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ON SOME QUESTIONS OF PARTITIO NUMERORUM: TRES CUBI*

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Abstract. This paper is concerned with the function $r_3(n)$, the number of representations of n as the sum of at most three positive cubes,

$$r_3(n) = \text{card}\{\mathbf{m} \in \mathbb{Z}^3 : m_1^3 + m_2^3 + m_3^3 = n,\, m_j \geq 1\}.$$

Our understanding of this function is surprisingly poor, and we examine various averages of it. In particular

$$\sum_{m=1}^{n} r_3(m), \ \sum_{m=1}^{n} r_3(m)^2$$

and

$$\sum_{\substack{n\leq x\\ n\equiv a \bmod q}} r_3(n).$$

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1. Introduction. This paper is concerned with the function $r_3(n)$, the number of representations of n as the sum of three positive cubes,

$$r_3(n) = \text{card}\{\mathbf{m} \in \mathbb{Z}^3 : m_1^3 + m_2^3 + m_3^3 = n,\, m_j \geq 1\}.$$

We use this opportunity to pay homage to Christopher Hooley (1928–2018) who, amongst many other things, made highly significant contributions to our understanding of $r_3(n)$ as well as to Waring's problem for cubes and more generally to cubic forms. Nevertheless, our understanding of this function still leaves much to be desired. Even on average. Let

$$\Delta(n) = \sum_{m=1}^{n} r_3(m) - \Gamma\left(\frac{4}{3}\right)^3 n.$$

Then it is readily seen, by the usual principle that the number of lattice points inside a d-dimensional convex body differs from its volume by an amount bounded by the $d-1$ dimensional surface volume, that

$$\Delta(n) \ll n^{\frac{2}{3}}.$$

This does not seem to have been otherwise studied very closely. However, we can prove the following by applying van der Corput's method.

*In Memoriam Christopher Hooley FRS, 1928–2018

THEOREM 1.1. *We have*

$$\sum_{m=1}^{n} r_3(m) = \Gamma\left(\frac{4}{3}\right)^3 n - \frac{\Gamma\left(\frac{4}{3}\right)^2}{2\Gamma\left(\frac{5}{3}\right)} n^{\frac{2}{3}} + O\left(n^{\frac{5}{9}}(\log n)^{\frac{1}{3}}\right).$$

The second-order term here is interesting. Of course, it can be varied by including some or all sums of three cubes on the boundary, that is with at least one cube being zero.

The proof of the above really only utilises two of the three variables and this suggests that there is some scope for further improvements, and perhaps, the error term is as small as $O(n^{\frac{1}{2}})$ or even $O(n^{\frac{1}{3}+\varepsilon})$. On the other hand, one would guess that

$$\sum_{m=1}^{n} r_3(m) - \Gamma\left(\frac{4}{3}\right)^3 n + \frac{\Gamma\left(\frac{4}{3}\right)^2}{2\Gamma\left(\frac{5}{3}\right)} n^{\frac{2}{3}} = \Omega\left(n^{\frac{1}{3}}\right)$$

in analogy with other three-dimensional lattice point problems. As far as I am aware the best that can be established in this direction is by the variant of the Erdős–Fuchs theorem [6], as generalised in Vaughan [22] and sharpened in Hayashi [12], which gives

$$\sum_{m=1}^{n} r_3(m) - \Gamma\left(\frac{4}{3}\right)^3 n + \frac{\Gamma\left(\frac{4}{3}\right)^2}{2\Gamma\left(\frac{5}{3}\right)} n^{\frac{2}{3}} = \Omega\left(n^{\frac{1}{4}}\right).$$

Possibly, the method of Montgomery and Vaughan [20] could be adapted to give this also.

Let

$$\Delta_3(x) = \sum_{m\le x} r_3(m) - \Gamma\left(\frac{4}{3}\right)^3 x + \frac{\Gamma\left(\frac{4}{3}\right)^2}{2\Gamma\left(\frac{5}{3}\right)} x^{\frac{2}{3}}. \tag{1.1}$$

One can ask about the mean square of $\Delta_3(x)$. One version of this would be to ascertain the abscissa of convergence σ_3 of

$$\int_1^\infty |\Delta_3(x)|^2 x^{-2\sigma-1} dx.$$

More generally, if $r_k(n)$ denotes the number of ways of writing n as the sum of k k-th powers of positive integers and

$$\Delta_k(x) = \sum_{m\le x} r_k(m) - \Xi_k(x),$$

where $\Xi_k(x)$ is an expected main term. This is a natural generalisation of the classical case $k=2$ which goes back to Hardy [10] who established $\sigma_2 = \frac{1}{4}$. The exact nature of $\Xi_k(x)$ is not entirely clear since there will surely be lower order terms as in (1.1). However, one can nevertheless ask about the abscissa of convergence σ_k of

$$\int_1^\infty |\Delta_k(x)|^2 x^{-2\sigma-1} dx. \tag{1.2}$$

More precisely, one might expect to be able to show that if

$$\Xi_k(x) = \Gamma\left(\frac{k+1}{k}\right)^k x + \sum_{j=1}^{J} C_j x^{\theta_j},$$

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where the $\{\theta_j\}$ is a strictly decreasing sequence of real number in $[0, 1)$ and the C_j are real numbers, then $\sigma_k \geq \frac{k-1}{2k}$ regardless of the choice of C_j and θ_j.

Several multiplicative generalisations of this have been considered. Let $r(n; K)$ be the number of integral ideals of norm n in an algebraic extension of the rational numbers of degree k. Then, the corresponding main term $M(x; K)$ is αx where α is the residue of the Dedekind zeta function $\zeta(s, K)$ at 1. Ayoub [1] has shewn that $\sigma_2 = \frac{1}{4}$ when K has degree 2 and Vaughan [25] has shewn that $\sigma_3 = \frac{1}{3}$ when K has degree 3 and that $\sigma_k \geq \frac{k-1}{2k}$ when $k \geq 4$. It is conjectured that equality should occur for all k.

In the concomitant problem in which $r(n; K)$ is replaced by $d_k(n)$, the number of ways of writing n as the product of k positive integers, it is also known that $\sigma_2 = \frac{1}{4}$ (Hardy [10]), $\sigma_3 = \frac{1}{3}$ (Cramér [5]) and $\sigma_k \geq \frac{k-1}{2k}$. In addition, it is known (Heath-Brown [13]) that equality occurs when $k = 4$.

In view of these results, one might guess that the abscissa of convergence of (1.2) is also $\sigma_k = \frac{k-1}{2k}$, although nothing is known when $k \geq 3$.

Our understanding of the second moment

$$M_2(n) = \sum_{m=1}^{n} r_3(m)^2$$

is even worse. Let $P = \lfloor n^{\frac{1}{3}} \rfloor$ and

$$f(\alpha) = \sum_{m \leq P} e(\alpha m^3). \tag{1.3}$$

By combining the work of Hooley [14] or Wooley [26] on the one hand with that of Vaughan [23] on the other, we have

$$\int_0^1 |f(\alpha)|^6 d\alpha \ll n^{\frac{7}{6}} \tag{1.4}$$

and thus, as

$$r_3(m) = \int_0^1 f(\alpha)^3 e(-\alpha m) d\alpha$$

for $1 \leq m \leq n$, it follows by Bessel's inequality that

$$\sum_{m=1}^{n} r_3(m)^2 \ll n^{\frac{7}{6}}.$$

Futhermore, Brüdern and Wooley [4] have deduced from Boklan's [2, 3] sharpening of Vaughan [23] that

$$\int_0^1 |f(\alpha)|^6 d\alpha \ll n^{\frac{7}{6}} (\log n)^{\varepsilon - \frac{3}{2}}.$$

This can be refined.

