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INNER FUNCTIONS AND ISOMETRIES

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Our objective in this note is to prove a theorem about the spectral behavior of certain multiplication operators. It was motivated by and yields an extension of part of a theorem of Lumer [3] concerning the possibility of imbedding the classical Hardy spaces in abstract ones.

We fix a σ-finite measure m on a measure space X and we regard $L^\infty(m)$ either as a space of functions or as the von Neumann algebra of multiplication operators on $L^2(m)$. Also, we fix a (closed) subspace $\mathfrak{M}$ of $L^2(m)$ and let $\mathfrak{U}$ denote the algebra of all $\phi \in L^\infty(m)$ such that $\varphi\mathfrak{M} \subseteq \mathfrak{M}$. A uni-modular function in $\mathfrak{U}$ will be called an *inner function*.

THEOREM. *Suppose that* (*a*) *there is a function f in $\mathfrak{M}$ which vanishes at most on a null set such that the linear manifold $\{\phi f \mid \phi \in \mathfrak{U}\}$ is dense in $\mathfrak{M}$ and* (*b*) *the only real-valued functions in $\mathfrak{U}$ are constants. Then for each nonconstant inner function θ in $\mathfrak{U}$, the spectral measure of θ regarded as a unitary operator on $L^2(m)$ is mutually absolutely continuous with respect to Lebesgue measure on the unit circle.*

The result of Lumer is contained in the following corollary whose proof we omit because it is easily constructed from the Theorem and straightforward arguments in spectral theory of the sort found in Chapitre III no 2 of [4]. Here, $L^\infty(\mathbf{T})$ and $H^\infty(\mathbf{T})$ are the usual Lebesgue and Hardy spaces on the unit circle $\mathbf{T}$.

COROLLARY. *Suppose that hypotheses* (a) *and* (b) *of the Theorem are satisfied and that θ is a nonconstant inner function in $\mathfrak{U}$. For each polynomial $p(z, \bar{z})$ in z and $\bar{z}$ on $\mathbf{T}$, let $p(\theta, \bar{\theta})$ be the same polynomial in θ and $\bar{\theta}$ on X. Then the correspondence $p(z, \bar{z}) \to p(\theta, \bar{\theta})$ extends to be an isometric $*$-isomorphism from $L^\infty(\mathbf{T})$ into $L^\infty(m)$ which is also continuous with respect to the weak-$*$ topologies on $L^\infty(\mathbf{T})$ and $L^\infty(m)$ and which carries $H^\infty(\mathbf{T})$ into $\mathfrak{U}$.*

To see that Lumer's result is contained in the Corollary, momentarily let X be a compact Hausdorff space, let A be a function algebra on X and let m be a representing measure for a point in the maximal ideal space of A. Then if $\mathfrak{M} = H^2(m)$, the closure of A in $L^2(m)$, $\mathfrak{U}$ is $H^\infty(m)$, the intersection $L^\infty(m) \cap H^2(m)$. It is clear that $\mathfrak{M}$ satisfies hypothesis (a) with f taken to be the constant function 1; and it is well-known that since m is multiplicative on A, $\mathfrak{M}$ satisfies hypothesis (b) as well.

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Hence, given a nonconstant inner function, the Corollary shows how to use it to imbed $L^\infty(\mathbf{T})$ and $H^\infty(\mathbf{T})$ into $L^\infty(m)$ and $H^\infty(m)$, resp.; and this is part of the result of Lumer. He also shows how to imbed the other L^p and H^p spaces on the circle into the corresponding ones on X and our results yield this part of his theorem too; but we shall not enter into the details.

The proof of the Theorem is based on two preparatory lemmas of independent interest.

If φ lies in $\mathfrak{U}$, then φ_+ will denote the restriction $\varphi \mid \mathfrak{M}$ of φ to $\mathfrak{M}$ and $\mathfrak{U}_+$ will denote the algebra of all such restrictions.

LEMMA 1. *If $\mathfrak{M}$ satisfies hypothesis* (a) *of the Theorem, then $\mathfrak{U}_+$ is a maximal abelian algebra of operators on $\mathfrak{M}$. If, in addition, $\mathfrak{M}$ satisfies hypothesis* (b), *then $\mathfrak{U}_+$ is irreducible.*

Proof. Let f be a function in $\mathfrak{M}$ satisfying the conditions of hypothesis (a) and let $d\mu=|f|^2\,dm$. Then μ is a finite measure and the map U defined by the formula $Ug=g/f$, $g \in L^2(m)$, is a Hilbert space isomorphism of $L^2(m)$ onto $L^2(\mu)$ which carries f to the constant function 1. It is clear that $UL^\infty(m)U^{-1}=L^\infty(\mu)$ and that if $\mathfrak{M}'=U\mathfrak{M}$, then $U\mathfrak{U}U^{-1}$ is the set of all $\varphi \in L^\infty(\mu)$ such that $\varphi\mathfrak{M}' \subseteq \mathfrak{M}'$. Likewise it is clear that $\mathfrak{M}'$ satisfies hypothesis (a) with $\mathfrak{U}$ replaced by $U\mathfrak{U}U^{-1}$ and f replaced by 1. Hence we may assume without loss of generality that m is finite and that $f=1$. In this case, hypothesis (a) implies that $\mathfrak{U}=L^\infty(m) \cap \mathfrak{M}$ and so if A is any operator on $\mathfrak{M}$ which commutes with $\mathfrak{U}_+$, then A must be the multiplication operator determined by the function $\varphi=A1$ in $\mathfrak{M}$. But then, standard arguments (see [2, pp. 212–213]) show that φ lies in $L^\infty(m)$ as well and thus $A=\varphi_+$ belongs to $\mathfrak{U}_+$.

