AGGREGATION AND CAPITAL ALLOCATION FORMULAS FOR BIVARIATE DISTRIBUTIONS

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Cossette, Marceau, and Perreault derived formulas for aggregation and capital allocation based on risks following two bivariate exponential distributions. Here, we derive formulas for aggregation and capital allocation for 18 mostly commonly known families of bivariate distributions. This collection of formulas could be a useful reference for financial risk management.

Keywords: closed form, risk, sum

1. INTRODUCTION

Results related to the sum of dependent risks are of interest in the calculation of the overall capital charge for a portfolio of risks, the evaluation and analysis of risk measures for decision making, and strategic planning require the knowledge of the cumulative distribution function of the sum of dependent random variables (Cossette, Marceau, and Perreault [11]). Risk measures like Value at Risk and Tail-Value at Risk can be used for these purposes.

In recent years, several closed-form expressions for the distribution of aggregate risks, its TVaR and TVaR-based allocations have been developed, based on an allocation method due to Tasche [21]. These expressions are based on a given joint distribution between the components of a portfolio. The joint distributions considered so far are the: multivariate normal distribution (Panjer [20]); multivariate elliptical distributions (Landsman and Valdez [16]; Dhaene et al. [13]; multivariate gamma distribution (Furman and Landsman [14]); multivariate Tweedie distribution (Furman and Landsman [15]); multivariate Pareto distribution (Chiragiev and Landsman [8]); Farlie-Gumbel-Morgenstern copula (Barges, Cossette and Marceau [3]); Farlie–Gumbel–Morgenstern copula with mixed Erlang marginals (Cossette et al. [9]); multivariate compound distributions (Cossette, Mailhot, and Marceau [10]); bivariate exponential and bivariate mixed Erlang distributions (Cossette, Marceau, and Perreault [11]).

The aim of this note is to derive expressions for the distribution of aggregate risks, its TVaR and TVaR based allocations for a comprehensive collection of bivariate distributions. We consider 18 families of bivariate distributions each defined on $(0, \infty) \times (0, \infty)$ or $(\beta, \infty) \times (\beta, \infty)$ for some $\beta > 0$. They include mixtures of independent exponential distributions, Crovelli [12] bivariate exponential distribution, Lawrance and Lewis' [17] bivariate

exponential distribution, Block and Basu [5] bivariate exponential distribution, Arnold and Strauss [1] bivariate exponential distribution, Becker and Roux [4] bivariate gamma distribution, Cheriyan [7] bivariate gamma distribution, bivariate Liouville distributions, bivariate equilibrium distributions due to Unnikrishnan Nair and Sankaran [22], Chacko and Thomas [6] bivariate Pareto distribution, bivariate Pareto distribution in Eq. (10.68) of Balakrishnan and Lai [2], bivariate Pareto distribution with equal scale parameters, generalized bivariate Pareto distribution, Lee and Cha [18] bivariate distribution and truncated bivariate normal distribution. We have not considered the bivariate normal or other distributions defined over $(-\infty, \infty) \times (-\infty, \infty)$, as they have already been considered by others.

2. MATHEMATICAL NOTATION

Let (X, Y) be non-negative continuous risks with joint probability density function $f(x, y)$. Let $S = X + Y$ denote the aggregated risk. We are interested in: the probability density function of S given by

$$f_S(s) = \int_0^s f(y, s - y)dy;$$

(1)

the cumulative distribution function of S given by

$$F_S(s) = \int_0^s f_S(y)dy;$$

(2)

the truncated expectation of S given by

$$E[S1_{\{S>b\}}] = \int_b^\infty s f_S(s)ds;$$

(3)

the contribution of each risk to the aggregated risk given by

$$E[X1_{\{S>b\}}] = \int_b^\infty g_{X,\,S}(s)ds,$$

(4)

where

$$g_{X,\,S}(s) = \int_0^s x f(x, s - x)dx.$$

(5)

We derive expressions for (1)–(5) for 18 bivariate distributions.

The derived expressions given in Section 3 involve several special functions, including the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} \exp{(-t)}dt;$$

the beta function defined by

$$B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1}dt;$$

the incomplete beta function defined by

$$B_x(a, b) = \int_0^x t^{a-1}(1 - t)^{b-1}dt;$$

the standard normal distribution function defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt;$$

the Gauss hypergeometric function defined by

$$_2F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!};$$

the Appell hypergeometric function of the first kind of two variables defined by

$$F_1(a,b,c,d;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n x^m y^n}{(d)_{m+n} m! n!};$$

and the degenerate hypergeometric series of two variables defined by

$$\Phi_1(a,b,c,x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_n x^m y^n}{(c)_{m+n} m! n!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial.

