J. Plasma Phys. (2019), *vol.* 85, 925850101
doi:10.1017/S0022377818001319

TUTORIAL

An introduction to the physics of the Coulomb logarithm, with emphasis on quantum-mechanical effects

J. A. Krommes†

Princeton Plasma Physics Laboratory, Princeton University, MS 28, P.O. Box 451, Princeton, NJ 08543–0451, USA

(Received 20 September 2018; revised 30 November 2018; accepted 3 December 2018)

An introduction to the physical interpretation of the Coulomb logarithm is given with particular emphasis on the quantum-mechanical corrections that are required at high temperatures. Excerpts from the literature are used to emphasize the historical understanding of the topic, which emerged more than a half-century ago. Several misinterpretations are noted. Quantum-mechanical effects are related to diffraction by scales of the order of the Debye screening length; they are not due to quantum uncertainty related to the much smaller distance of closest approach.

Key words: plasma properties, quantum plasma

1. Introduction

The Coulomb logarithm $\ln \Lambda$, often called the 'Spitzer logarithm' in honour of its discussion in the pioneering monograph of Spitzer (1962) (and earlier by Cohen, Spitzer & Routly (1950) in the section prepared by Spitzer), is one of the most fundamental quantities in basic plasma physics, as it quantifies the dominance of small-angle scattering in a weakly coupled plasma. In introductory discussions of collisions in plasmas, usually only classical physics is considered. However, for sufficiently high temperatures quantum-mechanical effects become important. Although the proper way of including those effects has been understood at least since the work of Cohen *et al.* (1950), there appears to be some ignorance of that early literature and, thus, some residual uncertainty about the interpretation of the quantum-mechanical corrections. Instead of presenting a full review of this subject, which would be quite lengthy, here I shall merely present a brief tutorial whose purpose is to provide a new student of the subject with some entry points to the literature relating to the basic physics of the Coulomb logarithm. Sometimes-lengthy excerpts from original papers and reviews are used to emphasize the historical development. Several interesting points of confusion are identified. The focus is on conceptual foundations, not practical applications.

† Email address for correspondence: krommes@princeton.edu

I shall address only the restricted problem of the calculation and interpretation of $\ln \varLambda$ in weakly coupled, many-body, charge-neutral plasmas with the usual statistical symmetries of homogeneity and isotropy; other issues, such as the effects of anisotropy considered in the interesting work by Mulser, Alber & Murakami (2014), are not discussed.

2. The Coulomb logarithm in classical kinetic theory – a brief review

One definition of $\varLambda$ is $\varLambda \doteq \lambda_D/b_{min}$ ($\doteq$ denotes a definition), where λ_D is the Debye screening length (the effective maximum impact parameter for two-body scattering) and b_{min} is a characteristic length to be determined. In classical scattering theory,[1] $b_{min} = b_0$, where

$$b_0 \doteq \frac{q_1 q_2}{\mu u^2} \tag{2.1}$$

is the impact parameter for 90° scattering between particles 1 and 2. Here q is the charge, μ is the reduced mass and u is the relative velocity. The distance of closest approach is $2b_0$. For a weakly coupled plasma, b_0 is very much smaller than λ_D; specifically, $b_0/\lambda_D = O(\epsilon_p)$, where $\epsilon_p \doteq 1/(\overline{n}\lambda_D^3)$ is the so-called plasma (expansion) parameter ($\overline{n}$ is the mean density).

The result $b_{min} = b_0$ arises in classical, weakly coupled plasma kinetic theory (Montgomery & Tidman 1964) as follows. One can determine the net scattering cross-section σ, or in more detail the velocity-dependent collision operator, from an approximate calculation of the pair correlation function in the limit of small ϵ_p. That leads to an integration over all possible impact parameters b or, equivalently, over wavenumber magnitude $k \doteq b^{-1}$. In particular, with $\boldsymbol{\epsilon}_{\boldsymbol{k}} \doteq -4\pi \mathrm{i}\boldsymbol{k}/k^2$ being the Fourier representation of the bare Coulomb field of a unit point charge, $\mathcal{D}(\boldsymbol{k}, \omega)$ being the Vlasov dielectric function and f being the one-particle distribution function, the Balescu–Lenard collision operator (which captures the effect of dielectric shielding[2] but not large-angle scattering) for collisions of species s on species $\overline{s}$ is[3]

$$\begin{aligned} C_{s\overline{s}}[f] \doteq & -\pi(\overline{n}m)_s^{-1}(\overline{n}q^2)_s(\overline{n}q^2)_{\overline{s}}\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{\cdot}\int \mathrm{d}\overline{\boldsymbol{v}}\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\frac{\boldsymbol{\epsilon}_{\boldsymbol{k}}\,\boldsymbol{\epsilon}_{\boldsymbol{k}}^*}{|\mathcal{D}(\boldsymbol{k}, \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})|^2}\delta(\boldsymbol{k}\boldsymbol{\cdot}(\boldsymbol{v}-\overline{\boldsymbol{v}})) \\ & \boldsymbol{\cdot}\left(\frac{1}{m_s}\frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_{\overline{s}}}\frac{\partial}{\partial \overline{\boldsymbol{v}}}\right) f_s(\boldsymbol{v}) f_{\overline{s}}(\overline{\boldsymbol{v}}). \end{aligned} \tag{2.2}$$

