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On the C^1 non-integrability of differential systems via periodic orbits

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We go back to the results of Poincaré [Poincare, H (1891) Sur lintegration des equations differentielles du premier ordre et du premier degre I and II, *Rendiconti del circolo matematico di Palermo* **5**, 161–191] on the multipliers of a periodic orbit for proving the C^1 non-integrability of differential systems. We apply these results to Lorenz, Rossler and Michelson systems, among others.

Key words: $\boldsymbol{C}^1$ integrability, differential systems, periodic orbits

1 Introduction and statements of main results

In these last years the Ziglin and the Morales–Ramis theories have been used for studying the non-meromorphic integrability of an autonomous differential system. In some sense the Ziglin theory is a continuation of Kovalevskaya's ideas used for studying the integrability of a rigid body because it relates the non-integrability of the considered system with the behaviour of some of its non-equilibrium solutions as a function of complex time using the monodromy group of their variational equations. Ziglin's theory was extended to the so-called Morales–Ramis' theory, which replaces the study of the monodromy group of variational equations by the study of the Galois differential group, which is easier to analyse (see [8] for more details and the references therein). But like the Ziglin theory, the Morales–Ramis theory can only study the non-existence of meromorphic first integrals.

Kovalevskaya's ideas and consequently the Ziglin and the Morales–Ramis theories go back to Poincaré's results (see Arnold [1]), who used the multipliers of the monodromy group of variational equations associated to periodic orbits for studying the non-integrability of autonomous differential systems. The main difficulty in applying Poincaré's non-integrability method to a given autonomous differential system is to find for such an equation periodic orbits having multipliers different from 1.

It seems that Poincaré's this result was forgotten by the mathematical community until modern Russian mathematicians (specially Kozlov) wrote on it (see [1, 10]).

We consider the autonomous differential system

$$\dot{x} = f(x), \tag{1.1}$$

where $f : U \to \mathbb{R}^n$ is C^2, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative with respect to time t. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that the solution $\phi(t, x_0)$ is *T-periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The *periodic orbit* associated to the periodic solution $\phi(t, x_0)$ is $\gamma = \{\phi(t, x_0), t \in [0, T]\}$. The *variational equation* associated to the T-periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\bigg|_{x=\phi(t,x_0)} \right) M, \tag{1.2}$$

where M is an $n \times n$ matrix. Of course, $\partial f(x)/\partial x$ denotes the Jacobian matrix of f with respect to x. The *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (1.2) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

The following proposition and theorem go back to Poincaré's results (see [20]). Since we cannot find their explicit proofs in the literature, we prove them in Section 2.

Proposition 1 *Let $\phi(t, x_0)$ be a T-periodic orbit of the C^2 differential system* (1.1). *The eigenvector tangent to the periodic orbit has an associated eigenvalue equal to* 1. *So the periodic orbit has at least one multiplier equal to* 1.

Let $F : U \to \mathbb{R}$ be a non-constant function of class C^1 such that

$$\nabla F(x) \cdot f(x) = 0.$$

Then F is called the *first integral* of f, because F is constant in the solutions of system (1.1). We note that $\cdot$ indicates the usual inner product of $\mathbb{R}^n$.

Given an $n \times n$-matrix N, we denote its transpose by N^T. The gradient of F is defined as

$$\nabla F(x) = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right).$$

We say that first two integrals $F : U \to \mathbb{R}$ and $G : U \to \mathbb{R}$ are *linearly independent* if their gradients are independent in all the points of U except into a set of the Lebesgue measure zero.

Theorem 2 *Let $f : U \to \mathbb{R}$ be the C^2 vector field associated to* (1.1)*, and let $F_k : U \to \mathbb{R}$ be the first integral for $k = 1, \dots, r$ with $r < n$. Assume that $F_1, \dots, F_r$ are linearly independent in U. Let γ be a T-periodic orbit of the vector field f such that at every point $x \in \gamma$ and the vectors $\nabla F_1(x), \dots, \nabla F_r(x)$ and $f(x)$ are linearly independent. Then* 1 *is a multiplier of the periodic orbit γ with multiplicity of at least $r + 1$.*

The following two results are the immediate consequences of Theorem 2.