In-built routines for computing these special functions are available in packages like Matlab, Maple and Mathematica. For example, Beta[a, b], Beta[x, a, b], Hypergeometric2F1[a, b, c, x] and AppellF1[a, b, c, d, x, y] in Mathematica compute the beta, incomplete beta, Gauss hypergeometric and Appell hypergeometric functions. Mathematica allows for arbitrary precision, so the accuracy of computations is not an issue.

3. THE COLLECTION

Here, we give expressions for (1)–(5) for 18 bivariate distributions. Their derivations are routine and are not presented. The detailed derivations can be obtained from the corresponding author.

Mixtures of independent exponential distributions:

$$f(x,y) = \sum_{k=1}^{m} C_k \exp\left(-\alpha_k x - \beta_k y\right),$$

$$f_S(s) = \sum_{k=1}^{m} \frac{C_k}{\beta_k - \alpha_k} \left[\exp\left(-\alpha_k s\right) - \exp\left(-\beta_k s\right)\right],$$

$$F_S(s) = \sum_{k=1}^{m} \frac{C_k}{\beta_k - \alpha_k} \left[\frac{1 - \exp\left(-\alpha_k s\right)}{\alpha_k} - \frac{1 - \exp\left(-\beta_k s\right)}{\beta_k}\right],$$

$$E\left[S1_{\{S>b\}}\right] = \sum_{k=1}^{m} \frac{C_k}{\beta_k - \alpha_k} \left[\frac{(b\alpha_k + 1)\exp\left(-b\alpha_k\right)}{\alpha_k^2} - \frac{(b\beta_k + 1)\exp\left(-b\beta_k\right)}{\beta_k^2}\right],$$

$$g_{X,\,S}(s) = \sum_{k=1}^{m} C_k \exp\left(-\beta_k s\right) \left[\frac{1 - \exp\left(\beta_k s - \alpha_k s\right)}{(\alpha_k - \beta_k)^2} - \frac{s\exp\left(\beta_k s - \alpha_k s\right)}{\alpha_k - \beta_k}\right],$$

$$E\left[X1_{\{S>b\}}\right] = \sum_{k=1}^{m} C_k \left[-\frac{(\alpha_k b + 1)\exp\left(-\alpha_k b\right)}{\alpha_k^2 (\alpha_k - \beta_k)} + \frac{\exp\left(-\beta_k b\right)}{\beta_k (\alpha_k - \beta_k)^2} - \frac{\exp\left(-\alpha_k b\right)}{\alpha_k (\alpha_k - \beta_k)^2}\right]$$

for $\alpha_k > 0$, $\beta_k > 0$, $x > 0$ and $y > 0$.

Crovelli [12] bivariate exponential distribution:

$$f(x,y) = \begin{cases} \alpha\beta \exp(-\beta y)\left[1 - \exp(-\alpha x)\right], & \text{if } 0 \le \alpha x \le \beta y, \\ \alpha\beta \exp(-\beta x)\left[1 - \exp(-\beta y)\right], & \text{if } 0 \le \beta y \le \alpha x, \end{cases}$$

$$f_S(s) = (\alpha + \beta)\exp\left(-\frac{\alpha\beta s}{\alpha + \beta}\right) + \frac{\alpha^2}{\alpha - \beta}\exp(-\beta s) - \frac{\beta^2}{\alpha - \beta}\exp(-\alpha s),$$

$$F_S(s) = \frac{(\alpha + \beta)^2}{\alpha\beta}\left[1 - \exp\left(-\frac{\alpha\beta s}{\alpha + \beta}\right)\right] + \frac{\alpha^2}{\beta(\beta - \alpha)}\left[1 - \exp(-\beta s)\right]$$
$$- \frac{\beta^2}{\alpha(\beta - \alpha)}\left[1 - \exp(-\alpha s)\right],$$

$$E\left[S1_{\{S>b\}}\right] = \frac{(\alpha + \beta)^3}{(\alpha\beta)^2}\exp\left(-\frac{\alpha\beta b}{\alpha + \beta}\right)\left[1 + \frac{\alpha\beta b}{\alpha + \beta}\right] + \frac{\alpha^2}{\beta(\alpha - \beta)}\exp(-\beta b)$$
$$- \frac{\beta^2}{\alpha(\alpha - \beta)}\exp(-\alpha b),$$