Theorem 1.2. *We have*

$$\int_0^1 |f(\alpha)|^6 d\alpha \ll n^{\frac{7}{6}} (\log n)^{\varepsilon - \frac{5}{2}}.$$

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COROLLARY 1.3. *We have*

$$\sum_{m=1}^{n} r_3(n)^2 \ll n^{\frac{7}{6}} (\log n)^{\varepsilon - \frac{5}{2}}.$$

This is, of course, some way from the bound $\ll_\varepsilon n^{1+\varepsilon}$ which is often conjectured. This author would not be surprised if the bound $\ll n$ holds. One can obtain the former conjecture provided that one is prepared to assume a rather exotic form of the Riemann hypothesis. Hooley [16] considers the cubic form

$$g(x_1, \ldots, x_6) = x_1^3 + \ldots + x_6^3$$

and the discriminant

$$\Delta(m_1, \ldots, m_6) = 3 \prod (m_1^{3/2} \pm m_2^{3/2} \pm \ldots \pm m_6^{3/2}),$$

For $\Delta(m_1, \ldots, m_6) \neq 0$, Hooley introduces the projective varieties $V(\mathbf{m})$ over $\mathbb{Q}$ which are given by the simultaneous equations

$$g(\xi_1, \ldots, \xi_6) = m_1 \xi_1 + \ldots + m_6 \xi_6 = 0.$$

This leads to the reduced non-singular varieties $V(m_1, \ldots, m_6; p)$ that are defined over $\mathbb{F}_p$ and in turn leads to

$$L(m_1, \ldots, m_6; p; T) = \exp\left(- \sum_{r=1}^{\infty} E(m_1, \ldots, m_6; p^r) T^r \right),$$

where the E are essentially errors defined in terms of the number of points in $\mathbb{F}_{p^r}$ less the expected number. These give the local factors,

$$L(m_1, \ldots, m_6; p; T) = \prod_{j=1}^{10} (1 - \lambda_{j,p} T)^{-1}$$

and the Hasse–Weil L-function is defined by

$$L(m_1, \ldots, m_6; s) = \prod_{p \nmid \Delta(m_1, \ldots, m_6)} L(m_1, \ldots, m_6; p; p^{-s}).$$

Then, Hooley [16, 18], has shewn on the Riemann hypothesis for this L-function that

$$\sum_{m=1}^{n} r_3(m)^2 \ll n^{1+\varepsilon}.$$

Apart from the ε, this is clearly best possible, as can be seen by considering solutions of

$$l_1^3 + l_2^3 + l_3^3 = l_4^3 + l_5^3 + l_6^3,$$

in which the l_j on the right are a permutation of those on the left.

One might ponder the possibility that there is a positive constant C such that,

$$\sum_{m=1}^{n} r_3(m)^2 \sim Cn. \tag{1.5}$$

The above argument shews that if true, then $C \geq 6$. Also, it is clear by the examination of major arcs in an appropriate version of the Hardy–Littlewood method, that those arcs alone contribute

$$\sim C_0 n,$$

where

$$C_0 = \Gamma\left(\frac{4}{3}\right)^6 \mathfrak{S}$$

and

$$\mathfrak{S} = \sum_{q=1}^{\infty} q^{-6} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S_3(q, a)|^6.$$

Hooley has shewn [17], Theorem 1, that

$$\sum_{m=1}^{n} r_3(n)^2 \geq Cn + o(n),$$

where

$$C > \max\{6, C_0\},$$

and thus were (1.5) to be true, it would have to hold with a C larger than the obvious guesses.

Consequently, it is of some interest to explore the behaviour of

$$\sum_{m=1}^{n} r_3(n) F(m)$$

for various interesting choices of arithmetical function F other than r_3. Let $r_2(n)$ denotes the number of ways of representing n as the sum of two squares of integers

$$r_2(n) = \mathrm{card}\{\mathbf{l} \in \mathbb{Z}^2 : l_1^2 + l_2^2 = n\}.$$

Then, the number of representations of n as the sum of at most two squares and three positive cubes is

$$\sum_{m=1}^{n} r_3(m) r_2(n-m)$$

and Hooley [15] has shewn that this is

$$\pi \Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}(n) n + O\left(n (\log n)^{-\delta}\right).$$

Here, δ is a small but fixed positive number, and the singular series $\mathfrak{S}(n)$ can be defined by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} q^{-5} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S_3(q, a)^3 S_2(q, a)^2 e(-an/q)$$

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with

$$S_k(q,a)=\sum_{r=1}^{q} e(ar^k/q),$$

or by

$$\prod_p \left(\lim_{t\to\infty} p^{-4t}M(p^t;n)\right),$$

where $M(q;n)$ is the number of solutions of

$$l_1^3+l_2^3+l_3^3+l_4^2+l_5^2\equiv n \pmod q.$$

Let

$$\theta(n)=\begin{cases}1 & m \text{ is prime},\\ 0 & \text{otherwise}.\end{cases}$$

The Hardy and Littlewood [11] Conjecture M states that

$$\sum_{m=1}^{n} r_3(m)\theta(m)\sim\Gamma\left(\frac{4}{3}\right)^3\mathfrak{S}\frac{n}{\log n},$$

as $n\to\infty$, where $\mathfrak{S}$ is the singular series for this problem and can be defined by

$$\mathfrak{S}=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{\mu(q)S_3(q,a)^3}{\phi(q)q^3}. \tag{1.6}$$

Hooley has shewn [18] that

$$\sum_{m=1}^{n} r_3(m)\theta(m)\le 4\Gamma\left(\frac{4}{3}\right)^3\mathfrak{S}\frac{n}{\log n}+o\left(\frac{n}{\log n}\right) \tag{1.7}$$

as $n\to\infty$, and that on an extended Riemann hypothesis, the constant 4 here can be replaced by 3. In both of these memoirs, Hooley makes extensive use of estimates for sums of the kind

$$\Upsilon(x;q,a)=\sum_{\substack{m\le x\\ m\equiv a \pmod q}} r_3(m), \tag{1.8}$$

at least on average over $q\le x^{\frac{1}{2}}$. The basic analysis is undertaken in Hooley [15], and this is a very substantial and deep paper of 46 pages, most of which is concerned with $\Upsilon(x;q,a)$. However, at no stage is an explicit estimate given for

$$\sum_{q\le Q}|E(x;q,a)|,$$

where

$$E(x;q,a)=\Upsilon(x;q,a)-\Gamma\left(\frac{4}{3}\right)^3 x\rho(q,a)q^{-3} \tag{1.9}$$

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and $\rho(q, a)$ denotes the number of solutions of the congruence $l_1^3 + l_2^3 + l_3^3 \equiv a \pmod q$. Nevertheless, the estimates given there can be adapted to prove the following theorems. We consider sequences $\{\mathcal{A}(q)\}$ where each $\mathcal{A}(q)$ is a set of residue classes modulo q. We call such a family multiplicative when each $\mathcal{A}(q)$ has the property that whenever $q = q_1 q_2$ with $(q_1, q_2) = 1$ each $a \in \mathcal{A}(q)$ can be written in the form $a_1 q_2 \overline{q}_1 + a_2 q_1 \overline{q}_2$ with $a_j \in \mathcal{A}(q_j)$. Here, $\overline{q}_j$ denotes any x such that $q_j x \equiv 1 \pmod{q_{3-j}}$.

