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One-dimensional multifluid plasma models. Part 1. Fundamentals†

P. BACHMANN and D. SÜNDER

Max-Planck-Institut für Plasmaphysik, Bereich Plasmadiagnostik, EURATOM Association, Mohrenstrasse 41, D-10117 Berlin, Germany (e-mail: bachmann@ipp.mpg.de)

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This paper is concerned with one-dimensional and time-dependent multifluid plasma models derived from multifluid MHD equations. In order to reduce the number of equations to be solved, the impurities are described in the framework of the average ion approach without restricting the impurity densities to be small compared with the hydrogen plasma density. Equalizing the plasma temperatures and adopting the condition of quasineutrality, we arrive at a three-fluid description of a current-carrying plasma, and analyse the ability of the self-consistent system of model equations thus obtained to support stationary solutions in a moving frame. This system is reduced to a currentless plasma description assuming at first different flow velocities of the particles and then a currentless, streaming plasma where all particles move with the same velocity. Introducing Lagrangian coordinates and adopting an equation of state, a single reaction–diffusion equation (RDE) for the temperature is obtained. The impurity density, which affects the radiation loss term and the heat conduction coefficient of the RDE, has to be calculated as a function of the temperature by solving additionally a first-order differential equation. This is demonstrated for carbon and high-Z impurities.

1. Introduction

Transport phenomena in multicomponent plasmas have been studied within the framework of multifluid models and codes in many papers (see e.g. Braginskii 1965; Shdanov 1982; Dnestrovskii and Kostomarov 1986; Igitkhanov et al. 1990; Radford 1993; Zagorski 1996). Impurities strongly influence the transport properties of the edge plasmas in tokamaks and stellarators. Therefore a self-consistent description of the dynamics of all species is necessary without restricting the impurity densities to be small compared with the plasma density. It is the aim of this paper to analyse multifluid plasma models that can be applied to investigate the effect of impurities on plasma properties. A variety of models are presented, arranged according to their level of complexity. We start with a comprehensive multifluid description and finish with simple models that can easily be used.

In this paper, we investigate a system of one-dimensional, time-dependent hydrodynamic equations describing the dynamics of hydrogen plasma ions (i), impurity

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ions (j) with charge state Z_j and electrons (e) along magnetic field lines, taking into account ionization, recombination, excitation, charge exchange processes and radiative cooling of the plasma by impurity ions. For each of the $Z_f + 1$ ion fluids, where Z_f is the charge state of the fully ionized impurity, we derive continuity equations governing the particle densities n_α and momentum balance equations describing the flow velocity of the particles v_α ($\alpha = e, i, Z_j$). The electron and ion temperatures T_e and T_i are governed by convection–conduction equations. Here it is assumed that the ion fluids may have different velocities but a common temperature T_i. This multifluid plasma description is outlined in Sec. 2. In order to simplify the treatment, the impurities are described by the average ion model (Sec. 3).

Equalizing the temperatures and adopting the condition of quasi charge neutrality, we arrive at a fluid description of a current-carrying plasma in Sec. 4.1. Stationary solutions in a moving reference frame of the obtained self-consistent system of equations are analysed. This system is reduced to a currentless plasma description (Sec. 4.2), assuming at first different flow velocities of the particles, and then a currentless, streaming plasma model where all particles move with the same velocity.

Introducing Lagrangian coordinates, adopting an equation of state, and including a differential equation that allows one to calculate the impurity density as a function of the temperature, a single reaction–diffusion equation for the temperature is derived. This reaction–diffusion model will be applied in a forthcoming paper to investigate effects related to impurity radiation phenomena.†

2. Multifluid plasma equations

The MHD equations describing the particle dynamics along the magnetic field lines are as follows. Continuity of hydrogen ions and impurity ions with the charge state Z_j:

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = S_{\alpha,n}, \qquad \alpha = i, Z_j. \tag{1}$$

Continuity of electrons:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) = S_{e,n}. \tag{2}$$

Momentum balance of plasma ions and impurity ions:

$$\frac{\partial}{\partial t}(m_\alpha n_\alpha v_\alpha) + \frac{\partial}{\partial x}\left(m_\alpha n_\alpha v_\alpha^2 + n_\alpha T_i - \eta_\alpha \frac{\partial v_\alpha}{\partial x}\right) = e n_\alpha E + R_\alpha + S_{\alpha,v}, \tag{3}$$

or, taking into account the continuity equations,

$$m_\alpha n_\alpha \left(\frac{\partial}{\partial t} + v_\alpha \frac{\partial}{\partial x}\right) v_\alpha + \frac{\partial}{\partial x}\left(n_\alpha T_i - \eta_\alpha \frac{\partial v_\alpha}{\partial x}\right) = e n_\alpha E + R_\alpha + S_{\alpha,v} - m_\alpha v_\alpha S_{\alpha,n}. \tag{4}$$

Simplified momentum equation for electrons:

$$\frac{\partial}{\partial x}(n_e T_e) = -e n_e E + R_e; \tag{5}$$

† A first application was presented at the 25th EPS Conference on Controlled Fusion and Plasma Physics, Prague 1998 (Bachmann and Sünder 1998b).

