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BUNDER'S PARADOX

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Abstract. Systems of *illative logic* are logical calculi formulated in the untyped λ-calculus supplemented with certain logical constants.[1] In this short paper, I consider a paradox that arises in illative logic. I note two prima facie attractive ways of resolving the paradox. The first is well known to be consistent, and I briefly outline a now standard construction used by Scott and Aczel that establishes this. The second, however, has been thought to be inconsistent. I show that this isn't so, by providing a nonempty class of models that establishes its consistency. I then provide an illative logic which is sound and complete for this class of models. I close by briefly noting some attractive features of the second resolution of this paradox.

§1. Illative paradoxes. We can characterize the terms of an *untyped λ-language* as follows.

DEFINITION 1.1. *Let Σ be a signature, i.e., a set of constants, and let V be a set of variables. The set of* untyped λ-terms, *given Σ and V, $\mathbb{T}_\Sigma^V$, is then defined as follows*:

- $x \in \mathbb{T}_\Sigma^V$, *for each* $x \in V$
- $X \in \mathbb{T}_\Sigma^V$, *for each* $X \in \Sigma$
- $XY \in \mathbb{T}_\Sigma^V$, *for each* $X, Y \in \mathbb{T}_\Sigma^V$
- $\lambda x.Y \in \mathbb{T}_\Sigma^V$, *for each* $Y \in \mathbb{T}_\Sigma^V$, *and each* $x \in V$.

An important relation amongst untyped λ-terms is that of β-equivalence. We can define this as follows.

DEFINITION 1.2. *We say that a term of the form $(\lambda x.Y)Z$ is a* β-redex *and $Y[x/Z]$ is its* β-contractum.[2]

DEFINITION 1.3. *We say that X is* β-equivalent *to X', $X =_\beta X'$, just in case X' results from X by changes of bound variables, and a finite number of substitutions of β-redexes for β-contractums and β-contractums for β-redexes.*

Let Σ be a signature containing a primitive or defined constant $\rightarrow$.[3] The well-known Curry Paradox shows that the following prima facie attractive principles entail every

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[1] Or systems formulated in an untyped combinatory language with additional logical constants. In this paper, I will primarily focus on versions formulated using the untyped λ-calculus. Note, though, that all of the main claims I make also hold, mutatis mutandis, for systems formulated in a combinatory language.

[2] Here $Y[x/Z]$ is the result of substituting Z for free occurrences of x in Y. It is assumed that Z is free for x in Y. For more details see Hindley & Seldin (2008).

[3] In what follows, we'll take $A \rightarrow B$ to be a definitional abbreviation for $\rightarrow AB$.

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term.[4] Here we take Γ to be a *set* of terms, and let X, Y etc. be terms. We let Γ, Γ' serve as an abbreviation for $\Gamma \cup \Gamma'$, and Γ, X as an abbreviation for $\Gamma \cup \{X\}$.

$$(\text{Ax})\ \frac{}{\Gamma, X \vdash X} \qquad (\text{Eq})\ \frac{\Gamma \vdash X \qquad X =_\beta Y}{\Gamma \vdash Y}$$

$$(\to E)\ \frac{\Gamma \vdash X \to Y \qquad \Gamma' \vdash X}{\Gamma, \Gamma' \vdash Y} \qquad (\to I)\ \frac{\Gamma, X \vdash Y}{\Gamma \vdash X \to Y}$$

Curry's Paradox: Given (Ax), (Eq), $(\to I)$, and $(\to E)$ we have $\vdash \bot$ for any arbitrary term $\bot$.

Proof. Let $C =_\beta C \to \bot$, where $\bot$ some arbitrary term.[5] By (Ax), we have $C \vdash C$. Since we have $C =_\beta C \to \bot$, by (Eq), we have $C \vdash C \to \bot$. Thus, by $(\to E)$, we have $C \vdash \bot$. And so, by $(\to I)$, we have $\vdash C \to \bot$, which, by (Eq), gives us $\vdash C$. And so finally, by $(\to E)$, we have $\vdash \bot$.

Now a natural assumption in developing an illative logical system is that some but not all terms in a λ-language with a constant $\to$ will express propositions. But, given this, it is also natural to think that certain prima facie plausible inference rules may need to be restricted in certain ways when some of the terms that they involve do not express propositions.

A natural conclusion, then, to draw from Curry's Paradox is that at least some of the rules: (Ax), (Eq), $(\to I)$, $(\to E)$, need to be restricted to allow for cases where some of the terms that they involve do not express propositions. This indeed was the response advocated by Curry and others.[6] In particular, it was suggested that the appropriate way to block the above derivation is to restrict the principle $(\to I)$.

Let Σ now be a signature containing, in addition to the constant $\to$, a constant *Prop*, which should be glossed as meaning: *is a proposition*. The suggestion advocated by Curry and others is that we should replace $(\to I)$ by the following:

$$(\to I_s)\ \frac{\Gamma, X \vdash Y \qquad \Gamma' \vdash Prop(X)}{\Gamma, \Gamma' \vdash X \to Y}$$

And indeed this restriction will suffice to block the derivation. For we can show that (Ax), (Eq), $(\to I_s)$, and $(\to E)$ are consistent.[7]

Of course, once we've introduced the constant *Prop* and appealed to it to restrict certain inference rules, we will want some principles that tell us when a term of the form $Prop(X)$ follows from some other terms Γ. And, at least prima facie, the following two principles seem attractive:

$$(\text{Prop 1})\ \frac{\Gamma \vdash X}{\Gamma \vdash Prop(X)} \qquad (\text{Prop 2})\ \frac{}{\Gamma \vdash Prop(Prop(X))}$$

4 See Curry (1942a).

5 Note that we can be assured that there is such a C given the existence of a λ-term Y such that, for every X, we have $YX =_\beta X(YX)$. A term with this functional behavior is called a Y-combinator. An example of such a term is: $\lambda z.(\lambda x.z(xx)\lambda x.z(xx))$.

6 See Curry, Hindley, & Seldin (1972).

7 This is a consequence of the existence of the nonempty class of models we will consider in §3.

The idea behind (Prop 1) is that the only terms that should be provable in an illative logic, given some set of assumptions, are those that express propositions, given those same assumptions. And the idea behind (Prop 2) is that while various terms may fail to express propositions, for any term X, the claim that X is a proposition should be a proposition. According to (Prop 2), then, *Prop* is a propositional function in the sense that we have $Prop(Prop(X))$ for every term X.

Despite their prima facie plausibility, however, Martin Bunder has shown that, while (Ax), (Eq), ($\rightarrow I_s$), and ($\rightarrow E$) are consistent, they are not jointly consistent with (Prop 1) and (Prop 2).[8]

> **Bunder's Paradox:** Given (Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$), (Prop 1) and (Prop 2), we have $\vdash \bot$ for any arbitrary term $\bot$.
>
> *Proof*. Let $B =_\beta Prop(B) \rightarrow (B \rightarrow \bot)$, where $\bot$ is some arbitrary term. By (Ax), we have $Prop(B), B \vdash B$. And so, by (Eq), we have $Prop(B), B \vdash Prop(B) \rightarrow (B \rightarrow \bot)$. And since, by (Ax), we have $Prop(B), B \vdash Prop(B)$, it follows, by ($\rightarrow E$), that we have $Prop(B), B \vdash \bot$. And so, by ($\rightarrow I_s$), we have $Prop(B) \vdash B \rightarrow \bot$. And since, by (Prop 2), we have $\vdash PropProp(B)$, it follows by ($\rightarrow I_s$) that we have $\vdash Prop(B) \rightarrow B \rightarrow \bot$, and so, by (Eq), we have $\vdash B$. And since, by (Prop 1), we have $\vdash Prop(B)$, it follows, by ($\rightarrow E$), that we have $\vdash \bot$.

I take it that (Ax) and (Eq) are nonnegotiable principles, if one wants to develop any reasonably strong illative logic. Given Bunder's Paradox, then, it seems that we are forced to either weaken the logic of $\rightarrow$ or the logic of *Prop*.