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Corollary 3 *Consider the C^2 differential system* (1.1). *If there is a periodic orbit γ having only $s+1$ multipliers equal to* 1, *then system* (1.1) *has at most $F_1, \ldots, F_s$ C^1 linearly independent first integrals defined in the neighbourhood of γ satisfying that vectors $\nabla F_1(x), \ldots, \nabla F_s(x)$ and $f(x)$ are linearly independent on the points $x \in \gamma$.*

Corollary 4 *Under the assumptions of Corollary* 3 *if $s = 0$, then system* (1.1) *has no C^1 first integrals F defined in a neighbourhood of γ such that vectors $\nabla F(x)$ and $f(x)$ are linearly independent on the points $x \in \gamma$.*

These two corollaries give us a tool for studying the C^1 non-integrability of system (1.1) in the neighbourhood of the periodic orbit γ. Note that Corollary 4 prevents the existence of C^1 first integrals of system (1.1) defined in U.

Using Corollary 4 we will prove, under convenient assumptions, the non-existence of C^1 first integrals for systems having a zero-Hopf bifurcation (see Theorem 5). Later, using Theorem 5 we will show the non-existence of C^1 first integrals for the Lorenz system (see Theorem 6) and for the Rössler system (see Theorem 7). Finally, we shall prove the non-existence of C^1 first integrals for the Michelson system (see Theorem 8), but for such a system we will not be able to apply Theorem 5.

The following four theorems are proved in Section 3.

Now we present four applications of Corollary 4 showing the C^1 non-integrability of some differential systems in $\mathbb{R}^3$.

Theorem 5 *Consider a C^3 differentiable system in $\mathbb{R}^3$ having the origin as a singular point with eigenvalues $\varepsilon a \pm ci$ and εd. Then such a system can be written as*

$$
\begin{aligned}
\dot{x} &= p(x,y,z) = \varepsilon a x - cy + \sum_{i+j+k=2} A_{ijk} x^i y^j z^k + O_3(x,y,z),\\
\dot{y} &= q(x,y,z) = cx + \varepsilon a y + \sum_{i+j+k=2} B_{ijk} x^i y^j z^k + O_3(x,y,z),\\
\dot{z} &= r(x,y,z) = \varepsilon d z + \sum_{i+j+k=2} C_{ijk} x^i y^j z^k + O_3(x,y,z),
\end{aligned}
\tag{1.3}
$$

where $O_3(x,y,z)$ denotes the terms of order at least three in x, y, z. Let

$$
\begin{aligned}
F &= A_{101} + B_{011},\\
G &= C_{020} + C_{200},\\
D &= c(-4aC_{002} + dF),\\
E &= D^2 + 8ac^2 F(-2aC_{002} + dF).
\end{aligned}
\tag{1.4}
$$

Assume that $(E - D^2)/(FG) > 0$ and $(D \pm \sqrt{E})/(2c^2 F) \neq 1$. Then system (1.3) *has a limit cycle γ_ε tending to the origin as ε tends to zero. Moreover, there exists $\varepsilon_0 > 0$ such that for either $\varepsilon \in (-\varepsilon_0, 0)$ or $\varepsilon \in (0, \varepsilon_0)$, system* (1.3) *has no C^1 first integrals F defined in* *the neighbourhood of γ_ε such that vectors $\nabla F(x,y,z)$ and $(p(x,y,z), q(x,y,z), r(x,y,z))$ are linearly independent on the points of γ_ε.*

We should apply Theorem 5 to the Lorenz and the Rossler systems.

Theorem 6 (Lorenz system) *Consider the Lorenz system*

$$\dot{x} = \sigma(y - z), \quad \dot{y} = rx - y - xz, \quad \dot{y} = -bz + xy, \tag{1.5}$$

with $(x,y,z) \in \mathbb{R}^3$ and the parameters $\sigma, r, b \in \mathbb{R}$. We change parameters b and r by parameters a and c through

$$\begin{aligned}
&b = -2a\varepsilon + \frac{(c + a\varepsilon)^2(\sigma - 1)}{c^2 + a^2\varepsilon^2 + 4a\varepsilon\sigma + 2\sigma(1+\sigma)},\\
&r = r_1/r_2,\\
&r_1 = (c^2 + a^2\varepsilon^2)^2 + (c^2(3 + 2a\varepsilon) + a\varepsilon(-4 + a\varepsilon(-5 + 2a\varepsilon)))\sigma + (c^2 + a\varepsilon(-4a\varepsilon))\sigma^2,\\
&r_2 = c^2(1 + 2a\varepsilon - \sigma) + a\varepsilon(2a^2\varepsilon^2 + 4\sigma(1+\sigma) + a(\varepsilon + 7\varepsilon\sigma)).
\end{aligned}$$