Ion and electron energy balance:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(3n_iT_i + m_in_iv_i^2 + 3\sum_{Z_j} n_{Z_j}T_i + \sum_{Z_j} m_{Z_j}n_{Z_j}v_{Z_j}^2\right)\\
&\quad+\frac{1}{2}\frac{\partial}{\partial x}\left(5n_iv_iT_i + 5\sum_{Z_j} n_{Z_j}v_{Z_j}T_i + m_in_iv_i^3 + \sum_{Z_j} m_{Z_j}n_{Z_j}v_{Z_j}^3\right)\\
&\quad+\frac{\partial}{\partial x}\left(q_i + \sum_{Z_j} q_{Z_j} - \eta_iv_i\frac{\partial}{\partial x}v_i - \sum_{Z_j}\eta_{Z_j}v_{Z_j}\frac{\partial}{\partial x}v_{Z_j}\right)\\
&\quad= JE + en_ev_eE - Q_{e,T} - Q_{e,R} + S_{i,E},
\end{aligned} \tag{6}
$$

$$
\frac{3}{2}\frac{\partial}{\partial t}(n_eT_e) + \frac{\partial}{\partial x}\left(\frac{5}{2}n_ev_eT_e + q_e\right) = -en_ev_eE + Q_{e,T} + Q_{e,R} + S_{e,E} + H_{\text{ext}}. \tag{7}
$$

Taking into consideration the particle and momentum balance equations and $m_{Z_j} = m_j$, we obtain for the ion temperature

$$
\begin{aligned}
&\frac{3}{2}n_i\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x}\right)T_i + \frac{3}{2}\sum_{Z_j} n_{Z_j}\left(\frac{\partial}{\partial t} + v_{Z_j}\frac{\partial}{\partial x}\right)T_i + n_iT_i\frac{\partial v_i}{\partial x}\\
&\quad+\sum_{Z_j} n_{Z_j}T_i\frac{\partial v_{Z_j}}{\partial x} - \eta_i\left(\frac{\partial v_i}{\partial x}\right)^2 - \sum_{Z_j}\eta_{Z_j}\left(\frac{\partial v_{Z_j}}{\partial x}\right)^2 + \sum_{Z_j}\frac{\partial q_{Z_j}}{\partial x} + \frac{\partial q_i}{\partial x}\\
&\quad= -Q_{e,T} - Q_{e,R} + S_{i,E} - v_i(S_{i,v} + R_i) - \sum_{Z_j} v_{Z_j}(S_{Z_j,v} + R_{Z_j})\\
&\quad+\left(\frac{1}{2}m_iv_i^2 - \frac{3}{2}T_i\right)S_{i,n} + \sum_{Z_j}\left(\frac{1}{2}m_jv_{Z_j}^2 - \frac{3}{2}T_i\right)S_{Z_j,n},
\end{aligned} \tag{8}
$$

and for the electron temperature

$$
\frac{3}{2}n_e\left(\frac{\partial}{\partial t}+v_e\frac{\partial}{\partial x}\right)T_e+n_eT_e\frac{\partial v_e}{\partial x}+\frac{\partial q_e}{\partial x} = Q_{e,T}+Q_{e,R}-v_eR_e+S_{e,E}-\frac{3}{2}T_eS_{e,n}+H_{\text{ext}}, \tag{9}
$$

where the total current density is

$$
J = en_iv_i + \sum_{Z_j} eZ_jn_{Z_j}v_{Z_j} - en_ev_e, \tag{10}
$$

and the sources and sinks of particles are

$$
S_{i,n} = n_e(k_{i,H}n_H - k_{r,i}n_i) + n_H\sum_{Z_j} n_{Z_j}k_{cx,Z_j,H} - n_0n_ik_{cx,i,0}, \tag{11}
$$

$$
\begin{aligned}
S_{Z_j,n} = {}& n_e[n_{Z_j-1}k_{i,Z_j-1} + n_{Z_j+1}k_{r,Z_j+1} - n_{Z_j}(k_{i,Z_j} + k_{r,Z_j})]\\
&+n_H(n_{Z_j+1}k_{\text{cx},Z_j+1,H} - n_{Z_j}k_{\text{cx},Z_j,H})\\
&+n_0(n_{Z_j+1}k_{\text{cx},Z_j+1,0} - n_{Z_j}k_{\text{cx},Z_j,0})\\
&+\delta_{Z_j1}n_0\left(n_ik_{\text{cx},i,0} + \sum_{Z_j^*} n_{Z_j^*}k_{\text{cx},Z_j^*,0}\right),
\end{aligned} \tag{12}
$$

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$$S_{e,n} = n_e \left[n_H k_{i,H} - n_i k_{r,i} + \sum_{Z_j} (n_{Z_j-1} k_{i,Z_j-1} - n_{Z_j} k_{r,Z_j}) \right], \tag{13}$$

the sources and sinks of momenta are

$$\begin{aligned} S_{i,v} = {} & -m_i n_H [k_{\mathrm{cx},H,i} n_i (v_i - v_H) - k_{i,H} n_e v_H] - m_i n_i n_e k_{r,i} v_i \\ & - m_i n_0 n_i k_{cx,i,0} v_i + m_i n_H \sum_{Z_j} n_{Z_j} k_{\mathrm{cx},Z_j,H} v_H - B_H n_H \frac{\partial T_H}{\partial x}, \end{aligned} \tag{14}$$