One option, then, is to try to weaken the principles concerning *Prop* in order to hold on to the principles concerning $\rightarrow$. Here the most natural option, I take it, is to reject (Prop 2) in order to hold on to (Prop 1), in addition to (Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$). For, insofar as we are forced to choose between (Prop 1) and (Prop 2), it seems more desirable to hold on to the inference from provability to propositionhood than to hold on to the claim that *Prop* is a propositional function.

In §3, I will briefly outline a construction due to Scott and Aczel which shows that this does provide one consistent response to this paradox.

Another prima facie attractive option, though, is to try to weaken the principles concerning $\rightarrow$ in order to hold on to the principles concerning *Prop*. Here the most natural option, I take it, is to further weaken ($\rightarrow I_s$) in order to hold on to ($\rightarrow E$) in addition to (Ax), (Eq), (Prop 1) and (Prop 2). Of course, for this option to be at all attractive, we would need there to be some other suitably principled introduction rule for $\rightarrow$ that was consistent with this set of principles. And the following seems like the most natural option:

$$(\rightarrow I_w)\ \frac{\Gamma, X \vdash Y \qquad \Gamma' \vdash Prop(X) \qquad \Gamma'' \vdash Prop(Y)}{\Gamma, \Gamma', \Gamma'' \vdash X \rightarrow Y}$$

Interestingly, this option has not, to my knowledge, been developed or shown to be consistent. Indeed, it has been claimed that a slight strengthening of the conjunction of (Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1) and (Prop 2) is inconsistent.[9] In particular, it has been

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[8] See Bunder (1976). For a related paradox see: Bunder (1970), Curry (1942b).

[9] See Bunder & Meyer (1978) for a putative proof of this inconsistency, and Czajka (2015) pp. 35–36 for brief discussion of the putative inconsistency of this package of principles.

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claimed that (Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1) and (Prop 2) are jointly inconsistent with the following additional principle:

$$(\text{Prop} \rightarrow)\ \frac{\Gamma \vdash Prop(X) \qquad \Gamma' \vdash Prop(Y)}{\Gamma, \Gamma' \vdash Prop(X \rightarrow Y)}$$

Since (Prop $\rightarrow$) seems extremely plausible, if these principles were all jointly inconsistent this would, I think, show that the correct response to Bunder's Paradox is to weaken the logic governing *Prop*. In §4, however, I will present a model construction technique that can be used to prove the consistency of (Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1), (Prop 2) and (Prop $\rightarrow$). Given the consistency of this package of principles, it strikes me as being at the very least a subtle question what the right response is to Bunder's Paradox.

§2. λ-Algebras.

In this section, we briefly describe a class of structures suitable for interpreting an untyped λ-language. The key feature of these structures is that terms that are β-equivalent are assigned the same denotation when interpreted in these structures.[10]

DEFINITION 2.1. *Let Σ be a signature, and let V be a set of variables. In addition to the members of Σ, we also take there to be two distinguished constants K and S. The set of* untyped combinatory-terms*, given Σ and V, $\mathbb{CT}^V_\Sigma$, is then defined as follows*:

- $x \in \mathbb{CT}^V_\Sigma$, *for each* $x \in V$
- $X \in \mathbb{CT}^V_\Sigma$, *for each* $X \in \Sigma$
- $X \in \mathbb{CT}^V_\Sigma$, *if either* $X = K$ *or* $X = S$
- $XY \in \mathbb{CT}^V_\Sigma$, *for each* $X, Y \in \mathbb{CT}^V_\Sigma$.

DEFINITION 2.2. *Let $\mathcal{C} = \langle C, \cdot \rangle$, where C is a set, and $\cdot$ is a binary operation on C. We say that $\mathcal{C}$ is a* combinatory algebra *just in case there are elements* $\mathbf{k}, \mathbf{s} \in C$ *such that*: (i) $\mathbf{k} \cdot a \cdot b = a$, *and* (ii) $\mathbf{s} \cdot a \cdot b \cdot c = (a \cdot c) \cdot (b \cdot c)$, *for all* $a, b, c \in C$.

Given a combinatory algebra, we can provide an interpretation for a set of untyped combinatory terms, relative to an assignment to variables, as follows.

DEFINITION 2.3. *Let $\mathcal{C} = \langle C, \cdot \rangle$ be a combinatory algebra. Let $\rho : V \rightarrow C$ and $[\![\cdot]\!] : \Sigma \rightarrow C$. We now define a function $[\![\cdot]\!]_\rho : \mathbb{CT}^V_\Sigma \rightarrow C$ as follows*:

- $[\![x]\!]_\rho = \rho(x)$, *if* $x \in V$
- $[\![X]\!]_\rho = [\![X]\!]$, *if* $X \in \Sigma$
- $[\![K]\!]_\rho = \mathbf{k}$
- $[\![S]\!]_\rho = \mathbf{s}$
- $[\![XY]\!]_\rho = [\![X]\!]_\rho \cdot [\![Y]\!]_\rho$.

We now describe a way of simulating λ-abstraction in a combinatory language.

DEFINITION 2.4. *For each $Y \in \mathbb{CT}^V_\Sigma$, we let $\lambda^*x.Y \in \mathbb{CT}^V_\Sigma$ be defined as follows*:

- $\lambda^*x.x = SKK$, *for* $x \in V$
- $\lambda^*x.Z = KZ$, *if Z does not contain any occurences of x*
- $\lambda^*x.ZQ = S(\lambda^*x.Z)(\lambda^*x.Q)$.

[10] Propositions listed without proof or a reference are standard results that may be found, together with more detailed discussion of such structures, in Barendregt (1984) or Hindley & Seldin (2008).

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Given this way of simulating λ-abstraction, we can now describe two mappings, one which takes untyped lambda terms to untyped combinatory terms, the other which takes untyped combinatory terms to untyped lambda terms.

DEFINITION 2.5. *We let* $CL : \mathbb{T}^V_\Sigma \to \mathbb{CT}^V_\Sigma$ *be the mapping such that*:

- $CL(x) = x$, *if* $x \in V$
- $CL(Y) = Y$, *if* $Y \in \Sigma$
- $CL(YZ) = CL(Y)CL(Z)$
- $CL(\lambda x.Y) = \lambda^* x.CL(Y)$.

DEFINITION 2.6. *We let* $\Lambda : \mathbb{CT}^V_\Sigma \to \mathbb{T}^V_\Sigma$ *be the mapping such that*:

- $\Lambda(x) = x$, *if* $x \in V$
- $\Lambda(Y) = Y$, *if* $Y \in \Sigma$
- $\Lambda(YZ) = \Lambda(Y)\Lambda(Z)$
- $\Lambda(K) = \lambda x \lambda y.x$
- $\Lambda(S) = \lambda x \lambda y \lambda z.xz(yz)$.

PROPOSITION 2.7. *Let* $X, Y \in \mathbb{T}^V_\Sigma$. *If* $X =_\beta Y$, *then* $\Lambda(CL(X)) =_\beta \Lambda(CL(Y))$.[11]

DEFINITION 2.8. *Given a combinatory algebra* $\mathcal{C} = \langle C, \cdot \rangle$, *we let* $\mathbb{T}^V_\mathcal{C}$ *and* $\mathbb{CT}^V_\mathcal{C}$ *be, respectively, the sets of untyped* λ*-terms and sets of untyped combinatory terms, where we* (*ambiguously*) *take* $\mathcal{C}$ *to be a signature such that each* $s_c \in \mathcal{C}$ *uniquely corresponds to some* $c \in C$.

We are now in a position to characterize the class of structures appropriate for interpreting the untyped λ-calculus.

DEFINITION 2.9. *Let* $\mathcal{C} = \langle C, \cdot \rangle$ *be a combinatory algebra. And let* $[\![\cdot]\!] : \mathbb{CT}^V_\mathcal{C} \to \mathcal{C}$ *be such that we have* $[\![s_c]\!] = c$, *for each* $c \in C$. *Then we say that* $\mathcal{C}$ *is a* λ-algebra *just in case*: *if* $\Lambda(X) =_\beta \Lambda(Y)$, *then* $[\![X]\!]_\rho = [\![Y]\!]_\rho$.