$$g_{X,\,S}(s) = \frac{\beta^2}{\alpha - \beta}s\exp(-\alpha s) + \frac{\beta^2(2\alpha - \beta)}{\alpha(\alpha - \beta)^2}\exp(-\alpha s) + \frac{\alpha^2(\alpha - 2\beta)}{\beta(\alpha - \beta)^2}\exp(-\beta s)$$
$$+ \beta s\exp\left(-\frac{\alpha\beta s}{\alpha + \beta}\right) + \frac{-\alpha^4 + 2\alpha^3 + \beta^4 - 2\alpha\beta^3}{\alpha\beta(\alpha - \beta)^2}\exp\left(-\frac{\alpha\beta s}{\alpha + \beta}\right),$$

$$E\left[X1_{\{S>b\}}\right] = \frac{\beta^2}{\alpha^2(\alpha - \beta)}(1 + \alpha b)\exp(-\alpha b) + \frac{\beta^2(2\alpha - \beta)}{\alpha^2(\alpha - \beta)^2}\exp(-\alpha b)$$
$$+ \frac{\alpha^2(\alpha - 2\beta)}{\beta^2(\alpha - \beta)^2}\exp(-\beta b) + \frac{(\alpha + \beta)^2}{\alpha^2\beta}\left(1 + \frac{\alpha\beta b}{\alpha + \beta}\right)\exp\left(-\frac{\alpha\beta b}{\alpha + \beta}\right)$$
$$+ \frac{(\alpha + \beta)\left(-\alpha^4 + 2\alpha^3 + \beta^4 - 2\alpha\beta^3\right)}{(\alpha\beta)^2(\alpha - \beta)^2}\exp\left(-\frac{\alpha\beta b}{\alpha + \beta}\right)$$

for $\alpha > 0$, $\beta > 0$, $x > 0$ and $y > 0$.

Lawrance and Lewis' [17] bivariate exponential distribution:

$$f(x,y) = I\left\{\beta y < x < y/\beta\right\}\frac{1}{1 + \beta}\exp\left(-\frac{x + y}{1 + \beta}\right) + \frac{1}{\beta}\exp\left(-\frac{x + y}{\beta}\right),$$

$$f_S(s) = \frac{(1 - \beta)s}{(1 + \beta)^2}\exp\left(-\frac{s}{1 + \beta}\right) + \frac{s}{\beta}\exp\left(-\frac{s}{\beta}\right),$$

$$F_S(s) = \frac{1 - \beta}{(1 + \beta)^2}\left[(1 + \beta)^2 - (1 + \beta)^2\exp\left(-\frac{s}{1 + \beta}\right) - (1 + \beta)s\exp\left(-\frac{s}{1 + \beta}\right)\right]$$
$$+ \frac{1}{\beta}\left[\beta^2 - \beta^2\exp\left(-\frac{s}{\beta}\right) - \beta s\exp\left(-\frac{s}{\beta}\right)\right],$$

$$E\left[S1_{\{S>b\}}\right] = \frac{(1 - \beta)(1 + \beta + b)}{1 + \beta}\exp\left(-\frac{b}{1 + \beta}\right) + (\beta + b)\exp\left(-\frac{b}{\beta}\right),$$

$$g_{X,\,S}(s) = \frac{\left(1 - \beta^2\right)s^2}{2(1+\beta)^3}\exp\left(-\frac{s}{1+\beta}\right) + \frac{s^2}{2\beta}\exp\left(-\frac{s}{\beta}\right),$$

$$E\left[X1_{\{S>b\}}\right] = \left(1-\beta^2\right)\left[1 + \frac{b}{1+\beta} + \frac{b^2}{2(1+\beta)^2}\right]\exp\left(-\frac{b}{1+\beta}\right)$$

$$+ \left(\beta^2 + b\beta + \frac{b^2}{2}\right)\exp\left(-\frac{b}{\beta}\right)$$

for $0 < \beta \le 1$, $x > 0$ and $y > 0$.

Block and Basu [5] bivariate exponential distribution:

$$f(x,y) = \begin{cases} C\exp\left(-\alpha x - \beta y\right), & \text{if } x > y, \\ C\exp\left(-\gamma x - \delta y\right), & \text{if } x \le y, \end{cases}$$

$$f_S(s) = \frac{C}{\beta - \alpha}\left[\exp(-\alpha s) - \exp\left(-\frac{(\alpha+\beta)s}{2}\right)\right]$$

$$- \frac{D}{\delta - \gamma}\left[\exp(-\delta s) - \exp\left(-\frac{(\delta+\gamma)s}{2}\right)\right],$$

$$F_S(s) = \frac{C}{\beta - \alpha}\left[\frac{1 - \exp(-\alpha s)}{\alpha} - \frac{2}{\alpha+\beta}\left(1 - \exp\left(-\frac{(\alpha+\beta)s}{2}\right)\right)\right]$$