THEOREM 1.4. *Let $n \in \mathbb{N}$. Suppose that $\{\mathcal{A}(q)\}$ is a multiplicative family with the property that there is a positive integer N with $\log\log\log N = o(\log\log n)$ as $n \to \infty$ such that for every $q \in \mathbb{N}$ and every $a \in \mathcal{A}(q)$, we have $(q, a)|N$. Then, there is a positive constant δ such that for every positive number ε and all $Q \le n^{\frac{1}{2}}(\log n)^{\delta}$, we have*

$$\sum_{q \le Q} \max_{a \in \mathcal{A}(q)} \sup_{x \le n} |E(x; q, a)| \ll_{\varepsilon} n^{\frac{8}{9}+\varepsilon} + n^{\frac{1}{3}} Q^{\frac{2}{9}} \left(Q^{\frac{10}{9}} + n^{\frac{5}{9}}\right) (\log n)^{-\delta}.$$

The simplest example of a multiplicative family which satisfies the requisite additional hypothesis of the theorem is the one in which each $\mathcal{A}(q)$ is the set of reduced residues modulo q.

When our interest includes, for example, the zero residue class for each q, we have a slightly weaker result. Note that $(q, 0) = q$ can be larger than the hypothetical N of the previous theorem.

THEOREM 1.5. *There is a positive constant C such that for all $Q \le n^{\frac{5}{9}}$, we have*

$$\sum_{q \le Q} \max_{a} \sup_{x \le n} |E(x; q, a)| \ll_{\varepsilon} n^{\frac{8}{9}+\varepsilon} + n^{\frac{1}{3}} Q^{\frac{2}{9}} \left(Q^{\frac{10}{9}} + n^{\frac{5}{9}}\right) (\log n)^{C}.$$

These theorems have the same general character as the Bombieri–Vinogradov theorem, for example in the form

$$\sum_{q \le Q} \max_{(a,q)=1} \sup_{x \le n} \left| \vartheta(x; q, a) - \frac{x}{\phi(q)} \right| \ll_A n(\log n)^{-A} + Q n^{\frac{1}{2}} (\log nQ)^4.$$

This gives a bound smaller than n when $Q = o(n^{1/2} \log^{-4} x)$. It is noteworthy that Theorem 1.4 does so for $Q = o(n^{1/2} \log^{3\delta/4} n)$. That is, by a small margin, it breaks the $n^{\frac{1}{2}}$ barrier. This is nevertheless most useful in applications.

The Hardy–Littlewood conjecture M has some similarity in character with the Goldbach binary conjecture, and the upper bound obtained in (1.7) bears the same relationship to Theorem 1.5 as the Goldbach binary conjecture does to the Bombieri–Vinogradov theorem. However, small further improvements in the latter case have been obtained using information from lower bound sieve estimates. The improvements are very small, and the best that this author is aware of is in Quarel's thesis [21], which also gives an overview of all previous work, where a constant a little bit larger than 3.9 is computed. The complications and circumlocutions occurring in these calculations in order to squeeze out the smallest contribution to the main term remind one somewhat of Dr Johnson's aphorism about a dog walking on hind legs. Much of this work also uses some version of a Chen inversion which does not seem possible in the situation considered here. Nevertheless, one can ponder the possibility of applying some of these methods to show that there is a positive constant δ such that

$$\sum_{m=1}^{n} r_3(m)\theta(m) \le (4-\delta)\Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}\frac{n}{\log n} + o\left(\frac{n}{\log n}\right).$$

It is clear that if $\theta_k(n; y)$ is the characteristic function of the set of natural numbers n which have at most k prime factors, all exceeding y, then lower bounds can be established for

$$\sum_{m=1}^{n} r_3(m)\theta_k(m; y)$$

for suitable k and y. We make an application of the weighted one-dimensional sieve to establish the following.

THEOREM 1.6. *There is a positive constant θ such that*

$$\sum_{m=1}^{n} r_3(m)\theta_3\left(m; n^{\theta}\right) \gg \frac{n}{\log n}.$$

Unfortunately, there seems to be no obvious way of undertaking a Chen inversion to reduce the θ_3 to a θ_2.

2. The proof of Theorem 1.1.

LEMMA 2.1 (van der Corput). *Suppose that $a < b$ and f has a continuous second derivative on $[a, b]$. Suppose also that $\mu > 0$, $\eta > 1$ and that for every $\alpha \in [a, b]$, we have $\mu \le |f''(\alpha)| \le \eta\mu$. Then*

$$\sum_{a<n\le b} e(f(n)) \ll \mu^{-\frac{1}{2}} + (b-a)\eta\mu^{\frac{1}{2}}.$$

This is Theorem 2.2 of Graham and Kolesnik [7].

For $\alpha \in \mathbb{R}$, we define

$$B_1(\alpha) = \alpha - \lfloor \alpha \rfloor - \frac{1}{2}, \quad B_2(\alpha) = \int_0^{\alpha} B_1(\beta)d\beta.$$

LEMMA 2.2. *Let $H \in \mathbb{N}$, $H \ge 2$ and $\alpha \in \mathbb{R}$. Then,*

$$B_1(\alpha) = -\sum_{0<|h|\le H} \frac{e(\alpha h)}{2\pi i h} + O\left(\min\left(1, \frac{1}{H\|\alpha\|}\right)\right) \tag{2.1}$$

and

$$\min\left(1, \frac{1}{H\|\alpha\|}\right) = \sum_{h=-\infty}^{\infty} c(h)e(\alpha h), \tag{2.2}$$

where

$$c(0) = \frac{2}{H}\left(1 + \log\frac{H}{2}\right), \quad c(h) \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right) \quad (h \ne 0). \tag{2.3}$$

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Proof. The expansion (2.1) is well known and follows easily from the expansion for $\log(1+z)$ and partial summation. The Fourier expansion (2.2) follows from any basic result on Fourier series, and the three estimates (2.3) for the coefficient

$$c(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \min\left(1, \frac{1}{H\|\alpha\|}\right) e(-\alpha h) d\alpha$$

follow by the bound $|c(h)| \le c(0)$, integrating by parts once, and integrating by parts twice respectively. □

LEMMA 2.3. *Let $r(n)$ denote the number of representations on n as the sum of two cubes of positive integers. Then,*

$$\sum_{n\le x} r(n) = x^{\frac{2}{3}} \frac{\Gamma\left(\frac{4}{3}\right)^2}{\Gamma\left(\frac{5}{3}\right)} + O\left(x^{\frac{2}{9}} (\log x)^{\frac{1}{3}}\right).$$

Proof. We begin with the observation that by dividing the lattice points l, m under the curve $\alpha^3 + \beta^3 = x$ according to whether l or m does not exceed $\left(\frac{x}{2}\right)^{\frac{1}{3}}$, we have

$$\begin{aligned}\sum_{n\le x} r(n) &= 2 \sum_{m\le (x/2)^{1/3}} \left\lfloor (x-m^3)^{\frac{1}{3}} \right\rfloor - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{3}} \right\rfloor^2 \\ &= 2 \sum_{m\le (x/2)^{1/3}} \left((x-m^3)^{\frac{1}{3}} - \frac{1}{2} \right) \\ &\quad - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{3}} \right\rfloor^2 - 2 \sum_{m\le (x/2)^{1/3}} B_1\left((x-m^3)^{\frac{1}{3}} \right).\end{aligned}$$

The first sum on the right here is

$$\int_0^{x^{1/3}} 2 \min\left(\lfloor \alpha \rfloor, \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{3}} \right\rfloor \right) (x-\alpha^3)^{-\frac{2}{3}} \alpha^2 d\alpha - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{3}} \right\rfloor.$$

The integral here is

$$\begin{aligned}&\int_0^{(x/2)^{1/3}} 2\left(\alpha - \frac{1}{2}\right) (x-\alpha^3)^{-\frac{2}{3}} \alpha^2 d\alpha \\ &+ \int_{(x/2)^{1/3}}^{x^{1/3}} 2 \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{3}} \right\rfloor (x-\alpha^3)^{-\frac{2}{3}} \alpha^2 d\alpha \\ &- \int_0^{(x/2)^{1/3}} 2B_1(\alpha)(x-\alpha^3)^{-\frac{2}{3}} \alpha^2 d\alpha.\end{aligned}$$