PROPOSITION 2.10. *Let* $\mathcal{C} = \langle C, \cdot \rangle$ *be a* λ*-algebra and let* $[\![\cdot]\!] : \mathbb{CT}^V_\mathcal{C} \to \mathcal{C}$ *be as above. Then if* $X, Y \in \mathbb{T}^V_\mathcal{C}$ *are such that* $X =_\beta Y$, *then* $[\![CL(X)]\!]_\rho = [\![CL(Y)]\!]_\rho$.

Proof. Assume that $X =_\beta Y$. Then by 2.7, we have $\Lambda(CL(X)) =_\beta \Lambda(CL(Y))$. And so it follows, since $\mathcal{C}$ is a λ-algebra that we have $[\![CL(X)]\!]_\rho = [\![CL(Y)]\!]_\rho$. □

As a consequence of the preceding, it follows that we can interpret untyped λ-terms by assigning denotations to constants in a λ-algebra.

DEFINITION 2.11. *Let* $\mathcal{C} = \langle C, \cdot \rangle$ *be a* λ*-algebra. Let* $\rho : V \to C$ *and* $[\![\cdot]\!] : \Sigma \to C$. *We take* $[\![\cdot]\!]_\rho : \mathbb{CT}^V_\Sigma \to C$ *to be as defined above. We now define* $[\![\cdot]\!]_\rho : \mathbb{T}^V_\Sigma \to C$ *as follows*:

- $[\![x]\!]_\rho = \rho(x)$, *if* $x \in V$
- $[\![X]\!]_\rho = [\![X]\!]$, *if* $X \in \Sigma$
- $[\![XY]\!]_\rho = [\![X]\!]_\rho \cdot [\![Y]\!]_\rho$
- $[\![\lambda x.Y]\!]_\rho = [\![CL(\lambda x.Y)]\!]_\rho$.

PROPOSITION 2.12. *Let* $\mathcal{C} = \langle C, \cdot \rangle$ *be a* λ*-algebra and let* $[\![\cdot]\!] : \mathbb{T}^V_\Sigma \to \mathcal{C}$ *be as above. Then if* $X, Y \in \mathbb{T}^V_\Sigma$ *are such that* $X =_\beta Y$, *then* $[\![X]\!]_\rho = [\![Y]\!]_\rho$.

[11] See Hindley & Seldin (2008) for a proof of this.

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PROPOSITION 2.13. *For each $X \in \mathbb{T}^V_\Sigma$, let $[X]$ be the β-equivalence class of members of $\mathbb{T}^V_\Sigma$. Let $C = \{[X] : X \in \mathbb{T}^V_\Sigma\}$. And let $[X] \cdot [Y] = [XY]$. Then $\langle C, \cdot \rangle$ is a λ-algebra.*

§3. Strong Kleene Illative Algebras.

We now describe a method for constructing models which satisfy (Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$), and (Prop 1).[12] In what follows, it will prove to be expeditious to work with a signature that contains primitive constants $\neg$, $\vee$. We can then take $A \rightarrow B$ to be defined as: $\neg A \vee B$.[13]

DEFINITION 3.1. *Let $\mathcal{C}$ be a λ-algebra, with distinguished elements*: **n**, **v**, **p**. *We say that $\langle \mathcal{C}, T, F \rangle$ is a* Strong Kleene Illative Algebra *just in case*:

(i) $T, F \subseteq C$, *and* $T \cap F = \emptyset$
(ii) $\mathbf{v} \cdot a \cdot b \in T$ *iff* $a \in T$ *or* $b \in T$
(iii) $\mathbf{v} \cdot a \cdot b \in F$ *iff* $a \in F$ *and* $b \in F$
(iv) $\mathbf{n} \cdot a \in T$ *iff* $a \in F$
(v) $\mathbf{n} \cdot a \in F$ *iff* $a \in T$
(vi) $\mathbf{p} \cdot a \in T$ *iff* $a \in T \cup F$.

DEFINITION 3.2. *Let Σ be such that it contains distinguished constants: $\neg$, $\vee$, Prop. Let $\mathcal{I} = \langle \mathcal{C}, T, F \rangle$ be a Strong Kleene Illative Algebra. Let $[\![\cdot]\!] : \Sigma \rightarrow C$ be such that*:

- $[\![\neg]\!] = \mathbf{n}$
- $[\![\vee]\!] = \mathbf{v}$
- $[\![Prop]\!] = \mathbf{p}$.

We let $[\![\cdot]\!]_\rho : \mathbb{T}^V_\Sigma \rightarrow C$ be defined as above. We say that $\langle \mathcal{I}, [\![\cdot]\!] \rangle$ is a Strong Kleene Illative Model *for* $\mathbb{T}^V_\Sigma$.

DEFINITION 3.3. *We let $\Gamma \models_{sk} X$ just in case, for every Strong Kleene Illative Model, and every $\rho : V \rightarrow C$, if $[\![Y]\!]_\rho \in T$, for every $Y \in \Gamma$, then $[\![X]\!]_\rho \in T$.*

THEOREM 3.4. *Let $A \rightarrow B =_{df} \neg A \vee B$. And let $\vdash_s$ be the smallest relation closed under (Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$), and (Prop 1). We have that if $\Gamma \vdash_s X$, then $\Gamma \models_{sk} X$.*

Proof. This can be proved by a simple induction. □

We now show that there are Strong Kleene Illative Algebras.

DEFINITION 3.5. *Let $\emptyset \subset P \subset \Sigma$, $\{\vee, \neg, Prop\} \subset \Sigma$ and $P \cap \{\vee, \neg, Prop\} = \emptyset$. For each $X \in \mathbb{T}^V_\Sigma$, we let $[X]$ be the β-equivalence class of members of $\mathbb{T}^V_\Sigma$. Then we let $\mathcal{C} = \langle C, \cdot \rangle$, where $C = \{[X] : X \in \mathbb{T}^V_\Sigma\}$ and $[X] \cdot [Y] = [XY]$.*

PROPOSITION 3.6. *By 2.13, we have that $\mathcal{C} = \langle C, \cdot \rangle$ is a λ-algebra.*

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12 The construction which leads to such models is described in Scott (1975) and Aczel (1980). The construction from Aczel (1980) is used to interpret first-order illative systems in Hindley & Seldin (1986), Chapter 17, while Bunder (1979) investigates how the construction developed in Scott (1975) may be used interpret illative systems. A similar construction also appears in Fitch (1981), and, in the context of truth theories, in Kripke (1975). More recently, such fixed-point constructions have been used in Czajka (2013) and Czajka (2015) to construct models of higher-order systems of illative logic.

13 Where $\neg A \vee B$ is itself a definitional abbreviation for $\vee(\neg A)B$.

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We now inductively define two sequences of subsets of C.

DEFINITION 3.7. *Let $T_0 \subset [P]$ and $F_0 \subset [P]$ be such that $T_0 \cap F_0 = \emptyset$ and $T_0 \cup F_0 = [P]$. We let $T_{\alpha+1}$ and $F_{\alpha+1}$ be defined as follows*:

- *If $p \in P$ and $[p] \in T_\alpha$, then $[p] \in T_{\alpha+1}$*
- *If $p \in P$ and $[p] \in F_\alpha$, then $[p] \in F_{\alpha+1}$*
- *If $[X] \in T_\alpha$ or $[Y] \in T_\alpha$, then $[\vee XY] \in T_{\alpha+1}$*
- *If $[X] \in F_\alpha$ and $[Y] \in F_\alpha$, then $[\vee XY] \in F_{\alpha+1}$*
- *If $[X] \in F_\alpha$, then $[\neg X] \in T_{\alpha+1}$*
- *If $[X] \in T_\alpha$, then $[\neg X] \in F_{\alpha+1}$*
- *If $[X] \in T_\alpha \cup F_\alpha$, then $[Prop(X)] \in T_{\alpha+1}$.*

Finally, where λ is a limit ordinal, we let $T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha$, and $F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$.