[1]The collision process in a weakly coupled plasma is discussed in the review article by Sivukhin (1966) and in numerous textbooks such as the one by Helander & Sigmar (2002). An equivalent background is helpful for the subsequent discussion.

[2]As a reminder, the notion of a Debye shielding cloud is a statistical concept. It arises from the spatial pair correlations between an ensemble of field particles and a specified test particle. That the Vlasov approximation to $\mathcal{D}(\boldsymbol{k}, \omega)$ is adequate is a standard result for weak coupling ($\epsilon_p \ll 1$, relevant for magnetic fusion). For that important regime, Rostoker (1964*a*) was led to his famous test-particle superposition principle. (Derivations of the superposition principle that are more compact than Rostoker's original can be found in Krommes (1976).) That principle, in turn, inspired the test-particle methods introduced by Rostoker (1964*b*); they have been extraordinarily successful. A representative and very clear textbook discussion of test-particle techniques can be found in Krall & Trivelpiece (1973). The resulting, well-justified formulas for the weakly coupled electric-field fluctuation spectrum at Debye scales and for the polarization drag on a moving test particle are manifested in the kernel of the Balescu–Lenard operator through the factors of $\boldsymbol{\epsilon}_{\boldsymbol{k}}/\mathcal{D}(\boldsymbol{k}, \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})$, which describe the dynamical shielding of the bare electric field of a test particle moving with velocity $\boldsymbol{v}$.

[3]Derivations of the Balescu–Lenard operator can be found in many textbooks, e.g. Ichimaru (1973, appendix B). See also Krommes (2018*b*, § G.1).

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

The Vlasov dielectric has the well-known form

$$\mathcal{D}(\boldsymbol{k},\omega)=1+\sum_{\bar{s}}\frac{\omega_{p\bar{s}}^2}{k^2}\int \mathrm{d}\bar{\boldsymbol{v}}\,\frac{\boldsymbol{k}\boldsymbol{\cdot}\partial_{\bar{\boldsymbol{v}}}f_{\bar{s}}(\bar{\boldsymbol{v}})}{\omega-\boldsymbol{k}\boldsymbol{\cdot}\bar{\boldsymbol{v}}+\mathrm{i}\epsilon}, \tag{2.3}$$

where $\omega_{p\bar{s}}$ is the plasma frequency. According to (2.2), this is to be evaluated at the characteristic streaming or transit frequency $\omega=\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v}$ (not a normal-mode frequency). Thus, the dependence on wavenumber magnitude cancels out under the $\bar{\boldsymbol{v}}$ integral in (2.3), and without further approximation the integration over wavenumber magnitude requires evaluation of

$$I(k)\doteq\int_0^{k_{max}}\frac{\mathrm{d}k}{k}\left(\frac{1}{[1+A(k\lambda_D)^{-2}]^2+[B(k\lambda_D)^{-2}]^2}\right), \tag{2.4}$$

where $A(\widehat{\boldsymbol{k}},\boldsymbol{v})$ and $B(\widehat{\boldsymbol{k}},\boldsymbol{v})$ are functions that at least for thermal particles are of order unity (Montgomery & Tidman 1964, Chap. 7). The integral (2.4) can be done exactly, but the details are not important for qualitative discussion; for $k_{max}\lambda_D\gg 1$, one finds