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We integrate the first and third integrals by parts and simply integrate the second to obtain

$$\int_0^{(x/2)^{1/3}} 2(x-\alpha^3)^{\frac13}d\alpha - 2\left[\left(\alpha-\frac12\right)(x-\alpha^3)^{\frac13}\right]_0^{(x/2)^{1/3}}$$
$$+2\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor(x/2)^{1/3} - \left[2B_2(\alpha)(x-\alpha^3)^{-\frac23}\alpha^2\right]_0^{(x/2)^{1/3}}$$
$$+\int_0^{(x/2)^{1/3}} 4B_2(\alpha)(x-\alpha^3)^{-\frac53}x\alpha d\alpha.$$

The last two expressions here are $\ll 1$ and the second expression is $-2\left(\frac{x}{2}\right)^{\frac23}+\left(\frac{x}{2}\right)^{\frac13}$. The first integral is twice the two-dimensional volume of the set of points α, β with $\alpha^3+\beta^3\le x$ and $\alpha\le(x/2)^{\frac13}$. Interchanging the variables α, β in one copy of this gives a cover of the whole region $\alpha^3+\beta^3\le x$ with a double cover of the square $[0, (x/2)^{1/3}]^2$. Hence, the integral is

$$\int_0^{x^{1/3}}(x-\alpha^3)^{\frac13}d\alpha+\left(\frac{x}{2}\right)^{\frac23}=\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}x^{\frac23}+\left(\frac{x}{2}\right)^{2/3}.$$

Thus, collecting together the various estimates, we have

$$\sum_{n\le x}r(n)=\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}x^{\frac23}$$
$$-2\left(\frac{x}{2}\right)^{\frac23}+\left(\frac{x}{2}\right)^{\frac23}+\left(\frac{x}{2}\right)^{\frac13}+2\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor\left(\frac{x}{2}\right)^{\frac13}$$
$$-\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor-\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor^2$$
$$-2\sum_{m\le(x/2)^{1/3}}B_1\left((x-m^3)^{\frac13}\right)+O(1).$$

This simplifies down to

$$\sum_{n\le x}r(n)=\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}x^{\frac23}-\left(\left(\frac{x}{2}\right)^{\frac13}-\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor\right)^2$$
$$+\left(\frac{x}{2}\right)^{\frac13}-\left\lfloor\left(\frac{x}{2}\right)^{\frac13}\right\rfloor-2\sum_{m\le(x/2)^{1/3}}B_1\left((x-m^3)^{\frac13}\right)+O(1)$$

and so

$$\sum_{n\le x}r(n)=\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}x^{\frac23}-2\sum_{m\le(x/2)^{1/3}}B_1\left((x-m^3)^{\frac13}\right)+O(1).$$

Let ν be a parameter at our disposal which satisfies $0<\nu\le(x/2)^{\frac13}$ and let

$$\mathcal{M}=\left\{\nu 2^j : j\ge 0,\ \nu 2^j\le(x/2)^{\frac13}\right\}.$$

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For each $M \in \mathcal{M}$, we use M' to denote

$$M' = \min\left(2M, (x/2)^{\frac{1}{3}}\right).$$

Then

$$\sum_{m \le (x/2)^{1/3}} B_1\left((x-m^3)^{\frac{1}{3}}\right) = \sum_{M \in \mathcal{M}} S(M) + O(\nu),$$

where

$$S(M) = \sum_{M < m \le M'} B_1\left((x-m^3)^{\frac{1}{3}}\right).$$

By (2.1) with H also at our disposal

$$S(M) = -\sum_{0<|h|\le H} \frac{T(M,h)}{2\pi i h} + O\left(\sum_{M<m\le M'} \min\left(1, \frac{1}{H\left\|(x-m^3)^{\frac{1}{3}}\right\|}\right)\right),$$

where

$$T(M,h) = \sum_{M<m\le M'} e\left(h\left(x-m^3\right)^{\frac{1}{3}}\right).$$

Moreover, by (2.2),

$$\sum_{M<m\le M'} \min\left(1, \frac{1}{H\left\|(x-m^3)^{\frac{1}{3}}\right\|}\right) = \sum_{h=-\infty}^{\infty} c(h)T(M,h).$$

We now consider $T(M,h)$ when $h \ne 0$. It suffices to suppose then that $h > 0$ for otherwise we can take the complex conjugate. Let

$$f(\alpha) = h\left(x-\alpha^3\right)^{\frac{1}{3}}.$$

Then, for $\alpha < x^{1/3}$,

$$f'(\alpha) = -h(x-\alpha^3)^{-\frac{2}{3}}\alpha^2$$

and

$$f''(\alpha) = -2h\alpha x(x-\alpha^3)^{-\frac{5}{3}}.$$

For $M \in \mathcal{M}$ and $\alpha \in [M, M']$, we have

$$f''(\alpha) \asymp hMx^{-\frac{2}{3}}.$$

Hence, by Lemma 2.1,

$$T(M,h) \ll x^{\frac{1}{3}}h^{-\frac{1}{2}}M^{-\frac{1}{2}} + h^{\frac{1}{2}}M^{\frac{3}{2}}x^{-\frac{1}{3}}.$$

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Hence,

$$\sum_{M<m\le M'} B_1\left((x-m^3)^{\frac13}\right) \ll \sum_{1\le h\le H}\left(x^{\frac13}h^{-2}M^{-\frac12}+h^{-\frac12}M^{\frac32}x^{-\frac13}\right)$$
$$+\frac{M\log(2H)}{H}+\sum_{h=1}^{\infty}|c(h)|\left(x^{\frac13}h^{-1}M^{-\frac12}+h^{\frac12}M^{\frac32}x^{-\frac13}\right)$$

and hence, by (2.3),

$$\sum_{M<m\le M'} B_1\left((x-m^3)^{\frac13}\right) \ll MH^{-1}\log(2H)+x^{\frac13}M^{-\frac12}+H^{\frac12}M^{\frac32}x^{-\frac13}.$$

A good choice for H is

$$H=x^{\frac29}M^{-\frac13}(\log x)^{\frac23}.$$

Thus,

$$\sum_{M<m\le M'} B_1\left((x-m^3)^{\frac13}\right) \ll M^{\frac43}x^{-\frac29}(\log x)^{\frac13}+x^{\frac13}M^{-\frac12}.$$

Summing over the elements of $\mathcal{M}$ gives

$$\sum_{\nu<m\le(x/2)^{1/3}} B_1\left((x-m^3)^{\frac13}\right) \ll x^{\frac29}(\log x)^{\frac13}+x^{\frac13}\nu^{-\frac12}.$$

Hence,

$$\sum_{m\le(x/2)^{1/3}} B_1\left((x-m^3)^{\frac13}\right) \ll \nu+x^{\frac29}(\log x)^{\frac13}+x^{\frac13}\nu^{-\frac12}.$$

The choice $\nu=x^{\frac29}$ gives the desired conclusion. □

We now apply Lemma 2.3 to prove Theorem 1.1. We have

$$\sum_{m=1}^{n} r_3(m)=\sum_{l\le n^{1/3}}\sum_{k=1}^{n-l^3} r(k)$$
$$=\sum_{l\le n^{1/3}}\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}(n-l^3)^{\frac23}+O\left(n^{\frac59}(\log n)^{\frac13}\right).$$