$$- \frac{D}{\delta - \gamma}\left[\frac{2}{\delta+\gamma}\left(1 - \exp\left(-\frac{(\delta+\gamma s)}{2}\right)\right) - \frac{1 - \exp(-\delta s)}{\delta}\right],$$

$$E\left[S1_{\{S>b\}}\right] = \frac{C}{\beta - \alpha}\left[\frac{(b\alpha+1)\exp(-b\alpha)}{\alpha^2} - \frac{4}{(\alpha+\beta)^2}\left(b\frac{\alpha+\beta}{2}+1\right)\exp\left(-b\frac{\alpha+\beta}{2}\right)\right]$$

$$- \frac{D}{\delta - \gamma}\left[\frac{4}{(\delta+\gamma)^2}\left(b\frac{\delta+\gamma}{2}+1\right)\exp\left(-b\frac{\delta+\gamma}{2}\right) - \frac{(b\delta+1)\exp(-b\delta)}{\delta^2}\right],$$

$$g_{X,\,S}(s) = C\left[\frac{\exp(-\beta s) - \exp(-\alpha s)}{(\alpha-\beta)^2} - \frac{s\exp(-\alpha s)}{\alpha-\beta}\right]$$

$$+ D\left[\frac{\exp(-\delta s) - \exp\left(-(\delta+\gamma)s/2\right)}{(\delta-\gamma)^2} - \frac{s\exp\left(-(\delta+\gamma)s/2\right)}{2(\delta-\gamma)}\right]$$

$$- C\left[\frac{\exp(-\beta s) - \exp\left(-\beta s - (\gamma-\delta)s/2\right)}{(\delta-\gamma)^2} - \frac{s\exp\left(-\beta s - (\gamma-\delta)s/2\right)}{2(\delta-\gamma)}\right],$$

$$E\left[X1_{\{S>b\}}\right] = C\left[-\frac{(b\alpha+1)\exp(-b\alpha)}{\alpha^2(\alpha-\beta)} + \frac{\exp(-b\beta)}{\beta(\alpha-\beta)^2} - \frac{\exp(-b\alpha)}{\alpha(\alpha-\beta)^2}\right]$$

$$+ D\left[\frac{\exp(-b\delta)}{\delta(\delta-\gamma)^2} - \frac{2\exp\left(-(\delta+\gamma)b/2\right)}{(\delta-\gamma)^2(\delta+\gamma)} - \frac{(2 + (\delta+\gamma)b)\exp\left(-(\delta+\gamma)b/2\right)}{(\delta+\gamma)^2(\delta-\gamma)}\right]$$

$$- C\left[\frac{\exp(-b\beta)}{\beta(\delta-\gamma)^2} - \frac{\exp\left(-b\beta - (\gamma-\delta)b/2\right)}{(\delta-\gamma)^2\left(\beta + (\gamma-\delta)/2\right)}\right]$$

$$+ C\frac{(2 + 2b\beta + b(\gamma-\delta))\exp\left(-b\beta - (\gamma-\delta)b/2\right)}{(\delta-\gamma)\left(2\beta+\gamma-\delta\right)^2}$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $x > 0$ and $y > 0$.

Arnold and Strauss [1] bivariate exponential distribution:

$$f(x,y) = C \exp\left(-\gamma x - \delta y - \theta x y\right),$$

$$f_S(s) = C\sqrt{\pi} p^{-1/2} \exp\left(-\delta s + \frac{q^2}{4p}\right) \left[\Phi\left(\sqrt{2ps} - (2p)^{-1/2}q\right) - \Phi\left(-(2p)^{-1/2}q\right)\right],$$

$$g_{X,\,S}(s) = 2^{-1} C\sqrt{\pi} p^{-3/2} q \exp\left(-\delta s + \frac{q^2}{4p}\right) \left[\Phi\left(\sqrt{2ps} - (2p)^{-1/2}q\right) - \Phi\left(-(2p)^{-1/2}q\right)\right]$$

$$- 2^{-1} C p^{-1} \exp\left(-\delta s + \frac{q^2}{4p}\right) \left[\exp\left(-ps^2 + qs\right) - 1\right]$$

for $-\infty < \gamma < \infty$, $-\infty < \delta < \infty$, $-\infty < \theta < \infty$, $x > 0$ and $y > 0$, where $p = -\theta$ and $q = \delta - \theta s$.