$$I(k)=\ln(k_{max}\lambda_D)+O(1). \tag{2.5}$$

To understand this result physically, note that the denominator inside the large parentheses in (2.4) represents the mean-square effect of dielectric shielding and makes the integral strongly convergent at $k\to 0$, effectively limiting the wavenumber integration to $k\gtrsim k_{min}\doteq k_D$, where $k_D\doteq\lambda_D^{-1}$; this corresponds to a maximum impact parameter $b_{max}\approx\lambda_D$. Because Debye shielding is ineffective at small scales, equation (2.4) is logarithmically divergent at large k and must be cut off at some k_{max} (or $b_{min}\doteq k_{max}^{-1}$); the dominant term in (2.5) obviously arises from evaluating $\int_{k_D}^{k_{max}}\mathrm{d}k\,k^{-1}=\int_{b_{min}}^{\lambda_D}\mathrm{d}b\,b^{-1}$. However, this large-$k$ or small-scale divergence is not physical. It arises because the Balescu–Lenard derivation is perturbative, based on zeroth-order straight-line trajectories; thus, large-angle scattering (the effect of impact parameters $b\lesssim b_0$) is misrepresented. There is no solution for this difficulty within the Balescu–Lenard framework. However, one can asymptotically match between an 'inner' solution that treats large-angle scattering correctly and an 'outer' solution appropriate for small-angle scattering, as was done, for example, by Frieman & Book (1963) for the classical regime. Such matching removes any apparent divergence at the small scales, in agreement with the result that the Rutherford cross-section is integrable as the scattering angle $\theta\to\pi$ or $b\to 0$. Of course, the scale b_0 remains in the asymptotically matched solution: it determines the two-body relationship between scattering angle and impact parameter,

$$\tan(\theta/2)=b_0/b; \tag{2.6}$$

it sets the basic size of the classical cross-section σ_{cl}, which is finite and proportional to b_0^2; and the asymptotic matching leads to $k_{max}=b_0^{-1}$ (this is clear on dimensional grounds). One is led to the dominant result[4] $\sigma_{cl}\sim b_0^2\ln\Lambda_{cl}$, where[5]

$$\ln\Lambda_{cl}\equiv L_{cl}\doteq\ln(\lambda_D/b_0)=O(\ln(\epsilon_p^{-1})). \tag{2.7}$$

[4] From (2.5), one sees that corrections to this result are of $O(1)$, i.e. of relative order $(\ln\Lambda_{cl})^{-1}\sim[\ln(\epsilon_p^{-1})]^{-1}\ll 1$. For the typical values $\ln\Lambda_{cl}\sim 10$–20, the corrections are thus in the range of 5–10 %. Note that if k_{max} is replaced by αk_{max}, where $\alpha=O(1)$, one must consider $\ln(\alpha k_{max}\lambda_D)=\ln(k_{max}\lambda_D)+\ln\alpha=\ln(k_{max}\lambda_D)+O(1)$. Thus, uncertainty in the choice of cutoff does not affect the dominant asymptotics.

[5] Conventional notation introduces $\lambda\doteq\ln\Lambda$. In this paper I use L instead of λ in order to avoid confusion with the symbols λ_D and λ_B.

Since this L_{cl} is independent of $\widehat{\boldsymbol{k}}$ and $\boldsymbol{v}$, one can perform the integration over solid angle in (2.2); one is ultimately led (Lenard 1960) to the Landau collision operator (Landau 1936), which is generally used in practice. The Landau operator is discussed in appendix B of Krommes (2018*a*) and in various textbooks.

An alternative way of arriving at $\ln \varLambda_{\text{cl}}$ is to begin with the Boltzmann collision operator (which does not incorporate Debye shielding but does handle large-angle scattering correctly), then to take its small-angle limit. (That amounts to considering a Fokker–Planck operator that does not incorporate shielding effects.) In that limit, one encounters the integral $\int_{k_{\min}}^{k_{\max}} \mathrm{d}k\, k^{-1} = \int_{b_{\min}}^{b_{\max}} \mathrm{d}b\, b^{-1}$, to be considered in the classical regime. Clearly, the logarithmic divergence at large b must be rectified by inserting the cutoff $b_{\max} = \lambda_{\text{D}}$, which as I discussed follows systematically from the Balescu–Lenard formalism.

In both of the above two procedures, which are correct only for small-angle scattering, the necessity for a small-scale cutoff remains. In order to deal with the misrepresentation of large-angle scattering in lieu of complicated asymptotic analysis, students are generally taught to merely insert the short-scale cutoff $b_{\min} = b_0$ or $k_{\max} = b_0^{-1}$, which describes the cross-over between large- and small-angle scattering. That is the correct result, and it is more than hand waving. It must be stressed that that recipe encapsulates the result of a systematic asymptotic matching between the classical regimes of large and small impact parameters, that the underlying physics is convergent for small impact parameters and that no discontinuity or exponentially rapid variation that would lead to diffraction occurs in the vicinity of b_0. These remarks expand upon some of those in the paper by Mulser *et al.* (2014).

It is insightful to transform the impact-parameter integration to an integration over scattering angle θ. When that is done, one sees according to (2.6) that the effect of dielectric shielding is to introduce a cutoff at small θ, so $\sigma \sim \int_{\theta_{\min}} \mathrm{d}\theta/\theta \sim \ln \theta_{\min}^{-1}$; see, for example, Landau & Lifshitz (1981, (41.9)). Classically, one has $\theta_{\min} \approx \theta_0 \doteq 2b_0/\lambda_{\text{D}}$ (which follows from the small-angle limit of (2.6)). In the next section, we shall see that quantum-mechanical diffraction effects on the Debye scale lead to a larger value of $\theta_{\min}$ for sufficiently high temperatures, and thus to a smaller value of $\ln \varLambda$.

