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PROPERTIES OF SECOND-ORDER REGULAR VARIATION AND EXPANSIONS FOR RISK CONCENTRATION

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The purpose of this study is two-fold. First, we investigate further properties of the second-order regular variation (2RV). These properties include the preservation properties of 2RV under the composition operation and the generalized inverse transform, among others. Second, we derive second-order expansions of the tail probabilities of convolutions of non-independent and identically distributed (i.i.d.) heavy-tail random variables, and establish second-order expansions of risk concentration under mild assumptions. The main results extend some ones in the literature from the i.i.d. case to non-i.i.d. case.

1. INTRODUCTION

Second-order regular variation (2RV) was originally studied in the extreme value theory and was used to study the speed of convergence of certain estimators; see [7] and [8]. The formal definition of 2RV will be given in Section 2. For a general theory of 2RV, we refer to [6]. 2RV provides a nice theoretical platform for studying second-order approximation of limiting properties. For example, Geluk et al. [11] discussed the equivalence of 2RV and asymptotic normality of Hill's estimator. Lin et al. [13] obtained the convergence rates of the distribution of the largest-order statistic under weaker conditions by using the properties of 2RV. Degen and Embrechts [4] highlighted the importance of the 2RV tail behavior of the underlying loss severity models in exploiting extreme value theory (EVT)-based estimation methodologies of high

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quantiles. Degen et al. [5] derived second-order approximations for the risk concentration and the diversification benefit by using the theory of 2RV under the assumption that the underlying risk variables are independent and identically distributed (i.i.d.). The definition of risk concentration is given by (4.2). Hua and Joe [12] studied some interesting properties of 2RV, and conducted asymptotic analysis on conditional tail expectation under the condition of 2RV.

The purpose of this study is two-fold. First, we investigate further properties of 2RV. These properties include the preservation properties of 2RV under the composition operation and the generalized inverse transform, among others. Second, we establish second-order approximations of risk concentration for non-i.i.d. risk variables. To the end of the second purpose, we derive second-order expansions of the tail probabilities of convolutions of non-i.i.d. heavy-tail random variables under mild assumptions. The main results extend some ones in [1] and [5] from the i.i.d. to non-i.i.d. case.

The whole study is organized as follows. Properties of the 2RV are presented in Section 2. The second-order approximations of tail probabilities of convolutions and of risk concentration are given in Sections 3 and 4, respectively.

Throughout, the terms "increasing" and "decreasing" mean "non-decreasing" and "non-increasing," respectively, and the notation "$\sim$" means asymptotic equivalence; that is, for functions g and h,

$$g(x) \sim h(x),\ x \to x_0 \iff \lim_{x\to x_0} \frac{g(x)}{h(x)} = 1.$$

2. PROPERTIES OF SECOND-ORDER REGULAR VARIATION

In this section, we will investigate further properties of 2RV. Although some properties of 2RV were mentioned in the literature, we state them here for completeness and also for easy reference, and give different proofs from those in the literature. First, we recall the notions of (first-order) regular variation and 2RV.

Standard references on regular variation are [2], [9], and [14].

DEFINITION 2.1: *A measurable function $h : \Re_+ \to \Re_+$ is said to be regularly varying at infinity with index $\alpha \in \Re\backslash\{0\}$, written $h \in \mathrm{RV}_\alpha$, if, for any $x > 0$,*

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\alpha. \tag{2.1}$$

If (2.1) *holds with $\alpha = 0$ for any $x > 0$, then h is said to be slowly varying at infinity and written as $h \in \mathrm{RV}_0$. If* (2.1) *holds with $\alpha = -\infty$ for any $x > 0$, then h is said to be rapidly varying at infinity and written as $h \in \mathrm{RV}_{-\infty}$.*

Similarly, one can define regular variation at 0^+ replacing $t \to \infty$ in (2.1) *by $t \to 0^+$. If h is regularly varying at 0^+ with index $\alpha \in \Re\backslash\{0\}$ [resp. slowly varying at 0^+, rapidly varying at 0^+], denote it by $h \in \mathrm{RV}_\alpha(0^+)$ [resp. $\mathrm{RV}_0(0^+)$, $\mathrm{RV}_{-\infty}(0^+)$].*

For the definition of 2RV, see [6, Sect. 2.3 and Appendix B] and [8].

DEFINITION 2.2: *Suppose that $h \in \mathrm{RV}_\alpha$ for some $\alpha \in \Re$. Then h is said to be of* 2RV *with first-order parameter α and second-order parameter $\rho \le 0$, denoted by $h \in 2\mathrm{RV}_{\alpha,\rho}$, if there exist some ultimately positive or negative function $A(t)$ and a constant $c \neq 0$ such that*

$$\lim_{t\to\infty} \frac{h(tx)/h(t) - x^\alpha}{A(t)} = cx^\alpha \int_1^x u^{\rho-1}du, \quad x > 0. \tag{2.2}$$

Here, $A(t)$ is referred to as the auxiliary function of h.

Similarly, suppose that $h \in \mathrm{RV}_\alpha(0^+)$ for some $\alpha \in \Re$. Then h is said to be of 2RV *with first-order parameter α and second-order parameter $\rho \ge 0$, denoted by $h \in 2\mathrm{RV}_{\alpha,\rho}(0^+)$, if there exist some ultimately positive or negative function $A(t)$ and a constant $c \neq 0$ such that*

$$\lim_{t\to 0^+} \frac{h(tx)/h(t) - x^\alpha}{A(t)} = cx^\alpha \int_1^x u^{\rho-1}du, \quad x > 0. \tag{2.3}$$

Here, $A(t)$ is also referred to as the auxiliary function of h.

In Definition 2.2, if the limit of the left-hand side of (2.2) exists and is not a multiple of x^α, then $A(t) \to 0$ as $t \to \infty$. Moreover, by Theorem 2.1 of [6], the limit must be the right-hand side of (2.2) and $|A(t)| \in \mathrm{RV}_\rho$. Therefore, $\rho \le 0$ in (2.2). In (2.3), $\rho \ge 0$ can be interpreted similarly. In both (2.2) and (2.3), by adjusting $A(t)$, we can always let $c = 1$. Throughout, we always assume that $A(t)$ is chosen such that $c = 1$. However, $A(t)$ is unique in both (2.2) and (2.3) in the sense of asymptotic equivalence as $t \to \infty$ and $t \to 0^+$, respectively.

We will use $\ell(x)$ to represent a slowly varying function (at infinity or at zero). If h is regular varying with index α, then h has a representation of the form

$$h(x) = x^\alpha \ell(x)$$

for some slowly varying function $\ell(x)$. It is easy to see that

$$h \in \mathrm{RV}_\alpha \Longleftrightarrow h_* \in \mathrm{RV}_{-\alpha}(0^+); \tag{2.4}$$

$$h \in 2\mathrm{RV}_{\alpha,\rho} \text{ with auxiliary } A(t) \Longleftrightarrow h_* \in 2\mathrm{RV}_{-\alpha,-\rho}(0^+) \text{ with auxiliary } A_*(t), \tag{2.5}$$

where $h_*(t) = h(1/t)$ and $A_*(t) = -A(1/t)$ for $t \in \Re_+$.

For any locally bounded function $h : \Re_+ \to \Re_+$, if $h(x)$ tends to infinity as x goes to infinity or if $h(x)$ tends to zero as x goes to zero, define its generalized

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inverse $h^{\leftarrow}$ by

$$h^{\leftarrow}(x) = \inf\{t \in \Re_+ : h(t) \geq x\}, \quad x \in \Re_+;$$

and if $h(x)$ tends to zero as x goes to infinity or if $h(x)$ tends to infinity as x goes to zero, define its generalized inverse $h^{\leftarrow}$ by

$$h^{\leftarrow}(x) = \sup\{t \in \Re_+ : h(t) \geq x\}, \quad x \in \Re_+.$$

The next two lemmas are elementary, which state the preservation properties of regularly varying functions under generalized inverse and composition operations.

LEMMA 2.3:

(i) If $h \in \mathrm{RV}_\alpha$ *with* $\alpha > 0$*, then* $h^{\leftarrow} \in \mathrm{RV}_{1/\alpha}$ *with*

$$h \circ h^{\leftarrow}(x) \sim h^{\leftarrow} \circ h(x) \sim x, \quad x \to \infty.$$

(ii) If $h \in \mathrm{RV}_{-\alpha}(0^+)$ *with* $\alpha > 0$*, then* $h^{\leftarrow} \in \mathrm{RV}_{-1/\alpha}$ *with*

$$h \circ h^{\leftarrow}(x) \sim x, \quad x \to \infty; \qquad h^{\leftarrow} \circ h(x) \sim x, \quad x \to 0^+.$$

(iii) If $h \in \mathrm{RV}_\alpha(0^+)$ *with* $\alpha > 0$*, then* $h^{\leftarrow} \in \mathrm{RV}_{1/\alpha}(0^+)$ *with*

$$h \circ h^{\leftarrow}(x) \sim h^{\leftarrow} \circ h(x) \sim x, \quad x \to 0^+.$$

(iv) If $h \in \mathrm{RV}_{-\alpha}$ *with* $\alpha > 0$*, then* $h^{\leftarrow} \in \mathrm{RV}_{-1/\alpha}(0^+)$ *with*

$$h \circ h^{\leftarrow}(x) \sim x, \quad x \to 0^+; \qquad h^{\leftarrow} \circ h(x) \sim x, \quad x \to \infty.$$

PROOF: (i) See Theorem 1.5.12 in [2].

