probably, this result could be extended to all other 'standard' realizations. In § 7.3, for any enhanceable realization D we describe an interesting new family of 'truncated realizations'; they correspond to 'forgetting' cohomology outside a given range of weights. Truncated realizations give a filtration of the complex that computes the given 'enhanced' realization of a motif Y. We obtain a spectral sequence S converging to $D^i(Y)$ (see (14)). Its E_n-terms for $n \geqslant 2$ have a nice description in terms of $t_{2n-2}(Y)$; in particular, E_1-terms are functorial in $t_0(Y)$. S is the *spectral sequence of motivic descent*. S gives a canonical integral weight filtration for 'enhanced' realizations of motives; for the 'standard' realizations this filtration coincides with the usual one (up to a shift of indices). S is 'motivically functorial', it is also functorial with respect to ('enhanced') transformation of functors.

In § 7.4 we prove that for a 'standard' H the Nth truncated realization computes $W_{l+N}H^l(X)/W_{l-1}H^l(X)$, while W equals the 'standard' weight filtration. A morphism f induces a zero morphism on cohomology if $t_0(f)$ is zero. We also prove (modulo certain 'standard' conjectures) that the rational length of a motif coincides with the 'range' of difference of l with the weights of H^l for all l; see Proposition 7.4.2. We conclude the section with a discussion of qfh-descent and motives of singular varieties (see § 7.5).

In § 8 we apply the general theory of [**6**] to describe any subcategory of $\mathfrak{H}$ that is generated by a fixed set of objects (see § 8.1). In particular, this method can be used to obtain the description of the triangulated category of effective Tate motives (i.e. the full triangulated subcategory of $\mathfrak{H}$ generated by $\mathbb{Z}(n)$ for $n > 0$).

In § 8.2 we describe the construction of 'localization of differential graded categories' (due to Drinfeld). This gives us a description of localizations of $\mathfrak{H}$. As an application, we prove that the motif of a smooth X/k is a mixed Tate one whenever the weight complex of X (defined in [**14**]) is.

In § 8.3 we verify that over an arbitrary perfect field one can apply our theory (at least) with rational coefficients. Moreover, over finite fields the Beilinson–Parshin conjecture holds if and only if $t\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q})$ is an equivalence. We also describe an idea for constructing a certain 'infinite integral' weight complex functor in finite characteristic.

In § 8.4 we prove that traces of endomorphisms of cohomology of motives induced by endomorphisms of motives do not depend on the choice of a Weil cohomology theory. In particular, this could be applied for the morphisms induced by 'open correspondences' (as described in Definition 3.1 of [**5**]). We obtain a generalization of Theorem 3.3 of [**5**] to the case of varieties which are not necessarily complements of smooth projective varieties by strict normal crossing divisors.

In § 8.5 we note that one can modify the description of DM^{s} so that $\mathbb{Z}(n)$ will have stupid length 0. Lastly we describe certain functors $m_N : \mathfrak{H}_N \to \mathrm{DM}^{\mathrm{eff}}_{-}$ (see § 8.6). t_N and m_N could be related to the (yet conjectural) weight filtration on $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

Notation

In this paper all complexes will be cohomological, i.e. the degree of all differentials is +1.

We recall that for any triangulated T there exists a unique category $T' \supset T$ that is obtained from T by 'adding the kernels of all projectors'; T' is called the idempotent completion of T (see [**1**]).

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For an additive category A we will denote by $A \otimes \mathbb{Q}$ its rational hull, i.e. $\operatorname{Obj} A \otimes \mathbb{Q} = \operatorname{Obj} A$ while morphisms are tensored by $\mathbb{Q}$. $A\mathbb{Q}$ will usually (except for some notation of § 4) denote the idempotent completion of $A \otimes \mathbb{Q}$.

We will call a realization of motives (usually of $\mathfrak{H}$) *enhanceable* if it has a differential graded enhancement (see § 7).

Other notation will be more or less standard. k will denote the ground field; we will assume (except in § 8.3) that the characteristic of k is zero. pt is a point, $\mathbb{A}^n$ is the n-dimensional affine space (over k), $\mathbb{P}^n$ is the projective space of dimension n.

For an additive category A we denote by $C^-(A)$ the category of complexes over A bounded from above; $C^{\mathrm{b}}(A) \subset C^-(A)$ is the subcategory of bounded complexes; $K^-(A)$ is the homotopy category of $C^-(A)$, i.e. the morphisms of complexes are considered up to homotopy equivalence; K^{b} denotes the homotopic category of bounded complexes; sometimes we will also need the unbounded categories $C(A)$ and $K(A)$; Ab is the category of abelian groups.

For a category $C, A, B \in \operatorname{Obj} C$, we denote by $C(A, B)$ the set of C-morphisms from A into B.

For categories C, D we write $C \subset D$ if C is a full subcategory of D.

We list the main definitions of this paper. Some basic motivic definitions (mostly coming from [**36**]) will be given in § 1.1. $C(X)$ will be defined in § 1.2; g^l will be defined in § 1.3; differential graded categories, $H(C)$ for a differential graded category C, $S(A)$, $S_N(A)$, $B^-(A)$, $B^{\mathrm{b}}(A)$, $B(A)$, and $C(A)$ for an additive category A will be defined in § 2.1; the categories of twisted complexes (Pre-Tr(C), Tr(C), Pre-Tr$^+(C)$, Tr$^+(C)$), arrows, $[P]$ and $P[i]$ for $P \in \operatorname{Obj} C$ will be defined in §§ 2.2 and 2.3; Tr(F), Pre-Tr(F), Tr$^+(F)$, and Pre-Tr$^+(F)$ for a differential graded functor F will be defined in Remark 2.3.3; J, $\mathfrak{H}$, and h will be defined in § 2.4; $\mathfrak{H}'$, h', j, and J' will be defined in § 2.5; C_- and different types of truncations of complexes ($\tau_{\leqslant b}$, $\tau_{[a,b]}$ and the canonical $[a, b]$-truncation) will be defined in § 2.7; m will be defined in § 3.1; $\mathrm{DM_{gm}}$ will be described in § 4; $C^N(P)$, $\mathfrak{H}_N$, and t_N will be defined in § 6.1; t, $t\mathbb{Q}$, and $\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}} \cong \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ will be defined in § 6.3; truncated realizations will be defined in § 7.3.

1. Cubical Suslin complexes

In this paper instead of the simplicial Suslin complex $\underline{C}(L(P))$ we consider its cubical version $C(P)$. In this section we prepare for the proof of the following fact: there exists a differential graded category J (it will be defined in § 2) whose objects are the (cubical) Suslin complexes of smooth projective varieties, while its morphisms are related to the morphisms between those complexes in $\mathrm{DM}^{\mathrm{eff}}_-$.

First we recall basic definitions of Voevodsky (along with some 'classical' motivic definitions).

1.1. Some definitions of Voevodsky: a reminder

We use much of the notation from [**36**]. We recall (some of) it here for the convenience of the reader. Those who remember Voevodsky's notation well (and agree to identify certain equivalent categories) could skip this subsection; note only that $\mathrm{DM^s}$ is the smallest strict triangulated subcategory of $\mathrm{DM}_-^{\mathrm{eff}}$ containing all motives of smooth varieties.

$\mathrm{Var} \supset \mathrm{SmVar} \supset \mathrm{SmPrVar}$ will denote the class of all varieties over k, respectively of smooth varieties, respectively of smooth projective varieties.

SmCor is the category of 'smooth correspondences', i.e. $\mathrm{Obj}\,\mathrm{SmCor} = \mathrm{SmVar}$, $\mathrm{SmCor}(X,Y) = \sum_U \mathbb{Z}$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X.

$\mathrm{Shv}(\mathrm{SmCor}) = \mathrm{Shv}(\mathrm{SmCor})_{\mathrm{Nis}}$ is the abelian category of additive cofunctors $\mathrm{SmCor} \to \mathrm{Ab}$ that are sheaves in the Nisnevich topology (when restricted to the category of smooth varieties); these sheaves are usually called 'sheaves with transfers'. Moreover, by default all sheaves will be sheaves in Nisnevich topology. By an abuse of notation we will also denote by Shv(SmCor) the set of all Nisnevich sheaves with transfers; $D^-(\mathrm{Shv}(\mathrm{SmCor}))$ is the derived category of Shv(SmCor).

For $Y \in \mathrm{SmVar}$ (more generally, for $Y \in \mathrm{Var}$, see § 4.1 of [**36**]) we consider $L(Y) = \mathrm{SmCor}(-,X) \in \mathrm{Shv}(\mathrm{SmCor})$. $L^{\mathrm{c}}(X)(Y) \supset L(X)(Y)$ denotes the group whose generators are the same as for $L(X,Y)$ except that U is only required to be quasi-finite over X. $L(X) = L^{\mathrm{c}}(X)$ for proper X. Note that $L^{\mathrm{c}}(X)$ is also a sheaf.

$M_{\mathrm{gm}}(X) = \underline{C}(L(X)) \cong C(L(X))$ is the Suslin complex of $L(X)$, see § 1.2 below for details; $M^{\mathrm{c}}_{\mathrm{gm}}(X) = \underline{C}(L^{\mathrm{c}}(X)) \cong C(L^{\mathrm{c}}(X))$.

$S \in \mathrm{Shv}(\mathrm{SmCor})$ is called homotopy invariant if for any $X \in \mathrm{SmVar}$ the projection $\mathbb{A}^1 \times X \to X$ gives an isomorphism $S(X) \to S(\mathbb{A}^1 \times X)$.

$\mathrm{DM}_-^{\mathrm{eff}} \subset D^-(\mathrm{Shv}(\mathrm{SmCor}))$ is the subcategory of complexes whose cohomology sheaves are homotopy invariant. It was proved in [**36**] that for any $F \in \mathrm{Shv}(\mathrm{SmCor})$ we have $\underline{C}(F) \in \mathrm{DM}_-^{\mathrm{eff}}$.

The functor $\mathrm{RC} : D^-(\mathrm{Shv}(\mathrm{SmCor})) \to \mathrm{DM}_-^{\mathrm{eff}}$ is given by taking total complexes of the Suslin bicomplex of a complex of sheaves (see § 3.2 of [**36**] for details).

$\mathrm{DM^s}$ will denote the full strict triangulated subcategory of $\mathrm{DM}_-^{\mathrm{eff}}$ generated by $M_{\mathrm{gm}}(X)$ for $X \in \mathrm{SmVar}$ (we do not add the kernels of projectors). $\mathrm{DM^s}$ has a natural tensor structure that can be defined using the relation $M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(Y) = M_{\mathrm{gm}}(X \times Y)$; tensor multiplication of morphisms is defined by means of a similar relation.

In [**36**] Voevodsky defined $\mathrm{DM^s}$ as a certain localization of $K^{\mathrm{b}}(\mathrm{SmCor})$ (note that he did not introduce any notation for $\mathrm{DM^s}$); then $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ was defined as the idempotent completion of DM^s. Yet Theorem 3.2.6 of [**36**] (essentially) states that 'his' DM^s is equivalent to those defined here. So we will denote by $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ the idempotent completion of 'our' DM^s.

$\mathrm{DM}_{\mathrm{gm}}$ in [**36**] was obtained from $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (considered as an abstract category, i.e. not as a subcategory of $\mathrm{DM}_-^{\mathrm{eff}}$) by the formal inversion of $\mathbb{Z}(1)$ with respect to $\otimes$. We will use the same definition; see § 4 below for details. $\mathrm{DM}_{\mathrm{gm}}$ is a rigid tensor triangulated category. We will also consider the idempotent completion $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ of $\mathrm{DM}_{\mathrm{gm}} \otimes \mathbb{Q}$.

One can easily check that $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ is the idempotent completion of $\mathrm{DM^s} \otimes \mathbb{Q}[\mathbb{Z}(-1)]$.

$\mathrm{Corr}_{\mathrm{rat}}$ will denote the (homological) category of rational correspondences. Its objects are smooth projective varieties; the morphisms are morphisms in SmCor up to homotopy equivalence. The category $\mathrm{Chow}^{\mathrm{eff}}$ is the idempotent completion of $\mathrm{Corr}_{\mathrm{rat}}$; it was shown

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in Proposition 2.1.4 of [**36**] that $\mathrm{Chow}^{\mathrm{eff}}$ is naturally isomorphic to the usual category of effective homological Chow motives.

Chow will denote the whole category of Chow motives, i.e. $\mathrm{Chow}^{\mathrm{eff}}[\mathbb{Z}(-1)]$.

Note (as it is well known already from the works on Tate motives that come from quadratic forms) that $\mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}} \neq \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q}$, i.e. on the rational level one gets more idempotents in $\mathrm{Corr}_{\mathrm{rat}}$. Certainly, the same is true for $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$.

We recall also that for categories of geometric origin (for example, for $\mathrm{Corr}_{\mathrm{rat}}$ and SmCor) the addition of objects is induced by the disjoint union of varieties operation.

1.2. The definition of the cubical complex

For any $P \in \mathrm{SmVar}$ we consider the sheaves

$$C'^i(P)(Y) = \mathrm{SmCor}(\mathbb{A}^{-i} \times Y, P), \quad Y \in \mathrm{SmVar}; \qquad C^i = 0 \quad \text{for } i > 0.$$

We will usually consider projective P.

By Yoneda's lemma,

$$C'^i(P)(Y) \cong \mathrm{Shv}(\mathrm{SmCor})(L(Y), L(P)) = \mathrm{Shv}(\mathrm{SmCor})(C'^0(Y), C'^i(P)).$$

For all $1 \leqslant j \leqslant -i$, $x \in k$, we define $d_{ijx} = d_{jx} : C'^i \to C'^{i+1}$ as $d_{jx}(f) = f \circ g_{jx}$, where $g_{jx} : \mathbb{A}^{-i-1} \times Y \to \mathbb{A}^{-i} \times Y$ is induced by the map $(x_1, \ldots, x_{-1-i}) \to (x_1, \ldots, x_{j-1}, x, x_j, \ldots, x_{-1-i})$. We define $C^i(P)(Y)$ as $\bigcap_{1\leqslant j\leqslant -i} \mathrm{Ker}\, d_{j0}$. One may say that $C^i(P)(Y)$ consists of correspondences that 'are zero if one of the coordinates is zero'. The boundary maps $\delta^i : C^i \to C^{i+1}$ are defined as $\sum_{1\leqslant j\leqslant -i}(-1)^j d_{j1}$. Again, $C^i = 0$ for positive i.

Since $C'^0 = C^0$, we have $C^i(P)(Y) \cong \mathrm{Shv}(\mathrm{SmCor})(C^0(Y), C^i(P))$.

Remark 1.2.1.

(1) The definition of the cubical Suslin complex can be easily extended to an arbitrary complex D over Shv(SmCor) (or over a slightly different abelian category). One should consider the total complex of the double complex whose terms are

$$D^{ij}(X) = \bigcap_{1\leqslant l\leqslant -i} \mathrm{Ker}\, g^*_{jl0} : D^j(\mathbb{A}^{-i} \times X) \to D^j(\mathbb{A}^{-i-1} \times X),$$

the boundaries are induced by δ^i.

(2) In the usual (simplicial) Suslin complex one defines $\underline{C}^i(F)(X) = F(D^{-i} \times X)$, where $D^{-i} \subset \mathbb{A}^{1-i}$ is given by $\sum_{1\leqslant l\leqslant 1-i} x_l = 1$; the boundaries come from restrictions to $x_l = 0$.

It is well known that cubical and simplicial complexes do not differ much; the main advantage of cubical complexes is that descriptions of (various) products become much nicer (see § 2.5 of [**28**]).

We formulate the main property of C.

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Proposition 1.2.2. *For any* $j \in \mathbb{Z}$, $Y \in \mathrm{SmVar}$, *and* $P \in \mathrm{SmPrVar}$ *there is a natural isomorphism* $H^j C(P)(Y) \cong A_{0,-j}(Y,P) \cong \mathrm{DM}^{\mathrm{eff}}_{-}(\underline{C}(Y), \underline{C}(P)[j])$.

Proof. $A_{0,-j}(Y,P) \cong \mathrm{DM}^{\mathrm{eff}}_{-}(\underline{C}(Y), \underline{C}^{\mathrm{c}}(P))$ by [**36**, Proposition 4.2.3]. Since P is projective, by [**36**, Proposition 4.1.5] we have $\underline{C}^{\mathrm{c}}(P) = \underline{C}(P)$.

The first isomorphism will be described in § 5 below.

All isomorphisms are natural. □

In particular the cohomology presheaves of $C(P)$ are homotopy invariant.

We denote the initial object of SmCor by 0. We define $C^i(0) = 0$ for all $i \in \mathbb{Z}$. We obtain

$$p(C(P)) \in \mathrm{Obj}\,\mathrm{DM}^{\mathrm{eff}}_{-} \subset \mathrm{Obj}\,D^{-}(\mathrm{Shv}(\mathrm{SmCor})),$$

where $p : K^{-}(\mathrm{Shv}(\mathrm{SmCor})) \to D^{-}(\mathrm{Shv}(\mathrm{SmCor}))$ is the natural projection.

1.3. The assignment $g \to (g^l)$

Let $P, Y \in \mathrm{SmVar}$. We construct a family of morphisms $C(Y) \to C(P)[i]$.

For any $f \in C'^i(P)(Y)$, $l \leqslant 0$, we define $f^l : C'^l(Y) \to C'^{l+i}(P)$ as follows. To the element $h \in \mathrm{SmCor}(Z \times \mathbb{A}^{-l}, Y)$, $Z \in \mathrm{SmVar}$, we assign $(-1)^{li} f \circ (\mathrm{id}_{\mathbb{A}^{-i}} \otimes h)$. It is easily seen that the same formula also defines the maps $f^l : C^l(Y) \to C^{l+i}(P)$ for $f \in C^i(P)(Y)$.

Proposition 1.3.1.

(1) *The assignment* $g \to G = (g^l)$ *defines a homomorphism* $\mathrm{Ker}\,\delta^i(P)(Y) \to K^{-}(\mathrm{Shv}(\mathrm{SmCor}))(C(Y), C(P)[i])$.

(2) *The assignment* $g \to G = (g^l)$ *induces an isomorphism* $H^i(C(P)(Y)) \cong \mathrm{DM}^{\mathrm{eff}}_{-}(C(Y), C(P)[i])$.

Proof. (1) For any $f \in C'^i(P)(Y)$, $h \in C'^l(Y)(Z)$, $Z \in \mathrm{SmVar}$ we have an equality

$$\delta^{i+l} f^l(h) = (-1)^i f^{l+1} \delta^l(h) + (\delta^i f)^l(h). \tag{1}$$

Hence if $\delta^i g = 0$, $g \in C^i(P)(Y)$ then G defines a morphism of complexes $C(Y) \to C(P)[i]$.

(2) Using (1) we obtain that the elements of $\delta^{i+1}(C^{i+1}(P)(Y))$ give homomorphisms $C(Y) \to C(P)[i]$ that are homotopy equivalent to 0. Hence we obtain a homomorphism $H^i C(P)(Y) \to \mathrm{DM}^{\mathrm{eff}}_{-}(p(C(Y)), p(C(P)[i]))$. The bijectivity of this homomorphism will be proved in § 5 below. □

2. Differential graded categories; the description of $\mathfrak{H}$ and $h : \mathfrak{H} \to K^{-}(\mathrm{Shv}(\mathrm{SmCor}))$

Categories of *twisted complexes* (defined in §§ 2.2 and 2.3) were first considered in [**6**]. Yet our notation differs slightly from that of [**6**]; some of the signs are also different.

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In §§ 2.4 and 2.5 we define and describe our main categories: J, $\mathfrak{H}$, J' and $\mathfrak{H}'$.

In § 2.6 we define a natural 'stupid' filtration on $\mathfrak{H}'$ that is 'close' to those on $C^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$; we prove its natural properties.

We will not need the formalism of § 2.7 till § 7.

2.1. The definition of differential graded categories

Recall that an additive category C is called graded if for any $P, Q \in \operatorname{Obj} C$ there is a canonical decomposition $C(P,Q) \cong \bigoplus_i C^i(P,Q)$ defined; this decomposition satisfies $C^i(*,*) \circ C^j(*,*) \subset C^{i+j}(*,*)$. A differential graded category (cf. [**6**] or [**10**]) is a graded category endowed with an additive operator $\delta : C^i(P,Q) \to C^{i+1}(P,Q)$ for all $i \in \mathbb{Z}$, $P, Q \in \operatorname{Obj} C$. δ should satisfy the equalities $\delta^2 = 0$ (so $C(P,Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in \operatorname{Obj} C$, $f \in C^i(P,Q)$, $g \in C(Q,R)$. In particular, $\delta(\mathrm{id}_P) = 0$.

We denote δ restricted to morphisms of degree i by δ^i.

For an additive category A one can construct the following differential graded categories. The notation introduced below will be used throughout the paper.

We denote the first one by $S(A)$. We set $\operatorname{Obj} S(A) = \operatorname{Obj} A$; $S(A)^i(P,Q) = A(P,Q)$ for $i = 0$; $S(A)^i(P,Q) = 0$ for $i \neq 0$. We take $\delta = 0$.

We also consider the category $B^-(A)$ whose objects are the same as for $C^-(A)$, whereas for $P = (P^i)$, $Q = (Q^i)$ we define $B^-(A)(P,Q)^i = \bigoplus_{j \in \mathbb{Z}} A(P^j, Q^{i+j})$. Obviously, $B^-(A)$ is a graded category.

We denote by $B^{\mathrm{b}}(A)$ the full subcategory of $B^-(A)$ whose objects are bounded complexes. $B(A)$ and $C(A)$ will denote the corresponding categories whose objects are unbounded complexes.

We set $\delta f = d_Q \circ f - (-1)^i f \circ d_P$, where $f \in B^i(P,Q)$, d_P and d_Q are the differentials in P and Q. Note that the kernel of $\delta^0(P,Q)$ coincides with $C(A)(P,Q)$ (the morphisms of complexes); the image of δ^{-1} are the morphisms homotopic to 0.

For any $N \geqslant 0$ one can define a full subcategory $S_N(A)$ of $B^{\mathrm{b}}(A)$ whose objects are complexes concentrated in degrees $[0,N]$. We have $S(A) = S_0(A)$.

$B^{\mathrm{b}}(A)$ can be obtained from $S(A)$ (or any $S_N(A)$) by means of the category functor Pre-Tr described below.

For any differential graded C we define a category $H(C)$; its objects are the same as for C; its morphisms are defined as

$$H(C)(P,Q) = \operatorname{Ker} \delta_C^0(P,Q) / \operatorname{Im} \delta_C^{-1}(P,Q).$$

2.2. Categories of twisted complexes (Pre-Tr(C) and Tr(C))

Having a differential graded category C one can construct two other differential graded categories Pre-Tr(C) and Pre-Tr$^+(C)$ as well as triangulated categories Tr(C) and Tr$^+(C)$. The simplest example of these constructions is Pre-Tr$(S(A)) = B^{\mathrm{b}}(A)$.

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Definition 2.2.1.

(1) The objects of Pre-Tr(C) are

$$\{(P^i),\ P^i \in \operatorname{Obj} C,\ i \in \mathbb{Z},\ q_{ij} \in C^{i-j+1}(P^i, P^j)\};$$

here almost all P^i are 0; for any $i, j \in \mathbb{Z}$ we have $\delta q_{ij} + \sum_l q_{lj} \circ q_{il} = 0$. We call q_{ij} *arrows* of degree $i - j + 1$. For $P = \{(P^i), q_{ij}\}$, $P' = \{(P'^i), q'_{ij}\}$ we set

$$\text{Pre-Tr}_l(P, P') = \bigoplus_{i,j \in \mathbb{Z}} C^{l+i-j}(P^i, P'^j).$$

For $f \in C^{l+i-j}(P^i, P'^j)$ (an arrow of degree $l + i - j$) we define the differential of the corresponding morphism in Pre-Tr(C) as

$$\delta_{\text{Pre-Tr}(C)} f = \delta_C f + \sum_m (q'_{jm} \circ f - (-1)^{(i-m)l} f \circ q_{mi}).$$

(2) $\operatorname{Tr}(C) = H(\text{Pre-Tr}(C))$.

It can be easily seen that Pre-Tr(C) is a differential graded category (see [**6**]). There is also an obvious translation functor on Pre-Tr(C). Note also that the terms of the complex Pre-Tr$(C)(P, P')$ do not depend on q_{ij} and q'_{ij}, whereas the differentials certainly do.

We denote by $Q[j]$ the object of Pre-Tr(C) that is obtained by putting $P^i = Q$ for $i = -j$, all other $P^j = 0$, all $q_{ij} = 0$. We will write $[Q]$ instead of $Q[0]$ (i.e. $Q[i]$ is the translation of $[Q]$ by $[i]$).