3. Quantum-mechanical corrections to $\ln \varLambda$

As noted by Spitzer and more clearly explained in the 149 page review article by Sivukhin (1966), at sufficiently high temperature the classical approximation fails, quantum-mechanical considerations apply, and one finds $b_{\min} = \lambda_{\text{B}} \doteq \hbar/\mu u$, the 'de Broglie wavelength of the test particle in the coordinate system in which the scattering center ... is at rest' (Sivukhin 1966). Interpolation between the classical and quantum-mechanical limits leads to the standard prescription

$$b_{\min} = \max\{\lambda_{\text{B}}, b_0\}. \tag{3.1}$$

Spitzer (1962, p. 128) said it this way:[6]

> When the electron temperature exceeds approximately 4×10^5 degrees K, $\varLambda$ must be somewhat reduced below the values obtained from [classical theory], because of quantum-mechanical effects. An electron wave passing through a circular aperture of radius b will be spread out by diffraction through an angle $\lambda/2\pi b$, where λ is the electron wavelength. If this

[6]To conform to my notation, I have changed Spitzer's p to b, and w to u.

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

> deflection exceeds the classical deflection $2b_0/b$, then the previous equations must be modified; in accordance with the results of Marshak (1941) the only change needed is to reduce Λ by the ratio $\alpha c/u$, where α is the fine structure constant, equal to $1/137$.

Essentially the same discussion appears in Cohen *et al.* (1950, p. 233). Spitzer referred to the choice of a deflection angle, called θ_{min} above. His words correspond to the fact that when $\lambda_B > b_0$ diffraction of the de Broglie wave by an opaque disc of radius λ_D (not the very much smaller radius λ_B) produces a diffraction angle θ_B that is larger than the classical deflection angle θ_0 for a particle incident with maximum impact parameter $b_{max} = \lambda_D$; thus, $\sigma_{qu} \sim \ln \theta_B^{-1} < \sigma_{cl} \sim \ln \theta_0^{-1}$.

A possible source of confusion is that Spitzer did not completely spell out the argument; he did not explicitly state that the diffraction angle should be evaluated with $b_{max} = \lambda_D$, although this is clearly implied by the fact that λ_D appears in the classical θ_0, and by Spitzer's discussion (on the page preceding the above quotation) of λ_D as the maximum impact parameter). Instead, he cited a paper by Marshak (1941). Tracing back through the historical record is interesting. Marshak stated,[7]

> Thus far we have not given any explicit form for I [I being the θ integral of the Rutherford differential scattering cross section $\sigma(\theta)$]. If we look at the integral expression for I [(3.8) below with $\varepsilon = 0$] we see that it diverges if we integrate between the limits 0 and π. However, there are physical grounds for extending this integration only to some small angle θ_{min}, in which case:
>
> $$2I = \log_e \frac{2}{(1 - \cos\theta_{min})}.$$
>
> Now it can be shown that $\theta_{min} = \lambda\!\!\!^{-}/a$* where $\lambda\!\!\!^{-}$ is the de Broglie wavelength of the electron participating in the collision, and 'a' is the screening radius . . . .

Marshak referred to the screening radius 'of the atom', but it is clear that in the present plasma context one should replace a by λ_D. Thus, we see more clearly that Spitzer was recapitulating and providing a physical interpretation of Marshak's argument, which was focused on the physics of θ_{min} or, equivalently, the physics associated with maximum impact parameter λ_D. In turn, Marshak's * footnote, which provides the background to his assertion 'it can be shown that . . . ', refers to p. 497 of the famous paper of Bethe (1933) on the quantum mechanics of one- and two-electron atoms. Bethe worked in the first Born approximation (for which he cited earlier literature) and discussed a form factor F that determines $\sigma(\theta)$. In modern notation, the formula for $\sigma(\theta)$ is given by (3.6). (This reduces to the Rutherford cross-section when $\theta_B \rightarrow 0$.) Bethe emphasized,[8]

> Der Atomformfaktor F selbst hängt vom Ablenkungswinkel θ – genauer von $(\sin\theta/2)/\lambda_B$ – ab.

(The last ratio is not dimensionless; obviously, the intended comparison is between $\sin\frac{1}{2}\theta$ and $\frac{1}{2}\theta_B$ or, for small angle, between θ and θ_B.) It is the quantum correction to the Rutherford cross-section, which is important at small scattering angle and was well

[7]To conform to my notation, I have replaced Marshak's θ_0 by θ_{min}.