(ii) Define $h_*(x) = h(1/x)$. Then, by (2.4), $h_* \in \mathrm{RV}_\alpha$. By part (i), we have $h_*^{\leftarrow} \in \mathrm{RV}_{1/\alpha}$ and

$$h_* \circ h_*^{\leftarrow}(x) \sim h_*^{\leftarrow} \circ h_*(x) \sim x, \quad x \to \infty.$$

Note that, for $x \in \Re_+$,

$$h_*^{\leftarrow}(x) = \inf\left\{t \in \Re_+ : h\left(\frac{1}{t}\right) \geq x\right\} = \inf\left\{\frac{1}{t} \in \Re_+ : h(t) \geq x\right\}$$

$$= (\sup\{t \in \Re_+ : h(t) \geq x\})^{-1} = \frac{1}{h^{\leftarrow}(x)}.$$

Then $h^{\leftarrow} \in \mathrm{RV}_{-1/\alpha}$,

$$x \sim h_* \circ h_*^{\leftarrow}(x) = h \circ h^{\leftarrow}(x), \quad x \to \infty$$

and

$$h_*^{\leftarrow} \circ h_*(x) = \left[h^{\leftarrow} \circ h\left(\frac{1}{x}\right)\right]^{-1} \sim x, \quad x \to \infty,$$

or, equivalently, $h^{\leftarrow} \circ h(x) \sim x, x \to 0^+$.

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(iii) It can be proved by a similar argument to that of the proof of Theorem 1.5.12 in [2].

(iv) The proof is similar to part (ii) by considering $h_*(x) = h(1/x)$ and applying part (iii). This completes the proof. ■

LEMMA 2.4:

(i) Let $g \in \mathrm{RV}_\alpha$ *and* $h \in \mathrm{RV}_\beta$ *with* $\alpha > 0$ *and* $\beta \in \Re$. *Then* $h \circ g \in \mathrm{RV}_{\alpha\beta}$.

(ii) Let $g \in \mathrm{RV}_{-\alpha}$ *and* $h \in \mathrm{RV}_\beta(0^+)$ *with* $\alpha > 0$ *and* $\beta \in \Re$. *Then* $h \circ g \in \mathrm{RV}_{-\alpha\beta}$.

(iii) Let $g \in \mathrm{RV}_{-\alpha}(0^+)$ *and* $h \in \mathrm{RV}_\beta$ *with* $\alpha > 0$ *and* $\beta \in \Re$. *Then* $h \circ g \in \mathrm{RV}_{-\alpha\beta}(0^+)$.

(iv) Let $g \in \mathrm{RV}_\alpha(0^+)$ *and* $h \in \mathrm{RV}_\beta(0^+)$ *with* $\alpha > 0$ *and* $\beta \in \Re$. *Then* $h \circ g \in \mathrm{RV}_{\alpha\beta}(0^+)$.

PROOF: (i) See Proposition 2.6(iv) in [14].

(ii) Define $g^*(x) = 1/g(x)$ and $h^*(y) = h(1/y)$. Then $g^* \in \mathrm{RV}_\alpha$ and $h^* \in \mathrm{RV}_{-\beta}$. Since $\alpha > 0$, it follows that $g^*(x) \to +\infty$ as $x \to \infty$. By part (i), we have

$$h \circ g = h^* \circ g^* \in \mathrm{RV}_{-\alpha\beta}.$$

(iii) It follows from part (i) and (2.4).

(iv) It follows from part (iii) by observing $1/g \in \mathrm{RV}_{-\alpha}(0^+)$ and $h(1/x) \in \mathrm{RV}_{-\beta}$. This completes the proof. ■

For any $h \in \mathrm{RV}_\alpha$ with $\alpha \in \Re$, $1/h \in \mathrm{RV}_{-\alpha}$. For 2RV, we have the next proposition.

PROPOSITION 2.5:

(i) If $h \in 2\mathrm{RV}_{\alpha,\rho}$ *with auxiliary function* $A(t)$, $\rho < 0$ *and* $\alpha \in \Re$, *then* $1/h \in 2\mathrm{RV}_{-\alpha,\rho}$ *with auxiliary function* $B(t) \sim -A(t)$ *as* $t \to \infty$.

(ii) If $h \in 2\mathrm{RV}_{\alpha,\rho}(0^+)$ *with auxiliary function* $A(t)$, $\rho > 0$ *and* $\alpha \in \Re$, *then* $1/h \in 2\mathrm{RV}_{-\alpha,\rho}(0^+)$ *with auxiliary function* $B(t) \sim -A(t)$ *as* $t \to 0$.

PROOF: (i) From the proof of Lemma 3 in [12], we know that $h(x)$ has the following representation

$$h(x) = kx^\alpha[1 - \eta(x)] \text{ with } |\eta| \in \mathrm{RV}_\rho \text{ and } A(t) \sim -\rho\eta(t),\ t \to \infty, \quad \textbf{(2.6)}$$

where $k > 0$ is some constant. Here, it should be pointed out that $|\eta| \in \mathrm{RV}_\rho$ with $\rho < 0$ implies that $\eta(t) \to 0$ as $t \to \infty$ and $\eta(x)$ is ultimately positive or negative.

Note that, for any fixed $x > 0$,

$$\frac{1}{-A(t)}\left[\frac{1/h(tx)}{1/h(t)} - x^{-\alpha}\right] = x^{-\alpha}\frac{\eta(tx)-\eta(t)}{A(t)(\eta(tx)-1)}$$
$$\sim x^{-\alpha}\frac{\eta(tx)-\eta(t)}{\rho\eta(t)} \sim x^{-\alpha}\frac{x^{\rho}-1}{\rho}$$

as $t \to \infty$. This means $1/h \in 2\text{RV}_{-\alpha,\rho}$ with auxiliary function $B(t) \sim -A(t)$ as $t \to \infty$.

(ii) From the proof of part (i) and (2.5), $h(x)$ also has the following representation:

$$h(x) = kx^{\alpha}[1-\eta(x)] \text{ with } |\eta| \in \text{RV}_{\rho}(0^+) \text{ and } A(t) \sim -\rho\eta(t),\ t \to 0^+, \quad \textbf{(2.7)}$$

where $k > 0$ is some constant. The rest of the proof is similar to the above paragraph and, hence, omitted. This completes the proof. ■

The next two results present the preservation property of the generalized inverse of an 2RV function.

Proposition 2.6:

(i) Suppose $h \in 2\text{RV}_{\alpha,\rho}$ *with auxiliary function* $A(t)$*,* $\alpha > 0$ *and* $\rho < 0$*. If* h *is continuous, then* $h^{\leftarrow} \in 2\text{RV}_{1/\alpha,\rho/\alpha}$ *with auxiliary function* $B(t) = -\alpha^{-2}A \circ h^{\leftarrow}(t)$*.*

(ii) Suppose $h \in 2\text{RV}_{-\alpha,\rho}$ *with auxiliary function* $A(t)$*,* $\alpha > 0$ *and* $\rho < 0$*. If* h *is continuous, then* $h^{\leftarrow} \in 2\text{RV}_{-1/\alpha,-\rho/\alpha}(0^+)$ *with auxiliary function* $B(t) = -\alpha^{-2}A \circ h^{\leftarrow}(t)$*.*

Proof: (i) From (2.6), we have

$$h \in 2\text{RV}_{\alpha,\rho},\ \rho < 0 \Longleftrightarrow |\eta(x)| \in \text{RV}_{\rho},\ \rho < 0 \text{ and } A(t) \sim -\rho\eta(t) \quad \textbf{(2.8)}$$

with $\eta(x) = 1 - k^{-1}x^{-\alpha}h(x)$ for some $k > 0$. Since $h \in \text{RV}_{\alpha}$, $h^{\leftarrow} \in \text{RV}_{1/\alpha}$ by Lemma 2.3(i). Define $\eta_*(t) = 1 - (t/k)^{-1/\alpha}h^{\leftarrow}(t)$ for $t \in \Re_+$. By (2.8), to prove $h^{\leftarrow} \in 2\text{RV}_{1/\alpha,\rho/\alpha}$ with auxiliary function $B(t)$, it suffices to prove that

$$|\eta_*(t)| \in \text{RV}_{\rho/\alpha}, \quad \textbf{(2.9)}$$

which implies $B(t) \sim -\rho\eta_*(t)/\alpha$ as $t \to \infty$.

To prove (2.9), denote $x = h^{\leftarrow}(t)$. Since h is continuous, $h(x) = t$. Note that $t \to \infty$ iff $x \to \infty$ since $h(x) \to \infty$ as $x \to \infty$. Thus, by substituting $t = h(x)$, we have

$$\eta_*(t) = 1 - k^{1/\alpha}[h(x)]^{-1/\alpha}x = 1 - \left[\frac{1}{k}x^{-\alpha}h(x)\right]^{-1/\alpha}$$
$$= 1 - (1-\eta(x))^{-1/\alpha} = -\frac{1}{\alpha}\eta(x) + o(\eta(x))$$
$$\sim -\frac{1}{\alpha}\eta \circ h^{\leftarrow}(t), \quad t \to \infty.$$

By Lemma 2.4(i), we conclude that $|\eta_*| \sim |\eta| \circ h^{\leftarrow}/\alpha \in \mathrm{RV}_{\rho/\alpha}$. Now,

$$B(t) \sim -\frac{\rho}{\alpha}\eta_*(t) = \frac{\rho}{\alpha^2}\eta \circ h^{\leftarrow}(t) \sim -\frac{1}{\alpha^2}A \circ h^{\leftarrow}(t), \quad t \to \infty.$$

(ii) The proof is similar to that of part (i). We outline the proof. From (2.7), h has a representation as follows: $h(x) = kx^{-\alpha}[1-\eta(x)]$ for some positive constant k, $|\eta| \in \mathrm{RV}_\rho$, and the auxiliary function $A(t) \sim -\rho\eta(t)$ as $t \to \infty$. Since $h \in \mathrm{RV}_{-\alpha}$, $h^{\leftarrow} \in \mathrm{RV}_{-1/\alpha}(0^+)$ by Lemma 2.3(iv). Define $\eta_*(t) = 1 - (t/k)^{1/\alpha}h^{\leftarrow}(t)$ for $t \in \Re_+$. By (2.8), to prove $h^{\leftarrow} \in 2\mathrm{RV}_{-1/\alpha,-\rho/\alpha}(0^+)$ with auxiliary function $B(t)$, it suffices to prove that

$$|\eta_*(t)| \in \mathrm{RV}_{-\rho/\alpha}(0^+), \tag{2.10}$$

implying $B(t) \sim \rho\eta_*(t)/\alpha$ as $t \to 0^+$.