LEMMA 3.8. *If $\alpha \leq \beta$, then $T_\alpha \subseteq T_\beta$ and $F_\alpha \subseteq F_\beta$.*

Proof. This may be proved by a simple induction. □

LEMMA 3.9. *There is some α such that $T_\alpha = T_{\alpha+1}$ and $F_\alpha = F_{\alpha+1}$.*

Proof. By 3.8, we have that the sequences T_β and F_β are both monotonically increasing sequences of subsets of C. It is a consequence of the Knaster-Tarski fixed-point theorem that each such sequence stabilizes. And since each sequence stabilizes, it follows that there is some α such that $T_\alpha = T_{\alpha+1}$ and $F_\alpha = F_{\alpha+1}$.[14] □

PROPOSITION 3.10. *If A and B are constants and $AX_1, \ldots, X_n =_\beta BY_1, \ldots, Y_k$, then $A = B$, $n = k$ and $X_i =_\beta Y_i$, for each i.*[15]

LEMMA 3.11. *For each ordinal α, $T_\alpha \cap F_\alpha = \emptyset$.*

Proof. Assume that for all $\beta < \alpha$ we have $T_\beta \cap F_\beta = \emptyset$. We will show that we have $T_\alpha \cap F_\alpha = \emptyset$. There are three cases to consider.

(i) $\alpha = 0$. In this case, by construction, we have $T_0 \cap F_0 = \emptyset$.

(ii) $\alpha = \gamma + 1$. By our Induction Hypothesis we have that $T_\gamma \cap F_\gamma = \emptyset$. Now assume that we have $[X] \in T_{\gamma+1} \cap F_{\gamma+1}$. Then, given our construction, we must have either (a) $X =_\beta p$, for some $p \in P$, (b) $X =_\beta \vee ZQ$, for some terms Z and Q, (c) $X =_\beta \neg Z$, for some term Z, or (d) $X =_\beta Prop(Z)$, for some term Z. And, given 3.10, it follows that exactly one of (a)–(d) can obtain.
If (a) uniquely obtains, then given our construction it follows that $[X] = [p] \in T_\gamma \cap F_\gamma$ which contradicts our Induction Hypothesis. If (b) uniquely obtains, then given our construction we must have $[Z] \in F_\gamma$ and $[Q] \in F_\gamma$, and either $[Z] \in T_\gamma$ or $[Q] \in T_\gamma$, which contradicts our Induction Hypothesis. If (c) uniquely obtains, then given our construction we must have $[Z] \in T_\gamma \cap F_\gamma$, contradicting our Induction

[14] See Tarski (1955). It's worth noting that the Knaster-Tarski fixed-point theorem, in full generality, is stronger than is needed here. For one can show by a simple induction that if there is some α such that $[X] \in T_\alpha$, then there is some $n \in \mathbb{N}$ such that $[X] \in T_n$, and similarly that if there is some α such that $[X] \in F_\alpha$, then there is some $n \in \mathbb{N}$ such that $[X] \in F_n$. It follows, given the monotonicity of these sequences, that we have $T_\omega = T_{\omega+1}$ and $F_\omega = F_{\omega+1}$, and so, more generally, $T_\omega = T_\alpha$ and $F_\omega = F_\alpha$, for each $\alpha \geq \omega$.

[15] See Hindley & Seldin (1986) Corollary 1.35.5 for a proof.

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Hypothesis. Finally, given our construction, if (d) holds then it cannot be that $[X] = [Prop(Z)] \in T_{\gamma+1} \cap F_{\gamma+1}$. For, given our construction, we have, in general, that $[Prop(Z)] \notin F_\alpha$. This may be proved by a simple induction, but the key fact is that there is no clause in our construction that allows $[Prop(Z)] \in F_\alpha$.

(iii) $\alpha = \lambda$ for some limit ordinal λ. In this case, given our construction, $T_\lambda \cap F_\lambda = \emptyset$ follows straightforwardly from the Induction Hypothesis that for all $\beta < \lambda$ we have $T_\beta \cap F_\beta = \emptyset$, together with 3.8, which tells us that our sequences increase monotonically. □

THEOREM 3.12. *The class of Strong Kleene Illative Algebras is nonempty.*

Proof. Let $\mathcal{I}_s = \langle \mathcal{C}, T, F \rangle$, where $T = T_\alpha = T_{\alpha+1}$ and $F = F_\alpha = F_{\alpha+1}$.[16] Letting $\mathbf{n} = [\neg]$, $\mathbf{v} = [\vee]$ and $\mathbf{p} = [Prop]$, we verify that this structure satisfies conditions (i)–(vi).

We have that $\mathcal{I}_s$ satisfies condition (i), by 3.11. For conditions (ii)–(vi) we note that, given our construction, we immediately have the right-to-left direction of these biconditionals. And the left-to-right directions follow from the fact that, given 3.10, at most one of (a) $X =_\beta p$, for some $p \in P$, (b) $X =_\beta \vee ZQ$, for some terms Z and Q, (c) $X =_\beta \neg Z$, for some term Z, or (d) $X =_\beta Prop(Z)$, for some term Z can obtain. Thus consider the case of (ii). Suppose that we have $[\vee ZQ] \in T = T_{\alpha+1}$. Then, given our construction and the above fact, the only way that this could obtain is if $[Z] \in T = T_\alpha$ or $[Q] \in T = T_\alpha$. And so we have the left-to-right direction of (ii). Cases (iii)–(vi) may be justified in the same manner. □

COROLLARY 3.13. *(Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$), and (Prop 1) are jointly consistent.*

Proof. Given the existence of Strong Kleene Illative Algebras, it follows that there are Strong Kleene Illative Models. The joint consistency of (Ax), (Eq), ($\rightarrow I_s$), ($\rightarrow E$), and (Prop 1) follows from 3.4. □

§4. Weak Kleene Illative Algebras.

We now describe a method for constructing models which satisfy (Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1), (Prop 2), and (Prop $\rightarrow$).[17] We will continue to work with a signature that contains primitive constants $\neg$, $\vee$, and take $A \rightarrow B$ to be defined as: $\neg A \vee B$.

DEFINITION 4.1. *Let $\mathcal{C}$ be a λ-algebra, with distinguished elements*: $\mathbf{n}$, $\mathbf{v}$, $\mathbf{p}$. *We say that $\langle \mathcal{C}, T, F \rangle$ is a* Weak Kleene Illative Algebra *just in case*:

(i) $T, F \subseteq C$, *and* $T \cap F = \emptyset$

(ii) $\mathbf{v} \cdot a \cdot b \in T$ *iff* $a, b \in T \cup F$ *and either* $a \in T$ *or* $b \in T$

(iii) $\mathbf{v} \cdot a \cdot b \in F$ *iff* $a \in F$ *and* $b \in F$

(iv) $\mathbf{n} \cdot a \in T$ *iff* $a \in F$

(v) $\mathbf{n} \cdot a \in F$ *iff* $a \in T$

(vi) $\mathbf{p} \cdot a \in T$ *iff* $a \in T \cup F$

(vii) $\mathbf{p} \cdot a \in F$ *iff* $a \notin T \cup F$.

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[16] The existence of such an ordinal is ensured by 3.9. Note that we can let α be ω or any other ordinal greater than ω.

[17] See Gupta & Martin (1984) for this construction in the context of truth theories.

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DEFINITION 4.2. *Let Σ be such that it contains distinguished constants*: $\neg, \vee$, *Prop*. *Let* $\mathcal{I} = \langle \mathcal{C}, T, F \rangle$ *be a Weak Kleene Illative Algebra. Let* $[\![\cdot]\!] : \Sigma \to C$ *be such that*:

- $[\![\neg]\!] = \mathbf{n}$
- $[\![\vee]\!] = \mathbf{v}$
- $[\![Prop]\!] = \mathbf{p}$.

We let $[\![\cdot]\!]_\rho : \mathbb{T}^V_\Sigma \to C$ *be defined as above. We say that* $\langle \mathcal{I}, [\![\cdot]\!] \rangle$ *is a* Weak Kleene Illative Model *for* $\mathbb{T}^V_\Sigma$.

DEFINITION 4.3. *We let* $\Gamma \models_{wk} X$ *just in case, for every Weak Kleene Illative Model, and every* $\rho : V \to C$, *if* $[\![Y]\!]_\rho \in T$, *for every* $Y \in \Gamma$, *then* $[\![X]\!]_\rho \in T$.

THEOREM 4.4. *Let* $A \to B =_{df} \neg A \vee B$. *And let* $\vdash_w$ *be the smallest relation closed under (Ax), (Eq),* $(\to E)$, $(\to I_w)$, *(Prop 1) (Prop 2), and (Prop* $\to$*). We have that if* $\Gamma \vdash_w X$, *then* $\Gamma \models_{wk} X$.