The sum here is

$$\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}\sum_{l\le n^{1/3}}\int_l^{n^{1/3}}(n-\alpha^3)^{-\frac13}2\alpha^2d\alpha=\frac{\Gamma\left(\frac43\right)^2}{\Gamma\left(\frac53\right)}\int_0^{n^{1/3}}\lfloor\alpha\rfloor(n-\alpha^3)^{-\frac13}2\alpha^2d\alpha.$$

The integral here is

$$\int_0^{n^{1/3}}\left(\alpha-\frac12\right)(n-\alpha^3)^{-\frac13}2\alpha^2d\alpha-\int_0^{n^{1/3}}B_1(\alpha)(n-\alpha^3)^{-\frac13}2\alpha^2d\alpha.$$

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By integration by parts, the first integral is

$$\left[-\left(\alpha-\frac{1}{2}\right)(n-\alpha^3)^{\frac{2}{3}}\right]_0^{n^{1/3}}+\int_0^{n^{1/3}}(n-\alpha^3)^{\frac{2}{3}}d\alpha.$$

This is

$$-\frac{1}{2}n^{\frac{2}{3}}+n\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}.$$

Let $\xi=n^{\frac{2}{3}}$. Then

$$\int_{(n-\xi)^{1/3}}^{n^{1/3}}B_1(\alpha)(n-\alpha^3)^{-\frac{1}{3}}2\alpha^2d\alpha\ll\xi^{\frac{2}{3}}=n^{\frac{4}{9}}$$

and

$$\int_0^{(n-\xi)^{1/3}}B_1(\alpha)(n-\alpha^3)^{-\frac{1}{3}}2\alpha^2d\alpha=\left[B_2(\alpha)(n-\alpha^3)^{-\frac{1}{3}}2\alpha^2\right]_0^{(n-\xi)^{1/3}}$$
$$-\int_0^{(n-\xi)^{1/3}}B_2(\alpha)(n-\alpha^3)^{-\frac{4}{3}}2\alpha(2n-\alpha^3)d\alpha.$$

and this is

$$\ll n^{\frac{2}{3}}\xi^{-\frac{1}{3}}=n^{\frac{4}{9}}.$$

3. The proof of Theorem 1.2. Let ρ be a sufficiently large but fixed real number and define

$$\mathcal{S}=\left\{r\in[1,P]:p|r\Rightarrow p\notin\left((\log P)^{\rho},P^{\frac{1}{7}}\right]\right\},$$

$$\mathcal{T}=\{r\in[1,P]:\text{There is a prime }p\in\left((\log P)^{\rho},P^{\frac{1}{7}}\right]\text{ such that }p|r\}$$

$$\hat{f}(\alpha)=\sum_{r\in\mathcal{S}}e(\alpha r^3)\tag{3.1}$$

and

$$\tilde{f}(\alpha)=\sum_{r\in\mathcal{T}}e(\alpha r^3).\tag{3.2}$$

Boklan [2], Lemma E, has shewn that the arguments of Vaughan [23] and [24] combined with the improved bounds of Hall and Tenebaum [9] for Hooley's Δ function give

$$\int_0^1|\hat{f}(\alpha)|^8d\alpha\ll n^{\frac{5}{3}}(\log n)^{\varepsilon-3}.\tag{3.3}$$

A routine sieve argument, such as one based on the Selberg sieve, for example, as in the exposition of Halberstam and Richert [8], demonstrates that

$$\text{card}\,\mathcal{S}\ll\frac{n^{\frac{1}{3}}\log\log n}{\log n}.$$

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Hooley [14] (see also Wooley [26]) has shewn that the number of solutions of

$$m_1^3 + m_2^3 = m_3^3 + m_4^3,$$

with $m_j \le P$ in which the m_j on the right are not a permutation of those on the left is

$$\ll P^{\frac{5}{3}+\varepsilon}.$$

Thus, the total number of solutions with $m_j \in \mathcal{S}$ is

$$\ll (\operatorname{card} \mathcal{S})^2 + n^{\frac{5}{9}+\varepsilon}.$$

Hence,

$$\int_0^1 |\hat{f}(\alpha)|^4 d\alpha \ll \frac{n^{\frac{2}{3}} \log^2 \log n}{\log^2 n}.$$

Therefore, by (3.3) and Schwarz's inequality

$$\int_0^1 |\hat{f}(\alpha)|^6 d\alpha \ll n^{\frac{7}{6}} \log^{\varepsilon-\frac{5}{2}} n. \tag{3.4}$$

Clearly from (1.3), (3.1), and (3.2), we have

$$f(\alpha) = \hat{f}(\alpha) + \tilde{f}(\alpha),$$

so that

$$\int_0^1 |f(\alpha)|^6 d\alpha \ll \int_0^1 |\hat{f}(\alpha)|^6 d\alpha + \int_0^1 |\tilde{f}(\alpha)|^6 d\alpha. \tag{3.5}$$

Thus, it suffices now to bound

$$\int_0^1 |\tilde{f}(\alpha)|^6 d\alpha,$$

the number of solutions of

$$m_1^3 + m_2^3 + m_3^3 = m_4^3 + m_5^3 + m_6^3, \tag{3.6}$$

with $m_j \in \mathcal{T}$. First of all consider the subset of all such solutions in which at least two of the variables have at least one common prime factor p with $(\log P)^\rho < p \le P^{\frac{1}{7}}$. Then, the total number N_1 of such solutions is bounded by

$$\sum_{(\log P)^\rho < p \le P^{1/7}} \int_0^1 |f(p^3\alpha; P/p)|^2 |f(\alpha)|^4 d\alpha,$$

here

$$f(\beta; Q) = \sum_{m \le Q} e(\beta m^3).$$

Thus, by Hölder's inequality

$$N_1 \ll \sum_{(\log P)^\rho < p \le P^{1/7}} \left(\int_0^1 |f(p^3\alpha; P/p)|^6 d\alpha \right)^{\frac{1}{3}} \left(\int_0^1 |f(\alpha)|^6 d\alpha \right)^{\frac{2}{3}}.$$

Hence, by (1.4), we have

$$N_1 \ll \sum_{(\log P)^\rho < p \le P^{1/7}} P^{\frac{7}{2}} p^{-\frac{7}{6}} \ll P^{\frac{7}{2}} (\log P)^{-\frac{\rho}{6}}.$$

Therefore, it remains to bound the number of solutions of (3.6) with $m_j \in \mathcal{T}$ and for which each prime p dividing m_j with $(\log P)^\rho < p \le P^{\frac{1}{7}}$ has the property that $p \nmid m_k$ when $k \ne j$. The number of such solutions is $\ll N_2$ where N_2 is the number of solutions of

$$p^3 l^3 + m_2^3 + m_3^3 = m_4^3 + m_5^3 + m_6^3.$$

with $(\log P)^\rho < p \le P^{\frac{1}{7}}$, $l \le P/p$, $m_j \le P$ and $(p, m_j) = 1$. Let

$$\mathcal{M} = \left\{ (\log P)^\rho 2^j : 0 \le j,\, 2^j < P^{\frac{1}{7}} (\log P)^{-\rho} \right\}$$

and for each $M \in \mathcal{M}$ let $M' = \min \left\{ 2M, P^{\frac{1}{7}} \right\}$. Then

$$N_2 \le \sum_{M \in \mathcal{M}} N(M),$$

where $N(M)$ is the number of solutions of

$$m^3 l^3 + m_2^3 + m_3^3 = m_4^3 + m_5^3 + m_6^3,$$

with $M < m \le M'$, $l \le P/M$, $m_j \le P$ and $(m, m_j) = 1$. Thus,

$$N(M) = \sum_{M < m \le M'} \int_0^1 f(m^3\alpha; P/M) f_m(\alpha)^2 f_m(-\alpha)^3 \, d\alpha,$$

where

$$f_m(\alpha) = \sum_{\substack{r \le P \\ (r,m)=1}} e(\alpha r^3).$$

Hence, by Hölder's inequality,

$$N(M) \le \left(\int_0^1 \sum_{M < m \le M'} |f(m^3\alpha; P/M)|^4 |f_m(\alpha)|^2 \right)^{\frac{1}{4}} \left(\sum_{M < m \le M'} |f_m(\alpha)|^6 d\alpha \right)^{\frac{3}{4}}$$

and this is at most

$$\left(\int_0^1 \sum_{M < m \le M'} |f(m^3\alpha; P/M)|^4 |f_m(\alpha)|^2 d\alpha \right)^{\frac{1}{4}} M^{\frac{3}{4}} \left(\int_0^1 |f(\alpha)|^6 d\alpha \right)^{\frac{3}{4}}.$$