Set

$$K = \frac{a(4 + c^2 + 2\sigma - 2\sigma^2)}{c^2 + \sigma - 2\sigma^2 + \sigma^3}.$$

If $K > 0$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ in the neighbourhood of singular point $q = (\sqrt{br - b}, \sqrt{br - b}, r - 1)$, and the Lorenz system has no limit cycles if $\varepsilon < 0$ and has a unique limit cycle γ_ε if $\varepsilon > 0$. Moreover, $\gamma_\varepsilon \to q$ if $\varepsilon \to 0$. For $K < 0$, the limit cycle γ_ε exists only for $\varepsilon \in (-\varepsilon_0, 0)$.

If $(D \pm \sqrt{E})/(2c^2F) \neq 1$, with D, E, F given in (1.4) with

$$\begin{aligned}
A_{101} &= \frac{Q_1}{(2 + c^2 + 2\sigma)\Delta^{3/2}Q} + O(\varepsilon),\\
B_{011} &= \frac{Q_2}{(2 + c^2 + 2\sigma)\Delta^{3/2}Q} + O(\varepsilon),\\
C_{200} &= -\frac{Q_3}{\Delta^{1/2}Q} + O(\varepsilon),\\
C_{020} &= 0,\\
C_{002} &= -\frac{Q_4}{(2 + c^2 + 2\sigma)\Delta^{3/2}Q} + O(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
&\Delta = \frac{c^2(c^2 + (1+\sigma)^2)}{c^2 + 2\sigma(1+\sigma)},\\
&Q = c^6 + 12c^2\sigma^2(1+\sigma)^2 + 4\sigma^2(1+\sigma)^4 + 4c^4\sigma(1 + 2\sigma),\\
&Q_1 = c^4(1+\sigma)(c^2 + (1+\sigma)^2)(c^4 + 8c^2\sigma + 4\sigma(1+\sigma)^2),\\
&Q_2 = c^4(-1+\sigma)(c^2 + (1+\sigma)^2)(c^4 - 2c^2\sigma(1+\sigma) - 4\sigma(1+\sigma)^3),\\
&Q_3 = \sigma^2(2 + c^2 + 2\sigma)(c^2 + \sigma(1+\sigma)^2),\\
&Q_4 = c^4\sigma(c^2 - 2(\sigma - 2)(1+\sigma))(c^2 + (1+\sigma)^2)^2,
\end{aligned}$$

then the Lorenz system for $\varepsilon \in (0, \varepsilon_0)$ when $K > 0$, and for $\varepsilon \in (-\varepsilon_0, 0)$ when $K < 0$ has no C^1 first integrals $F(x, y, z)$ defined in the neighbourhood of the zero-Hopf periodic orbit γ_ε satisfying that $\nabla F(x, y, z)$ and $(\sigma(y-z), rx - y - xz, -bz + xy)$ are linearly independent on the points of γ_ε.

The Lorenz system (1.5) was defined in [12]. This system has been intensively studied from the point of view of integrability using different integrability theories, and in particular the Darboux integrability and analytic integrability (for example, see [3, 5–7, 9, 13, 16, 22–27]), but never from the point of view of C^1 integrability.

The Rossler system (see (1.6)) was obtained in [21]. It is a well-known dynamical model that has been intensively investigated mainly with respect to the notion of dynamical chaos.