Immediately from definition we have Pre-Tr$(S(A)) \cong B^{\mathrm{b}}(A)$.

A morphism $h \in \operatorname{Ker} \delta^0$ (a closed morphism of degree 0) is called a *twisted morphism*. For a twisted morphism $h = (h_{ij}) \in \text{Pre-Tr}((P^i, q_{ij}), (P'^i, q'_{ij}))$, $h_{ij} \in C(P^i, P'^j)$, we define $\operatorname{Cone}(h) = (P''^i, q''_{ij})$, where $P''^i = P^{i+1} \oplus P'^i$,

$$q''_{ij} = \begin{pmatrix} q_{i+1,j+1} & 0 \\ h_{i+1,j} & q'_{ij} \end{pmatrix}.$$

We have a natural triangle of twisted morphisms

$$P \xrightarrow{f} P' \to \operatorname{Cone}(f) \to P[1], \tag{2}$$

the components of the second map are $(0, \mathrm{id}_{P'^i})$ for $i = j$ and 0 otherwise. This triangle induces a triangle in the category $H(\text{Pre-Tr}(C))$.

Definition 2.2.2. For distinguished triangles in $\operatorname{Tr}(C)$ we take the triangles isomorphic to those that come from the diagram (2) for any $P, P' \in \text{Pre-Tr}(C)$, f being twisted.

We summarize the properties of Pre-Tr and Tr of [**6**] that are most relevant for the current paper. We have to replace bounded complexes by complexes bounded from above. Part (II) (4) of the following proposition is new.

Proposition 2.2.3.

(I) $\mathrm{Tr}(C)$ *is a triangulated category.*

(II) *For any additive category A there are natural isomorphisms*

(1) $\text{Pre-Tr}(B^-(A)) \cong B^-(A)$;

(2) $\mathrm{Tr}(B^-(A)) \cong K^-(A)$;

(3) $\text{Pre-Tr}(B(A)) \cong B(A)$;

(4) $\mathrm{Tr}(S_N(A)) \cong B^{\mathrm{b}}(A)$.

Proof. (I) See Proposition 1, § 2 of [**6**].

(II) (1), (3) See Lemma II.II.1.2.10 of [**27**].

(II) (2) Immediate from assertion (II) (1).

(II) (4) We have natural full embeddings $S_0(A) \subset S_N(A) \subset B^{\mathrm{b}}(A)$. Since $\mathrm{Tr}(S_0(A)) \cong \mathrm{Tr}(B^{\mathrm{b}}(A)) \cong B^{\mathrm{b}}(A)$, we obtain the assertion. □

2.3. The categories Pre-Tr$^+(C)$ and Tr$^+(C)$

In [**6**] $\text{Pre-Tr}^+(C)$ was defined as a full subcategory of $\text{Pre-Tr}(\tilde{C})$, where $\tilde{C}$ was obtained from C by adding formal shifts of objects. Yet it can be easily seen that the category defined in [**6**] is canonically equivalent to the category defined below (see also [**10**]). So we adopt the notation $\text{Pre-Tr}^+(C)$ of [**6**] for the category described below.

The definitions of $\text{Pre-Tr}^+(C)$ and $\mathrm{Tr}^+(C)$ could also be found in § 2.4 of [**10**]; there these categories were denoted by $C^{\text{pre-tr}}$ and C^{tr}.

Definition 2.3.1.

(1) $\text{Pre-Tr}^+(C)$ is defined as a full subcategory of $\text{Pre-Tr}(C)$. $A = \{(P^i), q_{ij}\} \in \operatorname{Obj}\text{Pre-Tr}^+(C)$ if there exist $m_i \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ we have $q_{ij} = 0$ for $i + m_i \geqslant j + m_j$.

(2) $\mathrm{Tr}^+(C)$ is defined as $H(\text{Pre-Tr}^+(C))$.

The following statement is an easy consequence of the definitions above.

Proposition 2.3.2.

(1) $\mathrm{Tr}^+(C)$ *is a triangulated subcategory of* $\mathrm{Tr}(C)$.

(2) $\mathrm{Tr}^+(C)$ *as a triangulated category is generated by the image of the natural map* $\operatorname{Obj} C \to \operatorname{Obj}\mathrm{Tr}^+(C) : P \to [P]$.

(3) *There are natural embeddings of categories* $i : C \to \text{Pre-Tr}^+(C)$ *and* $H(C) \to \mathrm{Tr}^+(C)$ *sending P to $[P]$.*

(4) $\text{Pre-Tr}(i)$, $\mathrm{Tr}(i)$, $\text{Pre-Tr}^+(i)$, *and* $\mathrm{Tr}^+(i)$ *are equivalences of categories.*

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It can be also easily seen that assertion (2) characterizes $\mathrm{Tr}^+(C)$ as a full subcategory of $\mathrm{Tr}(C)$

Proof. (1) It is sufficient to check that the cone of a map in $\text{Pre-Tr}^+(C)$ belongs to $\text{Pre-Tr}^+(C)$. This is easy. Also see § 4 of [**6**].

(2) See Theorem 1, § 4 of [**6**].

(3) By definition of $\text{Pre-Tr}^+(C)$ (respectively of $\mathrm{Tr}^+(C)$) there exists a canonical isomorphism of bifunctors $C(-,-) \cong \text{Pre-Tr}^+(C)([-],[-])$ (respectively $HC(-,-) \cong \mathrm{Tr}^+(C)([-],[-])$). In remains to note that both of these isomorphisms respect addition and composition of morphisms; the first one also respects differentials.

(4) The proof was given in §§ 3 and 4 of [**6**]. □

Remark 2.3.3.

(1) Since Pre-Tr, Pre-Tr^+, Tr, and Tr^+ are functors on the category of differential graded categories, any differential category functor $F : C \to C'$ naturally induces functors $\text{Pre-Tr}\, F$, $\text{Pre-Tr}^+ F$, $\mathrm{Tr}\, F$, and $\mathrm{Tr}^+ F$. We will use this fact throughout the paper.

For example, for $X = (P^i, q_{ij}) \in \mathrm{Obj}\,\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}\, F(X) = (F(P^i), F(q_{ij}))$; for a morphism $h = (h_{ij})$ of $\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}\, F(h) = (F(h_{ij}))$. Note that the definition of $\text{Pre-Tr}\, F$ on morphisms does not involve q_{ij}; yet $\text{Pre-Tr}\, F$ certainly respects differentials for morphisms.

(2) Let $F : \text{Pre-Tr}^+(C) \to D$ be a differential graded functor. Then the restriction of F to $C \subset \text{Pre-Tr}^+(C)$ (see part (3) of Proposition 2.3.2) gives a differential graded functor $FC : C \to D$. Moreover, since $FC = F \circ i$, we have $\text{Pre-Tr}^+(FC) = \text{Pre-Tr}^+(F) \circ \text{Pre-Tr}^+(i)$; therefore $\text{Pre-Tr}^+(FC) \cong \text{Pre-Tr}^+(F)$.

2.4. Definition of $\mathfrak{H}$ and h

For $X, Y, Z \in \mathrm{SmPrVar}$, $i, j, l \leqslant 0$, $f \in C^i(X)(Y)$, $g \in C^j(Y)(Z)$ we have the equality

$$(f^j(g))^l = f^{j+l}(g^l). \tag{3}$$

Hence we can define a (non-full!) subcategory J of $B^-(\mathrm{Shv}(\mathrm{SmCor}))$ whose objects are $[P] = C(P)$, $P \in \mathrm{SmPrVar}$, the morphisms are defined as

$$J^i(C(P), C(Q)) = \left\{ \bigoplus_{l \leqslant 0} (g^l) : g \in C^i(Q)(P) \right\},$$

the composition of morphisms and the boundary operators are the same as for $B^-(\mathrm{Shv}(\mathrm{SmCor}))$. There is an obvious addition defined for morphisms; the operation of disjoint union of varieties gives us the addition on objects. It follows immediately from (1) that J is a differential graded subcategory of $B^-(\mathrm{Shv}(\mathrm{SmCor}))$.

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Note that $J^i(-,-) = 0$ for $i > 0$; this is a very important property! In particular, for any $i < 0$, $X, Y, Z \in \operatorname{Obj} J$, it implies that $dJ^i(Y,Z) \circ J^0(X,Y) \subset dJ^i(X,Z)$ and $J^0(Y,Z) \circ dJ^i(X,Y) \subset dJ^i(X,Z)$. This is crucial for the construction of truncation functors t_N (see § 6.1 below). We call categories that have no morphisms of positive degrees *negative* differential graded categories; this property will be discussed in § 2.7 below.

We define $\mathfrak{H}$ as $\operatorname{Tr}(J)$. Since $C^l = 0$ for $l > 0$, we have $\mathfrak{H} = \operatorname{Tr}^+(J)$ (we can take $m_i = 0$ for any object of $\operatorname{Tr}(J)$ in Definition 2.3.1). Now Proposition 2.3.2 implies the following statement immediately.

Proposition 2.4.1. *$\mathfrak{H}$ is generated by $[P], P \in \mathrm{SmPrVar}$, as a triangulated category. Here $[P]$ denotes the object of $\mathfrak{H}$ that corresponds to $[P] = C(P) \in \operatorname{Obj} J$.*

We consider the functor $h : \mathfrak{H} \to K^-(\mathrm{Shv}(\mathrm{SmCor}))$ that is induced by the inclusion $J \to B^-(\mathrm{Shv}(\mathrm{SmCor}))$.

We also note that any differential graded functor $J \to A$ induces a functor $\mathfrak{H} \to \operatorname{Tr}^+(A)$.

The definition of $\mathfrak{H}$ implies immediately that $\mathfrak{H}([P], Q[i]) = H^i(C(Q)(P))$ for $P, Q \in \mathrm{SmPrVar}$.

2.5. An explicit description of $\mathfrak{H}$ and h

For the convenience of the reader we describe $\mathfrak{H}$ and h explicitly. Since in this subsection we just describe the category of twisted complexes over J explicitly, we do not need any proofs here.

We define $J' = \text{Pre-Tr}^+(J)$. J' is an *enhancement* of $\mathfrak{H}$ (in the sense of [**6**]). The idea is that taking cones of (twisted) morphisms becomes a well-defined operation in J' (in $\mathfrak{H}$ it is only defined up to a non-canonical isomorphism).

We describe an auxiliary category $\mathfrak{H}'$. $\operatorname{Obj} \mathfrak{H}' = \operatorname{Obj} J' = \operatorname{Obj} \mathfrak{H}$, whereas $\mathfrak{H}'(X,Y) = \operatorname{Ker} \delta^0_{J'}(X,Y)$ for $X, Y \in \operatorname{Obj} \mathfrak{H}'$.

Hence the objects of $\mathfrak{H}'$ are $(P^i,\ i \in \mathbb{Z},\ f_{ij},\ i < j)$, where (P^i) is a finite sequence of (not necessarily connected) smooth projective varieties (we assume that almost all P^i are 0), $f_{ij} \in C^{i-j+1}(P^j)(P^i)$ for all $m, n \in \mathbb{Z}$ satisfy the condition

$$\delta^{m-n+1}(P^n)(f_{mn}) + \sum_{m<l<n} f_{ln}^{m-l+1}(f_{ml}) = 0. \tag{4}$$

Morphisms $g : A = (P^i, f_{ij}) \to B = (P'^i, f'_{ij})$ can be described as sets $(g_{ij}) \in C^{i-j}(P'^j)(P^i)$, $i \leqslant j$, where the g_{ij} satisfy

$$\delta^{i-j}_{P'^j}(g_{ij}) + \sum_{j \geqslant l \geqslant i} f'^{i-l}_{lj}(g_{il}) = \sum_{j \geqslant l \geqslant i} g^{i-l+1}_{lj}(f_{il}) \quad \forall i, j \in \mathbb{Z}. \tag{5}$$

We will assume that $g_{ij} = 0$ for $i > j$.

Note that $g_{ij} = 0$ if $P^i = 0$ or $P^j = 0$. Hence the morphisms for any pair of objects in $\mathfrak{H}'$ are defined by means of a finite set of equalities.

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The composition of $g = (g_{ij}) : A \to B$ with $h = (h_{ij}) : B \to C = (P'^i, f'_{ij})$ is defined as

$$l_{ij} = \sum_{i\leqslant r\leqslant j} h_{rj}^{i-r}(g_{ir}).$$

$\mathfrak{H}'$ has a natural structure of an additive category. The direct sum of objects is defined by means of a disjoint union of varieties.

The morphisms $g, h : A = (P^i, f_{ij}) \to B = (P'^i, f'_{ij})$ are called homotopic ($g \sim h$) if there exist $l_{ij} \in C^{i-j-1}(P'^j)(P^i)$, $i-1 \leqslant j$, such that

$$g_{ij} - h_{ij} = \delta_{P'^j}^{i-j-1} l_{ij} + \sum_{i-1\leqslant r\leqslant j} f'^{i-r-1}_{rj}(l_{ir}) + \sum_{i\leqslant r\leqslant j+1} l_{rj}^{i-r+1}(f_{ir}). \tag{6}$$

Now $\mathfrak{H}$ can be described as a category whose objects are the same as for $\mathfrak{H}'$, whereas $\mathfrak{H}(A, B) = \mathfrak{H}'(A, B)/\!\sim$. The translation on $\mathfrak{H}$ is defined by shifts of indices (for P^i, f_{ij}). For $g = (g_{ij}) \in \mathfrak{H}'(A, B)$ its cone is defined as $C = \operatorname{Cone}(g) \in \operatorname{Obj} \mathfrak{H}'$, the ith term of C is equal to $P''^i = P^{i+1} \oplus P'^i$, whereas

$$h_{ij} \in C^{i-j+1}(P^{j+1} \oplus P'^j)(P^{i+1} \oplus P'^i) = \begin{pmatrix} f_{i+1,j+1} & 0 \\ g_{i+1,j} & f'_{ij} \end{pmatrix}, \quad i < j;$$

we have obvious natural maps $B \to C \to A[1]$.

It is easily seen that $\mathfrak{H}$ coincides with the category defined in § 2.4. We denote the projection $\mathfrak{H}' \to \mathfrak{H}$ by j.

Moreover, as in Theorem 4.6 of [**20**] one can check (without using the formalism described above) that $\mathfrak{H}$ with the structures defined is a triangulated category. One can also check directly that $P[0]$ for $P \in \mathrm{SmPrVar}$ generate $\mathfrak{H}$ as a triangulated category.

For $A = (P^i, f_{ij}) \in \operatorname{Obj}(\mathfrak{H}')$ we define $h'(A) \in C^-(\mathrm{Shv}(\mathrm{SmCor}))$ as $(C_A^j, \delta_A^j : C_A^j \to C_A^{j+1}) C^-(\mathrm{Shv}(\mathrm{SmCor}))$. Here $C_A^j = \sum_{i\leqslant j} C^{i-j}(P^j)$, the component of δ_A^j that corresponds to the morphism of $C^{i-j}(P^j)$ into $C^{i-j'+1}(P'^j)$ equals $\delta_{P^j}^{i-j}$ for $j = j'$ and equals $f_{jj'}^{i-j}$ for $j' \neq j$.

Note that the condition (4) implies $d_{h'(A)}^2 = 0$.

Now we define h' on morphisms. For $(l_{ij}) : A \to B$, $s \in \mathbb{Z}$, we set $h'(l)_s = \bigoplus_{i,j} l_{ij}^{s-i}$.

One can check explicitly that h' induces an exact functor $h : \mathfrak{H} \to K^-(\mathrm{Shv}(\mathrm{SmCor}))$.

By abuse of notation we denote by h' also the functor $J' \to B^-(\mathrm{Shv}(\mathrm{SmCor}))$.

Remark 2.5.1. P^i should be thought about as of ‘stratification pieces’ of the motif $A = (P^i, f_{ij})$. In particular, let Z be closed in X, $Z, X \in \mathrm{SmPrVar}$, $Y = X - Z$; suppose that Z is everywhere of codimension c in X. If we adjoin $Z(c)[2c]$ to $\operatorname{Obj} J$ (see § 8.5), then $M_{\mathrm{gm}}(Y)$ could be presented in $\mathfrak{H}$ as $((X, Z(c)[2c]), g_Z)$, where g_Z is the Gysin morphism (see Proposition 3.5.4 of [**36**]). See also Proposition 6.5.1 for a nice explicit description of the motif with compact support of any smooth quasi-projective X.

The main distinction of $\mathfrak{H}$ from the motivic category $\mathcal{D}$ defined by Hanamura (see [**17**], [**20**], and § 4) is that the Bloch cycle complexes (used in the definition of $\mathcal{D}$) are replaced by the Suslin complexes; we never have to choose distinguished subcomplexes for our constructions (in contrast with [**17**]). Note also that our definition works on the integral level in contrast with those of [**17**].

2.6. 'Stupid filtration' for motives

As we will see several times below, the category $\mathfrak{H}$ (hence also DM^{s}, cf. Theorem 3.1.1) is very close to $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. In a certain sense, $\mathfrak{H}$ has 'stupid filtration' related to those of $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. Certainly, this filtration is only defined on the level of $\mathfrak{H}'$ (note that the stupid truncation of an object of $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ depends on its lift to $C^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$!).

Proposition 2.6.1. *Let $X = (P^i, f_{ij})$ (as in § 2.5).*

(1) *For any $a \leqslant b \in \mathbb{Z}$ the set $(P^i, f_{ij} : a \leqslant i, j \leqslant b)$ gives an object $X_{[a,b]}$ of $\mathfrak{H}'$ (and so also of $\mathfrak{H}$).*

(2) *If $P^i = 0$ for $i < a$, then $(\mathrm{id}_{P^i}, i \leqslant b)$ gives a morphism $X \to X_{[a,b]}$ (in $\mathfrak{H}'$ and $\mathfrak{H}$).*

(3) *If $P^i = 0$ for $i > b$, then $(\mathrm{id}_{P^i}, i \leqslant b)$ gives a morphism $X_{[a,b]} \to X$.*

(4) *If $P^i = 0$ for $i < a$ and for $i > c$, $a < b < c$, then we have a distinguished triangle $X_{[a,b]} \to X \to X_{[b,c]}$*

Proof. (1) We have to check that the equality (4) is valid for $X_{[a,b]}$. Yet all terms of (4) are zero unless $a \leqslant i \leqslant j \leqslant b$. Moreover, in the case $a \leqslant i \leqslant j \leqslant b$ the terms of (4) are the same as for X. Both of these facts follow immediately from the negativity of J.

(2), (3) We have to check the condition (5) for these cases; again this is obvious by the negativity of J.

(4) We should check that $X \to X_{[b,c]}$ is homotopy equivalent to the second morphism of the triangle corresponding to $X_{[a,b]} \to X$; this easily follows from (2) (see also the corresponding part of § 2.5). □

The definition of the stupid filtration and its properties are quite similar to those described in § 1 of [**18**] (see property (6) in the end of that section). Note that we only used the fact that there are no morphisms of positive degrees between objects of J. See [**7**] for a vast generalization of this observation.

2.7. Other generalities on differential graded categories

We describe some new differential graded categories and differential graded functors. We will need them in § 7 below.

2.7.1. *Differential graded categories of morphisms*

For an additive category A we denote by $MS(A)$ the category of morphisms of $S(A)$. Its objects are $\{(X, Y, f) : X, Y \in \mathrm{Obj}\, A,\ f \in A(X, Y)\}$;

$$MS_0((X, Y, f), (X', Y', f')) = \{(g, h) : g \in A(X, X'),\ h \in A(Y, Y'),\ f' \circ g = h \circ g\}.$$

As for $S(A)$, there are no morphisms of non-zero degrees in $MS(A)$; hence the differential for morphisms is zero.

We denote $\text{Pre-Tr}(MS(A))$ by $MB^{\mathrm{b}}(A)$. We recall that a twisted morphism is a closed morphism of degree 0, i.e. an element of the kernel of δ^0.

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Proposition 2.7.1.

(1) *$MB^{\mathrm{b}}(A)$ is the category of closed morphisms of $B^{\mathrm{b}}(A)$. That means that its objects are $\{(X,Y,f) : X,Y \in \operatorname{Obj} B(A),\ f \in \operatorname{Ker} \delta^0(B(A)(X,Y))\}$,*

$$MB^{\mathrm{b}i}((X,Y,f),(X',Y',f')) = \{(g,h) : g \in B(A)^i(X,X'),\ h \in B(A)^i(Y,Y'),\ f' \circ g = h \circ f\}.$$

(2) *Let $MB(A)$ denote the unbounded analogue of $MB^{\mathrm{b}}(A)$. Then $\text{Pre-Tr}(MB(A)) \cong MB(A)$.*

(3) *Let $\mathrm{Cone} : MB(A) \to B^{\mathrm{b}}(A)$ denote the natural cone functor. Then the functor $\text{Pre-Tr}(\mathrm{Cone})$ in naturally isomorphic to Cone.*

Proof. (1) Easy direct verification.

(2) The proof is very similar to those of part (II) (1) of Proposition 2.2.3. First we note that $\text{Pre-Tr}(MB^{\mathrm{b}}B(A)) \cong MB^{\mathrm{b}}(A)$, then extend this to the unbounded analogue.

(3) Obviously, Cone is a differential graded functor. Hence it remains to apply part (2) of Remark 2.3.3. □

We have obvious differential graded functors $p_1, p_2 : MB(A) \to B(A)$: $p_1(X,Y,f) = X$, $p_2(X,Y,f) = Y$.

Corollary 2.7.2. *Let $F : J \to MB(A)$ be a differential graded functor.*

(1) *$\text{Pre-Tr}(F)$ gives a functorial system of closed morphisms $\text{Pre-Tr}(p_1 \circ F)(X) \to \text{Pre-Tr}(p_2 \circ F)(X)$ in $B(A)$ for $X \in \operatorname{Obj} J' = \operatorname{Obj} \mathfrak{H}$.*

(2) *Let A be an abelian category. Suppose that for any $P \in \mathrm{SmPrVar}$ the complex $F([P])$ is exact. Then there exists a natural quasi-isomorphism $\mathrm{Tr}^+(p_1 \circ F)(X) \sim \mathrm{Tr}^+(p_2 \circ F)(X)$ for $X \in \operatorname{Obj} \mathfrak{H}$.*

Proof. (1) Obvious.

(2) We have to show that $\text{Pre-Tr}^+(\mathrm{Cone}(F))(X)$ is quasi-isomorphic to 0 for any $X \in \operatorname{Obj} J'$. We consider the exact functor $G = \mathrm{Tr}^+(\mathrm{Cone}(F))(X)$; it suffices to show that $G = 0$. Recall that $[P], P \in \mathrm{SmPrVar}$, generate $\mathfrak{H}$ as a triangulated category. Hence $G([P]) = 0$ for any $P \in \mathrm{SmPrVar}$ implies that $G = 0$. □

2.7.2. *Negative differential graded categories; truncation functors*

We recall that a differential graded category C is called *negative* if $C^i(X,Y) = 0$ for any $i > 0$, $X, Y \in \operatorname{Obj} C$.

Certainly in this case all morphisms of degree 0 are closed (i.e. satisfy $\delta f = 0$). This notion is very important for us since J is negative.

For any differential graded C there exist a unique 'maximal' negative subcategory C_- (it is not full unless C is negative itself!). The objects of C_- are the same as for C

whereas $C_{-,i}(X,Y)=0$ for $i>0$, $C_{-,i}(X,Y)=C^i(X,Y)$ for $i<0$, and $C_{-,i}(X,Y)=\operatorname{Ker}\delta^0(C(X,Y))$ for $i=0$.

Obviously, if $F:D\to C$ is a differential graded functor, D is negative, then F factorizes through the faithful embedding $C_-\to C$.

Suppose that A is an abelian category.

Then zeroth (or any other) cohomology defines a functor $B_-(A)\to S(A)$.

More generally, we define two versions of the canonical truncation functor for $B_-(A)$. We will need these functors in § 7.3 below.

Let X be a complex over A, $a,b\in\mathbb{Z}$, $a\leqslant b$. We define $\tau_{\leqslant b}$ as the complex

$$\cdots\to X_{b-2}\to X_{b-1}\to \operatorname{Ker}(X_b\to X_{b+1}),$$

here $\operatorname{Ker}(X_b\to X_{b+1})$ is put in degree b. $\tau_{[a,b]}(X)$ is defined as $\tau_{\leqslant b}(X)/\tau_{\leqslant a-1}(X)$, i.e. it is the complex

$$X_{a-1}/\operatorname{Ker}(X_{a-1}\to X_a)\to X_a\to X_{a+1}\to\cdots\to X_{b-1}\to\operatorname{Ker}(X_b\to X_{b+1}).$$

The canonical $[a,b]$-truncation of X for $a<b$ is defined as

$$X_{[a,b]}=X_a/dX_{a-1}\to X_{a+1}\to\cdots\to X_{b-1}\to\operatorname{Ker}(X_b\to X_{b+1}),$$

again $\operatorname{Ker}(X_b\to X_{b+1})$ is put in degree b; for $a=b$ we take $H^a(X)$. Recall that truncations preserve homotopy equivalence of complexes.