Hence, by Lemma 5 of Vaughan [**23**] and (1.4),

$$N(M) \ll P^{\frac{7}{2}} (\log P)^{\frac{15}{4}} M^{-\frac{3}{8}}$$

and so on summing M over the elements of $\mathcal{M}$, we have

$$N_2 \ll P^{\frac{7}{2}} (\log P)^{\frac{5}{2} - \frac{3}{8}\rho}.$$

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Thus, by choosing ρ sufficiently large, we have

$$\int_0^1 |\tilde{f}(\alpha)|^6 d\alpha \ll n^{\frac{7}{6}} (\log n)^{-\frac{5}{2}}$$

and so with (3.4) and (3.5) this establishes Theorem 1.2.

4. The proof of Theorems 1.4 and 1.5. Here, we follow very closely the work of Hooley [15]. This in turn makes important use of results of Milne, Dwork, Deligne and Katz in algebraic geometry. In order to ease the translation of estimates from Hooley's memoir, we adopt his notation, with some changes, and the reader who wishes to follow our proof in detail would be well advised to have a copy of Professor Hooley's work to hand. In general in defining functions, we usually write the modulus before any residue class, as in our definition of Υ, and we usually include all parameters which we plan to vary. Hooley reverses the order of modulus and residue and in the interest of simplicity suppresses some variables. Thus, we will rewrite many of his crucial statements in our usual format. Let

$$S(q;\, \mathbf{h}, a) = \sum_{\substack{\mathbf{m} \bmod q \\ m_1^3+m_2^3+m_3^3 \equiv a \bmod q}} e(\mathbf{h}.\mathbf{m}/q),$$

$$F(q;\, a) = \sum_{\mathbf{h} \bmod q} |S(q;\, \mathbf{h}, a)|, \tag{4.1}$$

$$\rho(q, a) = S(q;\, \mathbf{0}, a)$$

and

$$B = \Gamma\left(\frac{4}{3}\right)^3.$$

Thus, in Hooley's notation, these are $S(h_1, h_2, h_3;\, q)$ and $F(q)$, as defined in his introduction, B as defined in *ibidem* (15) and $\rho(a, q)$ as defined in *ibidem* between (22) and (23).

We suppose that $x \le n$ and with Hooley (34) define

$$H_q = q^{\frac{2}{9}} x^{\frac{2}{9}} (\log x)^{-g},$$

where g is a fixed positive number at our disposal. We further observe that the $\Upsilon(x;\, q, a)$ of our (1.8) is $\Theta(x;\, a, q)$ of *ibidem* (5).

Hooley generally supposes that the modulus, his k, our q does not exceed $n^{\frac{1}{2}}$. We relax this. For the time being, we will suppose that

$$q \le Q \le n^{\frac{5}{9}}.$$

In the final stages of the proof of Theorem 1.4, we will reduce this. In a few places, our relaxation on Q requires a minor change, which we will advert to at the appropriate moment. We follow Hooley verbatim as far as *ibidem* (33). Thus, for $x \le n$ we have

$$\Upsilon(x, q, a) = \frac{\rho(q, a)}{q^3}\Big(Bx + O\big(n^{\frac{2}{3}} H\big)\Big) + O\big(M(n, H, q, a)\big).$$

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The function M satisfies

$$M(n, H, q, a) = \sum_{j=1}^{3} \Lambda_j(n, H, q, a),$$

where the Λ_j are the three multiple sums occurring in the error term in the first line of the displayed formula above *ibidem* (31). Hooley now sums the Λ_j over q, leaving a fixed. Instead for each q, we will take the maximum over $a \in \mathcal{A}(q)$ where either $\mathcal{A}(q)$ is as in Theorem 1.4 or is the set of all residue classes modulo q. He then splits the sum for Λ_3 into three separate sums, and we follow that course. Thus, we write

$$\Sigma_{11} = \sum_{q \le Q} \max_{a \in \mathcal{A}(q)} \Lambda_1(n, H, q, a),$$

$$\Sigma_{12} = \sum_{q \le Q} \max_{a \in \mathcal{A}(q)} \Lambda_2(n, H, q, a),$$

$$\Sigma_{13} = \sum_{q \le n^{5/12}} \max_{a \in \mathcal{A}(q)} \Lambda_3(n, H, q, a),$$

$$\Sigma_{14} = \sum_{\substack{q \le Q \\ q_1 > n^{1/36}}} \max_{a \in \mathcal{A}(q)} \Lambda_3(n, H, q, a),$$

$$\Sigma_{15} = \sum_{\substack{x^{5/12} < q \le Q \\ q_1 \le n^{1/36}}} \max_{a \in \mathcal{A}(q)} \Lambda_3(n, H, q, a).$$

Here q_1 and for future reference q_2 are defined by

$$q_1 = q_1(q) = \prod_{\substack{p^t \| q \\ p \le \xi}} p^t, \quad q_2 = q_2(q) = q/q_1,$$

where

$$\xi = n^{1/(\log\log n)^2}.$$

as in *ibidem* (121) and (122). Thus, by (1.9),

$$\sum_{q \le Q} \max_{a \in \mathcal{A}(q)} \sup_{x \le n} |E(x; q, a)| \ll n^{\frac{8}{9}} (\log n)^{-g} \sum_{q \le Q} q^{-\frac{25}{9}} \max_{a \in \mathcal{A}(q)} \rho(q, a) + \sum_{j=11}^{15} \Sigma_j.$$

When $\mathcal{A}(q)$ is as in Theorem 1.4, the argument given in Section 6 of *ibidem* shews that

$$\begin{aligned} n^{\frac{8}{9}} (\log n)^{-g} \sum_{q \le Q} q^{-\frac{7}{3}} \max_{a \in \mathcal{A}(q)} \rho(q, a) &\ll n^{\frac{8}{9}} Q^{\frac{2}{9}} \frac{(\log\log n)^{A_4}}{(\log n)^g} \prod_{p | N} \left(1 - \frac{1}{p}\right)^{1 - A_3} \\ &\ll n^{\frac{8}{9}} Q^{\frac{2}{9}} (\log n)^{-g} (\log\log n)^{A_4} (\log\log N)^{A_3}. \end{aligned}$$

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In the case of Theorem 1.5, a cruder estimate gives

$$n^{\frac{8}{9}}(\log n)^{-g}\sum_{q\le Q} q^{-\frac{25}{9}} \max_{a\in\mathcal{A}(q)} \rho(q,a) \ll n^{\frac{8}{9}}(\log\log n)^{A_4}\sum_{q\le Q}\frac{d_{A_3}(q)}{q^{7/9}}$$
$$\ll n^{\frac{8}{9}}Q^{\frac{2}{9}}(\log n)^C.$$

Let

$$\eta(q)=\prod_{p^t\|q} p^{t\alpha_t}, \tag{4.2}$$

where (Lemma 13 ibidem)

$$\alpha_1=1,\ \alpha_2=\frac{3}{2},\ \alpha_3=\frac{5}{3},\ \alpha_4=\frac{7}{4},\ \alpha_5=\frac{9}{5},\ \alpha_t=\frac{11}{6}\ (t\ge 6). \tag{4.3}$$

This is Hooley's $w(q)$ defined in Lemma 14 *ibidem*. We have changed this to η because in computer modern when not juxtaposed ω and w are almost indistinguishable to our eyes.