Theorem 7 (Rossler system) *Consider the Rossler system*

$$\dot{x} = -(y+z), \quad \dot{y} = x + ay, \quad \dot{y} = b - cz + xz, \tag{1.6}$$

with $(x, y, z) \in \mathbb{R}^3$ and the parameters $a, b, c \in \mathbb{R}$. We change the parameters a, b, c by the parameters a, u, v through

$$b = -\frac{b_1 b_2}{(-1 + (a - \varepsilon u)^2 + v^2)^2},$$

$$c = -a - 2\varepsilon u + a\varepsilon^2 u^2 + av^2 + \frac{(a - 2\varepsilon u)(1 + a^2 - 2a\varepsilon u)}{-1 + (a - \varepsilon u)^2 + v^2},$$

$$b_1 = (-1 + \varepsilon u(a - \varepsilon u))^2 + (-2 + a^2 - 2a\varepsilon u + 2\varepsilon^2 u^2)v^2 + v^4,$$

$$b_2 = 2a^2\varepsilon u - a(1 + 4\varepsilon^2 u^2) + 2\varepsilon u(\varepsilon^2 u^2 + v^2).$$

Set

$$L = au(-2 + a^2 + v^2)\Lambda_2,$$

with

$$\Lambda_2 = -2 + 4a^2 - 4a^4 + a^6 + 5v^2 - 8a^2 v^2 + 3a^4 v^2 - 4v^4 + 3a^2 v^4 + v^6.$$

If $L > 0$, there exists $\varepsilon_0 > 0$ sufficiently small such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ in the neighbourhood of the singular point

$$s = \left(\frac{c + \sqrt{c^2 - 4ab}}{2}, -\frac{c + \sqrt{c^2 - 4ab}}{2a}, \frac{c + \sqrt{c^2 - 4ab}}{2a} \right),$$

the Rossler system (1.6) has no limit cycles if $\epsilon < 0$ and has a unique limit cycle γ_ε if $\epsilon > 0$ that tends to q when $\epsilon \to 0$. For $L < 0$, the limit cycle only exists for $\varepsilon \in (-\varepsilon_0, 0)$.

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If $(D \pm \sqrt{E})/(2c^2F) \neq 1$, *with* D, E, F *given in* (1.4) *with*

$$A_{101} = -\frac{a(\Delta_1 - 1)(v^2 + \Delta_1(a^2 + (-2 + a^2)v^2 + v^4))}{\Delta_1(1 + (-2 + a^2)v^2 + v^4)\Delta_2} + O(\varepsilon),$$

$$B_{011} = \frac{a(\Delta_1 - 1)^3(a^4 - v^2 + v^4 + 2a^2(-1 + v^2))}{\Delta_1(1 + (-2 + a^2)v^2 + v^4)\Delta_2} + O(\varepsilon),$$

$$C_{200} = \frac{a\Delta_1(1 + (-2 + a^2)v^2 + v^4)}{v^2(1 + (-2 + a^2)v^2 + v^4)\Delta_2} + O(\varepsilon),$$

$$C_{020} = 0,$$

$$C_{002} = \frac{a(\Delta_1 - 1)}{\Delta_2} + O(\varepsilon),$$

where

$$\Delta_1 = a^2 + v^2,$$

$$\Delta_2 = v^2 + (-2 + a^2 + v^2)(a^4 + v^4 + 2a^2(-1 + v^2)),$$

$$A_1 = 2u - 4uv^2 + 2a^2uv^2 + 2uv^4 + 2w - 4a^2w + 4a^4w - a^6w - 5v^2w + 8a^2v^2w - 3a^4v^2w + 4v^4w - 3a^2v^4w - v^6w,$$

$$A_3 = 1 - 2v^2 + a^2v^2 + v^4,$$

$$A_4 = 4a^2 - 4a^4 + a^6 + v^2 - 6a^2v^2 + 3a^4v^2 - 2v^4 + 3a^2v^4 + v^6,$$

then the Rossler system for $\varepsilon \in (0, \varepsilon_0)$ *when* $L > 0$, *and for* $\varepsilon \in (-\varepsilon_0, 0)$ *when* $L < 0$ *has no* C^1 *first integrals* $F(x, y, z)$ *defined in the neighbourhood of the zero-Hopf periodic orbit* γ_ε *satisfying that* $\nabla F(x, y, z)$ *and* $(-(y+z), x+ay, b-cz+xz)$ *are linearly independent on the points of* γ_ε.