[8]In English: 'The atomic form factor F itself depends on the scattering angle θ – more precisely on $\sin(\theta/2)/\lambda$.'

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

known to the pioneers of quantum mechanics, that underlies Marshak's conclusion and Spitzer's interpretation.

The first Born approximation does not, in and of itself, predict the interpolation recipe (3.1). Indeed, in that approximation the integral that defines the total momentum transfer can be done exactly, a result known to Sivukhin (see (3.6) in the excerpt below) and surely earlier workers as well. The inner length b_0 enters the resulting formula only multiplicatively (the total scattering cross-section is proportional to b_0^2). However, the first Born approximation is valid (at best; see later discussion) only at high energies. Sivukhin explains the issue clearly:[9]

> 3. The classical-mechanics analysis applies so long as $(2\pi/\lambda)b_0 \gg 1$, where $\lambda = h/\mu u$ is the de Broglie wavelength of the test particle in the coordinate system in which the scattering center (field particle) is at rest. . . . We can write this condition in the form
>
> $$u \ll \alpha c, \tag{3.2}$$
>
> where
>
> $$\alpha = \left| \frac{ee^*}{\hbar c} \right| \tag{3.3}$$
>
> ($\hbar = h/2\pi = 1.05 \cdot 10^{-27}$ erg s). If e and e^* are equal to the elementary charge the constant $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant.
>
> The classical analysis cannot be used if (3.2) is not satisfied. This result might appear strange at first glance since the exact quantum-mechanical solution for scattering of a charged particle in a Coulomb field yields an expression for $\sigma(\theta, u)$ which is exactly the same as the classical expression . . . (cf., for example, Davydov (1976) or any text on quantum mechanics). The essential point here, however, is that the results coincide only when the scattering field is a Coulomb field over all space. In the case of a cutoff Coulomb field the wave properties of the particle are appreciably different from those given by the classical analysis.
>
> When
>
> $$u \gg \alpha c, \tag{3.4}$$
>
> the quantum-mechanical scattering problem can be solved relatively easily by means of the Born approximation. The solution of the problem is simplified if the cutoff Coulomb field is replaced by a Debye field with potential
>
> $$\varphi = \frac{e^*}{r} \mathrm{e}^{-r/\lambda_D}. \tag{3.5}$$
>
> If (3.4) is satisfied, it is well known that the quantum-mechanical analysis leads to the result
>
> $$\sigma(\theta, u) = \left(\frac{ee^*}{2\mu u^2} \right)^2 \frac{1}{\left(\sin^2 \dfrac{\theta}{2} + \varepsilon^2 \right)^2}, \tag{3.6}$$
>
> where
>
> $$\varepsilon = \frac{\lambda}{4\pi\lambda_D} = \frac{\hbar}{2\mu u \lambda_D} \tag{3.7}$$
>
> (cf., for example, [any textbook discussion of quantum-mechanical scattering theory]).

[9]To conform to my notation, I have changed Sivukhin's D to λ_D.

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

By substituting (3.6) [into the formula for mean momentum transfer] we recover [the classical result (2.7)] with the sole difference that the classical value of the Coulomb logarithm is replaced by the quantum-mechanical value

$$\begin{aligned} L_{qu} &= \frac{1}{4}\int_0^{\pi} \frac{\sin^2\dfrac{\theta}{2}\sin\theta}{\left(\sin^2\dfrac{\theta}{2}+\varepsilon^2\right)^2}\,d\theta \\ &= \frac{1}{2}\ln\frac{1+\varepsilon^2}{\varepsilon^2} - \frac{1}{2(1+\varepsilon^2)}. \end{aligned} \tag{3.8}$$

In all cases of physical interest it is found that

$$\varepsilon = \frac{\lambda}{2\pi\lambda_D} \ll 1, \tag{3.9}$$

so that the square of ε can be neglected compared with unity. In this approximation

$$L_{qu} = \ln\frac{1}{\varepsilon} - \frac{1}{2} = \ln\frac{4\pi\lambda_D}{\lambda} - \frac{1}{2}. \tag{3.10}$$

If the term $-1/2$ is neglected this expression differs from the classical value [$\ln(\lambda_D/b_0)$] only in that the lower limit b_0 is replaced by $\lambda/4\pi$. This result is easily understood: the De Broglie wave associated with the incident particle is diffracted on the Debye sphere surrounding the scattering center. Diffraction theory or elementary interference considerations shows that to within a factor of order unity the mean value of the diffraction angle is $\theta = \lambda/2\lambda_D$. If this value exceeds the classical limit $\theta_{min} = 2b_0/\lambda_D$, the classical formula ... no longer applies and $\theta = \lambda/2\lambda_D$ is to be taken as the lower limit in the integral in (3.8). This procedure leads to $L = \ln(4\lambda_D/\lambda)$, which differs from (3.10) by the unimportant factor of π under the logarithm. It also follows from this qualitative description that scattering on a cutoff Coulomb field is essentially the same as scattering on a Debye field (3.5).