Becker and Roux [4] bivariate gamma distribution:

$$f(x,y) = \begin{cases} C x^{\alpha-1}(x+py)^{\beta-1} \exp(-\gamma x - \delta y), & \text{if } x < y, \\ C x^{c-1}(x+qy)^{d-1} \exp(-ex - fy), & \text{if } x < y, \end{cases}$$

$$f_S(s) = 2^{-\alpha}\alpha^{-1} C s^{\alpha+\beta-1} \exp(-\delta s)\Phi_1\left(\alpha, 1-\beta, 1+\alpha, (\delta-\gamma)\frac{s}{2}, -\frac{1-p}{2p}\right)$$

$$+ q^{d-1} c^{-1} D s^{c+d-1} \exp(-fs)\Phi_1\left(c, 1-d, 1+c, (f-e)s, \frac{q-1}{q}\right)$$

$$- 2^{-c} q^{d-1} D s^{c+d-1} \exp(-fs)\Phi_1\left(c, 1-d, 1+c, (f-e)\frac{s}{2}, \frac{q-1}{2q}\right),$$

$$g_{X,\,S}(s) = 2^{-\alpha+1}(\alpha+1)^{-1} C s^{\alpha+\beta} \exp(-\delta s)\Phi_1\left(\alpha+1, 1-\beta, 2+\alpha, (\delta-\gamma)\frac{s}{2}, -\frac{1-p}{2p}\right)$$

$$+ q^{d-1}(1+c)^{-1} D s^{c+d} \exp(-fs)\Phi_1\left(1+c, 1-d, 2+c, (f-e)s, \frac{q-1}{q}\right)$$

$$- 2^{-1-c} q^{d-1} D s^{c+d} \exp(-fs)\Phi_1\left(1+c, 1-d, 2+c, (f-e)\frac{s}{2}, \frac{q-1}{2q}\right)$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $c > 0$, $d > 0$, $e > 0$, $f > 0$, $p > 0$, $q > 0$, $x > 0$ and $y > 0$.

Cheriyan [7] bivariate gamma distribution in Eq. (8.31) of Balakrishnan and Lai [2]:

$$f(x,y) = C \exp(-x-y) \int_0^{\min(x,y)} (x-z)^{\theta_1}(y-z)^{\theta_2-1} z^{\theta_2-1} \exp(z) dz,$$

$$f_S(s) = CB\left(\theta_1+1, \theta_2\right) \exp(-s) \int_0^{s/2} x^{\theta_1+\theta_3}(s-x)^{\theta_2-1}\Phi_1$$

$$\times \left(\theta_3, 1-\theta_2, \theta_1+\theta_3+1, x, \frac{x}{1-x}\right) dx + CB\left(\theta_2, \theta_3\right) \exp(-s)$$

$$\times \int_{s/2}^1 x^{\theta_1}(s-x)^{\theta_2+\theta_3-1}\Phi_1\left(\theta_3, -\theta_1, \theta_2+\theta_3, s-x, \frac{s-x}{x}\right) dx,$$

$$g_{X,\,S}(s) = C B\left(\theta_1+1,\theta_2\right)\exp(-s)\int_0^{s/2} x^{\theta_1+\theta_3+1}(s-x)^{\theta_2-1}\Phi_1$$

$$\times\left(\theta_3,1-\theta_2,\theta_1+\theta_3+1,x,\frac{x}{1-x}\right)dx + CB\left(\theta_2,\theta_3\right)\exp(-s)$$

$$\times\int_{s/2}^1 x^{\theta_1+1}(s-x)^{\theta_2+\theta_3-1}\Phi_1\left(\theta_3,-\theta_1,\theta_2+\theta_3,s-x,\frac{s-x}{x}\right)dx$$

for $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$, $x > 0$ and $y > 0$.

Mardia [19] bivariate distribution in Eq. (5.77) of Balakrishnan and Lai [2]:

$$f(x,y) = C\left(\alpha x+1\right)^p\left(\beta x+1\right)^q\left(\gamma x+\delta y+1\right)^r,$$

$$f_S(s) = C\sum_{i=0}^p\sum_{j=0}^q\sum_{k=0}^r\sum_{m=0}^k\binom{p}{i}\binom{q}{j}\binom{r}{k}\binom{r}{m}\alpha^i\beta^j\delta^m\gamma^{k-m}$$

$$\times B\left(i+k-m+1,j+m+1\right)s^{i+j+k+1},$$

$$F_S(s) = C\sum_{i=0}^p\sum_{j=0}^q\sum_{k=0}^r\sum_{m=0}^k\binom{p}{i}\binom{q}{j}\binom{r}{k}\binom{r}{m}\alpha^i\beta^j\delta^m\gamma^{k-m}$$

$$\times B\left(i+k-m+1,j+m+1\right)\frac{s^{i+j+k+2}}{i+j+k+2},$$

$$g_{X,\,S}(s) = C\sum_{i=0}^p\sum_{j=0}^q\sum_{k=0}^r\sum_{m=0}^k\binom{p}{i}\binom{q}{j}\binom{r}{k}\binom{r}{m}\alpha^i\beta^j\delta^m\gamma^{k-m}$$

$$\times B\left(i+k-m+2,j+m+1\right)s^{i+j+k+2}$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $p > 0$, $q > 0$, $r > 0$, $x > 0$ and $y > 0$, provided that p, q and r are integers.