In bounding Σ_j when $11\le j\le 14$, Hooley Section 16 *ibidem* uses only estimates for $S(q,\mathbf{h},\mathbf{a})$ and $\rho(q,a)$ from Section 12 and Lemma 14 of *ibidem*, and these are independent of a (his N). Since we are allowing Q to be as large as $n^{\frac{5}{9}}$, some small changes need to be made which we detail as follows. Thus, in reference to Σ_{11}, we have

$$\Lambda_1(n,H_q,q,a)\ll\frac{nA_{12}^{\omega(q)}\eta(q)}{H_q q^2}\sum_{h_1=1}^{\infty}\frac{(h_1,q)}{h_1}\min\left(1,\frac{q}{n^{1/3}h_1}\right).$$

Now for $h_1>n^{\frac{1}{3}}$, we replace the min by $h_1^{-\frac{1}{3}}$ and otherwise proceed without change. Thus,

$$\Sigma_{11}\ll n^{\frac{7}{9}+\varepsilon}\sum_{q\le Q}\frac{\eta(q)}{q^2}\ll n^{\frac{7}{9}+2\varepsilon}.$$

In the treatment of Σ_{12}, a similar adjustment in which we now replace $\min(1,\frac{q^{1/2}}{n^{1/6}h_j^{1/2}})$ by $h_j^{-\frac{1}{6}}$ when $h_j>n^{\frac{1}{3}}$ leads to the bound

$$\Sigma_{12}\ll n^{\frac{5}{9}+\varepsilon}\sum_{q\le Q}\frac{\eta(q)}{q^{13/9}}\ll n^{\frac{8}{9}+2\varepsilon},$$

which is acceptable.

In $\Sigma_{13,}$ the sum over q is constrained by the bound $q\le n^{\frac{5}{12}}$, so no adjustments are needed. However, in $\Sigma_{14,}$ we are not so fortunate. Thus, in the initial examination of $\Lambda_3(n,H_q,q,a)$ in (128) *ibidem,* we divide the h_j at $n^{\frac{4}{9}}$ rather than $n^{\frac{1}{3}}$ and then the final line of (128) holds for all $q\le Q$. Then following (130) *ibidem,* we obtain

$$\Sigma_{14}\ll n^{\frac{1}{3}}Q^{\frac{4}{3}}(\log n)^{3g+3}\sum_{\substack{q\le Q\\ q_1>n^{1/36}}}\frac{A_{21}^{\omega(q)}\eta(q)}{q^2}.$$

Following the argument without further change gives

$$\Sigma_{14}\ll n^{\frac{1}{3}}Q^{\frac{4}{3}}(\log n)^C\exp\left(-\frac{1}{36}(\log\log n)^{3/2}\right)$$

for a suitable positive constant C. This is acceptable.

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It remains to deal with Σ_{15}. We follow Section 17 *ibidem* as far as (138). We observe that in the discussion between (136) and (137), the argument is unchanged for q (Hooley's k) no larger than $n^{\frac{1}{2}}(\log n)^{\frac{9g}{2}}$ and so we henceforth suppose that

$$Q \le n^{\frac{1}{2}}(\log n)^{\frac{9g}{2}}.$$

Thus, in our notation (138), *ibidem* gives for some positive constant C

$$\Lambda_3(n, H_q, q, a) \ll \frac{nF(q_1, a_1)C^{\omega(q_2)}\eta(q_2)}{H_q^3 q_1^3}.$$

Here, a_1 is such that $a_1 \equiv a \pmod{q_1}$. Note that if $(q, a)|N$, then $(q_1, a_1)|N$. At this stage, it is convenient also to point out that if q has a factor m with $(m, N) = 1$, then the condition on the elements a of $\mathcal{A}(q)$ in Theorem 1.4 permits us to deduce that $(m, a) = 1$. Thus,

$$\begin{aligned}\Sigma_{15} &\ll n^{\frac{1}{3}}(\log n)^{3g} \sum_{\substack{q \le Q \\ q_1 \le n^{1/36}}} \max_{a_1 \in \mathcal{A}(q_1)} F(q_1; a_1)\frac{C^{\omega(q_2)}\eta(q_2)}{q^{2/3}q_1^3} \\ &\ll n^{\frac{1}{3}}Q^{\frac{1}{3}}(\log n)^{3g} \sum_{q_1 \le n^{1/36}} \max_{a_1 \in \mathcal{A}(q_1)} \frac{F(q_1; a_1)}{q_1^4} \sum_{q_2 \le Q/q_1} \frac{C^{\omega(q_2)}\eta(q_2)}{q_2}.\end{aligned}$$

where F is given by (4.1). Following Hooley, the dexter sum is of the form

$$\sum_{q_2 \le u} \frac{C^{\omega(q_2)}\eta(q_2)}{q_2},$$

where $Qn^{-\frac{1}{36}} < u \le Q/q_1$ and it is shewn in (142) ibidem to be

$$\ll \frac{u(\log\log n)^{C'}}{\log n}.$$

Thus,

$$\Sigma_{15} \ll \frac{n^{\frac{1}{3}}Q^{\frac{4}{3}}(\log\log n)^{C'}}{\log^{1-3g} n} \sum_{q_1 \le n^{1/36}} \max_{a_1 \in \mathcal{A}(q_1)} \frac{F(q_1; a_1)}{q_1^5}.$$

In the case of Theorem 1.4, Hooley's argument, with N, not n, governing the divisibility by p, leads us through (144) to (147) *ibidem*, and allows us to infer that the final sum is

$$\ll (\log\log n)^{C''}(\log n)^{\sigma},$$

where $0 < \sigma < 1$. Observe that by our hypothesis on N, we have

$$\prod_{p|N}(1 - 1/p)^{-2A_3} \ll (\log\log 2N)^{2A_3} \ll (\log n)^{\varepsilon}.$$

Thus,

$$\Sigma_{15} \ll \frac{n^{\frac{1}{3}}Q^{\frac{4}{3}}(\log\log n)^{C'''}}{\log^{1-\sigma-3g} n}.$$

This completes the proof of Theorem 1.4.

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In the contrary case, so that the argument leading to (149) *ibidem* is not available to us, we obtain only

$$\Sigma_{15} \ll n^{\frac{1}{3}} Q^{\frac{4}{3}} (\log\log n)^{C'''} \log^{3g} n,$$

which is nevertheless sufficient to confirm Theorem 1.5.