The quantum-mechanical relation (3.10) can be written in the form

$$L_{qu} = L_{cl} + \ln\frac{2\alpha c}{u} - \frac{1}{2} \tag{3.11}$$

[which, since $\alpha c/u < 1$, shows that when the diffraction correction is valid the size of the Coulomb logarithm is reduced].

We recall that this formula is derived under the assumption that $u \gg \alpha c$, whereas the classical expression (2.7) applies when $u \ll \alpha c$.

4. The quantum-mechanical calculation becomes extremely complicated in the intermediate region. It would not be very meaningful to be concerned with this region because we are already using the binary-collision approximation with its artificial and, indeed, somewhat arbitrary truncation of the Coulomb forces so that any improvement in the values of the Coulomb logarithm obtained as a result of more complex calculations

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

> would be quite illusory. Instead, it is simpler and more in the spirit of the approximation used here to proceed as follows. The formulas for the limiting cases (2.7) and (3.10) show that the Coulomb logarithm contains the velocity u under the logarithm so that the latter is a slowly varying function of u. It is physically obvious that this slow variation also obtains in the intermediate region. Hence, without incurring any serious error we can extrapolate (2.7) and (3.10) into the intermediate region up to the value $u = u_{\mathrm{lim}}$, at which both expressions coincide. When $u < u_{\mathrm{lim}}$ the classical formula (2.7) is to be used; when $u \gg u_{\mathrm{lim}}$ the quantum-mechanical formulas (3.10) or (3.11) are used.
>
> ... If the relative velocity u is replaced by the equivalent temperature according to the relation $3T = \mu u^2$ ... [and upon] substituting the appropriate values of the reduced mass for a deuterium plasma, we obtain the following limiting temperatures for electron–electron, electron–ion, and ion–ion collisions, respectively:

$$\left\{\begin{array}{l} T^{ee}_{\mathrm{lim}} = 6.65\ \mathrm{eV}, \\ T^{ei}_{\mathrm{lim}} = 13.3\ \mathrm{eV}, \\ T^{ii}_{\mathrm{lim}} = 2.45 \cdot 10^4\ \mathrm{eV} = 24.5\ \mathrm{keV}. \end{array}\right. \tag{3.12}$$

A similar discussion can be found on p. 239 of the book by Kulsrud (2005), who cites Sivukhin. (Kulsrud arrives at somewhat different but qualitatively similar transition temperatures by using a different interpolation scheme.)

All of the above discussions are consistent in their physical interpretations. Interestingly, however, Mulser *et al.* (2014) challenged Spitzer's heuristic description. They stated,

> A special argument for [the prescription (3.1)] is by Spitzer (1962). He arrives at the limitation $b \geqslant \lambda_{\mathrm{B}}$ by observing that for impact parameters $b \leqslant \lambda_{\mathrm{B}}$ the Coulomb differential cross section leads to higher diffraction values than an opaque disc of the same radius, which is 'unphysical.' It seems that for numerous researchers this constitutes the basic argument. ... Although physically appealing at first glance, it is false and self contradictory. In the neighborhood of the Coulomb singularity, the author compares Rutherford scattering with optical diffraction from a diaphragm of diameter $2\lambda_{\mathrm{B}}$. ... Spitzer's setting of $b_{\mathrm{min}} = \lambda_{\mathrm{B}}$ is a prominent example of excellent physical intuition but mistaken proof.

This characterization of Spitzer's discussion is incorrect and appears to be a misunderstanding; nowhere in his argument did Spitzer mention 'a diaphragm of diameter $2\lambda_{\mathrm{B}}$' or discuss impact parameters smaller than λ_{B}. In fact, in agreement with the various authors cited above, Mulser *et al.* (2014) also concluded that the de Broglie correction was associated with large impact parameters. But although they asserted that this was a new and surprising result, we see that it has been understood for more than a half-century.

Although the basic ideas and results are clear, some discussions in the literature are incomplete; for example, Mulser *et al.* (2014) cited a number of references in which apparently the classical cutoff was used. However, closer inspection shows that in several of the papers cited by Mulser *et al.* (2014) the authors were, in fact, aware of the quantum correction. For example, Rosenbluth, Macdonald & Judd (1957) refer in their footnotes 6 and 3 to the discussion of cutoffs by Cohen *et al.* (1950), which, as we have seen, contains the original version of Spitzer's argument. And although Balescu (1988) described the classical situation, he noted (p. 130),

> Some authors (e.g. Braginskii 1965) consider semi-empirical corrections to $\ln \Lambda$ under various conditions of temperature and density; we do not discuss these minor points here.