Mardia [19] bivariate gamma distribution in Eq. (5.81) of Balakrishnan and Lai [2]:

$$f(x,y) = C\left(\alpha x+1\right)^p\left(\beta x+\gamma y+1\right)^q\exp(-ry),$$

$$f_S(s) = C\sum_{i=0}^p\sum_{j=0}^q\sum_{k=0}^{i+j}\binom{p}{i}\binom{q}{j}\binom{i+j}{k}\alpha^i(\beta-\gamma)^j(-1)^k k!r^{-k-1}s^{i+j-k}(1+\gamma s)^{q-j}$$

$$-C\sum_{i=0}^p\sum_{j=0}^q\binom{p}{i}\binom{q}{j}\alpha^i(\beta-\gamma)^j(-1)^{i+j}(i+j)!r^{-i-j-1}(1+\gamma s)^{q-j}\exp(-rs),$$

$$F_S(s) = C\sum_{i=0}^p\sum_{j=0}^q\sum_{k=0}^{i+j}\sum_{m=0}^{q-j}\binom{p}{i}\binom{q}{j}\binom{i+j}{k}\binom{q-j}{m}\alpha^i(\beta-\gamma)^j\gamma^m(-1)^k k!r^{-k-1}$$

$$\times\frac{s^{i+j-k+m+1}}{i+j-k+m+1} - C\sum_{i=0}^p\sum_{j=0}^q\sum_{m=0}^{q-j}\binom{p}{i}\binom{q}{j}\binom{q-j}{m}$$

$$\times\alpha^i(\beta-\gamma)^j\gamma^m(-1)^{i+j}(i+j)!r^{-i-j-m-2}\gamma\left(m+1,rs\right),$$

$$g_{X,\,S}(s) = C \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{i+j} \binom{p}{i} \binom{q}{j} \binom{i+j+1}{k} \alpha^i (\beta - \gamma)^j (-1)^k k! r^{-k-1} s^{i+j-k+1} (1+\gamma s)^{q-j}$$

$$- C \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i} \binom{q}{j} \alpha^i (\beta - \gamma)^j (-1)^{i+j+1} (i+j+1)! r^{-i-j-2}$$

$$\times (1+\gamma s)^{q-j} \exp(-rs)$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $p > 0$, $q > 0$, $r > 0$, $x > 0$ and $y > 0$, provided that p and q are integers.

Bivariate Liouville distribution on page 202 of Balakrishnan and Lai [2]:

$$f(x,y) = C x^{\alpha-1} y^{\beta-1} g(x+y),$$

$$f_S(s) = C B(\alpha, \beta) s^{\alpha+\beta-1} g(s),$$

$$F_S(s) = C B(\alpha, \beta) \int_0^s u^{\alpha+\beta-1} g(u) du,$$

$$E\left[S 1_{\{S>b\}}\right] = C B(\alpha, \beta) \int_b^\infty u^{\alpha+\beta} g(u) du,$$

$$g_{X,\,S}(s) = C B(\alpha+1, \beta) s^{\alpha+\beta} g(s),$$

$$E\left[X 1_{\{S>b\}}\right] = C B(\alpha+1, \beta) \int_b^\infty u^{\alpha+\beta} g(u) du$$

for $\alpha > 0$, $\beta > 0$, $x > 0$ and $y > 0$.

Bivariate Liouville distribution in Eq. (9.46) of Balakrishnan and Lai [2]:

$$f(x,y) = C x^{\alpha-1} y^{\beta-1} (x+y)^{-\gamma} g(x+y),$$

$$f_S(s) = C B(\alpha, \beta) s^{\alpha+\beta-\gamma-1} g(s),$$

$$F_S(s) = C B(\alpha, \beta) \int_0^s u^{\alpha+\beta-\gamma-1} g(u) du,$$

$$E\left[S 1_{\{S>b\}}\right] = C B(\alpha, \beta) \int_b^\infty u^{\alpha+\beta-\gamma} g(u) du,$$

$$g_{X,\,S}(s) = C B(\alpha+1, \beta) s^{\alpha+\beta-\gamma} g(s),$$

$$E\left[X 1_{\{S>b\}}\right] = C B(\alpha+1, \beta) \int_b^\infty u^{\alpha+\beta-\gamma} g(u) du$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $x > 0$ and $y > 0$.