Proposition 2.7.3.

(1) *$\tau_{\leqslant b}$, $\tau_{[a,b]}$ and the canonical $[a,b]$-truncation define differential graded functors $B_-(A)\to B_-(A)$.*

(2) *Let $F:J\to B(A)$ be a differential graded functor; we can assume that its target is $B_-(A)$. We consider the functors $\tau_{[a,b]}F$ and $F_{[a,b]}$ that are obtained from F by composing it with the corresponding truncations. Then there exists a functorial family of quasi-isomorphisms $\operatorname{Tr}^+(\tau_{[a,b]}F)(X)\to\operatorname{Tr}^+(F_{[a,b]})(X)$ for $X\in\operatorname{Obj}\mathfrak{H}$.*

Proof. (1) Note that all truncations give idempotent endofunctors on $C(A)$.

Hence it suffices extend truncations to all morphisms of $B_-(A)$ and prove that truncations respect δ.

The definition of truncations on morphisms of negative degree is very easy. The only morphisms in $B_-(A)$ of degree 0 are twisted ones, i.e. morphisms coming from $C(A)$.

It remains to verify that if a given truncation τ of a morphism $f=(f_i):(X^i)\to(Y^i)$ in $B_-(A)$ is zero then $\tau(\delta f)=0$.

First we check this for $\tau=\tau_{\leqslant b}$. $\tau_{\leqslant b}f=0$ means that $f(\tau_{\leqslant b}X)=0$ (i.e. the corresponding restrictions of f_i are zero). Since the boundary maps $\tau_{\leqslant b}X$ into itself, the definition of δ for $B(A)$ gives the result.

Now we consider the case $\tau=\tau_{[a,b]}$. $\tau_{[a,b]}(f)=0$ means that $f(\tau_{\leqslant b}X)\subset\tau_{\leqslant a}Y$. Again it suffices to note that the boundary maps $\tau_{\leqslant b}X$ and $\tau_{\leqslant a}Y$ into themselves.

The case of canonical truncation could be treated in the same way.

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(2) The natural morphism $m([P]): \tau_{[a,b]}F([P]) \to F_{[a,b]}([P])$ gives a functor $H : J \to MB(A)$ such that $p_1(H) = \tau_{[a,b]}F$ and $p_1(H) = F_{[a,b]}$. It remains to note that $m([P])$ is a quasi-isomorphism for any $P \in \mathrm{SmPrVar}$ and apply part (2) of Corollary 2.7.2. □

Remark 2.7.4.

(1) Another way to obtain new differential graded categories is to take 'tensor products' of differential graded categories. In particular, one can consider the categories $J \otimes J$ and $\mathrm{Tr}(J \otimes J)$ (which could be denoted by $\mathfrak{H} \otimes \mathfrak{H}$).

(2) If $H^i(C(X,Y)) = 0$ for some differential graded category C, any $X, Y \in \mathrm{Obj}\, C$, $i > 0$, then $\mathrm{Tr}^+(C_-) \sim \mathrm{Tr}^+(C)$. Indeed, the embedding $C_- \to C$ gives a functor $F : \mathrm{Tr}^+(C_-) \to \mathrm{Tr}^+(C)$. An easy argument (described in detail in the proof of Theorem 3.1.1 below) shows then that F is a full embedding. Lastly, since $\mathrm{Obj}\, C$ generates $\mathrm{Tr}^+(C)$ is a triangulated category, one can easily prove (by induction) that F gives an equivalence, i.e. that any $X \in \mathrm{Obj}\,\mathrm{Tr}^+(C)$ is isomorphic to some $Y \in F_*(\mathrm{Obj}\,\mathrm{Tr}^+(C_-))$.

(3) One can also define *positive* differential graded categories in a natural way. Positive differential graded categories seem to be connected with t-structures. Yet we will not study this issue in the current paper.

3. The main classification result

In this section we prove the equivalence of $\mathfrak{H}$ and DM^s. It follows that the presentation of a motif as $m(X)$ for $X \in \mathfrak{H}$ could be thought about as of a 'motivic injective resolution'.

Unfortunately, we do not know how to compare Voevodsky's $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (or just SmVar) with $\mathfrak{H}$ 'directly'.

3.1. The equivalence of categories $m : \mathfrak{H} \to \mathrm{DM}^s$

We denote the natural functor $K^-(\mathrm{Shv}(\mathrm{SmCor})) \to D^-(\mathrm{Shv}(\mathrm{SmCor}))$ by p, denote $p \circ h$ by m.

Theorem 3.1.1. *m is a full exact embedding of triangulated categories; its essential image is* DM^s.

Proof. Since h is an exact functor, so is m. Now we check that m is a full embedding. By part (2) of Proposition 1.3.1, m induces an isomorphism $\mathrm{DM}^{\mathrm{eff}}_-(m([P]), m(Q[i])) \cong \mathfrak{H}([P], Q[i])$ for $P, Q \in \mathrm{SmPrVar}$, $i \in \mathbb{Z}$. Since $[R], R \in \mathrm{SmPrVar}$, generate $\mathfrak{H}$ as a triangulated category (see Proposition 2.4.1), the same is true for any pair of objects of $\mathfrak{H}$ (cf. part (2) of Remark 2.7.4).

We explain this argument in more detail.

First we verify that for any smooth projective P/k and arbitrary $B \in \mathfrak{H}$ the functor m gives an isomorphism

$$\mathrm{DM}^{\mathrm{eff}}_-(m([P]), m(B)) \cong \mathfrak{H}([P], B). \tag{7}$$

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By Proposition 1.3.1, (7) is fulfilled for $B = P'[j]$, $j \in \mathbb{Z}$, $P' \in \mathrm{SmPrVar}$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{H}$ the functor m defines a morphism of long exact sequences

$$\begin{array}{ccccccc} \longrightarrow \mathfrak{H}([P],Y) & \longrightarrow & \mathfrak{H}([P],Z) & \longrightarrow & \mathfrak{H}([P],X[1]) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(m([P]),m(Y)) & \longrightarrow & \mathrm{DM}^{\mathrm{eff}}_{-}(m([P]),m(Z)) & \longrightarrow & \mathrm{DM}^{\mathrm{eff}}_{-}(m(P),m(X)[1]) & \longrightarrow \end{array} \tag{8}$$

Thus if m gives an isomorphism in (7) for $B = X[i]$ and $B = Y[i]$ for $i = 0, 1$, then m gives an isomorphism for $B = Z$. Since objects of the form $B = [P']$ generate $\mathfrak{H}$ as a triangulated category, (7) is fulfilled for any $B \in \mathfrak{H}$. Hence for all $i \in \mathbb{Z}$, $B \in \mathfrak{H}$, we have $\mathrm{DM}^{\mathrm{eff}}_{-}(m(P[i]), m(B)) \cong \mathfrak{H}(P[i], B)$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{H}$ the functor m defines a morphism of long exact sequences $(\cdots \to \mathfrak{H}(Z, B) \to \cdots) \to (\cdots \to \mathrm{DM}^{\mathrm{eff}}_{-}(m(Z), m(B)) \to \cdots)$ similar to (8). Now the same argument as above proves that m is a full embedding.

It remains to calculate the 'essential image' M of the map that is induced by m on $\mathrm{Obj}(\mathfrak{H})$ (we adjoin to M all objects that are isomorphic to those coming from $\mathfrak{H}$). According to Proposition 5.2.2 below we have $m([P]) = C(P) \cong M_{\mathrm{gm}}(P)$.

Since $\mathfrak{H}$ is generated by $[P]$ for $P \in \mathrm{SmPrVar}$ as a triangulated category, M is the strict triangulated subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-}$ that is generated by all $M_{\mathrm{gm}}(P)$. Since the tensor structure on $\mathrm{DM}^{\mathrm{eff}}_{-}$ is defined by means of the relation $M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(Y) = M_{\mathrm{gm}}(X \times Y)$ for $X, Y \in \mathrm{SmVar}$, M is a tensor subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-}$. Since $M_{\mathrm{gm}}(P) \in \mathrm{Obj}\,\mathrm{DM}^{\mathrm{s}}$ for any $P \in \mathrm{SmPrVar}$, we have $M \subset \mathrm{DM}^{s}$. It remains to prove that M contains DM^{s}. By definition (cf. [**36**, § 2.1]) $\mathbb{Z}(1)[2] \in \mathrm{DM}^{\mathrm{eff}}_{-}$ can be represented as the cone of the natural map $M_{\mathrm{gm}}(\mathrm{pt}) \to M_{\mathrm{gm}}(\mathbb{P}^1)$ (we will identify $\mathbb{Z}(1)$ with $M_{\mathrm{gm}}(\mathbb{Z}(1))$). Hence $\mathbb{Z}(n) \in \mathrm{Obj}\,M$ for any $n > 0$. Since M_{gm} is a tensor functor, if $M_{\mathrm{gm}}(Z) \in \mathrm{Obj}\,M$ for $Z \in \mathrm{SmVar}$, then $Z(c)[2c] \in \mathrm{Obj}\,M$ for all $c > 0$. Now we apply Proposition 3.5.4 of [**36**] as well as the Mayer–Vietoris triangle for motives [**36**, § 2]; similarly to Corollary 3.5.5 of [**36**] we conclude that $\mathrm{Obj}\,M$ contains all $M_{\mathrm{gm}}(X)$ for $X \in \mathrm{SmVar}$ (cf. the remark in [**36**] that precedes Definition 2.1.1). A more detailed version of this argument will be used in the proof of Theorem 6.2.1 below.

Since DM^{s} is the smallest triangulated subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-}$ containing motives of all smooth varieties, we prove the claim. □

Remark 3.1.2.

(1) In order to calculate $\mathrm{DM}^{\mathrm{eff}}_{-}(M, M')$ for $M, M' \in \mathrm{DM}^{\mathrm{s}}$ (using the explicit description of $\mathfrak{H}$ given in § 2.5) in terms of cycles one needs to know $m^{-1}(M)$ and $m^{-1}(M')$ (or the preimages of their duals). See § 6.5 for a nice result in this direction.

(2) Note that we do not construct any comparison functor $\mathrm{DM}^{\mathrm{s}} \to \mathfrak{H}$ (or $\mathrm{SmVar} \to \mathfrak{H}$) explicitly. Yet if we denote by $\mathrm{SmCor}_{\mathrm{pr}}$ the full subcategory of SmCor whose objects are smooth projective varieties, then we have obvious functors $\mathrm{SmCor}_{\mathrm{pr}} \to K^{\mathrm{b}}(\mathrm{SmCor}_{\mathrm{pr}}) \to \mathfrak{H}$. Note that the Voevodsky's description of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ also gives a canonical functor $K^{\mathrm{b}}(\mathrm{SmCor}_{\mathrm{pr}}) \to \mathrm{DM}^{\mathrm{s}} \subset \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

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3.2. On motives of (possibly) singular varieties

Recall (see § 4.1 of [**36**]) that for any $X \in \mathrm{Var}$ (not necessarily smooth!) there were certain objects $M_{\mathrm{gm}}(X)$ and $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ of $\mathrm{DM}^{\mathrm{eff}}_{-}$ defined. $M_{\mathrm{gm}}(X)$ was called the motif of X; $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ was called the motif of X with compact support.

The following statement follows easily.

Corollary 3.2.1. *For any (not necessarily smooth) variety X/k there exist $Z, Z' \in \mathfrak{H}$ such that $m(Z) \cong M_{\mathrm{gm}}(X)$, $m(Z') \cong M^{\mathrm{c}}_{\mathrm{gm}}(X)$.*

Proof. It is sufficient to verify that $M_{\mathrm{gm}}(X), M^{\mathrm{c}}_{\mathrm{gm}}(X) \in \mathrm{DM}^{\mathrm{s}}$. The proof of this fact is the same as for Corollaries 4.1.4 and 4.1.6 in [**36**]. Indeed, for the proofs in [**36**] one does not need to add the kernels of projectors. □

Remark 3.2.2. We obtain that for the Suslin complex of an arbitrary variety X there exists a quasi-isomorphic complex M 'constructed from' the Suslin complexes of smooth projective varieties; $M \in h_*(\mathrm{Obj}\,\mathfrak{H}) \subset \mathrm{Obj}\,K^-(\mathrm{Shv}(\mathrm{SmCor}))$ is unique up to a homotopy.

Moreover, as we have noted in the proof of Lemma 5.2.1, the cohomology of $C(P)$ as a complex of presheaves for any $P \in \mathrm{SmPrVar}$ coincides with its hypercohomology (the corresponding fact for $\underline{C}(P)$ was proved in [**36**]). Hence the same is true for any $M \in h_*(\mathrm{Obj}\,\mathfrak{H})$. Hence for $X \in \mathrm{SmVar}$ the quasi-isomorphism $C(X) \to M$ is given by an element of $H^0(M)(X)$.

This result shows that the presentation of a motif as $m(X)$ for $X \in \mathfrak{H}$ could be thought about as of a 'motivic' analogue of taking an injective resolution; here the Suslin complexes of smooth projective varieties play the role of injective objects.

4. Comparison of $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ with Hanamura's category of motives

We prove that $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ is anti-isomorphic to Hanamura's triangulated category of motives.

We recall that $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is a tensor category and the functor $\otimes\mathbb{Z}(1) : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is fully faithful. Following Voevodsky we define $\mathrm{DM}_{\mathrm{gm}}$ as $\varinjlim_{\otimes\mathbb{Z}(1)} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$: that is, $\mathrm{DM}_{\mathrm{gm}}$ is the 'union' of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(-i)$ while each $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(-i)$ is isomorphic to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

Masaki Hanamura has kindly informed the author that he has (independently) obtained an alternative proof of the anti-equivalence of motivic categories; his proof uses the extension of the functor constructed in [**19**]. Unfortunately, Hanamura's proof is not available to the public (in any form).

4.1. The plan

We will not recall Hanamura's definitions in detail here (they are rather long) so the reader should consult § 2 of [**17**] for the definition of $\mathcal{D}_{\mathrm{fin}(k)}$ and $\mathcal{D}(k)$ (see also § 4 of [**20**] and § 4.5 of [**28**]). One of unpleasant properties of Hanamura's construction is that it uses a certain composition operation for Bloch's complexes (see below) which is only partially defined. Yet on the target of our comparison functor the composition is always defined.

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Note that Hanamura's category is cohomological (i.e. the functor SmPrVar $\to \mathcal{D}_{\text{fin}(k)}$ is contravariant); so it is natural to consider contravariant functors $\mathfrak{H} \to \mathcal{D}_{\text{fin}(k)}$. For that reason we will consider the categories $\mathcal{D}_{\text{fin}}^{\text{op}} \subset \mathcal{D}^{\text{op}}$ that equal $\mathcal{D}_{\text{fin}}(k)^{\text{op}} \subset \mathcal{D}(k)^{\text{op}}$.

$\mathcal{D}_{\text{fin}}^{\text{op}}$ could 'almost' be described as $\text{Tr}(I)$ for a certain differential graded I. The problem is that composition of morphisms in I is only partially defined; overcoming this difficulty makes the construction rather complicated.

Moreover, in our definition of $\mathfrak{H}$ we use not necessarily connected $P \in$ SmPrVar. In Hanamura's notation P should be replaced by $\bigoplus P^i$, where P^i are connected components of P; yet we will ignore this distinction below.

The proof consists of three parts.

(I) Construction of a functor $F : \mathfrak{H} \otimes \mathbb{Q} \to \mathcal{D}_{\text{fin}}^{\text{op}}$.

(II) Proof that F is a full embedding.

(III) Proof that F extends to an equivalence $\text{DM}_{\text{gm}}\,\mathbb{Q} \to \mathcal{D}^{\text{op}}$.

4.2. Construction of a comparison functor *F*

First we modify $\mathfrak{H}\otimes\mathbb{Q}$ slightly. We define $J\mathbb{Q}$ as the category whose objects are the same as for J while the morphisms are given by rational alternated cubical Suslin complexes.

These are defined similarly to the alternated cubical Bloch complexes.

Let Σ_n for $n \geqslant 0$ denote the group of permutations of n elements; it acts on $\mathbb{A}^n$ by permuting coordinates.

We define

$$J\mathbb{Q}^i([P],[Q]) = \mathbb{Q} \otimes \{a_i(f) : f \in J^i([P],[Q])\} \subset \mathbb{Q} \otimes J^i([P],[Q])$$

for all $i \in \mathbb{Z}$, $P, Q \in$ SmPrVar. Here a_i is the idempotent $(\sum_{\sigma\in\Sigma_{-i}} \text{sgn}(\sigma)\sigma)/(-i)!$, $\text{sgn}(\sigma)$ is the sign of a permutation. One can easily see that (a_i) is an endomorphism of the complex of $\mathbb{Q} \otimes J([P],[Q])$, i.e. we can consider the boundaries of $J\mathbb{Q}$ induced by those of J (note that $J\mathbb{Q}$ is both a subcomplex and a factor-complex of J).

We define the composition of morphisms in $J\mathbb{Q}$ as in a factor-category of $J \otimes \mathbb{Q}$. It is possible since for any $i, j \in \mathbb{Z}$, $P, Q, R \in$ SmPrVar, $f \in J^i(P,Q)$, $g \in J^j(Q,R)$ we have $a_{i+j}(g \circ f) = a_{i+j}(a_j(g) \circ a_i(f))$ (an easy direct verification).

An easy standard argument (see Lemma 2.28 of [**28**]) shows immediately that the alteration procedure gives a quasi-isomorphism $J([P],[Q]) \otimes \mathbb{Q} \to J\mathbb{Q}([P],[Q])$ for any $P, Q \in$ SmPrVar. Hence there exists an equivalence of categories $G : \mathfrak{H} \otimes \mathbb{Q} \to \mathfrak{H}\mathbb{Q}$ where $\mathfrak{H}\mathbb{Q} = \text{Tr}(J\mathbb{Q})$.

Hence for our purposes it suffices to construct a natural embedding $H : \mathfrak{H}\mathbb{Q} \to \mathcal{D}_{\text{fin}}^{\text{op}}$. As we have said above, $\mathcal{D}_{\text{fin}}^{\text{op}}$ is 'almost equal' to $\text{Tr}(I)$ for a certain differential graded 'almost category' I. Hence it suffices to define a certain differential graded functor $G : J\mathbb{Q} \to I$ and define $H = \text{Tr}\,G$. Indeed, the 'image' of H will be a subcategory of $\mathcal{D}_{\text{fin}}^{\text{op}}$ (not necessarily full) which composition of morphisms is compatible with those of $\mathfrak{H}\mathbb{Q}$; we do not have to care of morphisms and object of $\mathcal{D}_{\text{fin}}^{\text{op}}$ that do not come from $\mathfrak{H}\mathbb{Q}$.

We describe I. Let $\mathcal{Z}^r(X)$ for $X \in \mathrm{SmVar}$, $r \geqslant 0$, denote the Bloch's alternated cubical cycle complex (we use the version of the definition described in § 1 of [**17**]). It is defined in the following way (note that we consider cohomological complexes; that forces us to reverse arrows in the usual notation).

One first defines a sequence of groups $\mathcal{Z}'^r(X)$ with

$$\mathcal{Z}'^{ri}(X) = \sum_{U \subset X \times \mathbb{A}^{-i}} \mathbb{Q},$$

where U runs through all integral closed subschemes of $X \times \mathbb{A}^{-i}$ of codimension $r-i$ that intersect faces properly; here 'faces' mean subvarieties of $X \times \mathbb{A}^{-i}$ defined by putting some of the last coordinates of $X \times \mathbb{A}^{-i}$ equal to 0 or 1. $\mathcal{Z}^r$ is obtained form $\mathcal{Z}'^r$ by alteration, see the beginning of this subsection.

The boundaries $\mathcal{Z}^{ri}(X) \to \mathcal{Z}^{ri+1}(X)$ are defined as $\sum_{0 \leqslant j \leqslant -i}(g_{j0*} - g_{j1*})$, g_{jx} are defined as in § 1.2.

We have a natural map $\mathcal{Z}^r(X) \otimes \mathcal{Z}^s(Y) \to \mathcal{Z}^{r+s}$ (as complexes) defined by applying a_{i+j} to the corresponding tensor product of cycles (i, j are the indices of terms; see the text after Lemma 2.28 in [**28**]).

The objects of I are pairs $(P, r) : P \in \mathrm{SmPrVar}$, $r \in \mathbb{Z}$.

Now $I((X, r), (Y, s)) = \mathcal{Z}^{\dim Y + r - s}(X \times Y)$. Composition of morphisms $f \in I^i(X, Y)$, $g \in I^j(Y, Z)$ is defined by the formula

$$g \circ f = \mathrm{pr}_{X \times Z \times \mathbb{A}^{-i-j}}(X \times l(Y) \times Z \times \mathbb{A}^{-i-j}) \cap (g \otimes f),$$

where $l : Y \to Y \times Y$ is the diagonal embedding, pr is the natural projection. Note that the composition is defined only if $g \otimes f$ intersects $l(Y) \times X \times Z \times \mathbb{A}^{-i-j}$ properly.

A more detailed description of $\mathcal{Z}^r$ (and the discussion of several other questions relevant for the results of this section) could be found in [**28**]. The reader is strongly recommended to look at §§ 2.5 and 4.3 in [**28**].

The construction of H uses two facts.

(i) For any connected $P, Q \in \mathrm{SmPrVar}$, $\dim Q = r$, we have a natural embedding of complexes $J\mathbb{Q}([P], [Q]) \to \mathcal{Z}^r(P \times Q)$.

This is obvious from the definition of $\mathcal{Z}^r$.

(ii) For any connected $P, Q, R \in \mathrm{SmPrVar}$, $\dim R = s$, the partially defined composition $\mathcal{Z}^r(P, Q) \times \mathcal{Z}^s(Q, R) \to \mathcal{Z}^s(P, R)$ (cf. Proposition 4.3 of [**20**]) for Bloch complexes restricted to $J\mathbb{Q}(P, Q) \times J\mathbb{Q}(Q, R)$ coincides with the map induced by the composition in $J\mathbb{Q}$. This is very easy since the composition of morphisms in $J\mathbb{Q}$ is exactly the one induced from the composition of Bloch complexes (described above). Note that the composition of morphisms coming from $J\mathbb{Q}$ is always well defined.

Hence sending $[P] \in \mathrm{Obj}\, J\mathbb{Q}$ to $(P, 0) \in \mathrm{Obj}\, I$ and embedding $J\mathbb{Q}([P], [Q]) \to \mathcal{Z}^r(P \times Q)$ we obtain a differential graded functor $G : J\mathbb{Q} \to I$. This gives $H = \mathrm{Tr}\, G$.

We note that in Hanamura's construction one often has to choose distinguished subcomplexes (and modify the choice of elements of Bloch complexes) in order to 'compute' compositions of arrows in his categories. Yet in the 'image' of H this problem never occurs.

4.3. F is a full embedding

We check that H (and so also F) is fully faithful.

We use a standard argument, the same as those used in the proof of Theorem 3.1.1 (see also the proof of Proposition 8.3.1); it can be used for any exact functor that has a differential graded enhancement. It suffices to check that H induces an isomorphism

$$\mathfrak{H}\mathbb{Q}([P],[Q][i]) \to \mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}((P,0),(Q,0)[i])$$

for all $P, Q \in \mathrm{SmPrVar}$, $i \in \mathbb{Z}$. Since in $\mathcal{D}(k)$ there is Poincaré duality (by definition, see also (4.5) of [**20**]), for Q of pure dimension n we have

$$\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}((P,0),(Q,0)[i]) = \mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}((P \times Q,0),(\mathrm{pt},n)[i]) = \mathrm{Ch}^r(P \times Q,-i).$$

Here $\mathrm{Ch}^r(P \times Q,-i)$ denotes the rational higher Chow group, the last equality follows from Theorems (4.10) and (1.1) of [**20**]. Since the same is true in $\mathfrak{H}\mathbb{Q}$ (see Propositions 4.2.3 and 4.2.9 of [**36**]; cf. also Proposition 12.1 of [**11**]), the isomorphism is compatible with the maps of corresponding complexes, we get the claim.

Another way to express the same argument is to say that $G : J\mathbb{Q} \to I$ induces a quasi-isomorphism on morphisms (considered as complexes of abelian groups). This easily implies that $\mathrm{Tr}\, G$ is a full embedding (even in our situation when I is 'not quite a category').