$$\begin{aligned} S_{Z_j,v} = {} & m_j n_e [n_{Z_j-1} k_{i,Z_j-1} v_{Z_j-1} + n_{Z_j+1} k_{r,Z_j+1} v_{Z_j+1} \\ & - n_{Z_j} (k_{i,Z_j} + k_{r,Z_j}) v_{Z_j}] \\ & + m_j n_H (n_{Z_j+1} k_{\mathrm{cx},Z_j+1,H} v_{Z_j+1} - n_{Z_j} k_{\mathrm{cx},Z_j,H} v_{Z_j}) \\ & + m_j n_0 (n_{Z_j+1} k_{cx,Z_j+1,0} v_{Z_j+1} - n_{Z_j} k_{\mathrm{cx},Z_j,0} v_{Z_j}) \\ & + \delta_{Z_j 1} n_0 \left(n_i k_{\mathrm{cx},i,0} + \sum_{Z_j^*} n_{Z_j^*} k_{\mathrm{cx},Z_j^*,0} \right) v_0, \end{aligned} \tag{15}$$

and the sources and sinks of energies are

$$S_{e,E} = -n_e \left(k_{i,H} n_H I_{i,H} + \sum_{Z_j} n_{Z_j-1} k_{i,Z_j-1} I_{i,Z_j-1} + \sum_{Z_j} n_{Z_j} k_{\mathrm{rad},Z_j} S_{\mathrm{rad},Z_j} \right), \tag{16}$$

$$\begin{aligned} S_{i,E} = {} & \frac{1}{2} n_e k_{i,H} n_H (3T_H + m_i v_H^2) - \frac{3}{2} n_i k_{\mathrm{cx},i,H} n_H (T_i - T_H) \\ & - \frac{1}{2} n_e k_{r,i} n_i (3T_i + m_i v_i^2) - m_i n_H k_{\mathrm{cx},i,H} n_i (v_i - v_H) v_H - B_H n_H \frac{\partial T_H}{\partial x} v_H \\ & - \frac{1}{2} n_e k_{i,0} n_0 (3T_0 + m_j v_0^2) - \frac{3}{2} n_1 k_{\mathrm{cx},1,0} n_0 (T_i - T_0) \\ & - \frac{1}{2} n_e k_{r,1} n_1 (3T_i + m_j v_1^2) - m_j n_0 k_{\mathrm{cx},1,0} n_1 (v_1 - v_0) v_0 - B_0 n_H \frac{\partial T_0}{\partial x} v_0. \end{aligned} \tag{17}$$

Here $Z_j = 1, 2, \ldots, Z_f$, n_H , v_H , T_H, n_0, v_0 and T_0 are the hydrogen and impurity atom density, velocity and temperature respectively. k_i k_{cx}, k_r and k_{rad} are the rate coefficients for ionization, charge exchange, recombination and radiation, $I_{i,H} = 13.56$ eV; I_{i,Z_j} is the ionization potential, S_{rad,Z_j} is the radiation function, B_H is approximately equal to 0.25 (cf. Helander et al. 1994), and B_0 has order of magnitude 1. These functions, which depend on the properties of the inelastic interaction of ions with atoms, will be calculated in a forthcoming paper. H_{ext} is the external energy input and E is the electric field.

The functions contained in the system of equations (1)–(17) are defined as follows (cf. Golant et al. 1980; Shdanov 1982; Dnestrovskii and Kostomarov 1986; Radford 1993; Bachmann and Sünder 1998c). The viscosities of the ion species are

$$\eta_i = n_i T_i C_1(Z_0) \frac{1}{\nu_{ii}}, \tag{18}$$

$$\eta_{Z_j} = n_{Z_j} T_i C_{2j}(Z_0) \frac{1}{\sum_{Z_k} \nu_{Z_j Z_k}}, \tag{19}$$

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where the collision frequencies $\nu_{\alpha\beta}$, with the Coulomb logarithm $\ln \Lambda_c$, are given by

$$\nu_{\alpha\beta} = \frac{4(2\pi)^{1/2} e^4 \ln \Lambda_c}{3} n_\beta Z_\beta^2 Z_\alpha^2 \left[\frac{m_\beta (m_\beta + m_\alpha)^2}{m_\alpha (T_\alpha m_\beta + T_\beta m_\alpha)^3} \right]^{1/2}, \tag{20}$$

$$C_1(x) = \frac{3.84}{3} \frac{1 + \sqrt{2}x}{(1 + 1.87x)(1 + 0,67x)}, \tag{21}$$

$$C_{2j}(x) = 0.3 \left(1 + \frac{4xn_i}{Z_j n_j} \right) \frac{1}{1+x}, \tag{22}$$

$$Z_0 = \frac{n_j}{n_i} \langle Z_j^2 \rangle, \qquad n_j = \sum_{Z_j} n_{Z_j}, \qquad \langle Z_j^2 \rangle = \frac{1}{n_j} \sum_{Z_j} n_{Z_j} Z_j^2. \tag{23}$$