Bivariate equilibrium distributions due to Unnikrishnan Nair and Sankaran [22]:

$$f(x,y) = \mu^{-1} g(x+y),$$

$$f_S(s) = \mu^{-1} s g(s),$$

$$F_S(s) = \mu^{-1} \int_0^s u g(u) du,$$

$$E\left[S 1_{\{S>b\}}\right] = \mu^{-1} \int_b^\infty s^2 g(s) ds,$$

$$g_{X,\,S}(s) = 2^{-1}\mu^{-1}s^2 g\left(s\right),$$

$$E\left[X1_{\{S>b\}}\right] = 2^{-1}\mu^{-1}\int_b^\infty s^2 g\left(s\right) ds,$$

where $g(\cdot)$ is the probability density function of a univariate random variable X say and $\mu = E(X)$.

Chacko and Thomas [6] bivariate Pareto distribution:

$$f(x,y) = C(x+y)^{-\alpha},$$

$$f_S(s) = Cs^{-\alpha}(s - 2\beta),$$

$$F_S(s) = \frac{C}{2-\alpha}\left[s^{2-\alpha} - (2\beta)^{2-\alpha}\right] - \frac{2\beta C}{1-\alpha}\left[s^{1-\alpha} - (2\beta)^{1-\alpha}\right],$$

$$E\left[S1_{\{S>b\}}\right] = \frac{Cb^{3-\alpha}}{\alpha - 3} - \frac{2\beta Cb^{2-\alpha}}{\alpha - 2},$$

$$g_{X,\,S}(s) = 2^{-1}Cs^{1-\alpha}(s - 2\beta),$$

$$E\left[X1_{\{S>b\}}\right] = \frac{Cb^{3-\alpha}}{2(\alpha-3)} - \frac{\beta Cb^{2-\alpha}}{\alpha-2}$$

for $\alpha > 0$, $x > \beta > 0$ and $y > \beta > 0$.

Bivariate Pareto distribution in Eq. (10.68) of Balakrishnan and Lai [2]:

$$f(x,y) = C\left(a + x + y\right)^{-\beta},$$

$$f_S(s) = Cs\left(a + s\right)^{-\beta},$$

$$F_S(s) = C\left[\frac{(a+s)^{2-\beta}}{2-\beta} - \frac{a(a+s)^{1-\beta}}{1-\beta} - \frac{a^{2-\beta}}{2-\beta} + \frac{a^{2-\beta}}{1-\beta}\right],$$

$$E\left[S1_{\{S>b\}}\right] = C\left[\frac{(a+b)^{3-\beta}}{\beta-3} + \frac{2a(a+b)^{2-\beta}}{2-\beta} + \frac{a^2(a+b)^{1-\beta}}{\beta-1}\right],$$

$$g_{X,\,S}(s) = 2^{-1}Cs^2\left(a+s\right)^{-\beta},$$

$$E\left[X1_{\{S>b\}}\right] = 2^{-1}C\left[\frac{(a+b)^{3-\beta}}{\beta-3} + \frac{2a(a+b)^{2-\beta}}{2-\beta} + \frac{a^2(a+b)^{1-\beta}}{\beta-1}\right]$$

for $a > 0$, $\beta > 0$, $x > 0$ and $y > 0$.

Bivariate Pareto distribution with equal scale parameters:

$$f(x,y) = Cx^{\alpha-1}y^{\beta-1}(1 + px + py)^{-\gamma},$$

$$f_S(s) = CB(\alpha,\beta)s^{\alpha+\beta-1}(1 + ps)^{-\gamma},$$

$$F_S(s) = \frac{CB(\alpha,\beta)}{\alpha+\beta}s^{\alpha+\beta}{}_2F_1\left(\gamma, \alpha+\beta; \alpha+\beta+1; -ps\right),$$

$$E\left[S1_{\{S>b\}}\right] = CB(\alpha,\beta)p^{-\alpha-\beta-1}B_{1/(1+pb)}\left(\gamma - \alpha - \beta - 1, \alpha + \beta + 1\right),$$

$$g_{X,\,S}(s) = CB(\alpha+1,\beta)s^{\alpha+\beta}(1 + ps)^{-\gamma},$$

$$E\left[X1_{\{S>b\}}\right] = CB(\alpha+1,\beta)p^{-\alpha-\beta-1}B_{1/(1+pb)}\left(\gamma - \alpha - \beta - 1, \alpha + \beta + 1\right)$$

for $\alpha > 0$, $\beta > 0$, $p > 0$, $\gamma > 0$, $x > 0$ and $y > 0$.