To prove (2.10), denote $x = h^{\leftarrow}(t)$. Note that $t \to 0^+$ if $x \to \infty$. Thus,

$$\eta_*(t) = 1 - \left[\frac{1}{k}x^{\alpha}h(x)\right]^{1/\alpha} = 1 - (1-\eta(x))^{1/\alpha}$$
$$= \frac{1}{\alpha}\eta(x) + o(\eta(x)) \sim \frac{1}{\alpha}\eta \circ h^{\leftarrow}(t), \quad t \to 0^+.$$

By Lemma 2.4(iii), we conclude that $|\eta_*| \sim |\eta| \circ h^{\leftarrow}/\alpha \in \mathrm{RV}_{-\rho/\alpha}(0^+)$. Now,

$$B(t) \sim \frac{\rho}{\alpha^2}\eta \circ h^{\leftarrow}(t) \sim -\frac{1}{\alpha^2}A \circ h^{\leftarrow}(t)$$

as $t \to 0^+$. This completes the proof. ∎

PROPOSITION 2.7:

(i) Suppose that $h \in 2\mathrm{RV}_{-\alpha,\rho}(0^+)$ with auxiliary function $A(t)$, $\alpha > 0$ and $\rho > 0$. If h is continuous, then $h^{\leftarrow} \in 2\mathrm{RV}_{-1/\alpha,-\rho/\alpha}$ with auxiliary function $B(t) = -\alpha^{-2}A \circ h^{\leftarrow}(t)$.

(ii) Suppose that $h \in 2\mathrm{RV}_{\alpha,\rho}(0^+)$ with auxiliary function $A(t)$, $\alpha > 0$ and $\rho > 0$. If h is continuous, then $h^{\leftarrow} \in 2\mathrm{RV}_{1/\alpha,\rho/\alpha}(0^+)$ with auxiliary function $B(t) = -\alpha^{-2}A \circ h^{\leftarrow}(t)$.

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PROOF: (i) Define $h_*(x) = h(1/x)$ for $x \in \Re_+$. By (2.5), $h_* \in 2\mathrm{RV}_{\alpha,-\rho}$ with auxiliary function $A_*(t) = -A(1/t)$. Since $h_*(t)$ is continuous, by Proposition 2.6(i), $h_*^{\leftarrow} \in 2\mathrm{RV}_{1/\alpha,-\rho/\alpha}$ with auxiliary function

$$B_*(t) = -\alpha^{-2} A_* \circ h_*^{\leftarrow}(t) = \alpha^{-2} A\left(\frac{1}{h_*^{\leftarrow}(t)}\right).$$

From the proof of Proposition 2.6(i), we know that

$$h_*^{\leftarrow}(t) = \left(\frac{t}{k}\right)^{1/\alpha} (1 - \eta_*(t))$$

for some positive k and $|\eta_*| \in \mathrm{RV}_{-\rho/\alpha}$. Since $\eta_*(t) \to 0$ as $t \to \infty$, $h_*^{\leftarrow} \in 2\mathrm{RV}_{1/\alpha,-\rho/\alpha}$ reduces to

$$\lim_{t\to\infty} \frac{\eta_*(t) - \eta_*(tx)}{B_*(t)} = \int_1^x u^{-\rho/\alpha-1} du. \tag{2.11}$$

It is shown in the proof of Lemma 2.3(ii), $h_*^{\leftarrow}(t) = 1/h^{\leftarrow}(t)$ for $t \in \Re_+$. So, we have

$$h^{\leftarrow}(t) = \frac{1}{h_*^{\leftarrow}(t)} = \left(\frac{t}{k}\right)^{-1/\alpha} \frac{1}{1 - \eta_*(t)} = \left(\frac{t}{k}\right)^{-1/\alpha} [1 + \eta_*(t) + o(\eta_*(t))]. \tag{2.12}$$

Combining (2.11) with (2.12), we conclude that $h^{\leftarrow} \in 2\mathrm{RV}_{-1/\alpha,-\rho/\alpha}$ with auxiliary function $B(t) = -B_*(t) = -\alpha^{-2} A \circ h^{\leftarrow}(t)$.

(ii) By a similar argument to that in the proof of part (i), the desired result follows from (2.5) and Proposition 2.6(i).

■

The 2RV property is preserved under the composition operation, as stated in the next four propositions.

PROPOSITION 2.8: *Suppose that $f \in 2\mathrm{RV}_{\alpha,\rho}(0^+)$ and $g \in 2\mathrm{RV}_{\beta,\gamma}$ with respective auxiliary functions $A(t)$ and $B(t)$, where $\beta < 0$, $\gamma < 0$, $\rho > 0$, and $\alpha \in \Re$. Then*

(i) for $\gamma > \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}$ with auxiliary function $\alpha B(t)$;

(ii) for $\gamma = \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}$ with auxiliary function $\alpha B(t) + \beta A \circ g(t)$;

(iii) for $\gamma < \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\rho\beta}$ with auxiliary function $\beta A \circ g(t)$.

PROOF: Since $f \in \mathrm{RV}_\alpha(0^+)$ and $g \in \mathrm{RV}_\beta$, it follows from Lemma 2.4(ii) that $f \circ g \in \mathrm{RV}_{\alpha\beta}$. From (2.8), we know that there exist some positive constants k_1 and k_2 and functions $\eta_1(x)$ and $\eta_2(x)$ such that

$$f(x) = k_1 x^{\alpha}(1 - \eta_1(x)) \text{ with } |\eta_1| \in \mathrm{RV}_{\rho}(0^+) \quad \text{and} \quad A(t) \sim -\rho\eta_1(t), \quad t \to 0^+$$

and

$$g(x) = k_2 x^{\beta}(1 - \eta_2(x)) \text{ with } |\eta_2| \in \mathrm{RV}_{\gamma} \quad \text{and} \quad B(t) \sim -\gamma\eta_2(t), \quad t \to \infty.$$

Then

$$f \circ g(x) = k_1 k_2^{\alpha} x^{\alpha\beta}(1 - \eta_2(x))^{\alpha}(1 - \eta_1 \circ g(x))$$

and

$$\begin{aligned}\eta(x) &\stackrel{\text{def}}{=} 1 - k_1^{-1}k_2^{-\alpha}x^{-\alpha\beta} f \circ g(x) = 1 - (1 - \eta_2(x))^{\alpha}(1 - \eta_1 \circ g(x)) \\ &= 1 - \big(1 - \alpha\eta_2(x) + o(\eta_2(x))\big)(1 - \eta_1 \circ g(x)) \\ &= \alpha\eta_2(x) + \eta_1 \circ g(x) + o(\eta_2(x)), \;\; x \to \infty,\end{aligned}$$

where the last equality follows from $\eta_2(x) \to 0$ and $\eta_1 \circ g(x) \to 0$ as $x \to \infty$. Again by (2.8), to prove $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\tau}$ with auxiliary function $D(t)$ for some $\tau < 0$, it suffices to prove that $|\eta| \in \mathrm{RV}_{\tau}$, where $B(t)$ can be chosen as $D(t) = -\tau\eta(t)$. Note that

$$\lim_{t\to\infty} \frac{\eta(xt)}{\eta(t)} = \lim_{t\to\infty} \frac{\alpha\eta_2(xt) + \eta_1 \circ g(xt)}{\alpha\eta_2(t) + \eta_1 \circ g(t)}, \qquad \textbf{(2.13)}$$

and that $|\eta_2| \in \mathrm{RV}_{\gamma}$ and $|\eta_1| \circ g \in \mathrm{RV}_{\beta\rho}$ by Lemma 2.4(ii).

If $\gamma > \beta\rho$, then (2.13) reduces to

$$\lim_{t\to\infty} \frac{\eta(xt)}{\eta(t)} = \lim_{t\to\infty} \frac{\alpha + \eta_1 \circ g(xt)/\eta_2(xt)}{\alpha + \eta_1 \circ g(t)/\eta_2(t)} \times \frac{\eta_2(xt)}{\eta_2(t)} = x^{\gamma}, \quad x > 0,$$

that is, $\tau = \gamma$. Thus, the auxiliary function $D(t) = -\gamma\eta(t) \sim -\gamma\alpha\eta_2(t) \sim \alpha B(t)$ as $t \to \infty$.

If $\gamma < \beta\rho$, then (2.13) reduces to

$$\lim_{t\to\infty} \frac{\eta(xt)}{\eta(t)} = \lim_{t\to\infty} \frac{\alpha\eta_2(xt)/\eta_1 \circ g(xt) + 1}{\alpha\eta_2(t)/\eta_1 \circ g(t) + 1} \times \frac{\eta_1 \circ g(xt)}{\eta_1 \circ g(t)} = x^{\beta\rho},$$

that is, $\tau = \beta\rho$. Thus, $D(t) = -\beta\rho\eta(t) \sim -\beta\rho\, \eta_1 \circ g(t) \sim \beta A \circ g(t)$ as $t \to \infty$.