One could quibble with Balescu's characterization of the issue as 'minor', and the first Born approximation does not deserve to be called 'semi-empirical'. In any event, it is instructive to consider the explanation of Braginskii (1965, p. 238), who said,[10]

> At large velocities, in which case $e^2/hv < 1$, where h is Planck's constant (i.e., $v/c > 1/137$), it is necessary to use a smaller value for the maximum impact parameter; specifically, we use the distance for which the scattering angle is of the same order as the quantum uncertainty, in which case $p_{max} \approx \lambda_D e^2/hv$.

Here it is asserted that instead of using the classical formula $\theta_{min} \sim b_0/b_{max}$ with $b_{max} = \lambda_D$, one should use $b_{max} = \lambda_D(b_0/\lambda_B)$ or $\theta_{min} \sim \lambda_B/\lambda_D$, the latter 'quantum uncertainty' being the diffraction angle of a wave with wavelength λ_B encountering an object of radius λ_D. But although Braginskii's argument leads to the correct θ_{min}, his heuristic introduction of a modified b_{max} is incorrect; apparently, it was devised in order to obtain agreement with the proper θ_{min} and the earlier discussion of Spitzer. However, Braginskii did appreciate the role of the quantum-mechanical uncertainty due to diffraction by Debye screening clouds of radius λ_D, and he understood that the classical cutoff was not to be used for large velocities.

The proper value of θ_{min} must follow from a systematic kinetic theory. Clearly, λ_B can enter the problem only when quantum-mechanical effects are included. An asymptotic matching between the Born approximation and the quasi-classical regime is described by Landau & Lifshitz (1981, § 46), where original references are cited. The physics content of that calculation agrees with the heuristic arguments given above. A recent and more refined (as yet unpublished) analysis by R. Kulsrud (private communication, 2018) corrects those estimates by a relatively small amount in the five per cent range.

4. Discussion and summary

Are the quantum-mechanical corrections to $\ln \Lambda$ significant? For definiteness, consider electron–ion collisions. For cold ions, Huba (2016) quotes the easy-to-remember formula[11]

$$\ln \Lambda_{ei} = \begin{cases} c_{cl} - \ln(n_e^{1/2} Z T_e^{-3/2}) & \text{for } T_e < 10\,Z^2 \text{ eV}, \\ c_{qu} - \ln(n_e^{1/2} T_e^{-1}) & \text{for } 10\,Z^2 \text{ eV} < T_e, \end{cases} \tag{4.1}$$

[10]This quote corrects a typographical error; the original article describes the large-velocity limit as $v/c < 1/137$. Also, I have changed Braginskii's δ_D to λ_D.

[11]With $\Delta \doteq c_{qu} - c_{cl} - \ln Z - 1 \approx 0$ (for $Z = 1$), formula (4.1) predicts the crossover temperature $T_{e,c} = e^{2(1+\Delta)} \approx 7.4 e^{2\Delta}$. Clearly, small changes in the precise value of Δ can change $T_{e,c}$ by an amount of order unity. But the basic point is that this temperature, of the order of 10 eV, is much lower than the approximately 10 KeV temperatures characteristic of fusion plasmas, for which evidently quantum corrections play a role.

https://doi.org/10.1017/S0022377818001319 Published online by Cambridge University Press

where $c_{\rm cl} \approx 23$, $c_{\rm qu} \approx 24$, Z is the atomic number, n_e is in units of $\rm cm^{-3}$ and T_e is in eV. For the ITER-like parameters (FusionWiki 2012) $n_e \approx 10^{14}\ \rm cm^{-3}$, $T_e \approx 8.8$ KeV and $Z = 1$, one finds that the relative quantum-mechanical correction is approximately 17 %. Although this is not entirely negligible, one might question the quantitative significance of the quantum effects given that there are already subdominant corrections to $\ln \varLambda$ (recall footnote 4 on p. 3). Of course, this is just one numerical example; there are other possibilities best left for the discussion of specific applications. In any event, such a concern misses the point. The quantum effects are conceptually interesting and introduce a new dimension to the physics of the collision processes in plasmas. Clarifying one's understanding of those processes is a worthy goal in itself.

In summary, excerpts from the literature provide historical perspective and explain the correct interpretation of the quantum-mechanical correction to $\ln \varLambda$. The $\lambda_{\rm B}$ in the ratio $\varLambda_{\rm qu} = \lambda_{\rm D}/\lambda_{\rm B}$ that appears in the first Born approximation (which is, roughly speaking, valid for sufficiently high temperatures) 'is the result of the charge distributions at large impact parameters' (Mulser *et al.* 2014), not quantum uncertainty related to the impact parameter b_0 for 90° scattering or to the distance of closest approach ($2b_0$). This conclusion was obtained early on by Cohen *et al.* (1950), Spitzer (1962) and Sivukhin (1966), and it has been usefully repeated in more modern texts such as those of Kulsrud (2005) and Wesson (2011, chap. 14.5). Proper understanding of the physics of the Coulomb logarithm is crucial, as that basic quantity figures in a multitude of important plasma-physics applications.