The forces R_β ($\beta = i, Z_j, e$) are given as sums of the friction forces $R_{\beta,F}$ and the thermal forces $R_{\beta,T}$:

$$R_{i,F} = -m_i n_i \left[C_3(Z_{\text{eff}}) \nu_{ie} (v_i - v_e) + C_3(Z_0) \sum_{Z_j} \nu_{iZ_j} (v_i - v_{Z_j}) \right], \tag{24}$$

$$R_{i,T} = n_i \left[C_4(Z_{\text{eff}}) \frac{\partial T_e}{\partial x} - C_5(Z_0) \frac{\partial T_i}{\partial x} \right], \tag{25}$$

$$\begin{aligned} R_{Z_j,F} = -m_j n_{Z_j} \bigg[& C_3(Z_{\text{eff}}) \nu_{Z_j e} (v_{Z_j} - v_e) + C_3(Z_0) \nu_{Z_j i} (v_{Z_j} - v_i) \\ & + 0.8 \sum_{Z_k} \nu_{Z_j Z_k} (v_{Z_j} - v_{Z_k}) \bigg], \end{aligned} \tag{26}$$

$$R_{Z_j,T} = n_{Z_j} \left\{ \left[C_4(Z_0) Z_j^2 + 0.6 \left(\frac{Z_j^2}{\langle Z_j^2 \rangle} - 1 \right) \right] \frac{\partial T_i}{\partial x} + C_4(Z_{\text{eff}}) Z_j^2 \frac{\partial T_e}{\partial x} \right\}, \tag{27}$$

$$R_{e,F} = -m_e n_e C_3(Z_{\text{eff}}) \left[\nu_{ei} (v_e - v_i) + \sum_{Z_j} \nu_{eZ_j} (v_e - v_{Z_j}) \right], \tag{28}$$

$$R_{e,T} = -n_e C_5(Z_{\text{eff}}) \frac{\partial T_e}{\partial x}, \tag{29}$$

with the functions

$$Z_{\text{eff}} = \frac{n_i}{n_e} (1 + Z_0), \tag{30}$$

$$C_3(x) = \frac{(1 + 0.24x)(1 + 0.95x)}{(1 + 2.65x)(1 + 0.28x)}, \tag{31}$$

$$C_4(x) = \frac{2.2(1 + 0.52x)}{(1 + 2.65x)(1 + 0.28x)}, \tag{32}$$

$$C_5(x) = x C_4(x). \tag{33}$$

In the same way, the thermal fluxes q_β are given as sums of $q_{\beta,F}$ and $q_{\beta,T}$:

$$q_{i,F} = T_i C_4(Z_0) \sum_{Z_j} n_{Z_j} Z_j^2 (v_i - v_{Z_j}), \tag{34}$$

$$q_{i,T} = -n_i T_i C_6(Z_0) \frac{1}{m_i \nu_{ii}} \frac{\partial T_i}{\partial x}, \tag{35}$$

$$q_{Z_j,F} = n_{Z_j} T_i \frac{\sum_{Z_k} n_{Z_k} Z_k^2}{n_i Z_0} (v_{Z_j} - v_{Z_k}), \tag{36}$$

$$q_{Z_j,T} = -n_{Z_j} T_i C_{7j}(Z_0) \frac{1}{m_j \nu_{Z_j Z_j}} \frac{\partial T_i}{\partial x}, \tag{37}$$

$$q_{e,F} = T_e C_4(Z_{\text{eff}}) \left[n_i (v_e - v_i) + \sum_{Z_j} n_{Z_j} Z_j^2 (v_e - v_{Z_j}) \right], \tag{38}$$

$$q_{e,T} = -n_e T_e C_6(Z_{\text{eff}}) \frac{1}{m_e \nu_{ee}} \frac{\partial T_e}{\partial x}, \tag{39}$$

with the functions

$$C_6(x) = \frac{3.9(1 + 1.7x)}{(1 + 2.65x)(1 + 0,28x)}, \tag{40}$$

$$C_{7j}(x) = \frac{n_{Z_j}(n_j Z_j^2 + 2n_i x)}{n_i^2 x^2}. \tag{41}$$

The energy terms $Q_{e,T}$ and $Q_{e,R}$ are given by

$$Q_{e,T} = -\frac{3m_e}{m_i} n_e \nu_{ei} (T_e - T_i) \left(1 + \frac{m_i}{m_j} Z_0 \right), \tag{42}$$

$$Q_{e,R} = -m_e n_e C_3(Z_{\text{eff}}) \left[\nu_{ei}(v_e - v_i) v_i + \sum_{Z_j} \nu_{eZ_j} (v_e - v_{Z_j}) v_{Z_j} \right] - n_e C_5(Z_{\text{eff}}) \frac{\partial T_e}{\partial x} v_i, \tag{43}$$

The relation between the electric field and the charge density ρ is governed by Poisson's equation

$$\frac{\partial E}{\partial x} = 4\pi\rho, \qquad \rho = e \left(n_i + \sum_{Z_j} Z_j n_{Z_j} - n_e \right). \tag{44}$$

Multiplying the continuity equation for the impurities with eZ_j and summing the resulting equation over all charge states, we get

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial J_j}{\partial x} = e S_{j,n}^{(Z)}, \tag{45}$$

with the charge and current densities of the impurities

$$\rho_j = e \sum_{Z_j} Z_j n_{Z_j}, \qquad J_j = e \sum_{Z_j} Z_j n_{Z_j} v_{Z_j}, \tag{46}$$

and

$$S_{j,n}^{(Z)} = n_e n_0 k_{i,0} + n_0 n_i k_{\text{cx},i,0} - n_H \sum_{Z_j} n_{Z_j} k_{\text{cx},Z_j,H} + n_e \sum_{Z_j} n_{Z_j} (k_{i,Z_j} - k_{r,Z_j}). \tag{47}$$