If $\gamma = \beta\rho$, then $|\eta| \in \mathrm{RV}_{\gamma}$. Thus, $\tau = \gamma$ and the auxiliary function

$$D(t) = -\gamma\eta(t) \sim -\gamma\alpha\eta_2(t) - \gamma\eta_1 \circ g(t) \sim \alpha B(t) + \beta A \circ g(t)$$

as $t \to \infty$. This completes the proof. ■

By a similar argument to that in the proof of Proposition 2.8 with minor modifications, we can easily obtain the next three propositions.

PROPOSITION 2.9: *Suppose that $f \in 2\mathrm{RV}_{\alpha,\rho}$ and $g \in 2\mathrm{RV}_{\beta,\gamma}$ with respective auxiliary functions $A(t)$ and $B(t)$, where $\beta > 0$, $\gamma < 0$, $\rho < 0$, and $\alpha \in \Re$. Then*

(i) for $\gamma > \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}$ with auxiliary function $\alpha B(t)$;
(ii) for $\gamma = \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}$ with auxiliary function $\alpha B(t) + \beta A \circ g(t)$;
(iii) for $\gamma < \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\rho\beta}$ with auxiliary function $\beta A \circ g(t)$.

PROPOSITION 2.10: *Suppose that $f \in 2\mathrm{RV}_{\alpha,\rho}$ and $g \in 2\mathrm{RV}_{\beta,\gamma}(0^+)$ with respective auxiliary functions $A(t)$ and $B(t)$, where $\beta < 0$, $\gamma > 0$, $\rho < 0$, and $\alpha \in \Re$. Then*

(i) for $\gamma > \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\rho\beta}(0^+)$ with auxiliary function $\beta A \circ g(t)$;
(ii) for $\gamma = \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}(0^+)$ with auxiliary function $\alpha B(t) + \beta A \circ g(t)$;
(iii) for $\gamma < \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}(0^+)$ with auxiliary function $\alpha B(t)$.

PROPOSITION 2.11: *Suppose that $f \in 2\mathrm{RV}_{\alpha,\rho}(0^+)$ and $g \in 2\mathrm{RV}_{\beta,\gamma}(0^+)$ with respective auxiliary functions $A(t)$ and $B(t)$, where $\beta > 0$, $\gamma > 0$, $\rho > 0$, and $\alpha \in \Re$. Then*

(i) for $\gamma > \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\rho\beta}(0^+)$ with auxiliary function $\beta A \circ g(t)$;
(ii) for $\gamma = \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}(0^+)$ with auxiliary function $\alpha B(t) + \beta A \circ g(t)$;
(iii) for $\gamma < \rho\beta$, $f \circ g \in 2\mathrm{RV}_{\alpha\beta,\gamma}(0^+)$ with auxiliary function $\alpha B(t)$.

For regular variation, there is a well-known monotone density theorem (see, e.g., Proposition B.1.9 in [6]). The theorem states that if $f \in \mathrm{RV}_\alpha$, $\alpha > 0$ [resp. $\alpha < 0$] and

$$f(t) = f(t_0) + \int_{t_0}^{t} \psi(s)ds \quad [\text{resp. } f(t) = \int_{t}^{\infty} \psi(s)ds]$$

for $t \geq t_0$ with ψ monotone, then $\psi \in \mathrm{RV}_{\alpha-1}$. However, there is no analogous result for 2RV, as shown by the following counterexample.

Counterexample 2.12: ($H \in 2\mathrm{RV}_{\alpha,\rho}$, $\alpha > 0, \rho < 0$, and H' is monotone $\not\Longrightarrow H' \in$ 2RV)
Suppose that $H \in 2\mathrm{RV}_{\alpha,\rho}$ is twice differentiable with $\alpha > 1$, $\rho < -1$, and auxiliary function $A(t)$. Then, from (2.6), we have

$$H(x) = kx^\alpha[1 - x^\rho \ell(x)] \tag{2.14}$$

with $A(x) \sim -\rho x^\rho \ell(x)$ as $t \to \infty$ and $|\ell| \in \mathrm{RV}_0$ for some $k > 0$. Note that

$$h(x) = H'(x) = k\alpha x^{\alpha-1}\left[1 - \left(1 + \frac{\rho}{\alpha} + \frac{x\ell'(x)}{\alpha\ell(x)}\right)x^\rho \ell(x)\right] \tag{2.15}$$

$$\overset{\text{sgn}}{=} \alpha x^{\alpha-1} - (\alpha+\rho)x^{\alpha+\rho-1}\ell(x) - x^{\alpha+\rho}\ell'(x) \text{ and}$$

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$$h'(x) \overset{\text{sgn}}{=} \alpha(\alpha-1)x^{\alpha-2} - (\alpha+\rho)(\alpha+\rho-1)x^{\alpha+\rho-2}\ell(x) \\ - 2(\alpha+\rho)x^{\alpha+\rho-1}\ell'(x) - x^{\alpha+\rho}\ell''(x) \\ \overset{\text{sgn}}{=} \alpha(\alpha-1) - (\alpha+\rho)(\alpha+\rho-1)x^{\rho}\ell(x) - 2(\alpha+\rho)x^{\rho+1}\ell'(x) - x^{\rho+2}\ell''(x).$$

Now, choose $\ell(x) = 1 + x^{-1}\sin x$. Since

$$t^{\rho+1}\ell'(t) = t^{\rho}\left(\cos t - \frac{\sin t}{t}\right) \longrightarrow 0, \quad t \to \infty,$$

$$t^{\rho+2}\ell''(t) = t^{\rho+1}\left(\frac{2\sin t}{t^2} - \sin t - \frac{2\cos t}{t}\right), \longrightarrow 0, \quad t \to \infty,$$

we have $h'(x) > 0$ for x large enough, that is, h is ultimately increasing. From (2.15), it follows that, for $x > 0$,

$$\frac{h(tx)}{h(t)} - x^{\alpha-1} \sim x^{\alpha-1}t^{\rho}\ell(t)\left\{\left(1 + \frac{\rho}{\alpha} + \frac{t\ell'(t)}{\alpha\ell(t)}\right) - \left(1 + \frac{\rho}{\alpha} + \frac{tx\ell'(tx)}{\alpha\ell(tx)}\right)x^{\rho}\frac{\ell(tx)}{\ell(t)}\right\}. \tag{2.16}$$

Since $t\ell'(t)/\ell(t) \sim \cos t$, which does not converge as $t \to \infty$, it does not exist an auxiliary function $B(t)$ such that $h \in 2\mathrm{RV}$.

From (2.16), it is easy to see that if ℓ in (2.14) satisfies that

$$\frac{t\ell'(t)}{\ell(t)} \longrightarrow 0, \quad t \to \infty,$$

then $|h| \in 2\mathrm{RV}_{\alpha-1,\rho}$ with auxiliary function $B(t) \sim (1 + \rho/\alpha)A(t)$ and with parameters $\rho < 0$ and $\alpha \neq 0$ such that $\alpha + \rho \neq 0$. ◁

For completeness, we state one result, due to [12], which extends Karamata's theorem to a second-order regular condition for the case with regular variation index $\alpha < -1$.

PROPOSITION 2.13: *[12] Let $g \in 2\mathrm{RV}_{\alpha,\rho}$ with an auxiliary function $A(t)$, $\rho < 0$ and $\alpha < -1$, and define $g^*(t) = \int_t^{\infty} g(x)dx$. Then $g^* \in 2\mathrm{RV}_{\alpha+1,\rho}$ with auxiliary function $A^*(t) = \frac{1+\alpha}{1+\alpha+\rho}A(t)$ and*

$$g^*(t) + \frac{1}{1+\alpha}tg(t) \sim \frac{A(t)}{\rho}\left(\frac{1}{1+\alpha} - \frac{1}{1+\alpha+\rho}\right)tg(t), \quad t \to \infty. \tag{2.17}$$

In the end of this section, a counterexample is given to show that 2RV property is not closed under linear combination.

Example 2.14: Let h_1 and h_2 be two functions defined by

$$h_1(x) = x^{-\alpha}\left(1 + \frac{\log x}{x}\right), \quad x > 1 \quad \text{and}$$

$$h_2(x) = x^{-\alpha}\left(1 - \frac{\log x}{x}\right), \quad x > 1,$$

with $\alpha > 0$. It is easy to see that $h_i \in 2\mathrm{RV}_{-\alpha,-1}$ for $i = 1$ and 2. However, $(h_1 + h_2)/2 = x^{-\alpha}$ does not possess 2RV property. Moreover, it can be checked that both $h_1(x)$ and $h_2(x)$ are decreasing when x is large enough. So this example also shows that the mixture of 2RV distributions may not possess the 2RV property. $\triangleleft$

3. SECOND-ORDER APPROXIMATION OF TAIL PROBABILITIES OF CONVOLUTIONS

Barbe and McCormick [1] derived the second-order approximation for tail probability of the convolution of finite i.i.d. random variables under a mild regularity condition that the underlying survival function is regular varying and asymptotically smooth. In this section, we will extend such results to non-i.i.d. case, in which the underlying survival functions of all random variables are asymptotically smooth and regularly varying with the same index α.

Before we state and prove the main results, we should recall from [1] the definitions of the asymptotical smoothness and the right-tail dominance, and give some notations and some useful lemmas.

A function $h : \Re_+ \to \Re_+$ is said to be asymptotically smooth with index $-\alpha$ if

$$\lim_{\delta\to 0} \limsup_{t\to\infty} \sup_{0<|x|\le\delta} \left| \frac{h(t(1-x)) - h(t)}{xh(t)} - \alpha \right| = 0.$$

It is shown that the class of asymptotically smooth functions with index $-\alpha$ coincides with that of normalized regularly varying ones with index $-\alpha$. For the definition of the normalized regularly varying functions, see [2, p. 15]. In particular, if a distribution function F with density f satisfies that

$$\lim_{t\to\infty} \frac{tf(t)}{\overline{F}(t)} = \alpha > 0,$$

then $\overline{F}$ is asymptotically smooth with index $-\alpha$.