The Rossler system has been studied in [17] from the viewpoint of the Darbouxian integrability and in [14] from the viewpoint of the analytic integrability but never from the view point of the C^1 integrability.

Note that Theorem 5 cannot be applied to the Michelson system defined in the next theorem.

Theorem 8 (Michelson system) *Consider the Michelson system*

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{x^2}{2}, \tag{1.7}$$

with $(x, y, z) \in \mathbb{R}^3$ *and the parameter* $c \in \mathbb{R}$. *The Michelson system for* $c > 0$ *being sufficiently small has no* C^1 *first integrals* $F(x, y, z)$ *defined in the neighborhood of the zero-Hopf periodic orbit* γ *satisfying that* $\nabla F(x, y, z)$ *and* $(y, z, c^2 - y - \frac{x^2}{2})$ *are linearly independent on the points of* γ.

The Michelson system has been studied in [15] from the viewpoint of the analytic and Darboux integrability but never from the view point of the C^1 integrability.

2 Proof of Proposition 1 and Theorem 2

Proof of Proposition 1 Let $\phi(t, x_0)$ with $t \in \mathbb{R}$ and $x_0 \in U$ be the T-periodic solution of the autonomous differential system (1.1). Clearly, we have

$$\phi(\tau, \phi(t, x_0)) = \phi(t + \tau, x_0). \tag{2.1}$$

Differentiating (2.1) with respect to t and setting $t = 0$ and $\tau = T$, we get

$$\frac{\partial \phi}{\partial x}(T, x_0)\dot{\phi}(0, x_0) = \dot{\phi}(T, x_0). \tag{2.2}$$

Since $\phi(T, x_0) = x_0$, we get $\dot{\phi}(T, x_0) = f(\phi(T, x_0)) = f(x_0)$. In a similar way we have $\dot{\phi}(0, x_0) = f(x_0)$, and thus we can rewrite (2.2) in the following form

$$\frac{\partial \phi}{\partial x}(T, x_0)f(x_0) = f(x_0). \tag{2.3}$$

Note that differentiating equation (1.1) with respect to x, we have

$$\frac{\partial}{\partial x}\frac{d\phi(t, x)}{dt} = \frac{\partial f(\phi(t, x))}{\partial x}\frac{\partial \phi(t, x)}{\partial x}.$$

By Schwartz's lemma, we can rewrite the above equation as

$$\frac{d}{dt}\frac{\partial \phi(t, x)}{\partial x} = \frac{\partial f(\phi(t, x))}{\partial x}\frac{\partial \phi(t, x)}{\partial x},$$

which implies that $\frac{\partial \phi}{\partial x}(t, x)$ is a solution of (1.2). Note that since $\phi(0, x) = x$, taking derivative with respect to x, we obtain $\frac{\partial \phi(t,x)}{\partial x} = \mathrm{Id}$. Thus, $\frac{\partial \phi}{\partial x}(T, x_0)$ is the monodromy matrix associated to $\phi(t, x_0)$. It follows from (2.3) that $\frac{\partial \phi}{\partial x}(T, x_0)$ has 1 as an eigenvalue with eigenvector $f(x_0) \neq 0$ because x_0 is not an equilibrium point. Furthermore, since $f(x_0) = \dot{\phi}(0, x_0)$, we get that the eigenvector $f(x_0)$ is tangent to the periodic orbit at point x_0. This completes the proof of the proposition. □

Proof of Theorem 2 Since for each $k = 1, \ldots, r$, F_k is the first integral of system (1.1), we have $F_k(\phi(t, x)) = F_k(x)$. Differentiating this relation with respect to x we get

$$\nabla F_k(\phi(t, x))\frac{\partial \phi(t, x)}{\partial x} = \nabla F_k(x).$$

Then taking $t = T$ and $x = x_0$ in the previous equality, and since $\phi(T, x_0) = x_0$, we obtain

$$\nabla F_k(x_0)\frac{\partial \phi(T, x_0)}{\partial x} = \nabla F_k(x_0),$$

or equivalently,

$$\left(\frac{\partial \phi(T, x_0)}{\partial x}\right)^T \nabla F_k(x_0) = \nabla F_k(x_0).$$