Taking (1) and (2) into account we obtain the charge-balance equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0. \tag{48}$$

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If we assume that the charge-neutrality condition $\rho = 0$ is fulfilled, the electron density and the electron velocity are given by the expressions

$$n_e = n_i + \sum_{Z_j} Z_j n_{Z_j}, \tag{49}$$

$$v_e = \frac{n_i v_i + \sum_{Z_j} Z_j n_{Z_j} v_{Z_j} - J/e}{n_i + \sum_{Z_j} Z_j n_{Z_j}}. \tag{50}$$

The electric field E results from (8):

$$E = \frac{1}{e n_e}\left[R_e - \frac{\partial}{\partial x}(n_e T_e)\right], \tag{51}$$

and for the total current density J we obtain

$$\frac{\partial J}{\partial x} = 0 \Rightarrow J = J(t). \tag{52}$$

The coefficients C_1, C_3, C_4, C_5 and C_6 that determine the effect of the impurities on the transport processes in the plasma are complicated functions of the masses and temperatures of the particles. In the above formulae, these coefficients are calculated for $T_e/m_e \gg T_i/m_H$ and the case where the impurity mass m_j is much larger than the hydrogen mass m_H ($m_H/m_j \ll 1$), which is valid for the high-Z impurities considered here. This gives us the opportunity to study their change in a large parameter range of Z_0 or Z_{eff}.

The influence of neutral particles on the multifluid plasma equations is considered in the corresponding source and sink functions. Their effect on the plasma transport coefficients is neglected. The dynamics of the neutral particles should be described within the framework of a kinetic model. Often a simplified diffusion approximation is used (see e.g. Duderstadt and Martin 1979; Golant et al. 1980; Helander et al. 1994; Bachmann and Sünder 1998c). For simplicity, the temperature, velocity and density of the neutrals are assumed here to be given functions.

3. Average ion model

The consideration of all ionization stages of the impurities considerably increases the number of equations to be solved and the indeterminacy of the results. Hence it is necessary to describe the impurities within the framework of a simplified model (see Bachmann and Sünder 1998c):

(i) a modified coronal τ-approximation that allows one to calculate quasistationary discrete density distributions;

(ii) a non-stationary model, applicable to high-Z impurities; or

(iii) an average ion model (Post et al. 1994), which is considered below.

Taking the sum of the continuity and momentum balance equations for the impurities over all charge states, we get

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j v_j) = S_{j,n}, \tag{53}$$

$$\frac{\partial}{\partial t}(m_j n_j v_j) + \frac{\partial}{\partial x}\left(m_j n_j G_j^{(1)} v_j^2 + n_j T_i - \eta_j^{(1)} \frac{\partial v_j}{\partial x}\right) = R_j^* + S_{j,v}, \tag{54}$$

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with

$$n_j = \sum_{Z_j} n_{Z_j}, \qquad v_j = \frac{\sum_{Z_j} n_{Z_j} v_{Z_j}}{\sum_{Z_j} n_{Z_j}}, \tag{55a, b}$$

$$G_j^{(1)} = \frac{n_j \sum_{Z_j} n_{Z_j} v_{Z_j}^2}{\left(\sum_{Z_j} n_{Z_j} v_{Z_j}\right)^2}, \tag{55c}$$

$$\eta_j^{(1)} = \frac{T_i C_2(Z_0)}{\nu_{0j} Z_0} \sum_{Z_j} \frac{n_{Z_j}}{Z_j^2} \left(\frac{\partial v_{Z_j}}{\partial x}\right) \Big/ \left(\frac{\partial v_j}{\partial x}\right), \tag{56a}$$

$$\nu_{0j} = \frac{\nu_{Z_j Z_k} n_i}{n_{Z_k} Z_j^2 Z_k^2}, \tag{56b}$$

$$R_j^* = e\langle Z_j\rangle n_j E + R_j, \qquad \langle Z_j\rangle = \frac{\sum_{Z_j} Z_j n_{Z_j}}{\sum_{Z_j} n_{Z_j}}. \tag{57}$$

$$S_{j,n} = n_e(n_0 k_{i,0} - n_1 k_{r,1}) - n_H n_1 k_{\mathrm{cx},1,H} - n_0 n_1 k_{\mathrm{cx},1,0} + n_0(n_i k_{\mathrm{cx},i,0} + n_j k_{\mathrm{cx},j,0}), \tag{58}$$

$$\begin{aligned} S_{j,v} = m_j[n_e(n_0 k_{i,0} v_0 - n_1 k_{r,1} v_1) - n_1(n_0 k_{\mathrm{cx},1,0} + n_H k_{\mathrm{cx},1,H}) v_1 \\ + n_0(n_i k_{\mathrm{cx},i,0} + n_j k_{\mathrm{cx},j,0}) v_0], \end{aligned} \tag{59}$$