Proof. This can be proved by a simple induction. □

We now show that there are Weak Kleene Illative algebras. Our proof proceeds as follows. We first construct a sequence of monotonically increasing subsets of C, H_α. This sequence will have a fixed-point, H. The role of this construction is to pick out the subset of C that we can informally think of as representing the members of C that are propositions. We will then construct two sequences of monotonically increasing subsets of C, T_α and F_α. These sequences will also have fixed-points, T and F, which can be easily shown to satisfy conditions (i)–(v). In addition, this construction ensures that we have $[Prop(X)] \in T$ just in case $[X] \in H$, and $[Prop(X)] \in F$ just in case $[X] \notin H$. We then show that $H = T \cup F$. This ensures that $[Prop(X)] \in T$ just in case $[X] \in T \cup F$ and $[Prop(X)] \in F$ just in case $[X] \notin T \cup F$, ensuring that T and F satisfy conditions (vi) and (vii).

As in the preceding section, we let $\emptyset \subset P \subset \Sigma$, $\{\vee, \neg, Prop\} \subset \Sigma$ and $P \cap \{\vee, \neg, Prop\} = \emptyset$, and we let $\mathcal{C} = \langle C, \cdot \rangle$, where $C = \{[X] : X \in \mathbb{T}^V_\Sigma\}$ and $[X] \cdot [Y] = [XY]$. We inductively define a sequence of subsets of C as follows.

DEFINITION 4.5. *Let* $[PROP]$ *be the set of elements of C of the form* $[Prop(X)]$, *for some term X. Let* $H_0 = [P] \cup [PROP]$.

We let $H_{\alpha+1}$ *be defined as follows*:

- *If* $p \in P$, *then* $[p] \in H_{\alpha+1}$
- $[Prop(X)] \in H_{\alpha+1}$, *for each* $X \in \mathbb{T}^V_\Sigma$
- *If* $[X] \in H_\alpha$ *and* $[Y] \in H_\alpha$, *then* $[\vee XY] \in H_{\alpha+1}$
- *If* $[X] \in H_\alpha$, *then* $[\neg X] \in H_{\alpha+1}$.

Finally, where λ *is a limit ordinal, we let* $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$.

LEMMA 4.6. *If* $\alpha \leq \beta$, *then* $H_\alpha \subseteq H_\beta$.

Proof. This may be proved by a simple induction. □

LEMMA 4.7. *There is some* α *such that* $H_\alpha = H_{\alpha+1}$.

Proof. This follows by 4.6 and the Knaster-Tarski fixed-point theorem.[18] □

[18] Note that one could also show by a simple induction, and without appeal to the Knaster-Tarski fixed-point theorem, that if there is some α such that $[X] \in H_\alpha$, then there is some $n \in \mathbb{N}$ such that $[X] \in H_n$. It follows, given the monotonicity of this sequence, that we have $H_\omega = H_{\omega+1}$, and so, more generally, $H_\omega = H_\alpha$, for each $\alpha \geq \omega$.

We now construct a pair of sequences of monotonically increasing subsets of C.

DEFINITION 4.8. *Let* $H =_{df} H_\alpha = H_{\alpha+1}$.[19] *Next, let* $T_0 \subset [P] \cup [PROP]$ *and* $F_0 \subset [P] \cup [PROP]$ *such that* $T_0 \cap F_0 = \emptyset$, $T_0 \cup F_0 = [P] \cup [PROP]$. *In addition, we assume that if* $[X] \in H$, *then* $[Prop(X)] \in T_0$, *and if* $[X] \notin H$, *then* $[Prop(X)] \in F_0$. *Note that the consistency of these assumptions follows from 3.10.*

We let $T_{\alpha+1}$ *and* $F_{\alpha+1}$ *be defined as follows*:

- *If* $p \in P$ *and* $[p] \in T_\alpha$, *then* $[p] \in T_{\alpha+1}$
- *If* $p \in P$ *and* $[p] \in F_\alpha$, *then* $[p] \in F_{\alpha+1}$
- *If* $[Prop(X)] \in T_\alpha$, *then* $[Prop(X)] \in T_{\alpha+1}$
- *If* $[Prop(X)] \in F_\alpha$, *then* $[Prop(X)] \in F_{\alpha+1}$
- *If* $[X], [Y] \in T_\alpha \cup F_\alpha$, *and either* $[X] \in T_\alpha$ *or* $[Y] \in T_\alpha$, *then* $[\vee XY] \in T_{\alpha+1}$
- *If* $[X] \in F_\alpha$ *and* $[Y] \in F_\alpha$, *then* $[\vee XY] \in F_{\alpha+1}$
- *If* $[X] \in F_\alpha$, *then* $[\neg X] \in T_{\alpha+1}$
- *If* $[X] \in T_\alpha$, *then* $[\neg X] \in F_{\alpha+1}$.

Finally, where λ *is a limit ordinal, we let* $T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha$, *and* $F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$.

LEMMA 4.9. *If* $\alpha \leq \beta$, *then* $T_\alpha \subseteq T_\beta$ *and* $F_\alpha \subseteq F_\beta$.

Proof. This may be proved by a simple induction. □

LEMMA 4.10. *There is some* α *such that* $T_\alpha = T_{\alpha+1}$ *and* $F_\alpha = F_{\alpha+1}$.

Proof. This follows by 4.9 and the Knaster-Tarski fixed-point theorem.[20] □

LEMMA 4.11. *For each* α, $H_\alpha = T_\alpha \cup F_\alpha$.

Proof. We assume that, for all $\beta < \alpha$, $H_\beta = T_\beta \cup F_\beta$. We will show that $H_\alpha = T_\alpha \cup F_\alpha$. There are three cases to consider.

(i) $\alpha = 0$. In this case, we have $H_0 = [P] \cup [PROP] = T_0 \cup F_0$.

(ii) $\alpha = \gamma + 1$. Then by our Induction Hypothesis we have: $H_\gamma = T_\gamma \cup F_\gamma$. We will show that: $H_{\gamma+1} = T_{\gamma+1} \cup F_{\gamma+1}$. To this end, we will show that: $H_{\gamma+1} \subseteq T_{\gamma+1} \cup F_{\gamma+1}$. The proof that: $T_{\gamma+1} \cup F_{\gamma+1} \subseteq H_{\gamma+1}$ proceeds mutatis mutandis.

Suppose that $[X] \in H_{\gamma+1}$. Then, given our construction and 3.10 we have that exactly one of the following conditions obtains: (a) $X =_\beta p$, for some $p \in P$, (b) $X =_\beta Prop(Y)$, for some Y, (c) $X =_\beta \vee ZQ$, for some Z and Q, or (d) $X =_\beta \neg Z$, for some Z.

(a) Since $[p] \in T_\alpha \cup F_\alpha$, for every α, we have that $[p] = [X] \in T_{\gamma+1} \cup F_{\gamma+1}$.

(b) Since $[Prop(Y)] \in T_\alpha \cup F_\alpha$, for every α, we have that $[Prop(Y)] = [X] \in T_{\gamma+1} \cup F_{\gamma+1}$.

(c) We have $[\vee ZQ] \in H_{\gamma+1}$. It follows from this given the construction and 3.10 that $[Z] \in H_\gamma$ and $[Q] \in H_\gamma$. By our Induction Hypothesis we have, then, that $[Z] \in T_\gamma \cup F_\gamma$ and $[Q] \in T_\gamma \cup F_\gamma$. And so we have: $[\vee ZQ] \in T_{\gamma+1} \cup F_{\gamma+1}$.

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19 The existence of such an ordinal is ensured by 4.7. In particular, we can let α be ω or any other ordinal greater than ω.

20 Again, one could also show by a simple induction, and without appeal to the Knaster-Tarski fixed-point theorem, that if there is some α such that $[X] \in T_\alpha$, then there is some $n \in \mathbb{N}$ such that $[X] \in T_n$, and similarly that if there is some α such that $[X] \in F_\alpha$, then there is some $n \in \mathbb{N}$ such that $[X] \in F_n$. It follows, given the monotonicity of these sequences, that we have $T_\omega = T_{\omega+1}$ and $F_\omega = F_{\omega+1}$, and so, more generally, $T_\omega = T_\alpha$ and $F_\omega = F_\alpha$, for each $\alpha \geq \omega$.