5. The proof of Theorem 1.6. The linear sieve applied directly to our problem would only achieve a θ_4 in place of the θ_3. The standard way of reducing to a θ_3 is by the use of weights. Many elaborate weight systems have been introduced in order to extract as large a lower bound as possible from the available methods. In the interest of concision, we use the simplest weights and consider

$$\sum_{\substack{m=1\\(m,P)=1}}^{n} r_3(m)\left(1-\sum_{\substack{p|m\\ z<p\le n^{1/3}}}\frac{1}{2}\right),$$

where

$$z=n^{1/8},\ P=\prod_{p\le z} p.$$

The terms in this sum can only be positive when m has at most one prime factor not exceeding $n^{1/3}$. But, there can never be more than two prime factors greater than $n^{1/3}$. We can rearrange this as

$$\sum_{\substack{m=1\\(m,P)=1}}^{n} r_3(m)-\frac{1}{2}\sum_{z<p\le n^{1/3}}\ \sum_{\substack{m\le n/p\\(m,P)=1}} r_3(mp)$$

and apply the lower bound linear sieve to the first sum over m and the upper bound linear sieve to the second, taking advantage of Theorem 1.5 above. We appeal to Theorem 1 of Iwaniec [19] with $X=\Gamma(4/3)^3 n$, $\omega(p)=\rho(p,0)p^{-2}$, $y=n^{1/2}(\log n)^{-A}$, $z=n^{1/8}$, so that $s=\frac{\log y}{\log z}$ in the first sum, and with $X=\Gamma(4/3)^3 n/p$, $s=s_p=\frac{y/p}{\log z}$ in the second. Thus,

$$s=4-\frac{8A\log\log n}{\log n},\ s_p=4-\frac{8\log p}{\log n}-\frac{8A\log\log n}{\log n}$$

and the above is

$$\ge n\Gamma\left(\frac{4}{3}\right)^3\prod_{p\le z}\left(1-\frac{\rho(p,0)}{p^3}\right)\left(f(s)-\frac{1}{2}\sum_{z<p\le n^{1/3}}\frac{F(s_p)}{p}\right)+O\left(\frac{n}{(\log n)^{9/8}}\right).$$

We have $\rho(p,0)=p^2$, ($p\equiv 2\pmod 3$ or $p=3$) and $\rho(p,0)\le 3p^2$ ($p\equiv 1\pmod 3$). Thus,

$$1-\frac{\rho(p,0)}{p^3}>0.$$

We also have

$$\rho(p,0)=\frac{1}{p}\sum_{a=1}^{p} S_3(p,a)^3=p^2+\frac{1}{p}\sum_{a=1}^{p-1} S_3(p,a)^3.$$

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Thus,

$$1-\frac{\rho(p,0)}{p^3}=\left(1-\frac{1}{p}\right)\left(\sum_{k=0}^{\infty}\sum_{\substack{a=1\\ p\nmid a}}^{p^k}\frac{\mu(p^k)S_3(p^k,a)^3}{\phi(p^k)p^{3k}}\right).$$

By classical estimates for Gauss sums

$$\sum_{a=1}^{p-1}S_3(p,a)^3\ll p^{5/2}.$$

Hence,

$$\prod_{p\le z}\left(\sum_{k=0}^{\infty}\sum_{\substack{a=1\\ p\nmid a}}^{p^k}\frac{\mu(p^k)S_3(p^k,a)^3}{\phi(p^k)p^{3k}}\right)=\mathfrak{S}+O(z^{-1/2}),$$

where $\mathfrak{S}$ is as in (1.6) and $\mathfrak{S}>0$.

In the ranges of interest here, we have

$$f(s)=\frac{2e^{\gamma}\log(s-1)}{s},\quad F(s)=\frac{2e^{\gamma}}{s}.$$

Thus, by prime number theory and the smoothness of f and F, this is

$$\Gamma\left(\frac{4}{3}\right)^3\frac{4\mathfrak{S}n}{\log n}\left(\log 3-\int_{n^{1/8}}^{n^{1/3}}\frac{1}{x(\log x)\big(2-4(\log x)/\log n\big)}dx\right)+O\left(\frac{n}{(\log n)^{9/8}}\right).$$

By the change of variable $y=(\log n)/\log x$, the integral here is

$$\int_3^8\frac{dy}{2y-4}=\frac{1}{2}\log 6.$$

Thus,

$$\sum_{\substack{m=1\\(m,P)=1}}^{n}r_3(m)\left(1-\sum_{\substack{p|m\\ z<p\le n^{1/3}}}\frac{1}{2}\right)\ge\Gamma\left(\frac{4}{3}\right)^3\frac{2\mathfrak{S}n\log(3/2)}{\log n}+O\left(\frac{n}{(\log n)^{9/8}}\right),$$

which satisfies our desideratum.

REFERENCES

1. R. Ayoub, A mean value theorem for quadratic fields, *Pacific J. Math.* **8** (1958), 23–27.

2. K. D. Boklan, A reduction technique in Waring's problem, I, *Acta Arithmetica* **65** (1993), 147–161.

3. K. D. Boklan, The asymptotic formula in Waring's problem, *Mathematika* **41** (1994), 329–347.

4. J. Brüdern and T. D. Wooley, The asymptotic formulae in Waring's problem for cubes, *J. für die reine und angewandte Mathematik* **647** (2010), 1–23.

5. H. Cramér, Über das Teilerproblem von Piltz, *Arkiv för Mat. Astr. och. Fysik* **16** (1922), 155–157.

https://doi.org/10.1017/S0017089520000142 Published online by Cambridge University Press

6. P. Erdős and W. H. J. Fuchs, On a problem of additive number theory, *J. London Math. Soc.* **31** (1956), 67–73.

7. S. W. Graham and G. Kolesnik, *Van Der Corput's method of exponential sums*, London Mathematical Society Lecture Note, vol. 126 (Cambridge University Press, 1991).

8. H. Halberstam and H.-E. Richert, *Sieve methods* (Academic Press, London, 1974).

9. R. Hall and G. Tenenbaum, *Divisors* (Cambridge University Press, London, 1988).

10. G. H. Hardy, The average order of the functions $P(x)$ and $\Delta(x)$, *Proc. London Math. Soc.* **15** (1923), 192–213.

11. G. H. Hardy and J. E. Littlewood, Some problems of 'partito numerorum'; III: On the expression of a number as a sum of primes, *Acta Mathematica* **44** (1923), 1–70.

12. E. K. Hayashi, An elementary method for estimating error terms in additive number theory, *Proc. Amer. Math Soc.* **52** (1975), 55–59.

13. D. R. Heath-Brown, Mean values of the zeta-function and divisor problems, in *Recent Progress in Analytic Number Theory, Durham Symposium 1979*, vol. 1 (Academic Press, London, 1981), 115–119.

14. C. Hooley, On the numbers that are representable as the sum of two cubes, *J. Reine Angew. Math.* **314** (1980), 146–173.

15. C. Hooley, On Waring's problem for two squares and three cubes, *J. für die reine und angewandte Mathemtik* **328** (1981), 161–207.

16. C. Hooley, On Waring's problem, *Acta Math.* **157** (1986), 49–97.

17. C. Hooley, On some topics connected with Waring's problem, *J. für die reine und angewandte Mathematik* **369** (1986), 110–153.

18. C. Hooley, On hypothesis K^* in warings problem, in *Sieve methods, exponential sums, and their applications in number theory* (G. Greaves, G. Harman and M. N. Huxley, Editors) (Cambridge University Press, 1996), 175–185.

19. H. Iwaniec, Rosser's sieve, *Acta Arithmetica* **36** (1980), 171–202.

20. H. L. Montgomery and R. C. Vaughan, On the Erdős-Fuchs theorems, in *A tribute to Paul Erdős* (A. Baker, B. Bollobás and A. Hajnal, Editors) (Cambridge University Press, 1990).

21. D. Quarel, On a numerical upper bound for the extended Goldbach conjecture, arXiv: 1801.01813 [math.HO]

22. R. C. Vaughan, On the addition of sequences of integers, *J. Number Theory* **4** (1972), 1–16.

23. R. C. Vaughan, On Waring's problem for cubes, *J. für die reine und angewandte Mathemtik* **365** (1986), 122–170.

24. R. C. Vaughan, On Waring's problem for smaller exponents, *Mathematika* **33** (1986), 6–22.

25. R. C. Vaughan, A mean value theorem for cubic fields, *J. Number Theory* **100** (2003), 169–183.

26. T. D. Wooley, Sums of two cubes, *Int. Math. Res. Not.* **1995** (1995), 181–185.