$$\begin{aligned} R_j = -m_j n_i Z_0[C_3(Z_{\mathrm{eff}})\nu_{je}(G_j^{(2)} v_j - v_e) + C_3(Z_0)\nu_{ji}(G_j^{(2)} v_j - v_i)] \\ + n_i Z_0 \left[C_4(Z_0)\frac{\partial T_i}{\partial x} + C_4(Z_{\mathrm{eff}})\frac{\partial T_e}{\partial x}\right], \end{aligned} \tag{60}$$

$$\nu_{j\beta} = \frac{\nu_{Z_j \beta}}{Z_j^2}, \qquad \beta = e, i, \tag{61a}$$

$$G_j^{(2)} = \frac{\sum_{Z_j} Z_j^2 n_{Z_j} v_{Z_j} \sum_{Z_j} n_{Z_j}}{\sum_{Z_j} Z_j^2 n_{Z_j} \sum_{Z_j} n_{Z_j} v_{Z_j}}, \tag{61b}$$

$$k_{\mathrm{cx},j,0} = \frac{\sum_{Z_j} n_{Z_j} k_{\mathrm{cx},Z_j,0}}{\sum_{Z_j} n_{Z_j}} \tag{61c}$$

where $Z_j = 1, 2, \ldots, Z_f$. Using the expression (51) for the electric field,

$$\begin{aligned} eE = \frac{1}{n_e}\frac{\partial}{\partial x}(n_e T_e) - C_5(Z_{\mathrm{eff}})\frac{\partial}{\partial x} T_e \\ - m_e C_3(Z_{\mathrm{eff}})[\nu_{ei}(v_e - v_i) + Z_0 \nu_{ej}(v_e - v_j)], \end{aligned} \tag{62a}$$

$$\nu_{\alpha j} = \frac{\nu_{\alpha Z_j} n_i}{Z_j^2 n_{Z_j}}, \tag{62b}$$

we obtain

$$\begin{aligned} R_j^* = n_j C_4(Z_{\mathrm{eff}})\frac{\partial T_e}{\partial x}(\langle Z_j^2\rangle - \langle Z_j\rangle Z_{\mathrm{eff}}) + n_j\langle Z_j^2\rangle C_4(Z_0)\frac{\partial T_i}{\partial x} \\ - \langle Z_j\rangle \frac{n_j}{n_e}\frac{\partial}{\partial x}(n_e T_e) - \langle Z_j\rangle m_e n_j C_3(Z_{\mathrm{eff}})\nu_{ei}(v_e - v_i) \\ - m_j n_j \langle Z_j^2\rangle \left[C_3(Z_{\mathrm{eff}})\nu_{je}\left(1 - \langle Z_j\rangle \frac{n_j}{n_e}\right)(G_j^{(2)} v_j - v_e)\right. \\ \left. + C_3(Z_0)\nu_{ji}(G_j^{(2)} v_j - v_i)\right], \end{aligned} \tag{63}$$

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$$n_e = n_i + \langle Z_j \rangle n_j. \tag{64}$$

Multiplying the continuity equation by Z_j and summing the resulting over all charge states, we get, in accordance with (45)–(47),

$$\frac{\partial}{\partial t}(n_j\langle Z_j\rangle) + \frac{\partial}{\partial x}(G_j^{(3)}\langle Z_j\rangle n_j v_j) = S_{j,n}^{(Z)}, \tag{65}$$

with

$$S_{j,n}^{(Z)} = n_e n_0 k_{i,0} + n_0 n_i k_{\mathrm{cx},i,0} - n_H n_j k_{\mathrm{cx},j,H} + n_e n_j (k_{i,j} - k_{r,j}), \tag{66a}$$

$$G_j^{(3)} = \frac{n_j \sum_{Z_j} Z_j n_{Z_j} v_{Z_j}}{\sum_{Z_j} Z_j n_{Z_j} \sum_{Z_j} n_{Z_j} v_{Z_j}}, \tag{66b}$$

$$k_{i,j} = \frac{\sum_{Z_j} n_{Z_j} k_{i,Z_j}}{\sum_{Z_j} n_{Z_j}}, \qquad k_{r,j} = \frac{\sum_{Z_j} n_{Z_j} k_{r,Z_j}}{\sum_{Z_j} n_{Z_j}}, \tag{66c, d}$$

$$k_{\mathrm{cx},j,H} = \frac{\sum_{Z_j} n_{Z_j} k_{\mathrm{cx},Z_j,H}}{\sum_{Z_j} n_{Z_j}}. \tag{66e}$$

The right-hand-sides of the continuity equations for hydrogen ions and electrons read

$$S_{i,n} = n_e(k_{i,H} n_H - k_{r,i} n_i) + n_H k_{\mathrm{cx},j,H} n_j - n_0 n_i k_{\mathrm{cx},i,0}, \tag{67}$$

$$S_{e,n} = n_e[n_H k_{i,H} - n_i k_{r,i} + n_0 k_{i,0} + n_j(k_{i,j} - k_{r,j})]. \tag{68}$$

Using the relation $S_{e,n} = S_{i,n} + S_{j,n}^{(Z)}$ and the condition of charge neutrality, we obtain

$$\frac{\partial J}{\partial x} = 0, \qquad J = e(n_i v_i + G_j^{(3)}\langle Z_j\rangle n_j v_j - n_e v_e), \tag{69}$$

from which it follows that the current density J in a plasma without external ion or electron sources is a function of t only (or is constant), and the electron velocity is given by (cf. (50))

$$v_e = \frac{n_i v_i + G_j^{(3)}\langle Z_j\rangle n_j v_j - J/e}{n_i + \langle Z_j\rangle n_j}. \tag{70}$$