4.4. Conclusion of the proof

First we check that any object of the full triangulated subcategory $\mathcal{D}^{\mathrm{op}+}_{\mathrm{fin}} \subset \mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ whose objects are 'positive' diagrams (i.e. those that contain only symbols of the form (P,r), $r \geqslant 0$) is equivalent to $\mathfrak{H} \otimes \mathbb{Q}$. We should check that any object of $\mathcal{D}^{\mathrm{op}+}_{\mathrm{fin}}$ is isomorphic to an object of the form $H(X), X \in \mathrm{Obj}\, \mathfrak{H} \otimes \mathbb{Q}$.

We define the 'zeroth part' of $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ as the full triangulated subcategory of $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ generated by $(P,0)$. We note $\mathcal{D}^{\mathrm{op}+}_{\mathrm{fin}}$ is generated by (P,r), $r \geqslant 0$, as a triangulated category. This is easy to see directly and also follows immediately from property (6) of § 1 of [**18**]. Hence it suffices to verify that any (P,r), $r \geqslant 0$, $P \in \mathrm{SmPrVar}$ is isomorphic to a certain object of the 'zeroth part' of $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$.

In fact, (P,r) belongs to the triangulated subcategory of $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ generated by $(\mathbb{P}^l) \times P$ for $0 \leqslant l \leqslant r$. It suffices to check that (P,r) is a direct summand of $(P \times \mathbb{P}^r, 0)$, its complement is $(P \times \mathbb{P}^{r-1}, 0)$. First we note that morphisms between 'pure' objects (i.e. objects of $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ of the type (P,r)) by definition equal to the morphisms between corresponding Chow motives. The last statement is just a well-known property of Chow motives; it follows easily from the fact that $\mathrm{Chow}^{\mathrm{eff}}$ is a tensor category (note that $\mathcal{D}^{\mathrm{op}}_{\mathrm{fin}}$ also is and the tensor

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multiplications are compatible) and $[\mathbb{P}^r] = [\mathbb{P}^{r-1}] \oplus \mathrm{pt}(r)$ in $\mathrm{Chow}^{\mathrm{eff}}$ (in the covariant notation).

A similar direct sum statement is verified in § 2.3 of [**20**]. The main difference is that Hanamura presented (P, r) as a direct summand of $(P \times (\mathbb{P}^1)^r, 0)$ and does not care about its complement. Since we will idempotent complete $\mathrm{DM}_{\mathrm{gm}} \otimes \mathbb{Q}$ in the end of the section, this version also fits our purposes.

Hence we get the equivalence of $\mathfrak{H} \otimes \mathbb{Q}$ with $\mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}$.

Note that $\mathbb{Q}(1) \in \mathrm{Obj}\, \mathfrak{H} \otimes \mathbb{Q}$ differs from $\mathrm{pt}(1)$ in $\mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}$ by a shift by [2]; cf. also the remark at the bottom of p. 139 of [**20**].

Now we prove that this equivalence can be extended to $\mathrm{DM}_{\mathrm{gm}}\, \mathbb{Q}$.

The definition of morphisms in $\mathcal{D}_{\mathrm{fin}}^{\mathrm{op}}$ immediately implies the cancellation theorem, i.e. $\mathcal{D}_{\mathrm{fin}}^{\mathrm{op}}(X, Y) = \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}}(X(1), Y(1))$. The cancellation theorem is also true in DM^{s} and $\mathrm{DM}_{\mathrm{gm}}$; see Theorem 4.3.1 of [**36**] (or [**37**] for the characteristic p case).

Now note that $J\mathbb{Q}$ is a DG tensor category (in the natural sense); hence $\mathfrak{H}\mathbb{Q} = \mathrm{Tr}\, J\mathbb{Q}$ is a tensor triangulated category (Proposition 1.13 of [**29**] yields the proof immediately). It is also easily seen that H is a tensor functor.

Hence we can extend F to a functor

$$F' : \varinjlim_{\otimes \mathbb{Q}(1)} \mathfrak{H} \otimes \mathbb{Q} \to \varinjlim_{\otimes F(\mathbb{Q}(1))} \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}.$$

Since $F(\mathbb{Q}(1)) \cong (\mathrm{pt}, 1)[2] \in \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}}$, we obtain that

$$\varinjlim_{\otimes F(\mathbb{Q}(1))} \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+} = \bigcup_{n>0} \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n).$$

Moreover, since all $\mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n)$ are equivalent, F' induces an equivalence $\mathfrak{H} \otimes \mathbb{Q}(-n) \to \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n)$ for any n. It remains to recall that $\mathrm{DM}_{\mathrm{gm}}\, \mathbb{Q}$ is isomorphic to the idempotent completion of $\bigcup_{n>0} \mathfrak{H} \otimes \mathbb{Q}(-n)$, whereas $\mathcal{D}^{\mathrm{op}}$ is the idempotent completion of $\bigcup_{n>0} \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n)$.

The proof is finished.

Another way of the proof is to note that the isomorphisms $\mathfrak{H} \otimes \mathbb{Q} \to \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n)$ are compatible for all n (up to an autoequivalence of categories). Hence we obtain again that $\bigcup_{n>0} \mathfrak{H} \otimes \mathbb{Q}(-n) \sim \bigcup_{n>0} \mathcal{D}_{\mathrm{fin}}^{\mathrm{op}+}(-n)$.

Remark 4.4.1. Hanamura defined $\mathcal{D}(k)$ as a triangulated subcategory of a certain 'infinite' analogue of $\mathcal{D}_{\mathrm{fin}}(k)$. Yet we do not need this definition for the proof of equivalence (above) since $\mathcal{D}(k)$ was defined as the idempotent completion of $\mathcal{D}_{\mathrm{fin}}(k)$; recall that the idempotent completion of a triangulated category is canonical (see [**1**]).

5. The properties of cubical Suslin complexes

The main result of this section is that the cubical complex $C(X)$ is quasi-isomorphic (as a complex of presheaves) to the simplicial complex $\underline{C}(X)$ that was used in [**36**]. This fact was mentioned by Levine [**28**, Theorem 2.25] yet no complete proof was given. One of the possible methods of the proof (proposed in [**28**]) is the use of a bicomplex method. Recall that this method yielded a similar result for motivic cohomology in § 4 of [**26**] and could certainly adjusted to yield the proof of the statement desired. Yet we use another method here. The reader not interested in the details of the proof should skip this section.

5.1. A certain adjoint functor for the derived category of presheaves

We denote by PreShv(SmCor) the category of presheaves (of abelian groups) on SmCor, by D^-(PreShv(SmCor)) the derived category of PreShv(SmCor) (complexes are bounded from above), by $\mathrm{DPM}^{\mathrm{eff}}$ we denote a full subcategory of D^-(PreShv(SmCor)) whose objects are complexes with homotopy invariant cohomology.

Lemma 5.1.1. $C(P) \in \mathrm{DPM}^{\mathrm{eff}}$.

Proof. The scheme of the proof is the same as for Proposition 3.6 in [**35**]. First we check that for $Y \in \mathrm{SmVar}$ and $P \in \mathrm{SmPrVar}$ the maps $i_0^*, i_1^* : C(P)(Y \times \mathbb{A}) \to C(P)(Y)$ are homotopic; here i_0^*, i_1^* are induced by the embeddings $i_x : Y \times \{x\} \to Y \times \mathbb{A}$, $x = 0, 1$. We consider the maps $\mathrm{pr}_i : C'(P)^i(Y) \to C'(P)^i(Y \times \mathbb{A})$ induced by the projections $Y \times \mathbb{A} \to Y$. We consider the maps $h'_i : C'(P)^i(Y \times \mathbb{A}) \to C'(P)^{i-1}(Y)$ induced by isomorphisms $Y \times \mathbb{A} \times \mathbb{A}^{-i} \cong Y \times \mathbb{A}^{-i+1}$, and also $h_i : C(P)^i(Y \times \mathbb{A}) \to C(P)^{i-1}(Y)$, $h_i = h'_i - \mathrm{pr}_{i-1} \circ i_0^*$. We have

$$\delta_*^{i-1} h_i + h_{i+1} \delta_*^i = (i_1^* - i_0^*)_i,$$

i.e. h_i gives the homotopy needed. Then i_0^*, i_1^* induce coinciding maps on cohomology. Let $Y = U \times \mathbb{A}$, $U \in \mathrm{SmVar}$. We consider the morphism $H = \mathrm{id}_U \times \mu : U \times \mathbb{A}^2 \to U \times \mathbb{A}$, where μ is given by multiplication. We proved that the maps induced by $\mu \circ i_0$ and $\mu \circ i_1$ on the cohomology of $C(P)(Y)$ coincide. Hence the composition $U \times \mathbb{A} \to U \xrightarrow{\mathrm{id}_U \times i_0} U \times \mathbb{A}$ induces an isomorphism on the cohomology of $C(P)(U \times \mathbb{A})$ for any $U \in \mathrm{SmVar}$, i.e. the cohomology presheaves of $C(P)$ are homotopy invariant. □

Now we formulate an analogue of Proposition 3.2.3 in [**36**]. By (F) we denote a complex concentrated in degree 0 whose non-zero term is F.

Proposition 5.1.2.

(1) *There exists an exact functor* $R : D^-(\mathrm{PreShv}(\mathrm{SmCor})) \to \mathrm{DPM}^{\mathrm{eff}}$ *right-adjoint to the embedding* $\mathrm{DPM}^{\mathrm{eff}} \to D^-(\mathrm{PreShv}(\mathrm{SmCor}))$. *Besides* $R((F)) \cong \underline{C}(F)$ *(see the definition of* $\underline{C}(F)$ *in [**36**, 3.2]).*

(2) *In* $D^-(\mathrm{PreShv}(\mathrm{SmCor}))$ *we have* $R(\underline{C}(L(P))) \cong C(P) \cong R((L(P)))$.

Proof. (1) The proof is similar to the proof of existence of the projection $\mathrm{RC} : D^-(\mathrm{Shv}(\mathrm{SmCor})) \to \mathrm{DM}_-^{\mathrm{eff}}$ in 3.2 of [**36**]. We consider the localizing subcategory $\mathcal{A}$ in $D^-(\mathrm{PreShv}(\mathrm{SmCor}))$ that is generated by all complexes $L(X \times \mathbb{A}) \to L(X)$ for $X \in \mathrm{SmVar}$. As in [**36**] we have $D^-(\mathrm{PreShv}(\mathrm{SmCor}))/\mathcal{A} \cong \mathrm{DPM}^{\mathrm{eff}}$ (cf. Theorem 9.32 of [**30**]).

Now as in the proof of Proposition 3.2.3 in [**36**] we should verify the following statements.

(1) For any $F \in \mathrm{PreShv(SmCor)}$ the natural morphism $\underline{C}(F) \to (F)$ is an isomorphism in $D^-(\mathrm{PreShv(SmCor)})/\mathcal{A}$.

(2) For all $T \in \mathrm{DPM}^{\mathrm{eff}}$ and $B \in \mathcal{A}$ we have $D^-(\mathrm{PreShv(SmCor)})(B, T) = 0$.

The proof of the first assertion may be copied word for word from the similar statement in 3.2.3 of [**36**]. This was noted in the proof of Theorem 3.2.6 of [**36**].

As in [**36**], for the second assertion we should check for any $X \in \mathrm{SmVar}$ the bijectivity of the map

$$D^-(\mathrm{PreShv(SmCor)})((L(X)), T) \to D^-(\mathrm{PreShv(SmCor)})((L(X \times \mathbb{A})), T)$$

induced by the projection $X \times \mathbb{A} \to X$. Since representable presheaves are projective in PreShv(SmCor) (obvious from Yoneda's Lemma, cf. [**30**, 2.7]), this follows immediately from the homotopy invariance of the cohomology of $\underline{C}(F)$.

(2) From part (2) of Lemma 5.1.3 below we obtain that the morphism $(L(P)) \to C(P)$ induces an isomorphism $R((L(P))) \cong R(C(P))$ in the category $D^-(\mathrm{PreShv(SmCor)})$. Using assertion (1) we obtain that the map $\underline{C}(P) \to (L(P))$ induces an isomorphism $R((L(P))) \cong R(\underline{C}(P))$. Since R is right-adjoint to an embedding of categories, it remains to note that $\underline{C}(L(P))$, $C(P) \in \mathrm{DPM}^{\mathrm{eff}}$. □

Lemma 5.1.3.

(1) $R(C^j(P)) = 0$ *for* $j < 0$.

(2) *The morphism* $i_P : (L(P)) \to C(P)$ *induces an isomorphism* $R((L(P))) \cong R(C(P))$ *in* $D^-(\mathrm{PreShv(SmCor)})$.

Proof. (1) We consider the same maps $h_i : C(P)^i(Y \times \mathbb{A}) \to C(P)^{i-1}(Y)$ as in the proof of Lemma 5.1.1. Obviously, h_i is epimorphic, besides $\operatorname{Ker} h_i \cong C(P)^i(Y)$. We obtain an exact sequence

$$0 \to C(P)^i(Y) \to C(P)^i(Y \times \mathbb{A}) \to C(P)^{i-1}(Y) \to 0. \qquad (9)$$

We prove the assertion by induction on j. The case $j = -1$ follows immediately from (9) applied for the case $i = 0$. If $R(C^j(P)) = 0$ for $j = m$, then R maps $C(P)_m$ and $C''(P)_m$ to 0, where $C''(P)_m(Y) = C(P)_m(Y \times \mathbb{A})$. Applying (9) for $i = m$ we obtain $R(C(P)_{m-1}) = 0$ (recall that R is an exact functor).

(2) Follows from assertion (1) immediately. □

Now we recall [**12**, Theorem 8.1] that the cohomology groups of $\underline{C}(L(P))(Y)$ are exactly $A_{0,-i}(Y, P)$. Hence we completed the proof of Proposition 1.2.2.

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5.2. Proof of Proposition 1.3.1

Lemma 5.2.1. *For all* $i \in \mathbb{Z}$, $P, Y \in \mathrm{SmPrVar}$, *the obvious homomorphism*

$$K^-(\mathrm{Shv}(\mathrm{SmCor}))((L(Y)), C(P)[i]) \to D^-(\mathrm{Shv}(\mathrm{SmCor}))((L(Y)), C(P)[i])$$

is bijective.

Proof. By definition the homomorphism considered in the map from the cohomology of $C(P) = C^c(P)$ into its hypercohomology. By Theorem 8.1 of [**12**] for $\underline{C}(P)$ $(= \underline{C}^c(P))$ the corresponding map is bijective. Hence the assertion follows from $\underline{C}(P) \cong C(P)$ in $D^-(\mathrm{PreShv}(\mathrm{SmCor}))$. □

Proposition 5.2.2. $i_P : (L(P)) \to C(P)$ *induces an isomorphism* $\mathrm{RC}((L(P))) \cong C(P)$ *in* $\mathrm{DM}_-^{\mathrm{eff}}$.

Proof. Literally repeating the argument of the proof of part (1) of Lemma 5.1.3 we obtain $\mathrm{RC}(C^j(P)) = 0$ for $j < 0$. Therefore, $\mathrm{RC}((L(P))) \cong \mathrm{RC}(C(P))$. It remains to note that $C(P) \in \mathrm{Obj}\, \mathrm{DM}_-^{\mathrm{eff}}$. □

Now we finish the proof of Proposition 1.3.1. The assignment $g \to G = (g^l)$ defines a homomorphism

$$K^-(\mathrm{Shv}(\mathrm{SmCor}))((L(Y)), C(P)[i]) \to K^-(\mathrm{Shv}(\mathrm{SmCor}))(C(Y), C(P)[i]).$$

Hence it is sufficient to verify that the map

$$\mathrm{DM}_-^{\mathrm{eff}}(\mathrm{RC}((L(Y))), C(P)[i]) \to \mathrm{DM}_-^{\mathrm{eff}}(C(Y), C(P)[i])$$

induced by this homomorphism coincides with the homomorphism induced by the map $i_{P*} : \mathrm{RC}(L(Y)) \cong \mathrm{RC}((C(Y)))$. Since $G \circ i_{P*} = g$, we are done.

6. Truncation functors, the length of motives, and $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}})$

In § 6.1 using the canonical filtration of the (cubical) Suslin complex we define the *truncation functors* t_N. These functors are new though certain very partial cases were (essentially) considered in [**14**] and [**16**] (there another approaches were used).

The target of t_0 is just $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ (complexes of rational correspondences, see § 1.1). In § 6.3 we prove that t_0 extends to $t : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$. In § 6.4 we prove that t induces an isomorphism $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}) \cong K_0(\mathrm{Chow}^{\mathrm{eff}})$ thus answering the question of 3.2.4 of [**14**]. In Corollary 6.4.3 we extend this result to an isomorphism $K_0(\mathrm{DM}_{\mathrm{gm}}) \cong K_0(\mathrm{Chow})$.

The functors t (Proposition 6.3.1) and all t_N (see Theorem 6.2.1) are conservative. t induces a natural functor $t_{\mathrm{num}} : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \to K^{\mathrm{b}}(\mathrm{Mot}_{\mathrm{num}}^{\mathrm{eff}})$. Over a finite field $t_{\mathrm{num}}\mathbb{Q}$ is (conjecturally) an equivalence, cf. Remark 8.3.2.

We define the *length* of a motif: *stupid* length in § 6.2, *fine* and *rational* length in § 6.3. The stupid length is not less than the fine length, the fine length is not less than the

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rational one. We prove (Theorem 6.2.1) that motives of smooth varieties of dimension N have stupid length less than or equal to N; besides $t_N(X)$ contains all information on motives of length less than or equal to N. The length of a motif is a natural motivic analogue of the length of weight filtration for a mixed Hodge structure.

For a smooth quasi-projective variety X we calculate $m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))$ explicitly (in § 6.5). Using this result we prove that the weight complex of Gillet and Soulé for a smooth quasi-projective variety X can be described as $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X)))$. Next we recall the cdh-topology of Voevodsky and prove this statement for arbitrary $X \in \mathrm{Var}$ (see § 6.6). Besides, $t_0(m^{-1}(M_{\mathrm{gm}}(X)))$ essentially coincides with the functor h described in Theorem 5.10 of [**16**].

In the next section we will verify that the weight filtration of 'standard' realizations is closely related to t_N; the rational length of a motif coincides with the (appropriately defined) length of the weight filtration of its singular realization.

6.1. Truncation functors of level N

For $N \geqslant 0$ we denote the $-N$th canonical filtration of $C(P)$ as a complex of presheaves (i.e. $C^{-N}(P)/d_P C^{-N-1}(P) \to C^{-N+1}(P) \to \cdots \to C^0(P) \to 0$) by $C^N(P)$.

We denote by J_N the following differential graded category. Its objects are the symbols $[P]$ for $P \in \mathrm{SmPrVar}$ whereas $J_N([P],[Q])^i = C^{Ni}(Q)(P)$. The composition of morphisms is defined similarly to those in J. For morphisms in J_N presented by $g \in C^i(Q)(P)$, $h \in C^j(R)(Q)$, we define their composition as the morphism represented by $h^i(g)$ for $i+j \geqslant -N$ and 0 for $i+j < -N$. Note that for $i+j=-N$ we take the class of $h^i(g) \mod d_R C^{-N-1}(R)(P)$; for $i=-N$, $j=0$, and vice versa, g is only defined up to an element of $d_Q C^{-N-1}(Q)(P)$ (respectively h is defined up to an element of $d_R C^{-N-1}(R)(Q)$) yet the composition is well defined. The boundary on morphisms is also defined as in J, i.e. for $g \in J_N(P,Q)$ we define $\delta g = d_Q g$. Certainly, all J_N are negative (i.e. there are no morphisms of degree greater than 0).

We have an obvious functor $J \to J_N$. As noted in Remark 2.3.3, this gives canonically a functor $t_N : \mathfrak{H} \to \mathrm{Tr}(J_N)$. We denote $\mathrm{Tr}(J_N) = \mathrm{Tr}^+(J_N)$ by $\mathfrak{H}_N$; note that $\mathfrak{H}_0$ is precisely $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$.

For any $m \leqslant N$ we also have an obvious functor $J_N \to J_m$. It induces a functor $t_{Nm} : \mathfrak{H}_N \to \mathfrak{H}_m$ such that $t_m = t_{Nm} \circ t_N$.

Certainly, one can give a description of $\mathfrak{H}_N$ that is similar to the description of $\mathfrak{H}$ given in § 2.5. Hence objects of $\mathfrak{H}_N$ could be represented as certain $(P^i, f_{ij} \in C^{Ni-j+1}(P^j)(P^i),\ i<j \leqslant i+N+1)$, the morphisms between (P^i, f_{ij}) and (P'^i, f'_{ij}) are represented by certain $g_{ij} \in C^{Ni-j}(P'^j, P^i)$, $i \leqslant j \leqslant i+N$, etc. The functor t_N 'forgets' all elements of $C^m([P],[Q])$ for $P, Q \in \mathrm{SmPrVar}$, $m < -N$, and factorizes $C^{-N}([P],[Q])$ modulo coboundaries. In particular, for $N=0$ we get ordinary complexes over $\mathrm{Corr}_{\mathrm{rat}}$.

6.2. 'Stupid' length of motives (in DM^{s}); conservativity of t_0

It was proved in [**36**] that the functor M_{gm} gives a full embedding of $\mathrm{Corr}_{\mathrm{rat}} \to \mathrm{DM}^{\mathrm{eff}}_{-}$. In this subsection we prove a natural generalization of this statement.

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We will say that $P = (P^i, f_{ij}) \in \operatorname{Obj}\mathfrak{H}'$ is concentrated in degrees $[l, m]$, $l, m \in \mathbb{Z}$, if $P^i = 0$ for $i < l$ and $i > m$. We denote the corresponding additive set of objects of $\mathfrak{H}'$ by $\mathfrak{H}'_{[a,b]}$. We denote by $\mathfrak{H}_{[a,b]}$ the objects of $\mathfrak{H}$ that are isomorphic to those coming from $\mathfrak{H}'_{[a,b]}$.

Obviously (from the description of distinguished triangles in $\operatorname{Tr}(C)$ for any differential graded D), if $A \to B \to C \to A[1]$ is a distinguished triangle, $A, C \in \mathfrak{H}_{[a,b]}$, then $B \in \mathfrak{H}_{[a,b]}$.

Theorem 6.2.1.

(1) *For any smooth variety Y/k of dimension less than or equal to N we have $m^{-1}(M_{\mathrm{gm}}(Y)) \in \mathfrak{H}_{[0,N]}$.*

(2) *For any smooth variety Y/k of dimension less than or equal to N we have $m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(Y)) \in \mathfrak{H}_{[-N,0]}$.*

(3) *If $A \in \mathfrak{H}_{[a,b]}$, $B \in \mathfrak{H}_{[c,d]}$, $N \geqslant d - a$, $N \geqslant 0$, then $\mathfrak{H}(A, B) \cong \mathfrak{H}_N(t_N(A), t_N(B))$.*

(4) *If $s \in \mathfrak{H}(X, X)$ for $X \in \operatorname{Obj}\mathfrak{H}$ is an idempotent and $t_0(s) = 0$ then $s = 0$.*

(5) *t_0 is conservative, i.e. for $Y \in \mathfrak{H}$ we have $Y = 0 \iff t_0(Y) = 0$.*

(6) *$f : A \to B$ is an isomorphism whenever $t_0(f)$ is.*

Proof. (1) Obviously, the statement is valid for smooth projective Y. We prove the general statement by induction on dimension.

By the projective bundle theorem (see Proposition 3.5.3 of [**36**]) for any $c \geqslant 0$ we have a canonical isomorphism $\mathbb{P}^c \cong \bigoplus_{0 \leqslant i \leqslant c} \mathbb{Z}(i)[2i]$. Hence $\mathbb{Z}(c)[2c]$ can be represented as a cone of the natural map $M_{\mathrm{gm}}(\mathbb{P}^{c-1}) \to M_{\mathrm{gm}}(\mathbb{P}^c)$. Therefore, $\mathbb{Z}(c)[2c] \in m(\mathfrak{H}_{[-1,0]})$.

One can easily show that for any $X \in \mathfrak{H}_{[e,f]}$, $e, f \in \mathbb{Z}$, $c > 0$ we have $X(c)[2c] \in \mathfrak{H}_{[e-1,f]}$. This could be done by presenting $X(n)[2n]$ as a cone of the (naturally defined) map $\mathbb{P}^{n-1} \otimes X \to \mathbb{P}^n \otimes X$; cf. Remark 1.14 of [**29**] and also § 8.5 below.

We recall the Gysin distinguished triangle (see Proposition 3.5.4 of [**36**]). For a closed embedding $Z \to X$, Z is everywhere of codimension c, it has the form

$$M_{\mathrm{gm}}(X - Z) \to M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(c)[2c] \to M_{\mathrm{gm}}(X - Z)[1]. \tag{10}$$

Suppose that the assertion is always fulfilled for $\dim Y = N' < N$.