Acknowledgements

I am grateful to R. Kulsrud for informative discussions, and to G. Hammett, M. Kunz and E. Startsev for critical comments on early drafts of the manuscript. This work was supported by the US Department of Energy Contract DE-AC02-09CH11466.

REFERENCES

BALESCU, R. 1988 *Transport Processes in Plasmas*, vol. 1. Elsevier.

BETHE, H. A. 1933 Quantenmechanik der Ein- und Zwei-Electronenprobleme. *Handbuch d. Phys.* **24** (1), 273–560.

BRAGINSKII, S. I. 1965 Transport processes in a plasma. In *Reviews of Plasma Physics, Vol. 1* (ed. M. N. Leontovich), pp. 205–311. Consultants Bureau.

COHEN, R. S., SPITZER, L. JR & ROUTLY, P. MCR. 1950 The electrical conductivity of an ionized gas. *Phys. Rev.* **80**, 230–238.

DAVYDOV, A. S. 1976 *Quantum Mechanics*. Pergamon (translated, edited, and with additions by D. ter Haar).

FRIEMAN, E. A. & BOOK, D. L. 1963 Convergent classical kinetic equation for a plasma. *Phys. Fluids* **6**, 1700–1706.

FUSIONWIKI 2012 http://fusionwiki.ciemat.es/fusionwiki/index.php?title=ITER&oldid=4090.

HELANDER, P. & SIGMAR, D. J. 2002 *Collisional Transport in Magnetized Plasmas*. Cambridge University Press.

HUBA, J. D. 2016 NRL Plasma Formulary. Naval Research Laboratory, https://www.nrl.navy.mil/ppd/content/nrl-plasma-formulary.

ICHIMARU, S. 1973 *Basic Principles of Plasma Physics – A Statistical Approach*. Benjamin.

KRALL, N. A. & TRIVELPIECE, A. W. 1973 *Principles of Plasma Physics*. McGraw-Hill.

KROMMES, J. A. 1976 Two new proofs of the test particle superposition principle of plasma kinetic theory. *Phys. Fluids* **19**, 649–655.

KROMMES, J. A. 2018*a* Projection-operator methods for classical transport in magnetized plasmas. Part 1. Linear response, the Braginskii equations and fluctuating hydrodynamics. *J. Plasma Phys.* **84**, 925840401.

KROMMES, J. A. 2018*b* Projection-operator methods for classical transport in magnetized plasmas. Part 2. Nonlinear response and the Burnett equations. *J. Plasma Phys.* **84**, 905840601.

KULSRUD, R. M. 2005 *Plasma Physics for Astrophysics*. Princeton University Press.

LANDAU, L. D. 1936 Kinetic equation in the case of the Coulomb interaction. *Phys. Z. Sowjetunion* **10**, 154 (JETP **7**, 203–9 (1937)).

LANDAU, L. D. & LIFSHITZ, E. N. 1981 *Physical Kinetics*. Pergamon.

LENARD, A. 1960 On Bogoliubov's kinetic equation for a spatially homogeneous plasma. *Ann. Phys.* **10**, 390–400.

MARSHAK, R. E. 1941 The radiative and conductive opacities under white dwarf conditions. *Ann. N.Y. Acad. Sci.* **41**, 49–60.

MONTGOMERY, D. C. & TIDMAN, D. A. 1964 *Plasma Kinetic Theory*. McGraw-Hill.

MULSER, P., ALBER, G. & MURAKAMI, M. 2014 Revision of the Coulomb logarithm in the ideal plasma. *Phys. Plasmas* **21**, 042103.

ROSENBLUTH, M. N., MACDONALD, W. & JUDD, D. L. 1957 Fokker–Planck equation for an inverse-square force. *Phys. Rev.* **107**, 1–6.

ROSTOKER, N. 1964*a* Superposition of dressed test particles. *Phys. Fluids* **7**, 479–490.

ROSTOKER, N. 1964*b* Test particle method in kinetic theory of a plasma. *Phys. Fluids* **7**, 491–498.

SIVUKHIN, D. V. 1966 Coulomb collisions in a fully ionized plasma. In *Reviews of Plasma Physics*, vol. 4, pp. 93–241. Consultants Bureau.

SPITZER, L. JR 1962 *Physics of Fully Ionized Gases*. Interscience.

WESSON, J. 2011 *Tokamaks*, 4th edn. Oxford University Press.