A distribution function F is said to be right-tail dominant if

$$\lim_{t\to\infty} \frac{F(-t\delta)}{\overline{F}(t)} = 0, \quad \forall\, \delta > 0.$$

For any distribution function F, denote the truncated mean of F by

$$\mu_F(t) = \int_{-t}^{t} x dF(x), \quad t \in \Re_+.$$

If the mean μ_F of F exists, then $\mu_F(t) \to \mu_F$ as $t \to \infty$, where μ_F can be written as

$$\mu_F = \int_0^{\infty} \overline{F}(x)dx - \int_{-\infty}^{0} F(x)dx.$$

Similarly, $\mu_G(t)$ and μ_G are defined.

3.1. Some Lemmas

LEMMA 3.1: *Let F and G be two distribution functions such that* $\overline{F} \in \mathrm{RV}_{-\alpha}$ *and* $\overline{G} \in \mathrm{RV}_{-\alpha}$ *with* $0 < \alpha \leq 1$. *Assume that* $\overline{F}$ *and* $\overline{G}$ *are right-tail dominant and*

$$\lim_{t\to\infty} \frac{\overline{F}(t)}{\overline{G}(t)} = c \in \Re_+. \tag{3.1}$$

If μ_F *and* μ_G *are infinite* (α *must be smaller than or equal to* 1), *then*

$$\lim_{t\to\infty} \frac{\mu_F(t)}{\mu_G(t)} = c.$$

PROOF: Note that

$$\int_{-t}^{0} xdF(x) = tF(-t) - \int_{-t}^{0} F(x)dx \tag{3.2}$$

and

$$\int_{0}^{t} xdF(x) = -t\overline{F}(t) + \int_{0}^{t} \overline{F}(x)dx. \tag{3.3}$$

Since F is right-tail dominant, it follows that the infinity of μ_F implies that at least one of $\int_{-\infty}^{0} F(x)dx$ and $\int_0^{\infty} \overline{F}(x)dx$ is infinite, and that if $\int_{-\infty}^{0} F(x)dx = \infty$ then $\int_0^{\infty} \overline{F}(x)dx = \infty$. Thus,

$$\lim_{t\to\infty} \frac{\int_{-t}^{0} F(x)dx}{\int_0^t \overline{F}(x)dx} = \lim_{t\to\infty} \frac{F(-t)}{\overline{F}(t)} = 0.$$

Moreover, by Karamata's theorem (see Theorem B.1.5 in [6]), we have

$$\frac{t\overline{F}(t)}{\int_0^t \overline{F}(x)dx} \longrightarrow 1-\alpha, \quad t \to \infty,$$

when $\alpha \leq 1$. Then, from (3.2) and (3.3), we get

$$\lim_{t\to\infty} \frac{\int_{-t}^{0} xdF(x)}{\int_0^t xdF(x)} = 0.$$

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Similarly,

$$\lim_{t\to\infty} \frac{\int_{-t}^{0} x dG(x)}{\int_0^t x dG(x)} = 0.$$

So

$$\lim_{t\to\infty} \frac{\mu_F(t)}{\mu_G(t)} = \lim_{t\to\infty} \frac{\int_0^t x dF(x)}{\int_0^t x dG(x)} = \lim_{t\to\infty} \frac{\int_0^t \overline{F}(x)dx}{\int_0^t \overline{G}(x)dx} = \lim_{t\to\infty} \frac{\overline{F}(t)}{\overline{G}(t)} = c.$$

This completes the proof. ■

It should be pointed out that the conclusion in Lemma 3.1 does not hold if μ_F and/or μ_G are finite.

LEMMA 3.2: *Let F and G be two distribution functions such that $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ and $\overline{G} \in 2\mathrm{RV}_{-\alpha,\rho}$ with $\alpha > 0$ and $\rho + \alpha < 0$. If (3.1) holds, then*

$$\lim_{t\to\infty} \frac{\overline{F}(t) - c\overline{G}(t)}{[\overline{G}(t)]^2} = 0.$$

PROOF: From (2.6), $\overline{F}(t)$ and $\overline{G}(t)$ have the following representations: there exist a constant $k > 0$ and slowing varying functions ℓ_F and ℓ_G such that

$$\overline{F}(t) = ckt^{-\alpha}\ell_F(t), \quad \lim_{t\to\infty} \ell_F(t) = 1, \quad |1 - \ell_F(t)| \in \mathrm{RV}_\rho$$

and

$$\overline{G}(t) = kt^{-\alpha}\ell_G(t), \quad \lim_{t\to\infty} \ell_G(t) = 1, \quad |1 - \ell_G(t)| \in \mathrm{RV}_\rho.$$

Since $\rho + \alpha < 0$, we have

$$t^\alpha |1 - \ell_F(t)| \longrightarrow 0, \qquad t^\alpha |1 - \ell_G(t)| \longrightarrow 0,$$

as $t \to \infty$. Then $t^\alpha |\ell_F(t) - \ell_G(t)| \to 0$ as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \frac{\overline{F}(t) - c\overline{G}(t)}{[\overline{G}(t)]^2} = \lim_{t\to\infty} \frac{ct^\alpha(\ell_F(t) - \ell_G(t))}{k\ell_G^2(t)} = 0.$$

This completes the proof. ■

LEMMA 3.3: *[1] Let F and G be two distribution functions such that $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ with $\alpha \wedge \beta \geq 1$. Assume that $\overline{F}$ and $\overline{G}$ are asymptotically smooth and right-tail dominant, with $\int_{-\infty}^0 x dF(x)$ and $\int_{-\infty}^0 x dG(x)$ both finite. Then $\overline{F * G}$ is asymptotically smooth with index $-\alpha \wedge \beta$, right-tail dominant, and*

$$\overline{F * G}(t) = \overline{F}(t) + \overline{G}(t) + \frac{1}{t}\left(\alpha\overline{F}(t)\mu_G(t) + \beta\overline{G}(t)\mu_F(t)\right)(1 + o(1)), \quad t \to \infty.$$

LEMMA 3.4: *[1] Let $\overline{F}$ and $\overline{G}$ be two asymptotically smooth survival functions such that $\overline{F} \in \mathrm{RV}_{-\alpha}$ and $\overline{G} \in \mathrm{RV}_{-\beta}$ with $\alpha \vee \beta < 1$. Then $\overline{F * G}$ is also asymptotically smooth with index $-\alpha \wedge \beta$, and*

$$\lim_{t\to\infty} \frac{\overline{F*G}(t) - \overline{F}(t) - \overline{G}(t)}{\overline{F}(t)\overline{G}(t)} = I(\alpha,\beta) + I(\beta,\alpha) + 2^{\alpha+\beta} - 2^{\alpha} - 2^{\beta},$$

where

$$I(\alpha,\beta) = \int_0^{1/2} ((1-x)^{-\alpha} - 1)\beta x^{-\beta-1} dx.$$

In Lemma 3.3, under the mild condition, the first-order term to approximate $\overline{F*G}(t)$ is $\overline{F}(t) + \overline{G}(t)$ as $t \to \infty$, while the second-order term is $\alpha\overline{F}(t)\mu_G(t)/t + \beta\overline{G}(t)\mu_F(t)/t = o(\overline{F}(t) + \overline{G}(t))$ as $t \to \infty$ since $\mu_F(t)/t \to 0$ and $\mu_G(t)/t \to 0$ as $t \to \infty$ if μ_F and μ_G exist. The same comment applies to Lemma 3.4. In the next two subsections, we give second-order expansions of survival functions of convolutions for n independent heavy-tail random variables.

3.2. Expansion for the Case $\alpha \geq 1$

PROPOSITION 3.5: *Let $F_1, \ldots, F_n$ be a sequence of right-tail dominant distribution functions such that all the $\overline{F}_i$s are asymptotically smooth with index $-\alpha$, and denote by $G = F_1 * \cdots * F_n$ the convolution of the F_is. Assume that there exist positive constants $c_1, \ldots, c_n$ such that*

$$\lim_{t\to\infty} \frac{t}{\mu_{F_j}(t)} \left(\frac{\overline{F}_i(t)}{\overline{F}_j(t)} - \frac{c_i}{c_j} \right) = 0, \quad \forall\, 1 \leq i \neq j \leq n. \tag{3.4}$$

(i) *If $\alpha = 1$ and $\mu_{F_\nu} = \infty$ for some ν, then, for each i,*

$$\lim_{t\to\infty} \frac{t}{\mu_{F_i}(t)} \left(\frac{\overline{G}(t)}{\overline{F}_i(t)} - \frac{\sigma_c}{c_i} \right) = \alpha\, \frac{\sigma_{cc}}{c_i^2}, \tag{3.5}$$

where

$$\sigma_c = \sum_{i=1}^{n} c_i \;\; \text{and} \;\; \sigma_{cc} = \sum_{k\neq j} c_k c_j. \tag{3.6}$$

(ii) *If $\alpha \geq 1$ and $\mu_{F_\nu} < \infty$ for some ν, then*

$$\lim_{t\to\infty} t \left(\frac{\overline{G}(t)}{\overline{F}_i(t)} - \frac{\sigma_c}{c_i} \right) = \alpha\, \frac{\sum_{k\neq j} c_k \mu_{F_j}}{c_i}. \tag{3.7}$$