Generalized bivariate Pareto distribution:

$$f(x, y) = \frac{C x^{\alpha-1} y^{\beta-1}}{(1 + px + qy + rxy)^{\delta}},$$

$$f_S(s) = CB(\alpha, \beta)(-r)^{-\delta}(uv)^{-\delta} s^{\alpha+\beta-1} F_1\left(\alpha, \delta, \delta, \alpha+\beta; \frac{s}{u}, \frac{s}{v}\right),$$

$$g_{X,\,S}(s) = CB(\alpha+1, \beta)(-r)^{-\delta}(uv)^{-\delta} s^{\alpha+\beta} F_1\left(\alpha+1, \delta, \delta, \alpha+\beta+1; \frac{s}{u}, \frac{s}{v}\right)$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $p > 0$, $q > 0$, $r > 0$, $x > 0$ and $y > 0$, where $u = \frac{p-q+rs}{2r} + \sqrt{\frac{1+qs}{r} + \frac{(p-q+rs)^2}{4r^2}}$ and $v = \frac{p-q+rs}{2r} - \sqrt{\frac{1+qs}{r} + \frac{(p-q+rs)^2}{4r^2}}$.

Lee and Cha [18] bivariate distribution:

$$f(x, y) = \begin{cases} C(x+\alpha)^{\gamma}(y+\beta)^{\delta}, & \text{if } x < y, \\ D(x+t)^{v}(y+u)^{w}, & \text{if } x > y, \end{cases}$$

$$f_S(s) = 3^{-1} C \alpha^{\gamma} s^{-1}(s+\beta)^{\delta} F_1\left(1, -\gamma, -\delta, 2; -\frac{s}{2\alpha}, \frac{s}{2(\alpha+\beta)}\right)$$

$$+ 3^{-1} D s^{-1} \left(\frac{s}{2}+t\right)^{v}\left(\frac{s}{2}+u\right)^{w} F_1\left(1, -v, -w, 2; -\frac{s}{s+2t}, \frac{s}{s+2u}\right),$$

$$g_{X,\,S}(s) = 3^{-1} C \alpha^{1+\gamma} s^{-1}(s+\beta)^{\delta} F_1\left(1, -1-\gamma, -\delta, 2; -\frac{s}{2\alpha}, \frac{s}{2(\alpha+\beta)}\right)$$

$$- 3^{-1} C \alpha^{1+\gamma} s^{-1}(s+\beta)^{\delta} F_1\left(1, -\gamma, -\delta, 2; -\frac{s}{2\alpha}, \frac{s}{2(\alpha+\beta)}\right)$$

$$+ 3^{-1} D s^{-1} \left(\frac{s}{2}+t\right)^{1+v}\left(\frac{s}{2}+u\right)^{w} F_1\left(1, -1-v, -w, 2; -\frac{s}{s+2t}, \frac{s}{s+2u}\right)$$

$$- 3^{-1} D t s^{-1} \left(\frac{s}{2}+t\right)^{v}\left(\frac{s}{2}+u\right)^{w} F_1\left(1, -v, -w, 2; -\frac{s}{s+2t}, \frac{s}{s+2u}\right)$$

for $\alpha > 0$, $\beta > 0$, $t > 0$, $u > 0$, $\delta < 0$, $v < 0$, $x > 0$ and $y > 0$.

Truncated bivariate normal distribution:

$$f(x, y) = C \exp\left(-\alpha x^2 - \beta y^2 - \gamma x - \delta y - \theta xy\right),$$

$$f_S(s) = C\sqrt{\pi} p^{-1/2} \exp\left(-\beta s^2 - \delta s + \frac{q^2}{4p}\right)\left[\Phi\left(\sqrt{2p}s - (2p)^{-1/2}q\right) - \Phi\left(-(2p)^{-1/2}q\right)\right],$$

$$g_{X,\,S}(s) = 2^{-1} C\sqrt{\pi} p^{-3/2} q \exp\left(-\beta s^2 - \delta s + \frac{q^2}{4p}\right)$$

$$\times \left[\Phi\left(\sqrt{2p}s - (2p)^{-1/2}q\right) - \Phi\left(-(2p)^{-1/2}q\right)\right]$$

$$- 2^{-1} C p^{-1} \exp\left(-\beta s^2 - \delta s + \frac{q^2}{4p}\right)\left[\exp\left(-ps^2 + qs\right) - 1\right]$$

for $\alpha > 0$, $\beta > 0$, $-\infty < \gamma < \infty$, $-\infty < \delta < \infty$, $-\infty < \theta < \infty$, $x > 0$ and $y > 0$, where $p = \alpha + \beta - \theta$ and $q = 2\beta s - \gamma + \delta - \theta s$.

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