Let X/k be smooth quasi-projective. Since k admits resolution of singularities, X can be represented as a complement to a $P \in \mathrm{SmPrVar}$ of a divisor with normal crossings $\bigcup_{i \geqslant 0} Q^i$. Then using (10) one proves by induction on j that the assertion is valid for all $Y^j = P \setminus (\bigsqcup_{0 \leqslant i \leqslant j} Q^i)$. To this end we check by the inductive assumption for $j \geqslant 0$ that

$$M_{\mathrm{gm}}\biggl(P \setminus \biggl(\bigcup_{0 \leqslant i \leqslant j} Q^i\biggr) \setminus \biggl(P \setminus \biggl(\bigcup_{0 \leqslant i \leqslant j+1} Q^i\biggr)\biggr)\biggr) = M_{\mathrm{gm}}\biggl(Q^{i+1} \setminus \biggl(\bigcup_{0 \leqslant i \leqslant j} Q^j\biggr)\biggr) \in m(\mathfrak{H}_{[0,N-1]}).$$

Hence $M_{\mathrm{gm}}(X) \in m(\mathfrak{H}_{[0,N]})$.

If X is not quasi-projective we can still choose closed $Z \subset X$ (of codimension greater than 0) such that $X - Z$ is quasi-projective. Hence the assertion follows from the inductive assumption by applying (10).

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(2) The proof is similar to those of the previous part. The difference is that we do not have to twist and should use the distinguished triangle of Proposition 4.1.5 of [**36**]:

$$M^{\mathrm{c}}_{\mathrm{gm}}(Z) \to M^{\mathrm{c}}_{\mathrm{gm}}(X) \to M^{\mathrm{c}}_{\mathrm{gm}}(X-Z) \to M^{\mathrm{c}}_{\mathrm{gm}}(Z)[1] \tag{11}$$

instead of (10).

(3) We can assume (by increasing d if needed) that $N = d - a$.

Let $A = (P^i, f_{ij})$, $B = (P'^i, f'_{ij})$. As we have seen in § 2.5, any $g \in \mathfrak{H}(A,B)$ is given by a certain set of $g_{ij} \in C^{i-j}(P'^j)(P^i)$, $i \leqslant j$. The same is valid for $h = (h_{ij}) \in \mathfrak{H}_N(t_N(A), t_N(B))$; the only difference is that h_{ad} is given modulo $d_{P'^d}C^{-N-1}(P'^d)(P^a)$. Both (g_{ij}) and (h_{ij}) should satisfy the conditions

$$\delta^{i-j}_{P'^j}(m_{ij}) + \sum_{j\geqslant l\geqslant i} f'^{i-l}_{lj}(m_{il}) = \sum_{j\geqslant l\geqslant i} m^{i-l+1}_{lj}(f_{il}) \quad \forall i,j \in \mathbb{Z}. \tag{12}$$

First we check surjectivity. We recall that the conditions (12) for g depend only on g_{ij} for $(i,j) \neq (a,d)$ and on $d_{P'^d}g_{ad}$. Hence if (h_{ij}) satisfies the conditions (12) then $h = t_N(r)$, where $r_{ij} = h_{ij}$ for all $(i,j) \neq (a,d)$, r_{ad} is an arbitrary element of $C^{-N}(P'^d)(P^a)$ satisfying $r_{ad} \mod d_{P'^d}C^{-N-1}(P'^d)(P^a) = h_{ad}$.

Now we check injectivity. Let $t_N(g) = 0$ for $g = (g_{ij}) \in \mathfrak{H}'(A,B)$. Note that $C^N(P)$ is a factor-complex of $C(P)$ for any $P \in \mathrm{SmPrVar}$. Hence similarly to § 2.5 one can easily check that there exist $l_{ij} \in C^{i-j-1}(P'^j)(P^i)$, $i \leqslant j$, such that

$$g_{ij} = \delta^{i-j-1}_{P'^j} l_{ij} + \sum_{i\leqslant r\leqslant j} (f'^{i-r-1}_{rj}(l_{ir}) + l^{i-r+1}_{rj}(f_{ir})) \tag{13}$$

for all $(i,j) \neq (a,d)$, for $i = a$, $j = d$ the equality (13) is fulfilled modulo $d_{P'^d}q$ for some $q \in C^{-N-1}(P'^d)(P^a)$. We consider (l'_{ij}), where $l'_{ij} = l_{ij}$ for all $(i,j) \neq (a,d)$, $l'_{ad} = l_{ad} + q$. Obviously, if we replace (l_{ij}) by (l'_{ij}) then (13) would be fulfilled for all i,j. Therefore, $g = 0$ in $\mathfrak{H}(A,B)$.

(4) Let $X = (P^i, f_{ij})$ be as in § 2.5; let s be given by a set of $s_{ij} \in J^{i-j}([P^i],[P^j])$, $i \leqslant j$; (s_{ij}) are defined up to a homotopy of the sort described in § 2.5. $t_0(s)$ is homotopic to zero. Since this homotopy can be represented by a set of $m_i \in \mathrm{SmCor}(P^i, P^{i-1}) = J^0([P^i],[P^{i-1}])$, we can lift this homotopy to $\mathfrak{H}$. This means that we take $l_{ij} = m_i$ for $j = i-1$, $l_{ij} = 0$ for $j \neq i-1$, where l is as in (6); this allows us to assume that $s_{ii} = 0$. Next, since $s^2 = s$ in $\mathfrak{H}$, we have $s^n = s$ for any $n > 0$. Now note that all degrees of components of s^n are less than or equal to $-n$. Hence $s^r = 0$ if $X \in \mathfrak{H}_{[a,b]}$ for $r > b - a$.

(5) Immediate from assertion (4) applied to $X = Y$ and $s = \mathrm{id}_Y$.

(6) Follows immediately from assertion (5) (recall that a morphism is an isomorphism whenever its cone is zero). □

In fact, for a smooth quasi-projective X one can compute $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ explicitly (see Proposition 6.5.1 below).

We say that $X \in \mathfrak{H}$ has *stupid length* less than or equal to N if for some $l \in \mathbb{Z}$, $m \leqslant l + N$, the motif $X \in \mathfrak{H}_{[l,m]}$. We will define the *fine length* of a motif below.

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Remark 6.2.2.

(1) In fact, surjectivity (but not injectivity) in part (3) is also valid for $d-a=N+1$. The proof is similar to those for the case $d-a=N$. We should choose r_{ad}, $r_{a+1,d}$, and $r_{a,d-1}$; the classes of $r_{a+1,d}$ and $r_{a,d-1}$ modulo coboundaries are fixed. This choice affects the equality (12) only for $i=a$, $j=d$. Note also that this equality only depends on $d_{P'^d}r_{ad}$. One can choose arbitrary values of $r_{a+1,d}$ and $r_{a,d-1}$ in the corresponding classes. Then the equality (12) with $r_{ad}=0$ will be satisfied modulo $d_{P'^d}q$ for some $q\in C^{-N-1}(P'^d)(P^a)$. Therefore, if we take $r_{ad}=q$ then $t_N(r)=h$.

(2) Let $A\in\mathfrak{H}_{[a,b]}$, $B\in\mathfrak{H}_{[c,d]}$, $N+1\geqslant d-a$, $N\geqslant 0$. Then one can check that $A\cong B$ if and only if $t_N(A)\cong t_N(B)$.

Indeed, if $f:A\to B$ is an isomorphism then $t_N(f)$ also is.

Conversely, let $f_N:t_N(A)\to t_N(B)$ be an isomorphism. Then, as was noted above, there exists an $f\in\mathfrak{H}(A,B)$ such that $f_N=t_N(f)$ (f is not necessarily unique). Since $t_N(F)$ is an isomorphism, $t_0(f)=t_{N0}(t_N(f))$ also is. From part (6) of Theorem 6.2.1 we obtain that f gives an isomorphism $A\cong B$.

It follows immediately that two objects $A,B\in\mathfrak{H}$ of stupid length less than or equal to $N+1$ are isomorphic whenever $t_N(A)\cong t_N(B)$.

(3) One could define $\mathfrak{H}_{N,[0,N]}\subset\mathfrak{H}_N$ similarly to $\mathfrak{H}_{[0,N]}$. Then t_N would give an equivalence of additive categories $\mathfrak{H}_{[0,N]}\to\mathfrak{H}_{N,[0,N]}$. Indeed, this restriction of t_N is surjective on objects; it is an embedding of categories by part (3) of Theorem 6.2.1.

6.3. *Fine* length of a motif (in $\mathbf{DM}^{\mathbf{eff}}_{\mathbf{gm}}$); conservativity of the weight complex functor $\boldsymbol{t}:\mathbf{DM}^{\mathbf{eff}}_{\mathbf{gm}}\to \boldsymbol{K}^{\mathbf{b}}(\mathbf{Chow}^{\mathbf{eff}})$

One can check (using the method of the proof of Proposition 6.4.1 below) that the stupid length of a motif $M\in\operatorname{Obj}\mathrm{DM}^{\mathrm{s}}$ coincides with the length of $t_0(M)\in K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. Yet replacing $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ by $K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ one can obtain a more interesting invariant which will be defined on the whole $\operatorname{Obj}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

Proposition 6.3.1.

(1) t_0 *can be extended to an exact functor* $t:\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}}\to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ *where* $\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}}$ *is the idempotent completion of* $\mathfrak{H}$.

(2) t *is conservative.*

Proof. (1) t_0 can be canonically extended to an exact functor from $\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}}$ to the idempotent completion of $K^b(\mathrm{Corr}_{\mathrm{rat}})$. It remains to note that the idempotent completion of $K^b(\mathrm{Corr}_{\mathrm{rat}})$ is exactly $K^b(\mathrm{Chow}^{\mathrm{eff}})$ (see, for example, Corollary 2.12 of [**1**]).

(2) Immediate from part (4) of Theorem 6.2.1. □

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Remark 6.3.2.

(1) Exactly the same arguments yield that there exists a functor $t_0 \otimes \mathbb{Q} : \mathfrak{H} \otimes \mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}} \otimes \mathbb{Q})$ that can be extended to a conservative $t\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}} \mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}} \mathbb{Q})$.

(2) Since $\mathfrak{H} \cong \mathrm{DM}^{\mathrm{s}}$, we have $\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}} \cong \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. For this reason we will usually identify these categories.

(3) Composing t (or $t\mathbb{Q}$) with the natural functor $\mathrm{Chow}^{\mathrm{eff}} \to \mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}}$ (respectively $\mathrm{Chow}^{\mathrm{eff}} \mathbb{Q} \to \mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}} \mathbb{Q}$) one gets a functor $t_{\mathrm{num}} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}})$ (respectively $t_{\mathrm{num}}\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}} \mathbb{Q})$). These functors can be easily extended to functors $\mathrm{DM}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Mot}_{\mathrm{num}})$ and $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Mot}_{\mathrm{num}} \mathbb{Q})$. Here $\mathrm{Mot}_{\mathrm{num}}$ (and $\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}}$) denotes the category of (effective) numerical motives; note that $\mathrm{Mot}_{\mathrm{num}} \mathbb{Q}$ is an abelian category (see [**23**]). Yet in order to obtain the 'correct' structure of $K^{\mathrm{b}}(\mathrm{Mot}_{\mathrm{num}} \mathbb{Q})$ (i.e. those compatible with 'standard' realizations) one needs Kunneth projectors for numerical motives; currently they are known to exist only over a finite field (see § 8.3). Moreover, one cannot prove that $t_{\mathrm{num}}\mathbb{Q}$ is conservative without assuming certain 'standard' conjectures (cf. Proposition 8.3.1). See also part (2) of Remark 8.3.2 below for further discussion in the case when k is finite.

Note also: if numerical equivalence of cycles coincides with homological equivalence (a standard conjecture!) then the cohomology of $t_{\mathrm{num}}\mathbb{Q}(X)$ computes the (pure) weight factors of étale (and singular) cohomology of X; see § 7.4 below. This generalizes to (weight complexes of) motives Remark 3.1.6 of [**14**].

Definition 6.3.3.

(1) For $M \in \mathrm{Obj}\, \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ we write $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b]}$ if $t(M) \cong W$ for a complex W of Chow motives concentrated in degrees $[a, b]$.

(2) We define the *fine* length of M as the smallest difference $b - a$ such that $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b]}$.

(3) The *rational length* of a motif M is the length of $t(M) \otimes \mathbb{Q} \in K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}} \mathbb{Q})$ ($\mathrm{Chow}^{\mathrm{eff}} \mathbb{Q}$ is the idempotent completion of $\mathrm{Chow}^{\mathrm{eff}} \otimes \mathbb{Q}$!).

Obviously, the fine length of $M \in \mathrm{Obj}\, \mathrm{DM}^{\mathrm{s}}$ is not greater than its stupid length and not less than its rational length. Note also that the fine length of $\mathbb{Z}(n)$ is 0. Besides, we also have the sets $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b],\mathbb{Q}} \subset \mathrm{Obj}\, \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \mathbb{Q}$ for each $a \leqslant b \in \mathbb{Z}$.

The length of a motif is a natural motivic analogue of the length of weight filtration of a mixed Hodge structure or of a geometric representation (i.e. of a representation coming from the étale cohomology of a variety). Note that even the stupid length of the motif of a smooth variety is not larger than its dimension; this is a motivic analogue of the corresponding statement for the weights of singular and étale cohomology.

The results of the next section along with Proposition 6.4.1 easily imply (see § 7.4) that the length of the weight filtration of the singular or étale cohomology of a motif

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M is not greater than the fine length of M. Moreover, the standard conjectures imply that $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b],\mathbb{Q}}$ if and only if for any l the weights of $H^l(M)$ (singular or étale cohomology) lie between $l+a$ and $l+b$ (see § 7.4 below).

6.4. The study of $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})$ and $K_0(\mathrm{DM}_{\mathrm{gm}})$

We recall some standard definitions (cf. [**14**, 3.2.1]). We define the Grothendieck group $K_0(\mathrm{Chow}^{\mathrm{eff}})$ as a group whose generators are of the form $[A], A \in \operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}$; the relations are $[C] = [A] + [B]$ for $C \cong A \oplus B \in \operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}$. Note that $A \oplus 0 \cong A$ implies $[A] = [B]$ if $A \cong B$. We use the same definition for $K_0(\mathrm{Chow})$.

The K_0-group of a triangulated category T is defined as the group whose generators are $[t], t \in \operatorname{Obj} T$; if $A \to B \to C \to A[1]$ is a distinguished triangle then $[B] = [A] + [C]$. Note that this also immediately implies $[A] = [B]$ if $A \cong B$.

The existence of t allows to calculate $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})$ easily. To this end we prove the following statement.

Proposition 6.4.1. *For $X \in \operatorname{Obj}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ if $X \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b]}$ then X is a direct summand of some $X' \in \mathfrak{H}_{[a,b]}$.*

Proof. The proof is similar (and generalizes) those of part (4) of Theorem 6.2.1.

Suppose that X is a direct summand of an object $Y = (P^i, f_{ij})$ of $\mathfrak{H}$ of length r; X is given by an idempotent endomorphism $s = (s_{ij})$ of Y. It suffices to verify that Y is a direct summand of $X' = Y_{[a,b]}$ (see Proposition 2.6.1).

Similarly to the proof of part (4) of Theorem 6.2.1, we can assume that $s_{ii} = 0$ for $i > b$ and $i < a$. Indeed, $s_{ii} = 0$ is 'homotopic to 0 outside of $[a,b]$', whereas we can alter s_{ii} by $f^0_{i-1i}(l_{ii-1}) + l^0_{i+1i}(f_{ii+1})$ for any set of $l_{kk-1} \in C^0(P^k, P^{k-1})$ (see (6)).

Again we have $s = s^n$ for any n. If we replace s by s^{r+1} we easily obtain that $s_{ij} = 0$ for $i > b$, $j > b$ and $i < a$, $j < a$.

Obviously, it suffices to check that (for the new choice of s):

(1) arrows $(s_{ij}, a \leqslant i \leqslant b)$ give a morphism $s' \in \mathfrak{H}(Y_{[a,b]}, Y)$;

(2) $(s_{ij}, a \leqslant j \leqslant b)$ give a morphism $s'' \in \mathfrak{H}(Y, Y_{[a,b]})$;

(3) $s = s''s'$ (in $\mathfrak{H}$).

(1) We have to check all equalities (5) for s'. Both sides of (5) belong to $J^{j-i+1}([P^i],[P^j])$ for some $i, j, a \leqslant i \leqslant b$. Since the components of s' are taken from s, we only have to compare the differences for both sides for $i, j \geqslant a$ (for other values of i, j both sides are zero for s'). The only summands in (5) for i, j that distinguish s from s' are those of the form $s^{i-l+1}_{lj}(f_{il})$ for $l > b$. Yet in these cases $s_{lj} = 0$ by our assumption.

(2) This is proved similarly.

(3) In J' we have $s^2 = s's'' + (s'' + s')d + d(s'' + s') + d^2$, where the components of d are morphisms in $J([P^i],[P^j])$ for $i < a$, $j > b$. It remains to note that all (possible) degrees of arrows in $(s'' + s')d + (s' + s'')d$ are positive and $d^2 = 0$. □

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Theorem 6.4.2. *t induces an isomorphism $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \cong K_0(\mathrm{Chow}^{\mathrm{eff}})$.*

Proof. Since t is an exact functor, it gives an abelian group homomorphism $a : K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \to K_0(K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}))$. By Lemma 3 of 3.2.1 of [**14**], there is a natural isomorphism $b : K_0(K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})) \to K_0(\mathrm{Chow}^{\mathrm{eff}})$. The embedding $i : \mathrm{Chow}^{\mathrm{eff}} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (see Proposition 2.1.4 of [**36**]) gives a homomorphism $c : K_0(\mathrm{Chow}^{\mathrm{eff}}) \to K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})$. The definitions of a, b, c imply immediately that $b \circ a \circ c = \mathrm{id}_{K_0(\mathrm{Chow}^{\mathrm{eff}})}$. Hence a is surjective, c is injective.

It remains to verify that c is surjective.

We claim that if $t(X) = P^i \to P^{i+1} \to \cdots \to P^j$, $P^l \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}$, then the class $[X] \in K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})$ equals $\sum(-1)^l[P^l]$.

We prove this fact by induction on the (fine) length of X. The length one case follows immediately from Proposition 6.4.1 (conservativity of t could be considered as a partial case of it).

To make the inductive step it suffices to show the existence of a morphism $l : P^j[-j] \to X$ in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ that gives the obvious morphism of complexes after we apply t to it. Indeed, then the length of the cone of l would be less than $j - i$ (cf. part (4) of Proposition 2.6.1).

By Proposition 6.4.1, X is a direct summand of some $X' \in \mathfrak{H}_{[i,j]}$. Hence the existence of l follows from part (3) of Theorem 6.2.1. □

Now we recall that $\mathrm{Chow}^{\mathrm{eff}}$ is a tensor category and the functor $\otimes\mathbb{Z}(1) : \mathrm{Chow}^{\mathrm{eff}} \to \mathrm{Chow}^{\mathrm{eff}}$ is fully faithful. Chow is usually defined as the 'union' of $\mathrm{Chow}^{\mathrm{eff}}(-i)$ while each $\mathrm{Chow}^{\mathrm{eff}}(-i)$ is isomorphic to $\mathrm{Chow}^{\mathrm{eff}}$, i.e. $\mathrm{Chow} = \varinjlim_{\otimes\mathbb{Z}(1)} \mathrm{Chow}^{\mathrm{eff}}$.

Besides, the same is true for $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \subset \mathrm{DM}_{\mathrm{gm}}$, see § 4. Hence the embedding $i : \mathrm{Chow}^{\mathrm{eff}} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ extends to $i' : \mathrm{Chow} \to \mathrm{DM}_{\mathrm{gm}}$.

Corollary 6.4.3. *i' induces an isomorphism $K_0(\mathrm{Chow}) \to K_0(\mathrm{DM}_{\mathrm{gm}})$.*

Proof. The definitions easily imply that $K_0(\mathrm{Chow}) = K_0(\mathrm{Chow}^{\mathrm{eff}})[\mathbb{Z}(1)]^{-1}$ and $K_0(\mathrm{DM}_{\mathrm{gm}}) = K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})[\mathbb{Z}(1)]^{-1}$. Now Theorem 6.4.2 yields the claim immediately. Note that $i'(\mathbb{Z}(-1)) = \mathbb{Z}(-1)$! □

Remark 6.4.4. Note that the categories $\mathrm{Chow}^{\mathrm{eff}} \subset \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ have compatible tensor categories structures. Hence their K_0-groups are actually rings, whereas the isomorphism constructed is an isomorphism of rings.

The same is true for $\mathrm{Chow} \subset \mathrm{DM}_{\mathrm{gm}}$.

6.5. Explicit calculation of $m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))$; the weight complex of smooth quasi-projective varieties

Let $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ for $X \in \mathrm{SmCor}$ denote the motif of X with compact support (cf. § 2.2 or § 4.1 of [**36**]).

Proposition 6.5.1. *For a smooth quasi-projective X/k let $j : X \to P$ be an embedding for $P \in \mathrm{SmPrVar}$, let $P \setminus X = \bigcup Y_i$, $1 \leqslant i \leqslant m$, be a smooth normal crossing divisor. Let $U_i = \bigsqcup_{(i_j)} Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_r}$ for all $1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant m$, $U_0 = P$. We have r natural maps $U_r \to U_{r-1}$. We denote by d_r their alternated sum (as a finite correspondence). We*

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consider $Q = (Q^i, f_{ij})$*, where* $Q^i = U_{-i}$ *for* $0 \leqslant i \leqslant -m$*,* $P^i = 0$ *for all other* i*;* $f_{ij} = d_i$ *for* $0 > i \geqslant -m$*,* $j = i+1$*, and* $f_{ij} = 0$ *for all other* (i, j)*. Then* $M^{\mathrm{c}}_{\mathrm{gm}}(X) \cong m(Q)$*.*

Proof. Let j^* denote the natural morphism $M^{\mathrm{c}}_{\mathrm{gm}}(P) = M_{\mathrm{gm}}(P) \to M^{\mathrm{c}}_{\mathrm{gm}}(X)$ (see [**36**, 4.1]). Then by Proposition 4.1.5 in [**36**] the cone of j^* is naturally isomorphic to $M^{\mathrm{c}}_{\mathrm{gm}}(R_0) = M_{\mathrm{gm}}(R_0)$, where $R_0 = \bigcup Y_i$ (note that R_0 is proper). Hence our assertion is equivalent to the statement that $C(R) \cong 0$, here C denotes the Suslin complex of R,

$$R = L(U_m) \xrightarrow{d_{m*}} L(U_{m-1}) \xrightarrow{d_{m-1*}} \cdots \xrightarrow{d_{2*}} L(U_1) \to L(R_0).$$

The acyclicity of $C(R)$ could be called 'multi-Mayer–Vietoris'. Its proof is quite similar to the corresponding part of the proof of Theorem 3.2.6 of [**36**]. By Theorem 5.9 of [**35**] it suffices to check that R is acyclic. We can verify this by applying Proposition 3.1.3 of [**36**] for the covering $\{Y_i \to R_0\}$. See also Lemma 7.1 of [**33**]. □

We also get an explicit presentation of $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ as a complex over SmCor (this corresponds to the first description of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ in [**36**]). The terms of the complex are (motives of) smooth projective varieties.

Remark 6.5.2.

(1) Applying Proposition 6.5.1 along with the statements of § 7.3 below we get a nice machinery for computing cohomology with compact support. Moreover, Proposition 6.5.1 appears to be connected with the Deligne's definition of (mixed) Hodge cohomology of smooth quasi-projective X. This is no surprise (cf. § 7.4 below); yet a deeper understanding of this matter could improve our understanding of cohomology.

(2) Using Proposition 6.5.1 along with Theorem 3.1.1 one can write an explicit formula for $\mathrm{DM}^{\mathrm{eff}}_{-}(M^{\mathrm{c}}_{\mathrm{gm}}(X), M^{\mathrm{c}}_{\mathrm{gm}}(Y))$ for smooth quasi-projective X, Y/k.