Suppose $\mathfrak{M}$ also satisfies hypothesis (b) and that P is the projection of $\mathfrak{M}$ onto a subspace which reduces $\mathfrak{U}_+$. Then P commutes $\mathfrak{U}_+$ and so by what was just proved $P=\varphi_+$ for some φ in $\mathfrak{U}$. Since P is idempotent, φ takes on only the values zero or one a.e.(m). Hence, by hypothesis (b), φ is constant and so P is either the identity or the zero operator. Whence $\mathfrak{U}_+$ is irreducible and the proof is complete.

Suppose U is a unitary operator on a Hilbert space $\mathscr{L}$ and let $E(\cdot)$ be its spectral measure. If $\mathscr{L}_{AC}$ (resp., $\mathscr{L}_S$) is the set of all f in $\mathscr{L}$ such that the scalar measure $(E(\cdot)f,f)$ is absolutely continuous (resp., singular) with respect to Lebesgue measure on $\mathbf{T}$, then, as is well-known, $\mathscr{L}_{AC}$ and $\mathscr{L}_S$ are orthogonal supplementary spectral subspaces of U and so reduce every operator commuting with U. The space $\mathscr{L}_{AC}$ (resp., $\mathscr{L}_S$) is called the absolutely continuous (resp., singular) spectral subspace of U and U is called absolutely continuous (resp., singular) in case $\mathscr{L}_{AC}=\mathscr{L}$ (resp., $\mathscr{L}_S=\mathscr{L}$). Suppose now that V is an isometry on a Hilbert space $\mathscr{H}$ and let W be its minimal unitary extension acting on the Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. Also, let $V=(V \mid \mathscr{E})\oplus(V \mid \mathscr{T})$ with $\mathscr{H}=\mathscr{E}\oplus\mathscr{T}$ be the Wold decomposition of V so that $V \mid \mathscr{E}$ is a pure isometry on $\mathscr{E}$ and $V \mid \mathscr{T}$ is a unitary operator $\mathscr{T}$ (see [2, p. 74]). Since the minimal unitary extension of a pure isometry is a bilateral shift (of suitable multiplicity) and since these are absolutely continuous, it follows

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that the singular spectral subspace of $V|\mathscr{T}$ on $\mathscr{T}$ coincides with the singular spectral subspace of W on $\mathscr{K}$. So, in particular, W is absolutely continuous if and only if $V|\mathscr{T}$ is. Finally, note that the results of [1] show that every operator on $\mathscr{H}$ which commutes with V is reduced by the singular spectral subspace of $V|\mathscr{T}$.

LEMMA 2. *Let $\mathfrak{M}$ satisfy the hypotheses* (a) *and* (b) *of the Theorem and let θ be a nonconstant inner function in $\mathfrak{U}$. Then the isometry θ_+ on $\mathfrak{M}$ is non-unitary and (provided it has one) the unitary summand in its Wold decomposition is absolutely continuous.*

Proof. If θ_+ were unitary, then $\mathfrak{M}$ would reduce the unitary multiplication operator θ on $L^2(m)$ and so θ and $\bar{\theta}$ would lie in $\mathfrak{U}$. The hypothesis (b) would then imply that θ is constant which is contrary to assumption. If the unitary summand in the Wold decomposition of θ_+ had a singular spectral subspace $\mathscr{E}_S$, then by the remarks in the preceding paragraph, $\mathscr{E}_S$ would reduce $\mathfrak{U}_+$. By Lemma 1, then, $\mathscr{E}_S$ must be either all of $\mathfrak{M}$ or the zero subspace. Since the first case is ruled out by the fact that θ_+ is not unitary, $\mathscr{E}_S=\{0\}$ and the proof is complete.

Proof of the Theorem. As in the proof of Lemma 1, we may assume without loss of generality that the constant function 1 lies in $\mathfrak{M}$. Since θ is unitary and lies in the abelian von Neumann algebra $L^\infty(m)$, its spectral projections lie in $L^\infty(m)$ also and so must be of the form $\mathscr{X}_M$ for measurable sets M. Let $\mathscr{X}_{M_s}$ be the spectral projection onto the singular spectral subspace of θ and let $\mathscr{K}$ be the span of the spaces $\bar{\theta}^n\mathfrak{M}$, $n=0, 1, 2, \ldots$. Then $\mathscr{K}$ reduces θ and $\theta \mid \mathscr{K}$ is the minimal unitary extension of θ_+ on $\mathfrak{M}$. By Lemma 2 and the remarks in the paragraph preceding it, $\theta \mid \mathscr{K}$ is absolutely continuous and, therefore, $\mathscr{K}$ and $\mathscr{X}_{M_s}L^2(m)$ are orthogonal. But then, since 1 lies in $\mathfrak{M}$ which in turn is contained in $\mathscr{K}$, this implies that $m(M_S)=(\mathscr{X}_{M_s}, 1)=0$. Thus the spectral measure $E(\cdot)$ of θ is absolutely continuous with respect to Lebesgue measure on **T**. To see that Lebesgue measure on **T** is absolutely continuous with respect to $E(\cdot)$ simply note that by Lemma 2, θ has a reducing subspace on which it is a bilateral shift. Since this implies that an $E(\cdot)$-null set is a Lebesgue null set, the proof is complete.

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