PROOF: Since $\mu_F(t)/t \to 0$ as $t \to \infty$ when μ_F exists, it follows from (3.4) that $\overline{F}_i(t)/\overline{F}_j(t) \to c_i/c_j$ for all $i \neq j$ as $t \to \infty$. This implies that if any one of $\mu_{F_1}, \ldots, \mu_{F_n}$ is infinite [finite], then all the μ_{F_i}s are infinite [finite]. We first establish that

$$\mu_{F_1*\cdots*F_k}(t) \sim \sum_{i=1}^{k} \mu_{F_i}(t) \ \text{ as } t \to \infty \tag{3.8}$$

for $k = 2, \ldots, n$ by using a similar argument to that in the proof of Proposition 2.4 in [1]. To see (3.8), first consider the case that all μ_{F_i}s are finite. Then $\mu_{F_i}(t) \sim \mu_{F_i}$ as $t \to \infty$ for each i and, hence,

$$\mu_{F_1*\cdots*F_k}(t) \sim \mu_{F_1*\cdots*F_k} = \sum_{i=1}^{k} \mu_{F_i} \sim \sum_{i=1}^{k} \mu_{F_i}(t) \ \text{ as } t \to \infty.$$

Now assume that all the μ_{F_i}s are infinite, which implies $\alpha = 1$. Since the regular variation property is closed under the convolution of identical distribution functions (see Lemma 1.3.1 in [9] or [10, p. 278]), we have $\overline{F_1 * \cdots * F_k} \in \mathrm{RV}_{-1}$ and

$$\overline{F_1 * \cdots * F_k}(t) \sim \sum_{i=1}^{k} \overline{F}_i(t) \ \text{ as } t \to \infty.$$

By Karamata's theorem, we have

$$t\,\overline{F_1 * \cdots F_k}(t) = o\left(\int_0^t \overline{F_1 * \cdots * F_k}(x)dx\right) \ \text{ as } t \to \infty.$$

By Lemma 3.3 and by induction, $\overline{F_1 * \cdots * F_k}$ is asymptotically smooth with index -1, and right-tail dominant for each k. Hence,

$$\begin{aligned}
\mu_{F_1*\cdots*F_k}(t) &\sim \int_0^t x dF_1 * \cdots * F_k(x) \\
&= -t\,\overline{F_1 * \cdots * F_k}(t) + \int_0^t \overline{F_1 * \cdots * F_k}(x)dx \\
&\sim \int_0^t \overline{F_1 * \cdots * F_k}(x)dx \\
&\sim \int_0^t \sum_{i=1}^{k} \overline{F}_i(x)\,dx = \sum_{i=1}^{k} \int_0^t \overline{F}_i(x)\,dx \\
&\sim \sum_{i=1}^{k} \mu_{F_i}(t) \ \text{ as } t \to \infty,
\end{aligned}$$

where the last asymptotic equivalence follows from $\int_0^t \overline{F}_i(x)dx \sim \mu_{F_i}(t) \to \infty$ as $t \to \infty$. This proves (3.8).

In views of (3.8), repeated application of Lemma 3.3 yields that

$$\begin{aligned}\overline{F_1 * \cdots * F_n}(t) &= \overline{F}_n(t) + \overline{F_1 * \cdots * F_{n-1}}(t) \\ &\quad + \frac{\alpha}{t}\left\{\overline{F}_n(t)\, \mu_{F_1 * \cdots * F_{n-1}}(t) + \overline{F_1 * \cdots * F_{n-1}}(t)\, \mu_{F_n}(t)\right\}(1 + o(1)) \\ &= \overline{F}_n(t) + \overline{F_1 * \cdots * F_{n-1}}(t) \\ &\quad + \frac{\alpha}{t}\left(\overline{F}_n(t) \sum_{i=1}^{n-1} \mu_{F_i}(t) + \overline{F_1 * \cdots * F_{n-1}}(t)\, \mu_{F_n}(t)\right)(1 + o(1)) \\ &= \cdots \\ &= \sum_{i=1}^{n} \overline{F}_i(t) + \frac{\alpha}{t} \sum_{i \neq j} \overline{F}_i(t)\, \mu_{F_j}(t) \times (1 + o(1)) \end{aligned} \quad \textbf{(3.9)}$$

as $t \to \infty$.

(i) Suppose $\alpha = 1$ and all the μ_{F_i}s are infinite. From (3.9), it follows that

$$\frac{t}{\mu_{F_i}(t)}\left(\frac{\overline{G}(t)}{\overline{F}_i(t)} - \frac{\sum_{k=1}^{n} \overline{F}_k(t)}{\overline{F}_i(t)}\right) = \alpha \frac{\sum_{k \neq j} \overline{F}_k(t) \mu_{F_j}(t)}{\overline{F}_i(t) \mu_{F_i}(t)} \times (1 + o(1)).$$

Because of (3.4), we have

$$\lim_{t \to \infty} \frac{t}{\mu_{F_i}(t)}\left(\frac{\sum_{k=1}^{n} \overline{F}_k(t)}{\overline{F}_i(t)} - \frac{\sigma_c}{c_i}\right) = 0.$$

By Lemma 3.1, $\overline{F}_j/\overline{F}_i \to c_j/c_i$ implies $\mu_{F_j}(t)/\mu_{F_i}(t) \to c_j/c_i$ as $t \to \infty$ for each pair $i \neq j$. Thus, (3.5) follows.

(ii) Suppose that $\alpha \geq 1$ and all the μ_{F_i}s are finite. From (3.9), we get

$$\overline{F_1 * \cdots * F_n}(t) = \sum_{i=1}^{n} \overline{F}_i(t) + \frac{\alpha}{t} \sum_{k \neq j} \overline{F}_k(t) \mu_{F_j} \times (1 + o(1)),$$

which implies that

$$t\left(\frac{\overline{G}(t)}{\overline{F}_i(t)} - \frac{\sum_{k=1}^{n} \overline{F}_k(t)}{\overline{F}_i(t)}\right) = \alpha \frac{\sum_{k \neq j} \overline{F}_k(t) \mu_{F_j}}{\overline{F}_i(t)} \times (1 + o(1)).$$

Since

$$\lim_{t \to \infty} t\left(\frac{\sum_{k=1}^{n} \overline{F}_k(t)}{\overline{F}_i(t)} - \frac{\sigma_c}{c_i}\right) = 0,$$

the desired result (3.7) follows. This completes the proof. ∎

Remark 3.6: In Proposition 3.5, Condition (3.4) can be replaced by

$$\lim_{t\to\infty} \frac{1}{\overline{F}_j(t)} \left(\frac{\overline{F}_i(t)}{\overline{F}_j(t)} - \frac{c_i}{c_j} \right) = d_{ij} \in \Re_+, \quad \forall\, 1 \le i \ne j \le n. \tag{3.10}$$

To prove that (3.10) implies (3.4), first assume that $\alpha = 1$ and $\mu_{F_i} = \infty$. Then, by Karamata's theorem,

$$t\overline{F}_i(t) = o\left(\int_0^t \overline{F}_i(x)dx \right) \quad \text{as } t \to \infty,$$

and hence $t\overline{F}_i(t) = o\big(\mu_{F_i}(t)\big)$ as $t \to \infty$, which implies that

$$\lim_{t\to\infty} \frac{t}{\mu_{F_j}(t)} \left(\frac{\overline{F}_i(t)}{\overline{F}_j(t)} - \frac{c_i}{c_j} \right) = \lim_{t\to\infty} \frac{t\overline{F}_j(t)}{\mu_{F_j}(t)} \times \frac{1}{\overline{F}_j(t)} \left(\frac{\overline{F}_i(t)}{\overline{F}_j(t)} - \frac{c_i}{c_j} \right) = 0.$$

Now assume that $\alpha \ge 1$ and μ_{F_i} is finite. Then $t\overline{F}_i(t) \to 0$ and hence $t\overline{F}_i(t) = o\big(\mu_{F_i}(t)\big)$ as $t \to \infty$. Therefore, the above equality holds. ◁

Remark 3.7: Another sufficient condition for (3.4) and (3.10) is that

$$\overline{F}_i \in 2\mathrm{RV}_{-\alpha,\rho}, \quad \lim_{t\to\infty} \frac{\overline{F}_i(t)}{\overline{F}_j(t)} = \frac{c_i}{c_j}, \quad \forall\, 1 \le i \ne j \le n, \tag{3.11}$$

where $\alpha \ge 1$, $\alpha + \rho < 0$ and the c_is are positive constants. This can be seen from Lemma 3.2 directly. ◁

A special consequence of Proposition 3.5 is the following corollary.

Corollary 3.8: *[1] Let F be a right-tail dominant distribution function which is also asymptotically smooth with index $-\alpha \le -1$, and denote by F^{*n} the n-fold convolution of F. Then*

$$\lim_{t\to\infty} \frac{t}{\mu_F(t)} \left(\frac{\overline{F^{*n}}(t)}{\overline{F}(t)} - n \right) = n(n-1)\alpha.$$

3.3. Expansion for the Case $0 < \alpha < 1$

Proposition 3.9: *Let $F_1, \ldots, F_n$ be a sequence of right-tail dominant distribution functions such that all the $\overline{F}_i$s are asymptotically smooth with index $-\alpha$, $0 < \alpha < 1$, and denote by $G = F_1 * \cdots * F_n$. If* (3.10) *holds or if* (3.11) *holds with $\alpha + \rho < 0$, then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}_k(t)} \left(\frac{\overline{G}(t)}{\overline{F}_k(t)} - \frac{\sigma_c}{c_k} \right) = J_\alpha \frac{\sigma_{cc}}{c_k^2} \tag{3.12}$$

for any k, where σ_c and σ_{cc} are given in (3.6), *and*

$$J_\alpha = I(\alpha,\alpha) + 2^{2\alpha-1} - 2^\alpha. \tag{3.13}$$

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Proof: By Lemma 3.2, (3.11) implies (3.10). Now suppose that (3.10) holds. By Lemma 3.4, $\overline{F_1 * \cdots * F_k}$ is right-tail dominant and asymptotically smooth with index $-\alpha$. Again repeated application of Lemma 3.4 yields that

$$\begin{aligned}\overline{F_1 * \cdots * F_n}(t) &= 2J_\alpha(1+o(1)) \times \overline{F_1 * \cdots * F_{n-1}}(t)\overline{F}_n(t) \\ &\quad + \overline{F_1 * \cdots * F_{n-1}}(t) + \overline{F}_n(t) \\ &= \cdots \\ &= \sum_{i=1}^{n} \overline{F}_i(t) + 2J_\alpha(1+o(1)) \sum_{1\le i<j\le n} \overline{F}_i(t)\overline{F}_j(t).\end{aligned}$$

The rest of the proof is similar to that of Proposition 3.5. ■

A special consequence of Proposition 3.9 is the following corollary.