Using § 4.3 of [**36**] one can also calculate $\mathrm{DM}^{\mathrm{eff}}_{-}(M_{\mathrm{gm}}(X), M_{\mathrm{gm}}(Y))$. Indeed, if $\dim X = m$, $\dim Y = n$, X, Y are smooth equidimensional, then (in the category of geometric motives $\mathrm{DM}_{\mathrm{gm}}$)

$$\begin{aligned}\mathrm{DM}^{\mathrm{eff}}_{-}(M_{\mathrm{gm}}(X), M_{\mathrm{gm}}(Y)) &= \mathrm{DM}_{\mathrm{gm}}(M_{\mathrm{gm}}(Y)^*, M_{\mathrm{gm}}(X)^*)\\ &= \mathrm{DM}_{\mathrm{gm}}(M^{\mathrm{c}}_{\mathrm{gm}}(Y)(-n)[-2n], M^{\mathrm{c}}_{\mathrm{gm}}(X)(-m)[-2m])\\ &= \mathrm{DM}_{\mathrm{gm}}(M^{\mathrm{c}}_{\mathrm{gm}}(Y)(m)[2m], M^{\mathrm{c}}_{\mathrm{gm}}(X)(n)[2n]),\end{aligned}$$

see § 8.5 for the discussion on $\mathbb{Z}(r)[2r]$.

In § 2 of [**14**] for any X/k a certain *weight complex* $W(X)$ of Chow motives was defined. In order to make the notation of [**14**] compatible with ours we reverse the arrows in the category of Chow motives. Thus we consider homological Chow motives instead of cohomological ones considered in [**14**]. We have $W(X) \in K^-(\mathrm{Chow}^{\mathrm{eff}})$.

Let m^{-1} denote the equivalence of $\mathrm{DM}^{\mathrm{s}} \subset \mathrm{DM}^{\mathrm{eff}}_{-}$ with $\mathfrak{H}$ inverse to m.

Corollary 6.5.3. *For any smooth quasi-projective X/k we have $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))) \in K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}}) \cong W(X)$ (in $K^{-}(\mathrm{Chow}^{\mathrm{eff}})$).*

Proof. By Proposition 2.8 of [**14**] the weight complex of X is isomorphic to the image in $K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}) \subset K^{-}(\mathrm{Chow}^{\mathrm{eff}})$ of the complex U defined in Proposition 6.5.1 (with arrows reversed). □

6.6. cdh-hypercoverings; the weight complex of Gillet and Soulé for arbitrary varieties

We recall one of the main tools of [**36**] (cf. Definition 4.1.9); it allows to do computations with motives of non-smooth varieties.

Definition 6.6.1. cdh-topology is the smallest Grothendieck's topology such that both Nisnevich coverings and coverings of the form $X' \coprod Z \to X$ are cdh-coverings; here $p : X' \to Z$ is a proper morphism, $i : Z \to X$ is a closed embedding, and the morphism $p^{-1}(X - i(Z)) \to X - i(Z)$ is an isomorphism.

By Lemma 12.26 of [**30**], proper cdh-coverings are exactly envelopes in the sense of 1.4.1 of [**14**]. Therefore, a hyperenvelope in the sense of [**14**] is exactly the same thing as a proper cdh-hypercovering. We recall that a cdh-hypercovering is an augmented simplicial variety $X.$ such that each $X_i \to (\mathrm{cosk}_{i-1}\mathrm{sk}_{i-1}(X))_i$ is a cdh-covering.

We introduce the category $\mathrm{Sch}^{\mathrm{prop}}$. Its objects are varieties over k, its morphisms are proper morphisms of varieties.

In [**14**] the weight complex functor $W : \mathrm{Sch}^{\mathrm{prop}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ was defined in the following way. The weight complex for a simplicial smooth projective variety T was defined (up to the reversion of arrows) as $T_0 \to T_1 \to T_2 \to \cdots$; the boundary maps were given by alternated sums of face maps. Recall that we reverse arrows in $W(T)$!

For $X \in \mathrm{Var}$ a proper $Y \supset X$ was chosen; $Z = Y - X$. It was shown in [**14**] that there exist hyperenvelopes $Z.$ of Z, $Y.$ of Y, and a simplicial closed embedding $Z. \to Y.$ extending the map $Z \to Y$, whereas the terms of $Z.$ and $Y.$ are smooth projective varieties. Then $W(X)$ was defined as the cone of $W(Z.) \to W(Y.)$ (if we reverse arrows). By means of comparing different hyperenvelopes Gillet and Soulé showed that $W(X)$ is well defined as an object of $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ and gives a functor $\mathrm{Sch}^{\mathrm{prop}} \to K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$.

Proposition 6.6.2. *The functor $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))) : \mathrm{Sch}^{\mathrm{prop}} \to K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ is equivalent (after we reverse all arrows) to the functor W.*

Proof. We recall that to compare $t_0 \circ m^{-1} \circ M^{\mathrm{c}}_{\mathrm{gm}}$ with the functor of Gillet and Soulé we should fix some choice of $m^{-1} \circ M^{\mathrm{c}}_{\mathrm{gm}}$ (*a priori* the latter one is only defined up to an isomorphism, cf. Remark 3.1.2). So first we should check that $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X)))$ is isomorphic to the weight complex of X defined in [**14**]; this does not require any choices.

Since $Y. \to Y$ is a cdh-hypercovering, the cdh-sheafification of the corresponding complex $L(Y.) \to L(Y)$ is quasi-isomorphic to 0. Then Theorem 5.5 of [**12**] shows that $C(Y.) \cong C(Y)$. Hence $C(Y.)$ calculates $M_{\mathrm{gm}}(Y)$. The same is true for Z.

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By Proposition 4.1.5 of [**36**] there exists a distinguished triangle

$$M_{\mathrm{gm}}(Z)(=M^{\mathrm{c}}_{\mathrm{gm}}(Z)) \to M_{\mathrm{gm}}(Y)(=M^{\mathrm{c}}_{\mathrm{gm}}(Y)) \xrightarrow{j} M^{\mathrm{c}}_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)[1]$$

in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Hence we obtain that $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X))) \cong W(X)$ (in $K^-(\mathrm{Chow}^{\mathrm{eff}})$).

The definitions imply that W and $t_0(m^{-1}(M^{\mathrm{c}}_{\mathrm{gm}}(X)))$ coincide as functors on the category of proper smooth varieties (cf. Remark 3.1.2). In order to compare the functors in general we can define $m^{-1} \circ M^{\mathrm{c}}_{\mathrm{gm}}$ using the method of § 2 of [**14**] (see the description above). It can be easily seen that this method allows to lift $W(X)$ to a functor $W' : \mathrm{Sch}^{\mathrm{prop}} \to \mathfrak{H}$ which (as we have just proved) can be identified with $m^{-1} \circ M^{\mathrm{c}}_{\mathrm{gm}}$. Certainly, in order to prove that W' is well defined one should replace the usage of the Gersten acyclicity (i.e. of Proposition 2 of 1.4.3 of [**14**]) in the proof of Theorem 2 of [**14**] by the usage of Theorem 5.5 of [**12**]. □

Remark 6.6.3.

(1) In Theorem 5.10 of [**16**] also a certain functor $h : \mathrm{Sch}_k \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ was constructed (Sch_k is the category of varieties over k). It can be shown that h is equivalent to the restriction of $t : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ to motives of varieties (see § 6.4 for the definition of t).

(2) In § 2 of [**14**] it was shown that any two different representatives W_i of $W(X)$ (considered as complexes over SmCor) could be connected by a chain of certain homomorphisms h_i of complexes of smooth projective varieties. Gillet and Soulé proved that h_i induce isomorphisms on the level of $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. The main technical tools were Proposition 2 and Theorem 1 of § 1 of [**14**] showing that hyperenvelopes give quasi-isomorphisms of complexes of Chow motives.

To any such W_i we can associate an object of $\mathfrak{H}$. Since $t_0(h_i)$ is an isomorphism, the corresponding map of motives will be an isomorphism too, see part (6) of Theorem 6.2.1.

Hence one can prove that the method of [**14**] gives a well-defined motif without using the cdh-descent reasoning above.

(3) More generally, one can easily define Voevodsky's motives of Deligne–Mumford stacks (i.e. stacks coming from quotients of varieties by finite groups) over k. For a finite G, $\#G = n$, acting on a variety X/k one can take

$$M_{\mathrm{gm}}(X/G)_{\mathbb{Q}} = a_{G*} M_{\mathrm{gm}}(X_{\mathbb{Q}}) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$$

and

$$M^{\mathrm{c}}_{\mathrm{gm}}(X/G)_{\mathbb{Q}} = a_{G*} M^{\mathrm{c}}_{\mathrm{gm}}(X_{\mathbb{Q}}) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}.$$

Here a_G is the idempotent correspondence

$$\frac{\sum_{g \in G} g}{n} : X \to X.$$

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As we have noted above there exists a conservative exact weight complex functor $t_{\mathbb{Q}} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to \mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q}$ (with properties similar to those of t). Certainly, for $G = \{e\}$ we will have $t_{\mathbb{Q}}(M_{\mathrm{gm}}(X/G)) = t(M_{\mathrm{gm}}(X))$ and $t_{\mathbb{Q}}(M^{\mathrm{c}}_{\mathrm{gm}}(X/G)) = t(M^{\mathrm{c}}_{\mathrm{gm}}(X))$. Besides, it is most probable that $t_{\mathbb{Q}}(M^{\mathrm{c}}_{\mathrm{gm}}(X/G))$ would coincide with the weight complex for X/G defined by Gillet and Soulé (see Proposition 14 of [**15**]). Indeed, both of these weight complexes could be calculated using proper hypercoverings of X/G. So it seems that the method of the proof of Proposition 6.6.2 could be extended to this case. Still the details of the proof of the isomorphness (as well as of [**15**]) have to be written out.

7. Realizations of motives; weight filtration; the spectral sequence of motivic descent

One of the main parts of the theory of motives is the problem of constructing and studying different *realizations*, i.e. exact functors $\mathrm{DM}^{\mathrm{s}} \to T$ for T being a triangulated category. Some authors consider functors from the category of (smooth) varieties to T, yet usually those functors can be factorized through DM^{s} (cf. [**21**, **27**]).

In § 7.1 we recall that any differential graded functor from J gives a realization of $\mathfrak{H}$ (and DM^{s}). This method of constructing realizations is a vast generalization of the method described in 3.1.1 of [**14**]. We call realizations that could be constructed from differential graded functors *enhanceable* realizations; this class seems to contain all 'standard' realizations as well as all representable functors for the category of motives (cf. § 7.2 and part (1) of Remark 7.3.1). In particular, in § 7.2 we verify that the étale cohomology realization is enhanceable; a reader who believes in this fact could skip this subsection.

For any enhanced realization D in § 7.3 we define a family of *truncated realizations*. One could say that truncated realizations correspond to 'forgetting cohomology outside a given range of weights'. In particular, for 'standard' realizations and motivic cohomology one obtains an interesting new family of realizations this way.

In § 7.3 we also prove that truncated realizations of *length* N could be factorized through t_N; they give a filtration on the natural complex that computes D. We obtain a spectral sequence S converging to $D(Y)$ for a motif Y. S could be called the *spectral sequence of motivic descent* (note that the usual cohomological descent spectral sequences compute cohomology of varieties only). For the cohomology with compact support of a variety S is very similar to the spectral sequence considered in 3.1.2 of [**14**]; yet the origin of S is substantially different from those of the mentioned one. Besides we do not need the sheaves to be torsion as one does for étale cohomology. $E_n(S)$ could be expressed in terms of $t_{2n-2}(Y)$ (see [**15**]); in particular, E_1-terms depend only on $t_0(Y)$ and have a nice description in terms of cohomology of smooth projective varieties. S gives a canonical weight filtration on a wide class of cohomological functors; for the 'standard' realizations this filtration coincides with the usual one (with indices shifted).

We note that (as an easy partial case of our results) we get a canonical 'weight' filtration on the motivic cohomology of any variety and the corresponding 'weight' spectral sequence for it. S (and the filtration) is 'motivically functorial'; they are also functorial with respect to 'enhanced' transformations of functors (this includes regulator maps).

In § 7.4 we verify that the our definition of weights gives classical weights for 'standard' realizations (at least, rationally). Moreover, if W denotes the weight filtration on $H^i(X)$ then $W_{l+N}H^i(X)/W_{l-1}H^i(X)$ is exactly the corresponding truncated realization; hence it factorizes through t_N. A morphism f induces a zero morphism on cohomology if $t_0(f)$ is zero. We also prove that the rational length of $X \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ coincides with the 'range' of difference of l with the weights of $H^l(X)$ for all l (cf. Proposition 7.4.2). If we assume certain 'standard' conjectures then there would be an equality, see Proposition 7.4.2.

We conclude the section by the discussion of qfh-descent cohomology theories and qfh-motives of (possibly) singular varieties. It turns out that a wide class of realizations (including 'standard' ones) are 'qfh-representable' (hence they are enhanceable realizations of DM^{s}). Moreover, the qfh-motif of a (not necessarily smooth) variety gives 'right values of standard realizations'.

7.1. Realizations coming from differential graded functors ('enhanceable' realizations)

We consider the problem of constructing and studying different *realizations* of motives, i.e. exact functors $\mathrm{DM}^{\mathrm{s}} \to T$ for T being a triangulated category. Our description of DM^s gives us a simple recipe for constructing realizations. Any differential graded functor $F : J \to X$ for a differential graded category X gives an exact functor $\mathrm{Tr}^+(F) : \mathfrak{H} \to \mathrm{Tr}^+(X)$ (and hence also a functor $\mathfrak{H} \to \mathrm{Tr}(X)$), cf. Remark 2.3.3. It can be easily seen that $\mathrm{Tr}^+(F)$ can be factorized through t_N if $t(J^l([Y],[Z])) = 0$ for any $Y, Z \in \mathrm{SmPrVar}$, $l < -N$. This is always true if $X^l = 0$ for $l < -N$. One can also note that all functors factorizing through t_N could be reduced (in a certain sense) to functors of such sort.

We will say that F gives an *enhancement* of the realization $\mathrm{Tr}^+(F)$; a realization that possesses an enhancement could be called *enhanceable*. Obviously, any differential graded transformation of enhancement induces an exact transformation of realizations. We will mostly consider contravariant functors F.

Note that for $N = 0$, X being equal to $S(A)$ for A an abelian category (see the definition of $S(A)$ in § 2.1), our construction of $\mathrm{Tr}^+(F)$ essentially generalizes to motives the recipe proposed in 3.1.1 of [**14**] for cohomology of varieties with compact support (cf. also [**16**]).

Note lastly that any exact functor $\mathrm{DM}^{\mathrm{s}} \to T$ can be uniquely extended to an exact functor from $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ to the idempotent completion of T.

7.2. 'Representable' contravariant realizations; étale cohomology

Now we verify that a large class of realizations are enhanceable; this includes étale cohomology.

To this end we describe a recipe for constructing a rich family of contravariant differential graded functors from J. Let A be a Grothendieck topology stronger than Nisnevich topology (for example, étale topology). We consider the category $\mathrm{Shv}(\mathrm{SmCor})_A$ (i.e. the morphisms are those of SmCor, coverings are those of A); let $C(\mathrm{Shv}(\mathrm{SmCor})_A)$ denote the category of (unbounded) complexes over $\mathrm{Shv}(\mathrm{SmCor})_A$. We suppose that for any $X \in \mathrm{SmCor}$ the representable presheaf $L(X) = \mathrm{SmCor}(-, X)$ is a sheaf. We denote by

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$D(\mathrm{Shv}(\mathrm{SmCor})_A)$ and $\mathrm{DM}_A^{\mathrm{eff}}$ the categories of unbounded complexes over $\mathrm{Shv}(\mathrm{SmCor})_A$ that are similar to the corresponding categories of [**36**] (i.e. derived category of complexes of sheaves, respectively derived category of complexes of sheaves with homotopy invariant cohomology). We also consider the categories $K(\mathrm{Shv}(\mathrm{SmCor})_A)$ and $B(\mathrm{Shv}(\mathrm{SmCor})_A)$ that are unbounded analogues of $K^-(\mathrm{Shv}(\mathrm{SmCor})_A)$ and $B^-(\mathrm{Shv}(\mathrm{SmCor})_A)$ (cf. § 2.1) respectively.

Now we verify that the étale cohomology realization is enhanceable in the case when K has finite étale cohomological dimension. Let $Y \in C^+(\mathrm{Shv}(\mathrm{SmCor})_A)$ be a complex of injective sheaves with transfers bounded from below with homotopy invariant hypercohomology (we need the hypercohomology condition if $A \neq \mathrm{Nis}$). Now we consider $C(L(X))$ for $X \in \mathrm{SmVar}$. Since $C(L(X))$ is quasi-isomorphic to $\underline{C}(L(X))$, for any $i \in \mathbb{Z}$ we have

$$D(\mathrm{Shv}(\mathrm{SmCor})_A)(C(L(X)), Y[i]) = D(\mathrm{Shv}(\mathrm{SmCor})_A)(\underline{C}(L(X)), Y[i]).$$

Since the correspondence $(F) \to \underline{C}(F)$ defines a functor RC_A which is left-adjoint to the embedding $\mathrm{DM}_A^{\mathrm{eff}} \to D(\mathrm{Shv}(\mathrm{SmCor})_A)$ (cf. Proposition 3.2.3 of [**36**]), $Y \in D(\mathrm{Shv}(\mathrm{SmCor})_A)$, we have

$$D(\mathrm{Shv}(\mathrm{SmCor})_A)(C(L(X)), Y[i]) = D(\mathrm{Shv}(\mathrm{SmCor})_A)(L(X), Y[i]).$$

Let $Z \in \operatorname{Obj} \mathfrak{H} = \operatorname{Obj} J'$ satisfy $m(Z) \cong M_{\mathrm{gm}}(X) = \underline{C}(L(X))$ (in $D^-(\mathrm{Shv}(\mathrm{SmCor}))$ and so also in $D(\mathrm{Shv}(\mathrm{SmCor})_A)$, cf. Corollary 3.2.1). Since the terms of Y are injective sheaves, we conclude that

$$H^{-i}(Y)(X) = K(\mathrm{Shv}(\mathrm{SmCor})_A)(L(X), Y[i]) \cong K(\mathrm{Shv}(\mathrm{SmCor})_A)(h(Z), Y).$$

Moreover, the complex $B(\mathrm{Shv}(\mathrm{SmCor})_A)(h'(Z'), Y)$ computes the complex $Y(X)$ up to a quasi-isomorphism (see the definitions of § 2.5).

Now we describe how the formalism of § 2 can be applied to the computation of $B(\mathrm{Shv}(\mathrm{SmCor})_A)(h'(Z), Y)$. We have a contravariant functor $Y^* : J \to C(\mathrm{Ab})$ that maps $[P] \in \operatorname{Obj} J$ to $B(\mathrm{Shv}(\mathrm{SmCor})_A)(C(P), Y)$. Let $a : J' = \text{Pre-Tr}\, J \to B(\mathrm{Shv}(\mathrm{SmCor})_A)$ denote the differential graded functor induced by the embedding $J \to B^-(\mathrm{Shv}(\mathrm{SmCor})_A)$, cf. Remark 2.3.3 and Proposition 2.2.3. Since $Y^* = B(\mathrm{Shv}(\mathrm{SmCor})_A)(-, Y) \circ a$, we obtain that

$$B(\mathrm{Shv}(\mathrm{SmCor})_A)(h'(Z), Y) \cong \text{Pre-Tr}(Y^*)(Z).$$

Here $\text{Pre-Tr}(Y^*)$ denotes the extension of Y^* to J', cf. Remark 2.3.3.

For example, we can take A being the étale site. Y could be an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ (or a resolution of any other étale complex C with transfers with homotopy invariant hypercohomology) by means of étale sheaves with transfers. By Proposition 3.1.8 and Remark 2 preceding Theorem 3.1.4 in [**36**] the cohomology of $Y(L(X))$ for $X \in \mathrm{SmVar}$ will compute the 'usual' étale hypercohomology (i.e. in the category of sheaves without transfers) of C restricted to X. Hence $\mathrm{Tr}(Y^*)$ gives the corresponding realization of motives. We obtain that in order to compute the étale realization of motives (with coefficients in $\mathbb{Z}/n\mathbb{Z}(r)$ for any $n > 0$, $r \geqslant 0$) it suffices to know the restriction of the corresponding 'representable functor' to the subcategory of $\mathrm{Shv}(\mathrm{SmCor})_A$ consisting of sheaves of the form $C^i(P)$ for $P \in \mathrm{SmPrVar}$. Note also that we can compute morphisms in the category of presheaves with transfers.

If C is a complex of cdh-sheaves (see § 6.6) with transfers then the cohomology of $Y(L(X))$ for $X \in \mathrm{SmVar}$ will compute the cdh-hypercohomology of C restricted to X for any X/k. Yet cdh-topology is not subcanonical; cdh-hypercohomology does not necessarily coincides with 'usual' hypercohomology of C. For the computation of 'usual' cohomology for singular varieties the qfh-topology seems to be more useful; see § 7.5 below.

One can check that the Galois action on $H^i_{\mathrm{et}}(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}, \mathbb{Z}/n\mathbb{Z})$, where $\bar{k}$ is the algebraic closure of k, n is prime to the characteristic of k, could be expressed in terms of our formalism. Indeed, isomorphisms in the derived category corresponding to the Galois action can be extended to an injective resolution of $\mathbb{Z}/n\mathbb{Z}$. Therefore, all statements of this section are valid for étale realization with values in Galois modules.

A similar method for constructing the étale realization (without using the formalism of differential graded categories) was described in [**21**] (see the reasoning following Proposition 2.1.2). In the next subsection we describe a general method of obtaining weight filtrations for realizations.

It seems that the same method can be applied to other 'classical' realizations including the 'mixed realization' one. Yet filling out the details is rather hard. Fortunately, one can avoid this by applying the *weight structure* formalism of [**7**]; see Remark 7.4.4 below.

7.3. The spectral sequence of motivic descent; weight filtration of realizations; the connection with t_N

Now we consider a contravariant functor $F : J \to B(A)$ for an abelian A. We denote the functor $\mathrm{Pre\text{-}Tr}(F) : J' \to B(A)$ by G, denote $\mathrm{Tr}(F) : \mathfrak{H} \to K(A)$ by E.

It seems very probable that one can 'enhance' all 'classical' realizations this way (possibly, for a 'large' A). Besides, it practice it usually suffices to consider functors whose targets are categories of complexes bounded (at least) from one side.

The constructions of this subsection use the results of § 2.7 heavily.

We recall that for a complex X over A, $a, b \in \mathbb{Z}$, $a \leqslant b$, its canonical $[a, b]$-truncation is the complex

$$X_a/dX_{a-1} \to X_{a+1} \to \cdots \to X_{b-1} \to \mathrm{Ker}(X_b \to X_{b+1}),$$

here $\mathrm{Ker}(X_b \to X_{b+1})$ is put in degree b; for $a = b$ we take $H^a(X)$. We also consider truncations of the type $\tau_{\leqslant b}$ (i.e. truncations from above).

For any $b \geqslant a \in \mathbb{Z}$ we consider the following functors (see § 2.7). By $F_{\tau_{\leqslant b}}$ we denote the functor that sends $[P]$ to $\tau_{\leqslant b}(F([P]))$. By $F_{\tau_{[a,b]}}$ we denote the functor that sends $[P]$ to $\tau_{\leqslant b}(F([P]))/\tau_{\leqslant a-1}(F([P]))$. For $N = a-b$ we consider the functor $F_{b,N}$ that sends $[P]$ to the $[a, b]$th canonical truncation of $F([P])$. These functors are differential graded; hence they extend to $G_b = \mathrm{Pre\text{-}Tr}(F_{\tau_{\leqslant b}}) : J' \to B^-(A)$, $G^{\mathrm{b}}_N = \mathrm{Pre\text{-}Tr}(F_{b,N}) : J' \to B^{\mathrm{b}}(A)$, and $G_{a,b} = \mathrm{Pre\text{-}Tr}(F_{\tau_{[a,b]}}) : J' \to B^{\mathrm{b}}(A)$. We recall that $G_{a,b}$ and G^{b}_N are connected by a canonical functorial quasi-isomorphism, see part (2) of Proposition 2.7.3. The reason

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for considering both of them is that the functors $G_{a,b}$ are more closely related to the spectral sequence (14) below whereas G_N^{b} behave better with respect to t_N.

We denote $\mathrm{Tr}(F_{b,N}) : \mathfrak{H} \to K^{\mathrm{b}}(A)$ by F_N^{b}. Since $F_{b,N}$ is concentrated in degrees $[b - N, b]$, $F_{b,N}$ maps all $J^m(X,Y)$ for $X, Y \in \mathrm{Obj}\, J$, $m < -N$, to 0. Hence one can present F_N^{b} as $b_{b,N} \circ t_N$ for a unique $b_{b,N} : \mathfrak{H}_N \to K(A)$. The set of F_N^{b} could be called *truncated realizations* for the realization E; N is the *length* of the realization. These realizations appear to be new even in the case when E is the étale cohomology. Note that for $X = [P]$, $P \in \mathrm{SmVar}$, the truncated realizations give exactly the corresponding truncations of $E([P])$ (i.e. of the corresponding 'cohomology' of P); that is what one usually expects from the weight filtration.