Therefore, $\nabla F_k(x_0)$ is an eigenvector of $(\frac{\partial \phi(T,x_0)}{\partial x})^T$ with eigenvalue 1. On the other hand,

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from the proof of Proposition 1 we have

$$\frac{\partial \phi(T, x_0)}{\partial x} f(x_0) = f(x_0).$$

Now ending the proof, since by assumption the vectors $\nabla F_1(x_0), \ldots, \nabla F_r(x)$ and $f(x_0)$ are linearly independent on the points of the periodic orbit γ, there are at least $r+1$ multipliers for the monodromy matrix equal to 1. □

3 Proof of Theorems 5, 6, 7 and 8

The proof of Theorem 5 is an immediate consequence of Corollary 4 and the following theorem (together with its proof which is given in [11]).

Theorem 9 *The following statements hold for system* (1.3).

(a) *If* $(E - D^2)/(FG) > 0$, *then the differential system* (1.3) *has a limit cycle* γ_ε *tending to the origin as* $\varepsilon \to 0$.

(b) *The multipliers of the limit cycle* γ_ε *are* 1 *and* $(D \pm \sqrt{E})/(2c^2 F)$.

Proof of Theorem 6 The part of the proof of Theorem 6 concerning the existence of γ_ϵ follows from Theorem 2 in [11]. Computing the eigenvalues at the singular point q and using Theorem 5 together with Corollary 4 the proof follows easily. □

Proof of Theorem 7 The part of the proof of Theorem 7 concerning the existence of γ_ϵ follows from Theorem 3 in [11]. Computing the eigenvalues at the singular point s and using Theorem 5 together with Corollary 4 the proof follows easily. □

The Michelson system (1.7) was obtained by Michelson [19] in the study of the travelling wave solutions of the Kuramoto–Sivashinsky equation. It is well known that system (1.7) is reversible with respect to the involution $R(x, y, z) = (-x, y, -z)$ and is volume-preserving under the flow of the system. It is easy to check that system (1.7) has two finite singularities,

$$p_1 = (-\sqrt{2}c, 0, 0) \quad \text{and} \quad p_2 = (\sqrt{2}c, 0, 0)$$

for $c \neq 0$, which are both saddle-foci. The singular point p_1 has a two-dimensional stable manifold and p_2 has a two-dimensional unstable manifold. Note that when $c = 0$. the Michelson system has a unique singular point at the origin with eigenvalues 0, $\pm i$. In [18] it is proved that for $c > 0$ being sufficiently small, the Michelson system (1.7) has a Hopf-zero bifurcation at the origin for $c = 0$. Here we shall reproduce the short proof of [18] because it is necessary for proving our result.

To prove Theorem 8 we need the following result essentially because of Malkin (1956) and Roseau (1966) (see [4]). In [2], a new and shorter proof is given.

Theorem 10 (Perturbations of an isochronous open set) *Consider a differential system*

$$\dot{x} = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \quad (t, x, \varepsilon) \in \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0), \tag{3.1}$$

where Ω is an open subset of $\mathbb{R}^n$, and F_0, F_1 and F_2 are C^2 smooth and T-periodic in time t. Let $x(t,z)$ be a solution of (3.1) when $\varepsilon = 0$ such that $x(0,z) = z$. Denoted by $M_z(t)$, the fundamental solution matrix of the variational equation

$$\dot{y} = D_x F_0(t, x(t,z))y,$$

such that $M_z(0) = Id$. Assume that there exists an open and bounded subset V with its closure $cl(V) \subset \Omega$ such that for each $z \in cl(V)$, the solution $x(t,z)$ is T-periodic. If $a \in V$ is a zero of the map $F : cl(V) \to \mathbb{R}^n$ defined by

$$F(z) = \int_0^T M_z^{-1}(t) F_1(t, x(t,z))\, dt, \tag{3.2}$$

and $\det(D_z F(a)) \neq 0$, then for $|\varepsilon| > 0$, sufficiently small system (3.1) has a T-periodic solution $\phi(t,\varepsilon)$ such that $\phi(0,\varepsilon) \to a$ as $\varepsilon \to 0$. Moreover, the periodic solution $\phi(t,\varepsilon)$ has the same stability type as the singular point at the origin of the linear differential system $\dot{y} = (D_z F(a))y$ if this singular point is hyperbolic.