Corollary 3.10: *[1] Let F be a right-tail dominant distribution function such that $\overline{F}$ is asymptotically smooth with index $-\alpha$, $0 < \alpha < 1$. Then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \left(\frac{\overline{F^{*n}}(t)}{\overline{F}(t)} - n \right) = n(n-1)J_\alpha.$$

Corollaries 3.8 and 3.10 are also summarized in Proposition 4.1 of [5].

4. SECOND-ORDER EXPANSIONS OF RISK CONCENTRATION

Let X be a random variable with distribution function F. The value-at-risk (VaR) with respect to the level $p \in (0,1)$ is defined as the generalized inverse of F:

$$\mathrm{VaR}_p[X] = F^{-1}(p) = \inf\{t \in \mathbb{R} : F(t) \ge p\}.$$

Let $X_1, \ldots, X_n$ be independent non-negative random variables with respective survival functions $\overline{F}_1, \ldots, \overline{F}_n$ satisfying that

$$\overline{F}_i \in \mathrm{RV}_{-\alpha}, \quad \lim_{t\to\infty} \frac{\overline{F}_i(t)}{\overline{F}_j(t)} = \frac{c_i}{c_j}, \quad \forall\, 1 \le i \ne j \le n, \tag{4.1}$$

with $\alpha > 0$ and the c_is being positive constants. From Theorem 5.1 in [3], it follows that

$$C(p) := \frac{\mathrm{VaR}_p\left[\sum_{i=1}^n X_i\right]}{\sum_{i=1}^n \mathrm{VaR}_p[X_i]} \longrightarrow \frac{\left(\sum_{i=1}^n c_i\right)^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}} \quad \text{as } p \to 1. \tag{4.2}$$

Here, $C(p)$ is termed by [5] as the risk concentration at level p, and $1 - C(p)$ as the diversification benefit. The term in the right-hand side of (4.2) is the first-order

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approximation of $C(p)$ as $p \to 1$. In this section, we identify conditions under which we establish the second-order approximation of $C(p)$.

PROPOSITION 4.1: *Let $X_1, \ldots, X_n$ be independent non-negative random variables with continuous and asymptotically smooth survival functions $\overline{F}_1, \ldots, \overline{F}_n$, respectively. Assume that*

$$U_i = \left(\frac{1}{\overline{F}_i}\right)^{\leftarrow} \in 2\mathrm{RV}_{1/\alpha,\rho}, \quad \forall\, i \tag{4.3}$$

with auxiliary function $a_i(t)$ for some $\alpha \ge 1$ and $\rho \le 0$. If (3.4) *holds, we have*

(i) *For $\rho < -1$, $\alpha = 1$, and $\mu_{F_\nu} = \infty$ for some ν,*

$$C(p) - 1 \sim \frac{\mu_{F_k}(F_k^{\leftarrow}(p))}{F_k^{\leftarrow}(p)} \times \frac{\sigma_{cc}}{\sigma_c^2} \tag{4.4}$$

as $p \to 1$ for each k, where σ_c and σ_{cc} are given in (3.6).

(ii) *For $\rho > -1$, $\alpha = 1$, and $\mu_{F_\nu} = \infty$ for some ν,*

$$C(p) - 1 \sim \frac{1}{\sigma_c} \sum_{i=1}^{n} c_i \frac{(\sigma_c/c_i)^{\rho} - 1}{\rho} \times a_i\left(\frac{1}{1-p}\right) \tag{4.5}$$

as $p \to 1$. Here and henceforth, for $\rho = 0$, the term $[1 - (\sigma_c/c_i)^{\rho}]/\rho$ is interpreted as $-\log(\sigma_c/c_i)$.

(iii) *For $\rho\alpha < -1$, $\alpha \ge 1$, and $\mu_{F_\nu} < \infty$ for some ν,*

$$C(p) - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^{n} c_i^{1/\alpha}} \sim \frac{1}{F_k^{\leftarrow}(p)} \times \frac{c_k^{1/\alpha}}{\sigma_c \sum_{i=1}^{n} c_i^{1/\alpha}} \sum_{l \ne j} c_l \mu_{F_j} \tag{4.6}$$

as $p \to 1$ for each k.

(iv) *For $\rho\alpha > -1$, $\alpha \ge 1$, and $\mu_{F_\nu} < \infty$ for some ν,*

$$C(p) - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^{n} c_i^{1/\alpha}} \sim \frac{\sigma_c^{1/\alpha}}{\left(\sum_{i=1}^{n} c_i^{1/\alpha}\right)^2} \times \sum_{i=1}^{n} c_i^{1/\alpha} \frac{(\sigma_c/c_i)^{\rho} - 1}{\rho} \times a_i\left(\frac{1}{1-p}\right) \tag{4.7}$$

as $p \to 1$.

PROOF: Let G denote the distribution function of $\sum_{i=1}^n X_i$. Since $U_i \in 2\mathrm{RV}_{1/\alpha,\rho}$ and $\overline{G}(t)/\overline{F}_i(t) \to \sigma_c/c_i$ as $t \to \infty$ for each i, it follows that

$$\lim_{t\to\infty} \frac{\frac{U_i(1/\overline{F}_i(t))}{U_i(1/\overline{G}(t))} - \left(\frac{\overline{G}(t)}{\overline{F}_i(t)}\right)^{1/\alpha}}{a_i(1/\overline{G}(t))} = \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \frac{(\sigma_c/c_i)^\rho - 1}{\rho}$$

by using the uniform convergence property of (2.2) with respect to $x \in [d_1, d_2]$ with $0 < d_1 < d_2 < \infty$. Setting $t = G^{\leftarrow}(p)$, we have

$$\begin{aligned}\frac{G^{\leftarrow}(p)}{F_i^{\leftarrow}(p)} &= \left(\frac{\overline{G}(t)}{\overline{F}_i(t)}\right)^{1/\alpha} + \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \frac{(\sigma_c/c_i)^\rho - 1}{\rho} \times a_i\left(\frac{1}{\overline{G}(t)}\right)(1+o(1)) \\ &= \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \Delta_i(t) + \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \frac{(\sigma_c/c_i)^\rho - 1}{\rho} \times a_i\left(\frac{1}{\overline{G}(t)}\right)(1+o(1)) \quad \textbf{(4.8)}\end{aligned}$$

as $t \to \infty$ or $p \to 1$, where the second equality follows from Proposition 3.5, and

$$\Delta_i(t) = \begin{cases} 1 + \dfrac{\mu_{F_i}(t)}{t} \dfrac{\sigma_{cc}}{c_i\sigma_c}(1+o(1)), & \text{if } \mu_{F_\nu} = \infty \text{ for some } \nu; \\ 1 + \dfrac{1+o(1)}{t\sigma_c} \displaystyle\sum_{l\neq j} c_l \mu_{F_j}, & \text{if } \mu_{F_\nu} < \infty \text{ for some } \nu. \end{cases}$$

Since $U_i \in 2\mathrm{RV}_{1/\alpha,\rho}$ with auxiliary function $a_i(t)$, we have $U_i \in \mathrm{RV}_{1/\alpha}$ and $|a_i(t)| \in \mathrm{RV}_\rho$. Hence, by Lemma 2.3, $\overline{F}_i \in \mathrm{RV}_{-\alpha}$. By Lemma 1.3.1 in [9] and Lemma 2.4, we have $\overline{G} \in \mathrm{RV}_{-\alpha}$ and $|a_i(1/\overline{G})| \in \mathrm{RV}_{\rho\alpha}$. Note that $\mu_{F_i}(t)/t \in \mathrm{RV}_{-(1\wedge\alpha)}$ and $\alpha \geq 1$. From (4.8), with $t = G^{\leftarrow}(p)$, it follows that:

$$\frac{G^{\leftarrow}(p)}{F_i^{\leftarrow}(p)} = \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \Delta_i(t), \quad t \to \infty, \quad \textbf{(4.9)}$$

when $\rho\alpha < -1$, and that

$$\frac{G^{\leftarrow}(p)}{F_i^{\leftarrow}(p)} = \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha} \left[1 + \frac{(\sigma_c/c_i)^\rho - 1}{\rho} \times a_i\left(\frac{1}{\overline{G}(t)}\right)(1+o(1))\right], \quad t \to \infty, \quad \textbf{(4.10)}$$

when $\rho\alpha > -1$.