Furthermore, we obtain an equation for the mean charge $\langle Z_j\rangle$:

$$\frac{\partial}{\partial t}\langle Z_j\rangle + v_j G_j^{(3)} \frac{\partial}{\partial x}\langle Z_j\rangle = \nu_Z, \tag{71}$$

with

$$\nu_Z = \frac{1}{n_j}\left[S_{j,n}^{(Z)} - \langle Z_j\rangle S_{j,n} - \langle Z_j\rangle \frac{\partial}{\partial x}(n_j v_j)(G_j^{(3)} - 1)\right]. \tag{72}$$

The sum of the ion and electron momentum equations is

$$\frac{\partial}{\partial t}(m_i n_i v_i) + \frac{\partial}{\partial x}\left(m_i n_i v_i^2 + n_i T_i + n_e T_e - \eta_i \frac{\partial v_i}{\partial x}\right) = -R_j^* + S_{i,v}. \tag{73}$$

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The ion and electron energy balance equations have the following forms:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}&\left(3n_iT_i + m_in_iv_i^2 + 3n_jT_i + m_jn_jG_j^{(1)}v_j^2\right)\\ &+\frac{1}{2}\frac{\partial}{\partial x}\left(5n_iv_iT_i + 5n_jv_jT_i + m_in_iv_i^3 + m_jn_jG_j^{(4)}v_j^3\right)\\ &+\frac{\partial}{\partial x}\left(q_i + q_j - \eta_iv_i\frac{\partial v_i}{\partial x} - \eta_j^{(2)}v_j\frac{\partial v_{Z_j}}{\partial x}\right)\\ &= JE + en_ev_eE - Q_{e,T} - Q_{e,R} + S_{i,E},\end{aligned} \tag{74}$$

$$\frac{3}{2}\frac{\partial}{\partial t}(n_eT_e) + \frac{\partial}{\partial x}\left(\frac{5}{2}n_ev_eT_e + q_e\right) = -en_ev_eE + Q_{e,T} + Q_{e,R} + S_{e,E}, \tag{75}$$

where

$$G_j^{(4)} = \frac{n_j^2\sum_{Z_j} n_{Z_j}v_{Z_j}^3}{\left(\sum_{Z_j} n_{Z_j}v_{Z_j}\right)^3}, \tag{76a}$$

$$\eta_j^{(2)} = \frac{T_iC_2(Z_0)}{\nu_jZ_0}\sum_{Z_j}\frac{n_{Z_j}v_{Z_j}}{Z_j^2}\frac{\partial v_{Z_j}}{\partial x}\bigg/\left(\frac{\partial v_j}{\partial x}v_j\right), \tag{76b}$$

$$q_j = -\sum_{Z_j}\frac{n_{Z_j}T_iC_{7j}(Z_0)}{m_j\nu_{Z_jZ_j}}\frac{\partial T_i}{\partial x}, \tag{77a}$$

$$q_e = T_eC_4(Z_{\text{eff}})n_i[(v_e - v_i) + Z_0(v_e - G_j^{(2)}v_j)] - n_eT_eC_6(Z_{\text{eff}})\frac{1}{m_e\nu_{ee}}\frac{\partial T_e}{\partial x}, \tag{77b}$$

$$q_i = T_iC_4(Z_0)n_iZ_0(v_i - G_j^{(2)}v_j) - n_iT_iC_6(Z_0)\frac{1}{m_i\nu_{ii}}\frac{\partial T_i}{\partial x}, \tag{77c}$$

$S_{i,E}$ is given by (17) and describes the interaction of the ions with the hydrogen atoms, and

$$S_{e,E} = -n_ek_{i,H}n_HI_{i,H} + Q_R, \tag{78a}$$

$$Q_R = -n_en_j(L_{\text{ion}} + L_{\text{rad}}) \tag{78b}$$

with the ionisation and radiation functions being given by

$$L_{\text{ion}}(T_e) = \frac{1}{n_j}\sum_{Z_j} n_{Z_j}k_{i,Z_j}S_{i,Z_j}. \tag{79a}$$

$$L_{\text{rad}}(T_e) = \frac{1}{n_j}\sum_{Z_j} n_{Z_j}k_{\text{rad},Z_j}S_{\text{rad},Z_j}. \tag{79b}$$

4. Reduced plasma models

In order to analyse the effect of impurities on transport properties in edge plasmas, we present in this section sucessively reduced plasma model equations, adopting the condition of quasineutrality (64). The impurity is considered as a single fluid by means of the average ion model. We assume the temperatures of the plasma species to be equal and neglect the dynamics of neutral particles. We start with a three-fluid description of a current-carrying plasma for electrons, plasma and

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impurity ions (Sec. 4.1), and analyse stationary solutions in a reference system moving with constant velocity relative to the laboratory frame. This system is then reduced to a currentless plasma description, assuming at first different flow velocities of the particles and then a streaming plasma where all particles move with the same velocity. Introducing Lagrangian coordinates and adopting an equation of state a single reaction–diffusion equation (RDE) for the temperature is obtained. A differential equation for the impurity density as a function of the temperature has to be included.