The complexes $G_b(X)$ give a filtration of $G(X)$ for any $X \in \mathrm{Obj}\, J'$; moreover $G_{a,b}(X) = G_b(X)/G_{a-1}(X)$.

Let $X = (P^i, q_{ij}) \in \mathrm{Obj}\, J' = \mathrm{Obj}\, \mathfrak{H}$. We obtain the spectral sequence of a filtered complex (see § III.7.5 of [**13**])

$$S : E_1^{ij}(S) \implies H^{i+j}(G(X)) \tag{14}$$

we call it the *spectral sequence of motivic descent*. Here $E_1^{ij}(S) = H^{i+j}G^{j,j}(X) = H^{i+j}(F_0^j(X))$. Note that $H^{i+j}(G(X)) = H^{i+j}(E(X))$, in the right-hand side we consider X as an object of $\mathfrak{H}$. It is easily seen that S is $\mathfrak{H}'$-contravariantly functorial with respect to X (starting from E_0). Besides starting from E_1 the terms of S are also $\mathfrak{H}$-functorial in X (for example, see (15)) below.

Moreover, if $h : F \to F'$ is a differential graded transformation of functors then the corresponding map of spectral sequences depends only on $\mathrm{Tr}(h)$ (starting from E_1). In particular, for the étale realization the spectral sequence does not depend on the choice of an injective resolution for the corresponding complex (see the previous subsection).

The spectral sequence S is similar to those coming from hypercoverings (and hyperenvelopes). Yet its terms are 'much more functorial'; it computes cohomology of any motif (not necessarily of a motif of a variety).

By definition, $E_1^{ij}(S) = H^{i+j}(F_0^j(Y))$ is the ith cohomology group of the chain complex $A_l = H^j(P^{-l})$. Hence the E_1-terms are functorial in the complex $(P^l) \in K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$, i.e. in $t_0(Y)$. S is convergent: if for X of (stupid) length N we choose a representative in $\mathfrak{H}'$ of length N then only $N+1$ rows of $E_1(S)$ would be non-zero. Besides if all $F_0^j(Y)$ are acyclic then $E(Y)$ is acyclic. We denote the filtration on $H^{\mathrm{s}}(E(Y))$ given by S by W_l; we call it the weight filtration of H^{s}.

For any b, N we also have a 'spectral subsequence'

$$S_N^{\mathrm{b}} : E_1^{ij}(S_N^{\mathrm{b}}) \implies H^{i+j}(G_{a,b}(X)) = H^{i+j}(G_N^{\mathrm{b}}(X)) = H^{i+j}(F_N^{\mathrm{b}}(Y)).$$

Its E_1-terms form a subset of the E_1-terms of S, the (non-zero) boundary maps are the same. We also have weight filtration on $H^{\mathrm{s}}(F_N^{\mathrm{b}}(Y))$.

For any $0 \leqslant l \leqslant N$ we have an obvious spectral sequence morphism $S_{N-l}^{b-l} \to S_N^{\mathrm{b}}$. It induces an epimorphism

$$\alpha_{l,b,N}^{\mathrm{s}} : H^{\mathrm{s}}(F_{N-l}^{b-l}(Y)) \to W_{b-l}(H^{\mathrm{s}}(F_N^{\mathrm{b}}(Y))).$$

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Now we use an argument that could be applied to any filtered complex. For any $N \geqslant 0$, $n \geqslant 1$, one easily sees that $E_n^{ij}(S_N^{\mathrm{b}}) = E_\infty^{ij}(S_N^{\mathrm{b}})$ if $b-n < j < b-N+n$. Moreover, if $b+1-n \geqslant j \geqslant b-N+n-1$ then $E_n^{ij}(S_N^{\mathrm{b}}) = E_n^{ij}(S)$. Therefore, we have

$$\begin{aligned} E_n^{ij}(S) &= \mathrm{Gr}_{n-1}^W H^{i+j}(F_{2n-2}^{j+n-1}(Y)) \\ &= W_{n-1}(H^{i+j}(F_{2n-2}^{j+n-1}(Y)))/W_{n-2}(H^{i+j}(F_{2n-2}^{j+n-1}(Y))) \\ &= \operatorname{Im} \alpha_{n-1,j+n-1,2n-2}^{i+j}/\operatorname{Im} \alpha_{n,j+n-1,2n-2}^{i+j}, \end{aligned} \tag{15}$$

i.e. it is the middle factor of the weight filtration of $H^{i+j}(F_{2n-2}^{j+n-1}(Y))$. A similar equality can be written for $E_n^{ij}(S_N^{\mathrm{b}})$ for any $b \in \mathbb{Z}$, $N \geqslant 0$, $n \geqslant 1$. Hence for any $n \geqslant 1$ the E_n-terms of S and all S_N^{b} depend only on $t_{2n-2}(Y)$.

Suppose that $X \in \mathfrak{H}'_{[c,d]}$. It can be easily verified (for example, using the spectral sequence (14)) that for any $j \in \mathbb{Z}$, $b-N-c \leqslant j \leqslant b-d$, the jth cohomology group of $F_N^{\mathrm{b}}(Y)$ coincides with $H^j(E(Y))$. Besides for any $j \in \mathbb{Z}$ the weights of $H^j(E(Y))$ lie between $j+c$ and $j+d$. In particular, by part (1) Theorem 6.2.1 the weights of $H^j(X)$ for $X \in \mathrm{SmVar}$ of dimension N lie between j and $j+N$, the weights of $H_c^j(X)$ (the cohomology with compact support) lie between $j-N$ and j.

Hence all 'cohomological information' of a motif of length less than or equal to N could be factorized through t_N (in order to prove this for fine length less than or equal to N one should use Proposition 6.4.1). This statement can be considered as the 'realization version' of Theorem 6.2.1 (parts (1) and (2)).

Remark 7.3.1.

(1) For an object $U \in \operatorname{Obj} J' = \operatorname{Obj} \mathfrak{H}' = \operatorname{Obj} \mathfrak{H}$ (recall that $J' = \text{Pre-Tr}(J)$) one can consider the differential (contravariant) graded functor $J_U : J \to B(\mathrm{Ab})$ that maps $[P]$ to $J'([P], U)$. Then $H^i(\mathrm{Tr}(U))(X) = \mathfrak{H}(X, U[i])$; this means that representable realizations are enhanceable.

In particular, we can take $U = \mathbb{Z}(n)$ for $n \geqslant 0$. Hence we obtain canonical 'weight' filtration on the motivic cohomology of any variety and the corresponding 'weight' spectral sequence for it. A simple example of this spectral sequence could be given by the Bloch's long exact localization sequence for higher Chow groups (see [**4**]).

Indeed, let $Z \subset X \in \mathrm{SmPrVar}$, Z is everywhere of codimension c, let $Y = X - Z$. Then (cf. the proof of part (1) of Theorem 6.2.1) the motif of Y is a cone of $M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(c)[2c]$. So the length of $M_{\mathrm{gm}}(Y)$ is 1; hence for any realization S reduces to a long exact sequence that relates cohomology of $X, Z(c)[2c]$, and Y. If the realization is motivic cohomology then it would certainly equal the exact sequence of Bloch. This example shows that the weight filtration obtained this way is non-trivial in general; it appears not to be mentioned in the literature. The filtration is compatible with the regulator maps (whose targets are 'classical' cohomology theories).

Using the spectral sequence relating algebraic K-theory to the motivic cohomology (see [**11**] and [**33**]) one can also obtain a new filtration on the K-theory of a smooth variety X.

(2) The spectral sequence functor described is additive (here we fix F and consider S as a functor from $\mathfrak{H}$). Hence it is easily seen that it could be uniquely extended to the whole $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (starting from E_1).

(3) It seems to be interesting to study the truncations and the weight spectral sequence for the cohomology of the sheaf $Y \to G_m(X \times Y)$ for a fixed variety Y (G_m is the multiplicative group). These things seem to be related with the Deligne's one-motives of varieties as they were described in [**2**].

More generally, for any $X \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ the functor $\underline{\mathrm{Hom}}(-, X) : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to \mathrm{DM}^{\mathrm{eff}}_{-}$ is enhanceable.

(4) In Remark 6.4.1 of [**7**] we verify that the truncated realizations coming from representable realizations are representable also. The truncated realizations are represented by t-truncations of the objects representing the original realizations with respect to a certain *Chow* t-structure. Note that this is the case for motivic cohomology and for 'classical' realizations of motives (with values in Ab).

More generally, one could define a t-structure on the category $\mathrm{Tr}(\text{DG-Fun}(J, B(A)))$ (differential graded functors) that corresponds to the canonical truncation of A-complexes. Then the realizations of the type considered here correspond to some objects of this category; truncations of a realization with respect to this t-structure would be exactly its truncated realizations.

(5) Another important source of differential graded functors from J (generalizing representable functors considered in part (1)) are those coming from localizations of $\mathfrak{H}$ (or of DM^{s} which is the same thing). It will be discussed in § 8.2 below.

7.4. Comparison with 'classical weights'; comparison of the rational length with the 'Hodge length'

Suppose now that there are no maps between different weights: that is, for any $P, P', Q, Q' \in \mathrm{SmPrVar}$, $f \in \mathrm{SmCor}(P, P')$, $g \in \mathrm{SmCor}(Q, Q')$, $i \neq j$, we have

$$A(\mathrm{Ker}(H^i(E([P])) \xrightarrow{f^*} H^i(E([P'])))), \mathrm{Coker}(H^j(E([Q])) \xrightarrow{g^*} H^j(E([Q'])))) = 0.$$

Recall that this condition is fulfilled for the étale and Hodge realizations with rational coefficients. Then S and all S^{b}_N degenerate at E_1. Therefore,

$$H^j F^{\mathrm{b}}_N(X) = W_b(H^j(E(Y)))/W_{b-N-1}(H^j(E(Y))).$$

Besides, if there are no maps between different weights, then for any $H^l(X)$ there cannot exist more than one filtration W_j on $H^l(X)$ such that W_j/W_{j-1} is of weight j. Hence for the étale and Hodge realizations our weight filtration coincides with the usual one (with indices shifted). Therefore, t_N can be called the weight functors.

The existence of the weight spectral sequence easily implies that the length of the weight filtration of H^l is not larger than the stupid length of $M \in \mathrm{Obj}\,\mathrm{DM}^{\mathrm{s}}$. Moreover, Proposition 6.4.1 gives the same inequality for the fine length of any object of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and also for the rational length (on the rational level). Besides, if $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b],\mathbb{Q}}$ then the weights of $H^l \otimes \mathbb{Q}(M)$ lie between $l + a$ and $l + b$.

Conjecture 7.4.1. *The converse implication is true also, i.e. if for all l the weights of $H^l \otimes \mathbb{Q}(M)$ (here H^* is the singular cohomology) lie between $l + a$ and $l + b$ then $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b],\mathbb{Q}}$.*

Now we show that Conjecture 7.4.1 follows from certain 'standard' conjectures.

Proposition 7.4.2. *Suppose that following statements are valid.*

(1) *Hodge standard conjecture.*

(2) *Any morphism of Chow motives that induces an isomorphism on singular cohomology is an isomorphism.*

Then Conjecture 7.4.1 is also valid.

Proof. Indeed, we should check that if $M \notin \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}[a,b],\mathbb{Q}}$ then at least for one l the weights of $H^l \otimes \mathbb{Q}(M)$ do not lie between $l + a$ and $l + b$. Using Proposition 6.4.1 we obtain that we have the motivic descent spectral sequence for the singular realization of objects of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$; its E_1^{ij}-term is the ith cohomology group of the chain complex $A_l = H^j(P^{-l})$. Therefore, one should check

(1) if the 'first term' of $t_{\mathbb{Q}}(M) = (P^i)$ is at cth place and the map $g : P^c \to P^{c+1}$ is not a projection onto a direct summand then for some l the $(l + c)$th weight component of $H^l \otimes \mathbb{Q}(M)$ is non-zero;

(2) the dual statement.

We verify (1); (2) is similar (and follows from (1) by duality). Suppose that for any l the $(l + c)$th weight piece of $H^l \otimes \mathbb{Q}(M)$ is zero. Note that this piece equals $\mathrm{Coker}(g_l^* : H^l(P^{c+1}) \to H^l(P^c))$; hence all g_l^* are surjective. Since the category of rational pure Hodge structures is semisimple; we can choose a family of splittings for g_l^* (that respect the Hodge structures). Since the splittings are Hodge, by Hodge standard conjecture (for $P^c \times P^{c+1}$) we obtain that these splittings can be realized by a morphism $h : P^{c+1} \to P^c$ of Chow motives. We obtain that $H^*(h)$ gives a splitting of $H^*(g)$; the condition (2) (applied to $h \circ g$) implies that g is a projection onto a direct summand. □

Remark 7.4.3.

(1) Note that Conjecture 7.4.1 implies the conservativity of the singular realization, which certainly implies condition (2) of Proposition 7.4.2.

(2) In order to get a similar result for the étale realization one should replace the Hodge conjecture by the Tate conjecture.

(3) One could also count the (minimal) number of non-zero terms of $t(M)$.

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We also note that for any morphism $f : Y \to Z$ for $Y, Z \in \mathfrak{H}$ the morphisms $H^l(f) : H^l(E(Z)) \to H^l(E(Y))$ for $Y, Z \in \mathfrak{H}$ are strictly compatible with the weight filtration. Therefore, $H^l(f)$ is zero if and only if the corresponding map of E_1-terms in $S(Y) \to S(Z)$ is zero. Hence the map $\mathfrak{H}(Y, Z) \to A(H^l(E(X)), H^l(E(Y)))$ factorizes through t_{0*}.

For cohomology with integral coefficients one may apply the previous statement for rational cohomology to obtain that f^* is zero on $H^l \otimes \mathbb{Q}$ if $t_0(f) = 0$. Hence if $t_0(f) = 0$ then f^* is zero on cohomology modulo torsion.

Using the results of § 6.6 one can compute $\mathfrak{H}(Z, T)$ for $Z = m^{-1}(M_{\mathrm{gm}}(X))$, $T = m^{-1}(M_{\mathrm{gm}}(Y))$, and also compute

$$M_N(X, Y) = \operatorname{Im} t_{N*}(\mathfrak{H}(Z, T)) \to \mathfrak{H}_N(t_N(Z), t_N(T)).$$

Remark 7.4.4. Formally all our filtrations and spectral sequences depend on the choices of enhancements for realizations. One could check that they are independent in fact; yet in order to use the theory described above it is necessary (at least) to prove that enhancements exist. This seems to be true for all reasonable cases; yet proofs could be difficult (see § 7.5 below).

There is a way to avoid these difficulties completely; it is studied in [**7**]. The idea is to consider a set of axioms of so-called *weight structures* for a triangulated category C; the axioms are somewhat similar to those for t-structures (yet the consequences of the axioms are quite distinct!). One could say that any object of C has a *weight decomposition* which is not unique but is 'unique up to a homotopy' (in a certain sense). In the case when $C = \mathfrak{H}$ one could consider the decompositions given by the 'stupid filtration' (see Proposition 2.6.1). Then for any (covariant) functor $H : C \to A$, where A is an abelian category, one could consider the filtration of $H(X)$ by $H(X_{[a,+\infty]})$ for $a \in \mathbb{Z}$. Note that while X does not determine $X_{[a,+\infty]}$ uniquely, it does determine the image of $H(X_{[a,+\infty]})$ in $H(X)$. If H is contravariant then one should consider the image $H(X_{[-\infty,b]}) \to H(X)$. The objects $X_{[a,+\infty]}$, $X_{[-\infty,b]}$ give a Postnikov tower for X; this yields a spectral sequence T converging to $H(X)$. Its E_1-terms are $H_i(P^j)$; hence they are not determined by X. Yet starting from E_2 all terms of the spectral sequence are canonical and functorial in X. In fact, the filtration induced by G_b on G could be obtained from the filtration for T by Deligne's decalage. Hence the spectral sequences and the filtration for T coincide (up a certain change of indexes) with the terms of S defined in (14).

This approach for constructing weight filtrations of realizations of Hanamura's motives was described in [**18**]; yet the proof of Proposition 3.5 in [**18**] relies heavily on (sort of) enhancements for realizations.

The advantage of our alternative approach is that enhancements are no longer needed; in particular, it could be applied to the stable homotopy category for which no (differential graded) enhancements exists. Yet in this abstract setting it is difficult to define truncation functors (especially the 'higher' ones). The reason for that is (as was noted by several authors) that the axiomatics of an (abstract) triangulated category is not 'rigid enough'.

7.5. qfh-descent cohomology theories; motives of singular varieties

Some 'standard' cohomology theories are difficult to represent by a complex of sheaves with transfers (in the way described in § 7.2). One of the ways to do this is to use qfh-topology.

We recall that the qfh-topology is the topology on the set of all varieties whose coverings are quasi-finite universal topological coverings (see [**34**] for a precise definition). In particular, the qfh-topology is stronger than the flat topology and the cdh-topology. There is a natural functor from $\mathrm{DM}_{-}^{\mathrm{eff}}$ to the derived category of qfh-sheaves with homotopy invariant cohomology (it is denoted as $\mathrm{DM}_{\mathrm{qfh}}(k)$); this functor is surjective on objects. Note also that any 'ordinary' topological sheaf restricted to Var gives a qfh-sheaf. Moreover, qfh-descent follows from proper descent combined with Zariski descent (see § 2 of [**34**]).

Let C be a complex of presheaves (possibly without transfers) whose hypercohomology satisfies qfh-descent. Then the qfh-hypercohomology of the qfh-sheafification of C coincides with the hypercohomology of C (for example, a similar statement was proved in the proof of Theorem 5.5 of [**12**]). Therefore, if the cohomology of C is homotopy invariant then it could be presented by means of 'representable' functors on DM^{s} as it was described in § 7.2 above. In particular, this shows that Betti and Hodge cohomology theories could be enhanced to differential graded realizations.

Now let C be a complex of qfh-sheaves. It was proved in [**34**] (see Theorems 3.4.1 and 3.4.4) that the qfh-hypercohomology of a variety X with coefficients in C coincides with the étale hypercohomology of C in the cases when either C is a $\mathbb{Q}$-vector space sheaf complex and X is normal or C is a locally constant étale sheaf complex. Hence in this cases the étale hypercohomology of C also gives a 'representable' realization.

Note that these realization compute qfh-hypercohomology with coefficients in C of any (not necessarily smooth) variety X. Hence $M_{\mathrm{gm}}(X)_{\mathrm{qfh}}$ (i.e. the image of $M_{\mathrm{gm}}(X)$ in $\mathrm{DM}_{\mathrm{qfh}}(k)$) seems to be the natural choice for the qfh-motif of a (possibly) singular variety. In particular, its 'standard' realizations have the 'right' values of $H^i_{\mathrm{et}}(X, \mathbb{Z}/l\mathbb{Z}(m))$ (at least, in the case when k has finite étale cohomological dimension).

8. Concluding remarks

In § 8.1 we give a general description of subcategories of $\mathfrak{H}$ generated by fixed sets of objects. In particular, this method can be used to obtain the description of the category of effective Tate motives (i.e. the full triangulated subcategory of DM^{s} generated by $\mathbb{Z}(n)$ for $n > 0$).

In § 8.2 we describe the construction of 'localization of differential graded categories' (due to Drinfeld). This gives us a description of localizations of $\mathfrak{H}$. All such localizations come from differential graded functors. As an application, we prove that the motif of a smooth variety is a mixed Tate one whenever its weight complex (as defined in [**14**], cf. §§ 6.5 and 6.6) is.

In § 8.3 we verify that over an arbitrary perfect field one can apply our theory (at least) with rational coefficients. One of the main tools is the Poincaré duality in characteristic p proved by Beilinson and Vologodsky. Moreover, over finite fields the Beilinson–Parshin conjecture (that $H^i(P,\mathbb{Q}(n))$ for smooth projective P could be non-zero only for $i = 2n$) holds if and only if $t\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q})$ is an equivalence (note that here $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$, $\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q}$, and $t\mathbb{Q}$ denote the corresponding idempotent completions); see Proposition 8.3.1. We also describe an idea for constructing certain 'infinite integral' weight complex in finite characteristic (see § 7.3 of [**7**] for a complete proof).

In § 8.4 we prove that traces of endomorphisms of cohomology of motives induced by endomorphisms of motives do not depend on the choice of a Weil cohomology theory. This result generalizes Theorem 3.3 of [**5**].

In § 8.5 we remark that one can easily add direct summands of objects to J. In particular, one could include $[P][2i](i)$ into J.

In § 8.6 we consider a functor $m_N : \mathfrak{H}_N \to \mathrm{DM}^{\mathrm{eff}}_{-}$ that maps $[P]$ into the Nth canonical truncation of $C(P)$ (as a complex of sheaves).

8.1. Subcategories of $\mathfrak{H}$ that are generated by a fixed set of objects

Let B be a set of objects of $\mathfrak{H}$; we assume that B is closed with respect to direct sums.

Let B' denote some full additive subcategory of $J' = \text{Pre-Tr}(J)$ such that the corresponding objects of $\mathfrak{H}$ are exactly elements of B (up to isomorphism).

Let $\mathfrak{B}$ denote the smallest triangulated category of $\mathfrak{H}$ containing B.

Proposition 8.1.1. *$\mathfrak{B}$ is canonically isomorphic to* $\mathrm{Tr}^+(B')$.

Proof. Follows immediately from Theorem 1 in § 4 of [**6**]. □

Remark 8.1.2.

(1) It follows immediately that in order to calculate the smallest triangulated category of containing an *arbitrary* fixed set of objects in $\mathfrak{H}$ it is sufficient to know morphisms between the corresponding objects in J' (i.e. certain complexes) as well as the composition rule for those morphisms.

(2) We obtain that for any triangulated subcategory of $D \subset \mathfrak{H}$ the embedding $D \to \mathfrak{H}$ is isomorphic to $\mathrm{Tr}^+(E)$ for some differential graded functor $E : F \to G$. Here G is usually equal to J' (though sometimes it suffices to take $G = J$); F depends on D.

It follows that for any $h \in \mathrm{Obj}\,\mathfrak{H}$ the representable contravariant functor $h^* : D \to \mathrm{Ab} : d \to \mathfrak{H}(d,h)$ can be represented as $H^0(u)$ for some contravariant differential graded functor $u : F \to B^-(\mathrm{Ab})$. See part (2) of Proposition 8.2.1 below for a similar statement for localization functors.

(3) Using this statement one can easily calculate the triangulated category of (mixed effective) Tate motives (cf. [**28**]). It is sufficient to take $B = \sum_{a_i \geqslant 0}[(\mathbb{P}^1)^{a_i}]$, i.e. the additive category generated by motives of non-negative powers of the projective line. This gives a certain extension of the description of [**32**] to the case of integral coefficients. Note that the description of the category of effective Tate motives immediately gives a description of the whole category of Tate motives since $\mathbb{Z}(1)$ is quasi-invertible with respect to $\otimes$. Alternatively, one could expand J, see § 8.5 below.

8.2. Localizations of $\mathfrak{H}$

Let C be a differential graded category satisfying the homotopical flatness condition, i.e. for any $X, Y \in \operatorname{Obj} C$ all $C^i(X, Y)$ are torsion free. Note that both J and $J' = \text{Pre-Tr}^+(J)$ satisfy this condition.

In [**10**] Drinfeld has proved (modifying a preceding result of Keller) that for C satisfying the homotopical flatness condition and any full differential graded subcategory B of C there exists a differential graded quotient C/B of C modulo B. This means that there exists a differential graded $g : C \to C/B$ that is surjective on objects such that $\operatorname{Tr}^+(g)$ induces an equivalence $\operatorname{Tr}^+(C)/\operatorname{Tr}^+(B) \to \operatorname{Tr}^+(C/B)$ (i.e. $\operatorname{Tr}^+(C)/\operatorname{Tr}^+(B) \cong \operatorname{Tr}^+(C/B)$, $\operatorname{Tr}^+(g)$ is zero on $\operatorname{Tr}^+(B)$ and induces this equivalence).

The objects of C/B are the same as for C whereas for $C_1, C_2 \in \operatorname{Obj} C = \operatorname{Obj}(C/B)$, $i \in \mathbb{Z}$, we define

$$\begin{aligned}(C/B)^i&(C_1, C_2)\\ &= C^i(C_1, C_2) \bigoplus \bigoplus_{j \geqslant 0} \bigoplus_{\substack{B_1, \dots, B_j \in \operatorname{Obj} B,\\ \sum a_i = i+j}} C^{a_1}(C_1, B_1) \otimes \varepsilon_{B_1} \otimes C^{a_2}(B_1, B_2)\\ &\qquad \otimes \varepsilon_{B_2} \otimes \cdots \otimes \varepsilon_{B_j} \otimes C_{a_j}(B_j, C_2). \qquad (16)\end{aligned}$$

Here $\varepsilon_b \in (C/B)^{-1}(b, b)$ for each $b \in \operatorname{Obj} B \in \operatorname{Obj}(C/B)$ is a 'canonical new morphism' such that $d_b \varepsilon_b = \mathrm{id}_b$; ε_b spans a canonical direct summand $\mathbb{Z}\varepsilon_b \subset (C/B)^{-1}(b, b)$. From this condition one recovers the differential on morphisms of C/B.