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(d) We have $[\neg Z] \in H_{\gamma+1}$. It follows from this given the construction and 3.10 that $[Z] \in H_\gamma$. By our Induction Hypothesis, it follows that we have $[Z] \in T_\gamma \cup F_\gamma$. And so we have: $[\neg Z] \in T_{\gamma+1} \cup F_{\gamma+1}$.

We've shown then that $H_{\gamma+1} \subseteq T_{\gamma+1} \cup F_{\gamma+1}$. And a parallel proof will establish: $T_{\gamma+1} \cup F_{\gamma+1} \subseteq H_{\gamma+1}$. Thus we have: $H_{\gamma+1} = T_{\gamma+1} \cup F_{\gamma+1}$.

(iii) $\alpha = \lambda$, for some limit ordinal λ. We show that $H_\lambda = T_\lambda \cup F_\lambda$. To this end we show: $H_\lambda \subseteq T_\lambda \cup F_\lambda$. Again, the proof of $T_\lambda \cup F_\lambda \subseteq H_\lambda$ proceeds mutatis mutandis.

Suppose that $[X] \in H_\lambda$. Then we have that there is some $\alpha < \lambda$ such that $[X] \in H_\alpha$. By our Induction Hypothesis it follows that $[X] \in T_\alpha \cup F_\alpha$. And so, it follows that $[X] \in T_\lambda \cup F_\lambda$. We thus have that $H_\lambda \subseteq T_\lambda \cup F_\lambda$. And we can provide a parallel proof that $T_\lambda \cup F_\lambda \subseteq H_\lambda$. Thus we have: $H_\lambda = T_\lambda \cup F_\lambda$. □

LEMMA 4.12. *For each ordinal α, $T_\alpha \cap F_\alpha = \emptyset$.*

Proof. The proof for this is a simple variant, mutatis mutandis, of the proof of 3.11. □

THEOREM 4.13. *The class of Weak Kleene Illative Algebras is nonempty.*

Proof. Let $\mathcal{I}_w = \langle \mathcal{C}, T, F \rangle$, where $T = T_\alpha = T_{\alpha+1}$ and $F = F_\alpha = F_{\alpha+1}$.[21] Letting $\mathbf{n} = [\neg]$, $\mathbf{v} = [\vee]$ and $\mathbf{p} = [Prop]$, we verify that this structure satisfies conditions (i)–(vii).

We have that $\mathcal{I}_w$ satisfies condition (i), by 4.12. For conditions (ii)–(v) we note that, given our construction, we immediately have the right-to-left direction of these biconditionals. And the left-to-right directions follow from the fact that, given 3.10, at most one of (a) $X =_\beta p$, for some $p \in P$, (b) $X =_\beta \vee ZQ$, for some terms Z and Q, (c) $X =_\beta \neg Z$, for some term Z, or (d) $X =_\beta Prop(Z)$, for some term Z can obtain. Thus consider the case of (ii). Suppose that we have $[\vee ZQ] \in T = T_{\alpha+1}$. Then, given our construction and the above fact, the only way that this could obtain is if $[Z], [Q] \in T \cup F = T_\alpha \cup F_\alpha$ and either $[Z] \in T = T_\alpha$ or $[Q] \in T = T_\alpha$. And so we have the left-to-right direction of (ii). Cases (iii)–(v) may be justified in the same manner.

To see that (vi)–(vii) hold note that, given 4.11, it follows that $H = T \cup F$. Moreover it follows given our construction and 3.10 that we have $[Prop(X)] \in T$ just in case $[X] \in H$ and $[Prop(X)] \in F$ just in case $[X] \notin H$. But then we have that $[Prop(X)] \in T$ just in case $[X] \in T \cup F$ and $[Prop(X)] \in F$ just in case $[X] \notin T \cup F$. Thus conditions (vi)–(vii) are satisfied. □

COROLLARY 4.14. *(Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1), (Prop 2), and (Prop $\rightarrow$) are jointly consistent.*

Proof. Given the existence of Weak Kleene Illative Algebras, it follows that there are Weak Kleene Illative Models. The joint consistency of (Ax), (Eq), ($\rightarrow E$), ($\rightarrow I_w$), (Prop 1), (Prop 2), and (Prop $\rightarrow$) follows from 4.4. □

§5. Weak Kleene Illative Logic.

Let $\vdash_{wk}$ be the smallest relation satisfying the following postulates:

[21] The existence of such an ordinal is ensured by 4.10. In particular, we can let α be ω or any other ordinal greater than ω.

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$$(\text{Ax})\ \frac{}{\Gamma, X \vdash_{wk} X} \qquad (\text{Eq})\ \frac{\Gamma \vdash_{wk} X \qquad X =_{\beta} Y}{\Gamma \vdash_{wk} Y}$$

$$(\text{Cut})\ \frac{\Gamma \vdash_{wk} A \qquad \Gamma', A \vdash_{wk} X}{\Gamma, \Gamma' \vdash_{wk} X}$$

$$(\neg E)\ \frac{\Gamma \vdash_{wk} A \qquad \Gamma' \vdash_{wk} \neg A}{\Gamma, \Gamma' \vdash_{wk} X}$$

$$(\text{DN 1})\ \frac{}{\Gamma, \neg\neg X \vdash_{wk} X} \qquad (\text{DN 2})\ \frac{}{\Gamma, X \vdash_{wk} \neg\neg X}$$

$$(\vee E)\ \frac{\Gamma \vdash_{wk} X \vee Y \qquad \Gamma', X, Y \vdash_{wk} Z \qquad \Gamma'', X, \neg Y \vdash_{wk} Z \qquad \Gamma''', \neg X, Y \vdash_{wk} Z}{\Gamma, \Gamma', \Gamma'', \Gamma''' \vdash_{wk} Z}$$

$$(\vee I1)\ \frac{\Gamma \vdash_{wk} X \qquad \Gamma' \vdash_{wk} Prop(Y)}{\Gamma, \Gamma' \vdash_{wk} X \vee Y} \qquad (\vee I2)\ \frac{\Gamma \vdash_{wk} Y \qquad \Gamma' \vdash_{wk} Prop(X)}{\Gamma, \Gamma' \vdash_{wk} X \vee Y}$$

$$(\vee/\neg 1)\ \frac{\Gamma \vdash_{wk} \neg(X \vee Y)}{\Gamma \vdash_{wk} \neg X} \qquad (\vee/\neg 2)\ \frac{\Gamma \vdash_{wk} \neg(X \vee Y)}{\Gamma \vdash_{wk} \neg Y}$$

$$(\vee/\neg 3)\ \frac{\Gamma \vdash_{wk} \neg X \qquad \Gamma' \vdash_{wk} \neg Y}{\Gamma, \Gamma' \vdash_{wk} \neg(X \vee Y)}$$

$$(\text{Prop 1})\ \frac{\Gamma \vdash_{wk} X}{\Gamma \vdash_{wk} Prop(X)} \qquad (\text{Prop 2})\ \frac{}{\Gamma \vdash_{wk} Prop(Prop(X))}$$

$$(\text{Prop 3})\ \frac{\Gamma, A \vdash_{wk} \neg(X \vee \neg X) \qquad \Gamma', \neg A \vdash_{wk} \neg(X \vee \neg X)}{\Gamma, \Gamma' \vdash_{wk} \neg Prop(A)}$$

$$(\text{Prop 4})\ \frac{\Gamma \vdash_{wk} Prop(X)}{\Gamma \vdash_{wk} X \vee \neg X} \qquad (\text{Prop 5})\ \frac{\Gamma \vdash_{wk} Prop(X \vee Y)}{\Gamma \vdash_{wk} Prop(X)}$$

$$(\text{Prop 6})\ \frac{\Gamma \vdash_{wk} Prop(X \vee Y)}{\Gamma \vdash_{wk} Prop(Y)}$$

$$(\text{Prop 7})\ \frac{\Gamma \vdash_{wk} Prop(X) \qquad \Gamma' \vdash_{wk} Prop(Y)}{\Gamma, \Gamma' \vdash_{wk} Prop(X \vee Y)}$$

In this section we will show that $\vdash_{wk}$ is sound and complete with respect to $\models_{wk}$, i.e., the semantic consequence relation characterized by the (nonempty) class of Weak Kleene Illative Models.[22]

THEOREM 5.1. *If* $\Gamma \vdash_{wk} X$*, then* $\Gamma \models_{wk} X$.