(i) First, assume that $\rho < -1, \alpha = 1$, and $\mu_{F_\nu} = \infty$ for some ν. Then, from (4.9), we get that

$$\frac{F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} = \frac{c_i}{\sigma_c}\left(1 - \frac{\mu_{F_i}(t)}{t} \frac{\sigma_{cc}}{c_i\sigma_c}(1+o(1))\right), \quad t \to \infty \quad \textbf{(4.11)}$$

and

$$\frac{\sum_{i=1}^n F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} - 1 \sim -\sigma_c^{-2}\sigma_{cc} \sum_{i=1}^n \frac{\mu_{F_i}(t)}{t}.$$

Then, for some k,

$$\lim_{t\to\infty} \frac{t}{\mu_{F_k}(t)} \left[\frac{\sum_{i=1}^n F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} - 1 \right] = -\sigma_c^{-2}\sigma_{cc} \sum_{i=1}^n \frac{c_i}{c_k} = -\frac{\sigma_{cc}}{\sigma_c c_k},$$

which implies that

$$\lim_{t\to\infty} \frac{t}{\mu_{F_k}(t)} \left[\frac{G^{\leftarrow}(p)}{\sum_{i=1}^n F_i^{\leftarrow}(p)} - 1 \right] = \frac{\sigma_{cc}}{\sigma_c c_k}. \tag{4.12}$$

Since $\mu_{F_k}(t)/t \in \mathrm{RV}_{-1}$, from (4.11), we get

$$\lim_{p\to 1} \frac{\mu_{F_k}(G^{\leftarrow}(p))/G^{\leftarrow}(p)}{\mu_{F_k}(F_k^{\leftarrow}(p))/F_k^{\leftarrow}(p)} = \lim_{p\to 1} \frac{F_k^{\leftarrow}(p)}{G^{\leftarrow}(p)} = \frac{c_k}{\sigma_c}. \tag{4.13}$$

Therefore, the desired result (4.4) follows from (4.12) and (4.13).

(ii) Assume that $\rho > -1$, $\alpha = 1$, and $\mu_{F_\nu} = \infty$ for some ν. The proof is similar to that of part (iv).

(iii) Assume that $\rho\alpha < -1$, $\alpha \ge 1$, and $\mu_{F_\nu} < \infty$ for some ν. Then, from (4.9), we get that

$$\frac{F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} = \left(\frac{c_i}{\sigma_c}\right)^{1/\alpha} \left(1 - \frac{1+o(1)}{\sigma_c t} \sum_{l\neq j} c_l \mu_{F_j}\right), \quad t\to\infty$$

and

$$\frac{\sum_{i=1}^n F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} - \frac{\sum_{i=1}^n c_i^{1/\alpha}}{\sigma_c^{1/\alpha}} \sim -\frac{1}{t\sigma_c^{1/\alpha+1}} \left(\sum_{i=1}^n c_i^{1/\alpha}\right) \sum_{l\neq j} c_l \mu_{F_j},$$

which implies that

$$\begin{aligned}
\frac{G^{\leftarrow}(p)}{\sum_{i=1}^n F_i^{\leftarrow}(p)} - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}} &\sim \left(\frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}}\right)^2 \frac{1}{t\sigma_c^{1/\alpha+1}} \left(\sum_{i=1}^n c_i^{1/\alpha}\right) \sum_{l\neq j} c_l \mu_{F_j} \\
&= \frac{1}{t} \frac{\sigma_c^{1/\alpha-1}}{\sum_{i=1}^n c_i^{1/\alpha}} \sum_{l\neq j} c_l \mu_{F_j} \\
&\sim \frac{1}{F_k^{\leftarrow}(p)} \frac{c_k^{1/\alpha}}{\sigma_c \sum_{i=1}^n c_i^{1/\alpha}} \sum_{l\neq j} c_l \mu_{F_j}
\end{aligned}$$

since $1/t = 1/G^{\leftarrow}(p) \sim (c_k/\sigma_c)^{1/\alpha}/F_k^{\leftarrow}(p)$ as $p\to 1$. This proves (4.6).

(iv) Assume that $\rho\alpha > -1$, $\alpha \geq 1$, and $\mu_{F_\nu} < \infty$ for some ν. Then, from (4.10), we get that

$$\frac{F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} = \left(\frac{c_i}{\sigma_c}\right)^{1/\alpha}\left[1 - \frac{(\sigma_c/c_i)^\rho - 1}{\rho} \times a_i\left(\frac{1}{\overline{G}(t)}\right)(1+o(1))\right], \quad t \to \infty$$

and

$$\frac{\sum_{i=1}^n F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} - \frac{\sum_{i=1}^n c_i^{1/\alpha}}{\sigma_c^{1/\alpha}} \sim \sum_{i=1}^n \left(\frac{c_i}{\sigma_c}\right)^{1/\alpha} \frac{1-(\sigma_c/c_i)^\rho}{\rho} \times a_i\left(\frac{1}{1-p}\right), \quad t \to \infty,$$

implying that

$$\frac{G^{\leftarrow}(p)}{\sum_{i=1}^n F_i^{\leftarrow}(p)} - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}} \sim \left(\frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}}\right)^2 \times \sum_{i=1}^n \left(\frac{c_i}{\sigma_c}\right)^{1/\alpha} \frac{(\sigma_c/c_i)^\rho - 1}{\rho} \times a_i\left(\frac{1}{1-p}\right), \quad t \to \infty.$$

This proves (4.7).

■

Remark 4.2: (i) In Proposition 4.1, the result for the boundary case $\rho\alpha = -1$ is not included because, in this case, the situation is more complicated. Furthermore, in Proposition 4.1, the second-order parameters ρ_i of all U_is are assumed to be the same as ρ. In fact, a refinement of the proof of Proposition 4.1 allow us to derive second-order approximations of $C(p)$ for the different ρ_is.

(ii) By Propositions 2.5 and 2.6, $U_i \in 2\mathrm{RV}_{1/\alpha,\rho}$ with auxiliary function $a_i(t)$ for $\alpha > 0$ and $\rho < 0$ is equivalent to

$$\overline{F}_i \in 2\mathrm{RV}_{-\alpha,\rho\alpha} \text{ with auxiliary function } b_i(t) = \alpha^2 a_i\left(\frac{1}{\overline{F}(t)}\right).$$

PROPOSITION 4.3: *Let $X_1, \ldots, X_n$ be independent non-negative random variables with continuous and asymptotically smooth survival functions $\overline{F}_1, \ldots, \overline{F}_n$, respectively. Assume that* (4.3) *holds with $0 < \alpha < 1$ and $\rho \leq 0$. If* (3.10) *holds, then we have*

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(i) *For* $\rho < -1$,

$$C(p) - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}} \sim (1-p)\frac{J_\alpha \sigma_{cc}\sigma_c^{1/\alpha-2}}{\alpha \sum_{i=1}^n c_i^{1/\alpha}}, \quad p \to 1,$$

where σ_c *and* σ_{cc} *are given in* (3.6)*, and* J_α *is defined by* (3.13)*.*

(ii) *For* $\rho > -1$, (4.7) *holds.*

Proof: In view of Proposition 3.9, note that (4.8) also holds for $G^{\leftarrow}(p)/F_i^{\leftarrow}(p)$ with $\Delta_i(t)$ replaced by

$$\Delta_i(t) = 1 + \frac{J_\alpha}{\alpha} \times \frac{\sigma_{cc}}{c_i\sigma_c}\overline{F}_i(t)(1+o(1)).$$

(i) Assume that $\rho < -1$. Since $\overline{F}_i \in \mathrm{RV}_{-\alpha}$ and $|a_i(1/\overline{G})| \in \mathrm{RV}_{\rho\alpha}$, from (4.8), we have

$$\frac{G^{\leftarrow}(p)}{F_i^{\leftarrow}(p)} = \left(\frac{\sigma_c}{c_i}\right)^{1/\alpha}\left(1 + \frac{J_\alpha}{\alpha} \times \frac{\sigma_{cc}}{c_i\sigma_c}\overline{F}_i(t)(1+o(1))\right)$$

with $t = G^{\leftarrow}(p)$. So

$$\frac{\sum_{i=1}^n F_i^{\leftarrow}(p)}{G^{\leftarrow}(p)} - \frac{\sum_{i=1}^n c_i^{1/\alpha}}{\sigma_c^{1/\alpha}} \sim -\frac{J_\alpha\sigma_{cc}}{\alpha\sigma_c^{1+1/\alpha}}\sum_{i=1}^n c_i^{1/\alpha-1}\overline{F}_i(t),$$

which implies that

$$\frac{G^{\leftarrow}(p)}{\sum_{i=1}^n F_i^{\leftarrow}(p)} - \frac{\sigma_c^{1/\alpha}}{\sum_{i=1}^n c_i^{1/\alpha}} \sim \frac{J_\alpha\sigma_{cc}\sigma_c^{1/\alpha-1}}{\alpha\left(\sum_{i=1}^n c_i^{1/\alpha}\right)^2}\sum_{i=1}^n c_i^{1/\alpha-1}\overline{F}_i(t)$$

$$\sim (1-p)\frac{J_\alpha\sigma_{cc}\sigma_c^{1/\alpha-2}}{\alpha\sum_{i=1}^n c_i^{1/\alpha}},$$

where the last equivalence follows from the fact that

$$\frac{\overline{F}_i(G^{\leftarrow}(p))}{1-p} = \frac{\overline{F}_i(G^{\leftarrow}(p))}{\overline{F}_i(F_i^{\leftarrow}(p))} \sim \left(\frac{G^{\leftarrow}(p)}{F_i^{\leftarrow}(p)}\right)^{-\alpha} \sim \frac{c_i}{\sigma_c}, \quad p \to 1.$$

This proves part (i).

(ii) For $\rho > -1$, the proof is the same as that of part (iv) of Proposition 4.1. This completes the proof. ■

Degen et al. [5] derived second-order approximations of $C(p)$ for i.i.d. heavy-tail random variables $X_1, \ldots, X_n$ by using the theories of second-order regular variation and second-order subexponentiality. Their main result, Theorem 3.1, is a direct consequence of Propositions 4.1 and 4.3.

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