Proof of Theorem 8 For any $\varepsilon \neq 0$ we take the change of variables

$$x = \varepsilon\bar{x}, \quad y = \varepsilon\bar{y}, \quad z = \varepsilon\bar{z} \quad \text{and} \quad c = \varepsilon d.$$

Then the Michelson system (1.7) becomes

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -y + \varepsilon d^2 - \varepsilon\frac{x^2}{2}, \tag{3.3}$$

where we still use x, y, z instead of $\bar{x}, \bar{y}, \bar{z}$. Now doing the change of variables

$$x = x, \quad y = r\sin\theta \quad \text{and} \quad z = r\cos\theta,$$

system (3.3) goes over to

$$\dot{x} = r\sin\theta, \quad \dot{r} = \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta, \quad \dot{\theta} = 1 - \frac{\varepsilon}{2r}(2d^2 - x^2)\sin\theta. \tag{3.4}$$

This system can be written as

$$\begin{aligned} \frac{dx}{d\theta} &= r\sin\theta + \frac{\varepsilon}{2}(2d^2 - x^2)\sin^2\theta + \varepsilon^2 f_1(\theta, r, \varepsilon), \\ \frac{dr}{d\theta} &= \frac{\varepsilon}{2}(2d^2 - x^2)\cos\theta + \varepsilon^2 f_2(\theta, r, \varepsilon), \end{aligned} \tag{3.5}$$

where f_1 and f_2 are analytic functions in their variables.

For any given x_0 and r_0, system (3.5) in $\varepsilon = 0$ has the 2π-periodic solution

$$x(\theta) = r_0 + x_0 - r_0\cos\theta, \quad r(\theta) = r_0, \tag{3.6}$$

such that $x(0) = x_0$ and $r(0) = r_0$. It is easy to see that the variational equation of (3.5) on $\varepsilon = 0$ along with solution (3.6) is

$$\begin{pmatrix} \dfrac{dy_1}{d\theta} \\ \dfrac{dy_2}{d\theta} \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

It has the fundamental matrix solution

$$M = \begin{pmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{pmatrix},$$

which is independent of the initial condition (x_0, r_0). Using Theorem 10 we have

$$F(x_0, r_0) = \frac{1}{2}\int_0^{2\pi} M^{-1} \begin{pmatrix} (2d^2 - x^2)\sin^2\theta \\ (2d^2 - x^2)\cos\theta \end{pmatrix}\Bigg|_{(3.6)} d\theta.$$

Then $F(x_0, r_0) = (g_1(x_0, r_0), g_2(x_0, r_0))$ with

$$g_1(x_0, r_0) = \tfrac{1}{4}\left(4d^2 - 5r_0^2 - 6r_0x_0 - 2x_0^2\right), \quad g_2(x_0, r_0) = \tfrac{1}{2}r_0(x_0 + r_0).$$

Solving $F(x_0, r_0) = 0$ we get that it has a unique non-trivial solution $x_0 = -2d$ and $r_0 = 2d$. The eigenvalues of the Jacobian matrix on this solution are

$$\lambda_1 = -\frac{d}{2}(1+\sqrt{5}) \neq 1 \quad \text{and} \quad \lambda_2 = -\frac{d}{2}(1-\sqrt{5}) \neq 1,$$

taking $d \neq 2/(1 \pm \sqrt{5})$. Therefore, by Theorem 10 the multipliers of this zero-Hopf periodic orbit γ are

$$1, \quad -\frac{d}{2}(1+\sqrt{5}) \neq 1 \quad \text{and} \quad -\frac{d}{2}(1+\sqrt{5}) \neq 1,$$

if $d \neq 2/(1 \pm \sqrt{5})$. Consequently, by Corollary 4, system (1.7) has no C^1 first integrals F defined in the neighborhood of γ satisfying that

$$\nabla F(x, y, z) \quad \text{and} \quad \left(y, z, c^2 - y - \frac{x^2}{2}\right)$$

are linearly independent on the points of γ. □

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