Proof. The can be proved by a simple induction. □

Next we will prove the converse of this result. To this end, we first first introduce some definitions and prove a lemma.

[22] See Czajka (2015) for sound and complete semantics for other illative logical systems.

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DEFINITION 5.2. *We say that* $\Gamma \subseteq \mathbb{T}_{\Sigma}^{V}$ *is* closed (*with respect to* $\vdash_{wk}$) *just in case if* $\Gamma \vdash_{wk} X$*, then* $X \in \Gamma$.

DEFINITION 5.3. *We say that* $\Gamma \subseteq \mathbb{T}_{\Sigma}^{V}$ *is* consistent (*with respect to* $\vdash_{wk}$) *just in case if* $\Gamma \nvdash_{wk} \neg(X \vee \neg X)$.

DEFINITION 5.4. *We say that* $\Gamma \subseteq \mathbb{T}_{\Sigma}^{V}$ *is* full (*with respect to* $\vdash_{wk}$) *just in case if* $\Gamma \nvdash_{wk} Y$ *and* $\Gamma \nvdash_{wk} \neg Y$*, then* $\Gamma, Y \vdash_{wk} \neg(X \vee \neg X)$ *and* $\Gamma, \neg Y \vdash_{wk} \neg(X \vee \neg X)$.

DEFINITION 5.5. *We say that* $\Gamma \subseteq \mathbb{T}_{\Sigma}^{V}$ *is* disjunctively complete (*with respect to* $\vdash_{wk}$) *just in case if* $\Gamma \vdash_{wk} X \vee Y$*, then either* $\Gamma \vdash_{wk} X$ *or* $\Gamma \vdash_{wk} Y$.

LEMMA 5.6. *Let* $\Gamma \nvdash_{wk} N$*. Then there is some closed, consistent, full and disjunctively complete* $\Gamma' \supseteq \Gamma$ *such that* $\Gamma' \nvdash_{wk} N$.

Proof. Consider the set: $\{\Psi \supseteq \Gamma : \Psi \nvdash_{wk} N\}$. Every chain C in this set has an upper bound, viz., $\cup C$. By Zorn's Lemma, then, there is a maximal element of this set. Call it Γ'. We will show that Γ' is closed, consistent, full and disjunctively complete.

(a) Γ' is closed. Let $\Gamma' \vdash_{wk} X$. We can show that $\Gamma', X \nvdash_{wk} N$. Suppose, then, that $\Gamma', X \vdash_{wk} N$. Then since $\Gamma' \vdash_{wk} X$ and since $\vdash_{wk}$ satisfies (Cut), we have $\Gamma' \vdash N$. It follows that $\Gamma', X \nvdash_{wk} N$. Since Γ' is maximal amongst $\{\Psi \supseteq \Gamma : \Psi \nvdash_{wk} N\}$, we have $X \in \Gamma'$.

(b) Γ' is consistent. Since $\Gamma' \nvdash_{wk} N$ it follows that $\Gamma' \nvdash_{wk} \neg(X \vee \neg X)$. For suppose $\Gamma' \vdash_{wk} \neg(X \vee \neg X)$. Then we have $\Gamma' \vdash_{wk} Prop(\neg(X \vee \neg X))$. And so we have $\Gamma' \vdash_{wk} Prop(X \vee \neg X)$, which gives us $\Gamma' \vdash_{wk} Prop(X)$, and so $\Gamma' \vdash_{wk} X \vee \neg X$, which together with $\Gamma' \vdash_{wk} \neg(X \vee \neg X)$, gives us $\Gamma' \vdash_{wk} N$, which cannot obtain.

(c) Γ' is full. Suppose that $\Gamma \nvdash_{wk} Y$ and $\Gamma \nvdash_{wk} \neg Y$. We will show $\Gamma, Y \vdash_{wk} \neg(X \vee \neg X)$ and $\Gamma, \neg Y \vdash_{wk} \neg(X \vee \neg X)$. Since $\Gamma \nvdash_{wk} Y$ and $\Gamma \nvdash_{wk} \neg Y$, we have $Y \notin \Gamma'$ and $\neg Y \notin \Gamma'$. Since Γ' is maximal in $\{\Psi \supseteq \Gamma : \Psi \nvdash_{wk} N\}$, it follows that $\Gamma', Y \vdash_{wk} N$ and $\Gamma', \neg Y \vdash_{wk} N$. Now we can prove by a simple induction that, in general, if $\Gamma', Y \vdash_{wk} N$ and $\Gamma', \neg Y \vdash_{wk} N$, and $\Gamma' \nvdash_{wk} N$, then $\Gamma, Y \vdash_{wk} \neg(X \vee \neg X)$ and $\Gamma, \neg Y \vdash_{wk} \neg(X \vee \neg X)$. Thus it follows that we have $\Gamma, Y \vdash_{wk} \neg(X \vee \neg X)$ and $\Gamma, \neg Y \vdash_{wk} \neg(X \vee \neg X)$.

(d) Γ' is disjunctively complete. Let $\Gamma \vdash_{wk} X \vee Y$. Suppose $\Gamma \nvdash_{wk} X$ and $\Gamma \nvdash_{wk} Y$. We have then that $X \notin \Gamma'$ and $Y \notin \Gamma'$. Since Γ' is maximal in $\{\Psi \supseteq \Gamma : \Psi \nvdash_{wk} N\}$, it follows that $\Gamma', X \vdash_{wk} N$ and $\Gamma', Y \vdash_{wk} N$. But then it follows that we have $\Gamma', X \vee Y \vdash_{wk} N$, and so, by (Cut), we have $\Gamma' \vdash_{wk} N$, which cannot hold. □

THEOREM 5.7. *If* $\Gamma \models_{wk} X$*, then* $\Gamma \vdash_{wk} X$.

Proof. We will prove this by proving the contrapositive: If $\Gamma \nvdash_{wk} X$, then $\Gamma \not\models_{wk} X$. To this end let us assume: $\Gamma \nvdash_{wk} X$. We will show that there is a Weak Kleene Illative model which satisfies every member of Γ and fails to satisfy X.

By 5.6, it follows from our assumption that $\Gamma \nvdash_{wk} X$ that there is a consistent, closed, full and disjunctively complete $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash_{wk} X$. Let C, then be the set of β-equivalence classes of $\mathbb{T}_{\Sigma}^{V}$, and let $[Y] \cdot [Z] = [YZ]$. We let $T = \{[Z] : \Gamma' \vdash_{wk} Z\}$, and $F = \{[Z] : \Gamma' \vdash_{wk} \neg Z\}$. We let $\mathbf{n} =_{df} [\neg]$, $\mathbf{v} =_{df} [\vee]$, and $\mathbf{p} =_{df} [Prop]$.

We first show $\langle C, \cdot, T, F\rangle$ is a Weak Kleene Illative Algebra. By 2.13 we have that $\langle C, \cdot\rangle$ is a λ-algebra. We then simply need to show that T and F satisfy conditions (i)–(vii) of 4.1.

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(i) $T, F \subseteq C$, and $T \cap F = \emptyset$. This follows from the consistency of Γ'.