For example, this construction (for $C = J$) gives an explicit description of the localization of $\mathfrak{H}$ by the triangulated category generated by all $[Q]$, $Q \in \mathrm{SmPrVar}$, $\dim Q < n$, for a fixed n (and hence also of the corresponding localization of DM^{s}).

We will only need (for part (3) of Proposition 8.2.1 below) the following obvious property of the construction: if $C^i(-,-) = 0$ for $i > 0$ then the same is true for C/B. Note also that in this case $C^0(X, Y) = (C/B)^0(X, Y)$ for all $X, Y \in \operatorname{Obj} C$; $H(C/B)(X, Y)$ is a certain easily described factor of $HC(X, Y)$ (yet we will not need this statement).

More generally, for localizations of $\mathfrak{H}$ modulo some $A \subset \mathfrak{H}$ it is sufficient to know the complexes $C/B([P], [Q])$ and the composition law for a certain B and all $P, Q \in \mathrm{SmPrVar}$; here either $C = J$ or $C = J'$. In the case when A is not generated by objects of (stupid) length 0 we are forced to take $C = J'$; this makes the direct sum in (16) huge.

Proposition 8.2.1.

(1) *If $F : \mathfrak{H} \to T$ is a certain localization functor (T is a triangulated category) then $F \cong \operatorname{Tr}^+(G)$ for a certain differential graded functor G from J.*

(2) *For any $t \in T$ the contravariant functor $t^* : \mathfrak{H} \to \mathrm{Ab} : X \to T(F(x), t)$ can be represented as $H^0(u)$ for some contravariant differential graded functor $u : J \to B^-(\mathrm{Ab})$.*

(3) *Let B be a full additive subcategory of J, $\operatorname{Obj} B = T \subset \mathrm{SmPrVar}$. Let $\mathfrak{B}$ denote the smallest triangulated subcategory of $\mathfrak{H}$ that contains all objects of B and is closed with respect to taking direct summands (in B). Then for $M \in \operatorname{Obj} \mathfrak{H}$ we have $M \in \operatorname{Obj} \mathfrak{B}$ whenever $t_0(M)$ is a direct summand of a complex all whose terms have the form $[Q]$, $Q \in T$.*

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Proof. (1) Let $A = \{X \in \operatorname{Obj}\mathfrak{H},\ F(X) = 0\} \subset ObjJ' = \operatorname{Obj}\mathfrak{H}$, we denote the corresponding full subcategory of J' by B'. By Proposition 8.1.1, $\mathrm{Tr}^+(B')$ is isomorphic to the categorical kernel of F. Let $H : J' \to J'/B'$ denote the functor given by Drinfeld's construction. Since $\mathrm{Tr}^+(J') = \mathfrak{H}$, we obtain $\mathrm{Tr}^+(H) \cong F$. Hence by part (2) of Remark 2.3.3 we can take G being the restriction of H to $J \subset J'$.

(2) Let w denote some element that corresponds to t in $\text{Pre-Tr}^+(J'/B')$. Then it suffices to take $u([P]) = \text{Pre-Tr}^+(J'/B')(\text{Pre-Tr}^+(G)([P], w))$.

(3) If $M \in \operatorname{Obj}\mathfrak{B}$ then $t_0(M)$ belongs to the triangulated subcategory of $\mathfrak{H}_0 = K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$ that contains all $[Q], Q \in T$ and is closed with respect to taking direct summands. This subcategory consists exactly of complexes described in the assertion (3).

We prove the converse implication.

We can assume that $t_0(M)$ is homotopy equivalent to a complex all whose terms have the form $[Q], Q \in T$.

Let I denote J/B, let $K = \mathrm{Tr}^+(I)$. Let $S : J \to J/B$ be the localization functor of [**10**]; let $u = \mathrm{Tr}^+(S)$. Since $I^i(-,-) = 0$ for $i > 0$, we have natural functors $I \to S(HI)$ (see § 2.1) and $v : K \to K_0$ where $K_0 = K^{\mathrm{b}}(HI)$ (the weight complex functor for this case). Hence we have functors $u_0 : \mathfrak{H}_0 \to K_0$ and $v : K \to K_0$ such that $u_0 \circ t_0 = v \circ u$. We obtain $v(u(M)) = 0$.

Suppose that $M = j(X)$ for some $X \in \operatorname{Obj}\mathfrak{H}'$. We can also assume that $X \in \operatorname{Obj} K$.

For any object $U \in \operatorname{Obj} K = \operatorname{Obj}\text{Pre-Tr}^+(I)$ we consider the (contravariant) differential graded functor $U^* : J \to B(\mathrm{Ab})$ that maps $[P]$ to $\text{Pre-Tr}^+(I)([P], U)$. Then we can consider the spectral sequence (14) for $\mathrm{Tr}^+(U^*) : E_1^{ij}(S) \implies K(X, U[i+j])$. As in § 7.3 we note that its E_1-terms depend only on $u_0(t_0(X))$ (they are just $K_0(u_0(t_0(X)), v(U)[i+j]))$. Since $u_0(t_0(X)) = v(u(X)) = 0$, we have $\mathrm{Tr}^+(U^*)(X) = 0$ for any U. Since $\mathrm{Tr}^+(U^*)(X) = K(X, U)$, we obtain that $X \cong 0$ in K. Hence $M \in \operatorname{Obj}\mathfrak{B}$. □

Part (3) is a generalization of part (5) of Theorem 6.2.1 (there $T = \{0\}$).

Corollary 8.2.2. *Let $X \in \mathrm{SmVar}$. Then $M_{\mathrm{gm}}(X)$ is a mixed Tate motif (as described in part (3) of Remark 8.1.2) in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (i.e. we add direct summands) whenever the complex $t_0(U) \in K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$ is.*

Proof. We apply part (3) of Proposition 8.2.1 for $T = \{\bigsqcup_{a_i \geqslant 0}(\mathbb{P}^1)^{a_i}\}$.

We obtain that $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ is a mixed Tate motif whenever $t_0(M^{\mathrm{c}}_{\mathrm{gm}}(X)) = W(X)$ is mixed Tate as an object of $K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}})$.

On the category of geometric of Voevodsky's motives $\mathrm{DM}_{\mathrm{gm}} \supset \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (see § 4 of this paper and § 4.3 of [**36**]) we have a well-defined duality such that $\mathbb{Z}(n)^* = \mathbb{Z}(-n)$. Therefore, the category of mixed Tate motives is a self-dual subcategory of $\mathrm{DM}_{\mathrm{gm}}$. Since $M_{\mathrm{gm}}(X)^* = M^{\mathrm{c}}_{\mathrm{gm}}(X)(-n)[-2n]$, $n = \dim X$ (see Theorem 4.3.7 of [**36**]; we can assume that X is equidimensional), we obtain that $M_{\mathrm{gm}}(X)$ is a Tate motif if and only if $M^{\mathrm{c}}_{\mathrm{gm}}(X)$ is. □

One can translate this statement into a certain condition on the motives $M_{\mathrm{gm}}(Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_r})$ (in the notation of Proposition 6.5.1).

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8.3. Motives in finite characteristic; $t \otimes \mathbb{Q}$ is (conditionally) an isomorphism over a finite field

In this subsection we consider Voevodsky'a motives over an arbitrary perfect k.

In the proof of Theorem 3.1.1 we used two facts.

(i) The cohomology of Suslin complex of a smooth projective variety coincides with its hypercohomology.

(ii) Motives of smooth projective varieties generate DM^{s}.

To the author's knowledge, neither of these facts is known in finite characteristic.

Yet for an covering of any variety over any perfect k one has the Mayer–Vietoris triangle (see Proposition 4.1.1 of [**36**]); one also has the blow-up distinguished triangle (see Proposition 3.5.3 of [**36**] and Proposition 5.21 of [**35**]). Besides, for a closed embedding of smooth varieties one has the Gysin distinguished triangle (see [**9**]).

8.3.1. *Certain integral arguments*

In [**3**] it was proved unconditionally that DM^{s} has a differential graded enhancement. In fact, this follows rather easily from the localization technique of Drinfeld described in § 8.2 along with Voevodsky's description of DM^{s} as a localization of $K^{\mathrm{b}}(\mathrm{SmCor})$. Moreover, Proposition 6.7 of [**3**] extends the Poincaré duality for Voevodsky motives to our case. Therefore, for $P, Q \in \mathrm{SmPrVar}$ we obtain

$$\mathrm{DM}^{\mathrm{s}}(M_{\mathrm{gm}}(P), M_{\mathrm{gm}}(Q)[i]) = \begin{cases} \mathrm{Corr}_{\mathrm{rat}}([P],[Q]) & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

Hence the triangulated subcategory $\mathrm{DM}_{\mathrm{pr}}$ of DM^{s} generated by $[P], P \in \mathrm{SmPrVar}$ could be described as $\mathrm{Tr}(I)$ for a certain negative differential graded I (see part (2) of Remark 2.7.4). Note also that the complexes $I(P, Q)$ compute motivic cohomology of $P \times Q$.

If we define the categories I_n similarly to J_n (see § 6.1) then $I_0 = \mathrm{Corr}_{\mathrm{rat}}$. Hence there exists a conservative weight complex functor $t_0 : \mathrm{DM}_{\mathrm{pr}} \to K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}})$. Moreover, for any 'enhanceable' realization of $\mathrm{DM}_{\mathrm{pr}}$ (again one can easily check that these include motivic cohomology and étale cohomology) and any $X \in \mathrm{Obj}\,\mathrm{DM}_{\mathrm{pr}}$ one has the motivic descent spectral sequence (14).

The problem is that (to the knowledge of the author) at this moment there is no way to prove that $\mathrm{DM}_{\mathrm{pr}}$ contains the motives of all varieties (though it certainly contains the motives of varieties that have 'smooth projective stratification').

Yet for any $X \in \mathrm{Var}$ one could consider the restriction X^*_{pr} of the functor X^* that equals $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(-, M_{\mathrm{gm}}(X))$ restricted to $\mathrm{DM}_{\mathrm{pr}}$. It has a differential graded 'enhancement'; in § 7.3 of [**7**] we prove that X^*_{pr} is representable by (at least) an object of a certain infinite analogue of $\mathrm{DM}_{\mathrm{pr}}$. This gives a (possibly, infinite) weight complex for X.

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8.3.2. *Rational arguments*

For rational coefficients (ii) was proved in Appendix B of [**22**]: it suffices to note that if there exists an étale finite morphism $U \to V$ for smooth U, V then $M_{\mathrm{gm}}(V) \otimes \mathbb{Q}$ is a direct summand of $M_{\mathrm{gm}}(U) \otimes \mathbb{Q}$. Hence the existence of de Jong's alterations (see [**8**]) yields (ii) for motives with rational coefficients (up to an idempotent completion of categories).

Hence we obtain that in the characteristic p case the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$ is the idempotent completion of $\mathrm{Tr}(I)$ for a certain negative differential graded I (see above). Moreover (by the same arguments as in the characteristic 0 case), there exists a conservative weight complex functor $t\mathbb{Q} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q})$.

Now we show that, in contrast to the characteristic 0 case, over finite fields the Beilinson–Parshin conjecture (see below) implies that $t\mathbb{Q}$ is an equivalence.

Proposition 8.3.1. *Let k be a finite field.*

Suppose that for any $X \in \mathrm{SmVar}$ the only non-zero cohomology group $H^i(X, \mathbb{Q}(n))$ is H^{2n}.

Then $t\mathbb{Q}$ is an equivalence of categories.

Proof. By Theorem 4.2.2 of [**36**] (note that it is valid with rational coefficients in our case also!) for any $P, Q \in \mathrm{SmPrVar}$, $i \in \mathbb{Z}$ we have

$$\mathrm{DM}^{\mathrm{eff}}_{-}(M_{\mathrm{gm}}(P) \otimes \mathbb{Q}, M_{\mathrm{gm}}(Q) \otimes \mathbb{Q}[i]) = H^{i+2r}(P \times Q, \mathbb{Q}(r)), \tag{17}$$

where r is the dimension of Q. Therefore, if the Beilinson–Parshin conjecture holds then

$$\mathrm{DM}^{\mathrm{eff}}_{-}(M_{\mathrm{gm}}(P) \otimes \mathbb{Q}, M_{\mathrm{gm}}(Q) \otimes \mathbb{Q}[i]) = \begin{cases} \mathrm{Corr}_{\mathrm{rat}} \otimes \mathbb{Q}([P],[Q]) & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$\mathrm{DM}^{\mathrm{s}} \otimes \mathbb{Q}([P],[Q][i]) \cong K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}} \otimes \mathbb{Q})(t_0 \otimes \mathbb{Q}([P]), t_0 \otimes \mathbb{Q}([Q][i]))$$

for all $i \in \mathbb{Z}$. Since $[P]$ for $P \in \mathrm{SmPrVar}$ generate $\mathrm{DM}^{\mathrm{s}} \otimes \mathbb{Q}$ as a triangulated category, the same easy standard reasoning as the one used in the proof of Theorem 3.1.1 shows that $t_0 \otimes \mathbb{Q}$ induces a similar isomorphism for any two objects of $\mathrm{DM}^{\mathrm{s}} \otimes \mathbb{Q}$; hence $t_0 \otimes \mathbb{Q}$ is a full embedding. Since $[P], P \in \mathrm{SmPrVar}$ generate $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}} \otimes \mathbb{Q})$ as a triangulated category, we obtain that $t_0 \otimes \mathbb{Q}$ is an equivalence of categories. Lastly, since $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}$ is the idempotent completion of $\mathrm{DM}^{\mathrm{s}} \otimes \mathbb{Q}$ and $K^{\mathrm{b}}(\mathrm{Chow}^{\mathrm{eff}}\,\mathbb{Q})$ is the idempotent completion of $K^{\mathrm{b}}(\mathrm{Corr}_{\mathrm{rat}} \otimes \mathbb{Q})$, $t\mathbb{Q}$ is an equivalence also. □

Remark 8.3.2.

(1) Obviously, the converse statement holds also: if $t\mathbb{Q}$ is an isomorphism then $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q}(M_{\mathrm{gm}}(P) \otimes \mathbb{Q}, M_{\mathrm{gm}}(Q) \otimes \mathbb{Q}[i]) = 0$ for $i \neq 0$, $P, Q \in \mathrm{SmVar}$; hence the Beilinson–Parshin conjecture holds.

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(2) As it was noted in Remark 6.3.2, using $t\mathbb{Q}$ one can obtain a functor $t_{\mathrm{num}}\mathbb{Q}$: $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,\mathbb{Q} \to K^{\mathrm{b}}(\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}}\,\mathbb{Q})$ (the 'wrong' homotopy category of effective numerical motives). Over a finite field we can use Kunneth projectors for numerical motives (cf. § 2 of [**31**]; note that the projectors are functorial!) to replace $K^{\mathrm{b}}(\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{num}}\,\mathbb{Q})$ by the 'correct' category $K^{\mathrm{b}}(\mathcal{M})$ such that $\mathcal{M}$ is Tannakian, $H_i(P)$ (the ith homological component) is put in degree i. Moreover, the 'standard' conjectures assumed in Theorem 5.3 of [**31**] (Tate conjecture plus numerical equivalence coincides with rational equivalence) imply that t_{num} can be extended to an equivalence of $\mathrm{DM}_{\mathrm{gm}}\,\mathbb{Q}$ with the motivic t-category of Theorem 5.3 of [**31**] (see also Theorem 56 of [**24**]). The relation between different conjectures is discussed in 1.6.5 of [**24**].

8.4. An application: independence on l for traces of open correspondences

Our formalism easily implies that for any $X \in \mathrm{Obj}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and any $f \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(X, X)$ the trace of the map f^* induced on the (rational) l-adic étale cohomology $H(X)$ does not depend on l. This gives a considerable generalization of Theorem 3.3 of [**5**] (see below).

Now we formulate the main statement more precisely. In this subsection we do not demand the characteristic of k to be 0.

Proposition 8.4.1. *Let H denote any of rational l-adic étale cohomology theories or the singular cohomology corresponding to any embedding of k into $\mathbb{C}$ (in characteristic 0). Denote by $\mathrm{Tr}\, f^*_H$ the sum $(-1)^i\,\mathrm{Tr}\, f^*_{H^i(X)}$.*

Then $\mathrm{Tr}\, f^*_H$ does not depend on the choice of H.

Proof. This statement is well known for Chow motives. Now we reduce everything to this case.

We consider the weight spectral sequence S for $H(X)$ (see § 7.3 and part (2) of Remark 7.3.1). Recall that S is functorial in H; it is functorial in X starting from E_1. Hence any f induces a certain endomorphism f^*_S of S starting from E_0 which is uniquely determined by f starting from E_1. To construct f^*_S one could also use the spectral sequence T described in Remark 7.4.4. Hence to reduce the statement to the case of Chow motives it suffices to apply the following formula (that follows immediately from the general spectral sequence formalism):

$$\mathrm{Tr}\, f^*_H = \sum_{i,j} -1^{i+j} f^*_{H^i(P^j)}; \tag{18}$$

see III.7.4 c of [**13**]. □

Note that, as we have shown above, these rational arguments also work in finite characteristic.

Now, Theorem 3.3 of [**5**] states the independence from the choice of H of traces of maps induced by *open correspondences* on the cohomology of $U \in \mathrm{SmVar}$ with compact support. Open correspondences and the corresponding cohomology maps were described in Definition 3.1 of [**5**]. We will not recall this definition here; we will only note that the map Γ_* of [**5**] comes from a certain $f \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(M^{\mathrm{c}}_{\mathrm{gm}}(U), M^{\mathrm{c}}_{\mathrm{gm}}(U))$. Indeed, Γ_* was defined

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as a composition $(p_1)_* \circ (p_2)^*$ (in the notation of [**5**]). One can define the morphism of motives corresponding to $(p_2)^*$ using the functoriality of M^c_{gm} with respect to proper morphisms (see § 4.1 of [**36**]). To define the morphism corresponding to $(p_1)_*$ one should use Corollary 4.2.4 of [**5**].

It follows that we have generalized Theorem 3.3 of [**5**] considerably. Indeed, we do not demand U to be a complement of a smooth projective variety by a strict normal crossing divisor. This is especially important in the characteristic p case. Besides, it seems that open correspondences of the type considered in Theorem 2.8 of [**5**] also give endomorphisms of $M^c_{gm}(U)$.

Remark 8.4.2.

(1) Certainly, exactly the same argument proves the independence from H of $n_\lambda(H) = (-1)^i n_\lambda f^*_{H^i(X)}$; here $n_\lambda f^*_{(H^i(X))}$ for a fixed algebraic number λ denotes the algebraic multiplicity of the eigenvalue λ for the operator $f^*_{H^i(X)}$.

(2) In fact, all these statements follow easily from the following statement: the appropriately defined group $K_0(\operatorname{End}\mathrm{Chow}^{\mathrm{eff}})$ surjects onto $K_0(\operatorname{End}\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}})$; see § 5.4 of [**7**] for details.

8.5. Adding kernels of projectors to J

The description of the derived category of Tate motives would be nicer if $\mathbb{Z}(i)$, $i \geqslant 0$, would be motives of length 0 (see part (3) of Remark 8.1.2 and § 6.2). To this end we show that one can easily add direct summands of objects to J.

Indeed, if the cohomology of a complex of sheaves coincides with its hypercohomology, the same it true for any direct summand of this complex. Therefore, if D, D' are direct summands (in $C^-(\mathrm{Shv}(\mathrm{SmCor}))$) of $C(P)$, $C(P')$ respectively, $P, P' \in \mathrm{SmPrVar}$, then the natural analogue of Proposition 1.3.1 will be valid for $\mathrm{DM}^{\mathrm{eff}}_-(p(D), p(D'))$. Hence any such D can be naturally added to J; then an analogue of Theorem 3.1.1 would be valid with DM^{s} extended by adding the corresponding direct summands of objects.

In particular, let $P \subset Q \in \mathrm{SmPrVar}$ and let there exist a section $j : Q \to P$ of the inclusion. Then one can add the cone of j to J (note that it is isomorphic to $m^{-1}(M^c_{gm}(Q-P)))$.

For example, one can present $[\mathbb{P}^1]$ as $[\mathrm{pt}] \oplus [\mathbb{Z}(1)[2]]$. Hence for $P \in \mathrm{SmPrVar}$, $i \geqslant 0$, one could include $[P][2i](i)$ into J (cf. the reasoning in the proof of part (1) of Theorem 6.2.1 and part (3) of Remark 8.1.2).

Yet this method certainly cannot give a (canonical) differential enhancement of the whole $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (or $\mathrm{DM}^{\mathrm{eff}\prime}_{\mathrm{gm}}$). To obtain an enhancement for it one could apply the 'infinite diagram' method of Hanamura (see § 2 of [**17**]).

8.6. The functors m_N

We consider the functor $J_N \to B^-(\mathrm{Shv}(\mathrm{SmCor}))$ that maps $[P]$ into $SC^N(P)$. Here $SC^{Ni}(P)$ is the Nisnevich sheafification of the presheaf $C^{Ni}(P)(-)$ (they coincide for

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$i \neq -N$). We consider the corresponding functor $h_N : \mathfrak{H}_N \to K^-(\mathrm{Shv}(\mathrm{SmCor}))$ and $m_N = p \circ h_N$. Note that for any $X \in \mathrm{Obj}\,\mathfrak{H}_N$ we have $m_N(X) \in \mathrm{DM}_-^{\mathrm{eff}}$.

By Proposition 2.7.2 (1) the natural morphisms $C_P \to SC^N(P)$ in $K^-(\mathrm{Shv}(\mathrm{SmCor}))$ induce a transformation of functors $\mathrm{Tr}_N : m \to m_N$. Besides Tr_N for any $X \in \mathfrak{H}$ is induced by a canonical map in $K^-(\mathrm{Shv}(\mathrm{SmCor}))$.

It seems that no nice analogue of part (3) of Theorem 6.2.1 is valid for m_N. Yet for low-dimensional varieties m_N coincides with m.

Proposition 8.6.1. *Suppose that the Beilinson–Soulé vanishing conjecture holds over k. Then $m_{2n}(X) \cong m(X)$ in $\mathrm{DM}_-^{\mathrm{eff}}$ if the dimension of X is less than or equal to n.*

Proof. We check by induction on n that Tr_{2n} is the identity for $m(X)$. This is obviously valid for $n = 0$.

Applying the same reasoning as in the proof of part (1) of Theorem 6.2.1 we obtain that it is sufficient to prove the assertion for smooth projective X of dimension less than or equal to n.

It is sufficient to check that $\mathrm{DM}_-^{\mathrm{eff}}(M_{\mathrm{gm}}(Y)[N], M_{\mathrm{gm}}(X)) = 0$ for any $Y \in \mathrm{SmVar}$, $N \geqslant 2n$.

By Theorem 4.3.2 of [**36**] if the dimension of X equals n then we have

$$\underline{\mathrm{Hom}}_{\mathrm{DM}_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X), \mathbb{Z}(n)[2n]) \cong M_{\mathrm{gm}}(X).$$

Hence

$$\begin{aligned}\mathrm{DM}_-^{\mathrm{eff}}(M_{\mathrm{gm}}(Y)[N], M_{\mathrm{gm}}(X)) &= \mathrm{DM}_-^{\mathrm{eff}}(M_{\mathrm{gm}}(Y \times X)[N], \mathbb{Z}(n)[2n]) \\ &= \mathrm{DM}_-^{\mathrm{eff}}(M_{\mathrm{gm}}(Y \times X)[N-2n], \mathbb{Z}(n)).\end{aligned}$$

It remains to note that by the Beilinson–Soulé conjecture,

$$\mathrm{DM}_-^{\mathrm{eff}}(M_{\mathrm{gm}}(Y \times X), \mathbb{Z}(n)[i]) = 0 \quad \text{for } i < 0.$$

□

Remark 8.6.2.

(1) We also see that for k of characteristic 0 and any N there exist $P, Q \in \mathrm{SmPrVar}$ such that $\mathfrak{H}(P[N], [Q]) \neq 0$. Hence none of t_N and m_N are full functors.

(2) It could be also easily checked that Tr_{2n} being identical for all X of dimension less than or equal to n, $n \in \mathbb{Z}$, implies the Beilinson–Soulé conjecture.

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