(ii) $[\vee AB] \in T$ iff $[A], [B] \in T \cup F$ and either $[A] \in T$ or $[B] \in T$. We have that $[\vee AB] \in T$ if and only if $\Gamma' \vdash_{wk} A \vee B$. And we have $\Gamma' \vdash_{wk} A \vee B$ if and only if either $(\Gamma' \vdash_{wk} A$ and $\Gamma' \vdash_{wk} Prop(B))$ or $(\Gamma' \vdash_{wk} B$ and $\Gamma' \vdash_{wk} Prop(A))$. The right-to-left direction of this biconditional is obvious. To see that the left-to-right direction holds assume $\Gamma' \vdash_{wk} A \vee B$. Then, since Γ' is disjunctively complete we have either $\Gamma' \vdash_{wk} A$ or $\Gamma' \vdash_{wk} B$. And we have $\Gamma' \vdash_{wk} Prop(A \vee B)$, and so $\Gamma' \vdash_{wk} Prop(A)$, and $\Gamma' \vdash_{wk} Prop(B)$, which suffices to establish the left-to-right direction. Finally, we have $(\Gamma' \vdash_{wk} A$ and $\Gamma' \vdash_{wk} Prop(B))$ or $(\Gamma' \vdash_{wk} B$ and $\Gamma' \vdash_{wk} Prop(A))$ if and only if either $([A] \in T$ and $[B] \in T \cup F]$ or $([B] \in T$ and $[A] \in T \cup F)$, which holds if and only if $[A], [B] \in T \cup F$ and either $[A] \in T$ or $[B] \in T$.

(iii) $[\vee AB] \in F$ if and only if $[A] \in F$ and $[B] \in F$. We have $[\vee AB] \in F$ if and only if $\Gamma' \vdash_{wk} \neg(A \vee B)$. And we have $\Gamma' \vdash_{wk} \neg(A \vee B)$ if and only if $\Gamma' \vdash_{wk} \neg A$ and $\Gamma' \vdash_{wk} \neg B$, which holds if and only if $[A] \in F$ and $[B] \in F$.

(iv) $[\neg A] \in T$ if and only if $[A] \in F$. We have $[\neg A] \in T$ if and only if $\Gamma' \vdash_{wk} \neg A$, which holds if and only if $[A] \in F$.

(v) $[\neg A] \in F$ if and only if $[A] \in T$. We have $[\neg A] \in F$ if and only if $\Gamma' \vdash_{wk} \neg\neg A$, which holds if and only if $\Gamma' \vdash_{wk} A$, which hold if and only if $[A] \in T$.

(vi) $[Prop(A)] \in T$ iff $[A] \in T \cup F$. We have $[Prop(A)] \in T$ if and only if $\Gamma' \vdash_{wk} Prop(A)$, which hold if and only if $\Gamma' \vdash_{wk} A \vee \neg A$, which, since Γ' is disjunctively complete, holds if and only if either $\Gamma' \vdash_{wk} A$ or $\Gamma' \vdash_{wk} \neg A$ which holds if and only if $[A] \in T \cup F$.

(vii) $[Prop(A)] \in F$ iff $[A] \notin T \cup F$. We have $[Prop(A)] \in F$ if and only if $\Gamma' \vdash_{wk} \neg Prop(A)$. Will show that this holds if and only if $\Gamma' \nvdash_{wk} A$ and $\Gamma' \vdash_{wk} \neg\neg A$, and so if and only if $[A] \notin T \cup F$. First we establish that if $\Gamma' \vdash_{wk} \neg Prop(A)$, then $\Gamma' \nvdash_{wk} A$ and $\Gamma' \vdash_{wk} \neg\neg A$, by establishing the contrapositive. To this end assume that either $\Gamma' \vdash_{wk} A$ or $\Gamma' \vdash_{wk} \neg A$. From this it follows that we have $\Gamma' \vdash_{wk} Prop(A)$. And so, since Γ' is consistent we have $\Gamma' \nvdash_{wk} \neg Prop(A)$. Next assume $\Gamma' \nvdash_{wk} A$ and $\Gamma' \vdash_{wk} \neg\neg A$. Since Γ' is full, we have $\Gamma', A \vdash_{wk} \neg(X \vee \neg X)$ and $\Gamma', \neg A \vdash_{wk} \neg(X \vee \neg X)$. And so we have $\Gamma' \vdash_{wk} \neg Prop(A)$. □

§6. Conclusion. We've seen that there are at least two prima facie plausible ways of responding to Bunder's Paradox. On the one hand, we can hold on to certain principles concerning the logic of $\rightarrow$ while weakening certain principles concerning *Prop*. On the other hand, we can weaken certain principles concerning $\rightarrow$ in order to hold on to certain prima facie plausible principles concerning *Prop*.

To fully assess the merits of these two possible responses to this paradox, we would need to consider what sorts of stronger illative logical systems these respective responses are compatible with. In the case of the first response, this question has been addressed in Czajka (2015). In the case of the second response, this question remains open, and must be left for future work.

In closing, let us note some prima facie attractive and unattractive features of the second response to Bunder's Paradox.

This response can be seen as following from a general theory—the Weak Kleene Illative Logic—that has a number of attractive features. First, according to this account, the Boolean operators are, roughly put, such as to make propositions from, but only from, other propositions. More precisely, given (Prop 1)–(Prop 7), we have that $Prop(X \vee Y)$

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will be provable, given some assumptions, just in case both *Prop*(X) and *Prop*(Y) are so provable, and *Prop*(X) will similarly be provable, given some assumptions, just in case *Prop*($\neg X$) is so provable. We can see this theory, then, as making precise a certain, to my mind, attractive picture according to which, with respect to the Boolean operators, failing to be a proposition is contagious. Within an illative logical framework, this strikes me as a simple and principled picture of how logically complex propositions may be built up from logically simpler propositions.

In addition, given the logic validated by the Weak Kleene Illative Models, we have as a validity a principle we might call *Propositional Excluded Middle*: *Prop*(X) $\vee$ $\neg$*Prop*(X). This principle, however, is not consistent with the logic validated by the Strong Kleene Illative Models.

One disconcerting feature of the Strong Kleene Illative Models is that there is an apparent gap between the notion of propositionhood as used in the model and the notion of propositionhood expressed by *Prop* given such models. Such a discrepancy, however, does not arise in the case of the Weak Kleene Illative Models. To see this, first note that, in working with either the Strong or the Weak Kleene Illative Models, it is natural to interpret the set $T \cup F$ as the set of propositions. And so, it is natural to say that, for each $x \in T \cup F$, x is a proposition, and for each $y \notin T \cup F$, y is not a proposition. In the case of Strong Kleene Illative Models, however, there can be no element of the underlying algebra e, which is a propositional function—that is, is such that in general we have $e \cdot a \in T \cup F$—and is such that in general we have $e \cdot a \in T$ just in case $a \in T \cup F$.[23] For this reason, if we interpret *Prop* in a Strong Kleene Illative Model, it cannot denote a element of the model that results in a true proposition when applied to a proposition, i.e., a member of $T \cup F$, and that results in a false proposition when applied to a nonpropositions, i.e., a nonmember of $T \cup F$. As we've seen, though, the same is not true in the case of Weak Kleene Illative Models. In this case, there can be a propositional function that maps every member of $T \cup F$ to a member of T, and every nonmember of $T \cup F$ to a member of F, and we can take such a propositional function to be the denotation of *Prop*. This strikes me as being an attractive feature of this class of models, and so gives us some reason to like the theory that such a class of models gives rise to.

While all of these strike me as being attractive features of the second response to Bunder's Paradox there are, of course, some downsides to this account. First, the logic governing the Boolean connectives validated by the Strong Kleene Illative Models is stronger than the logic validated by our Weak Kleene Illative Models. For example, the former validates ($\rightarrow I_s$), while the latter only validates the weaker ($\rightarrow I_w$). And there is a notable difference between ($\rightarrow I_s$) and ($\rightarrow I_w$). For the latter but not the former demands, in addition to a propositional typing restriction on the antecedent, a propositional typing restriction on the consequent. If, then, one is inclined to minimize such typing restrictions in formulating an illative logic, then one has reason to prefer ($\rightarrow I_s$) to ($\rightarrow I_w$). Finally, it is worth noting that the (Cut) rule may be derived in an illative logic with ($\rightarrow I_s$).[24] In our illative logic with ($\rightarrow I_w$), however, it needs to be postulated separately.

It seems to me, then, to be a delicate question what the right response is to Bunder's Paradox. I think, though, that there is good reason to take seriously the idea that the correct response to this paradox is that we should weaken certain principles concerning $\rightarrow$ in order to hold on to a number of plausible principles concerning *Prop*.

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23 See Aczel (1980) for a proof of a closely related result in the context of his Frege Structures.

24 See Czajka (2015).

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