4.1. *Current-carrying plasma model*

4.1.1. System of partial differential equations. Under the condition $T_i = T_e = T$, neglecting the neutral-particle dynamics ($n_H = n_0 = 0,\ k_{r,i} = k_{r,1} = 0$), the ion heat conductivity and the ion viscosity, we obtain the following system of equations for $n_i, n_j, v_i, v_j, J, \langle Z_j\rangle$ and T:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0, \tag{80}$$

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j v_j) = 0, \tag{81}$$

$$m_i n_i\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x}\right)v_i + \frac{\partial}{\partial x}[(n_i + n_e)T] = -R_j^*, \tag{82}$$

$$m_j n_j\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x}\right)v_j + \frac{\partial}{\partial x}(n_j T) = R_j^*, \tag{83}$$

$$\sum_\alpha \frac{3}{2}\left(\frac{\partial}{\partial t} + v_\alpha\frac{\partial}{\partial x}\right)(n_\alpha T) + \sum_\alpha \frac{5}{2} n_\alpha T\frac{\partial}{\partial x}v_\alpha - \frac{\partial}{\partial x}\left(\kappa_e\frac{\partial T}{\partial x}\right) = H_{\text{ext}} + H_j^* - Q_R, \tag{84}$$

$$\frac{\partial}{\partial t}\langle Z_j\rangle + v_j\frac{\partial}{\partial x}\langle Z_j\rangle = \nu_Z^{(*)}, \tag{85}$$

$$\frac{\partial J}{\partial x} = 0, \tag{86}$$

where $\alpha = e, i, j$. Here n_e is given by (64),

$$v_e = \frac{n_i v_i + \langle Z_j\rangle n_j v_j - J/e}{n_i + \langle Z_j\rangle n_j}, \tag{87}$$

$$\nu_Z^{(*)} = (n_i + \langle Z_j\rangle n_j)(k_{i,j} - k_{r,j}), \tag{88}$$

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$$R_j^* = n_j \frac{\partial T}{\partial x}\left[C_4(Z_{\text{eff}})(\langle Z_j^2\rangle - \langle Z_j\rangle Z_{\text{eff}}) + \langle Z_j^2\rangle C_4(Z_0) - \langle Z_j\rangle\right]$$
$$-n_j\left[\langle Z_j\rangle T \frac{\partial}{\partial x}\ln n_e + \langle Z_j\rangle m_e C_3(Z_{\text{eff}})\nu_{ei}(v_e - v_i)\right]$$
$$-n_j m_j \langle Z_j^2\rangle\left[C_3(Z_{\text{eff}})\nu_{je}\left(1 - \langle Z_j\rangle\frac{n_j}{n_e}\right)(v_j - v_e)\right.$$
$$\left. + C_3(Z_0)\nu_{ji}(v_j - v_i)\right], \tag{89}$$

the electron heat conduction coefficient

$$\kappa_e = C_6(Z_{\text{eff}})\frac{n_e T}{m_e \nu_{ee}} = \kappa_e(T, n_j/n_i), \tag{90}$$

H_{ext} is the external heat source, H_j^* is given by

$$H_j^* = n_e C_5(Z_{\text{eff}})(v_e - v_i)\frac{\partial T}{\partial x}$$
$$+ n_e m_e C_3(Z_{\text{eff}})\left[\nu_{ei}(v_e - v_i)^2 + \nu_{ej} Z_0 (v_e - v_j)^2\right]$$
$$+ m_i n_i Z_0 C_3(Z_0)\nu_{ij}(v_j - v_i)^2$$
$$-\frac{\partial}{\partial x}\left\{n_i T\left[C_5(Z_0)(v_i - v_j) + C_4(Z_{\text{eff}})(v_e - v_i + Z_0 v_e - Z_0 v_j)\right]\right\}, \tag{91}$$

and the impurity radiation loss term

$$Q_R = (n_i + \langle Z_j\rangle n_j) n_j L_{\text{rad}}(T), \tag{92}$$

where we have assumed $G_j^{(1)} = G_j^{(2)} = G_j^{(3)} = G_j^{(4)} = 1$. The assumption $T_i = T_e$ is only valid in plasmas with high electron–ion collision frequencies. Sometimes it is more useful to discuss the total momentum balance rather than (82):

$$m_i n_i\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x}\right)v_i + m_j n_j\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x}\right)v_j + \frac{\partial}{\partial x}(n_i + n_e + n_j)T = 0. \tag{93}$$

The effect of the impurities on the transport processes can easily be demonstrated for the simple case $\xi_j = n_j/n_i = \text{const}$, which is represented in Fig. 1, where the normalized functions $E, q_{i,T}, q_{e,T}$, and η_i (Fig. 1a), and $R_{i,T}, R_{e,T}, R_{j,T} = \sum_{Z_j} R_{Z_j,T}$ and $R_T = R_{e,T} + R_{i,T}$ (Fig. 1b) are plotted versus the mean charge $\langle Z_j\rangle$ for $\xi_j = 0.05$, assuming $\partial T/\partial x < 0$. It can be seen that all functions depend strongly on the mean impurity charge. For large $\langle Z_j\rangle$, these functions behave as follows: $\eta_i, q_{i,T}, q_{e,T}$ tend to zero, E and the ion thermal force, which changes sign at $\langle Z_j\rangle = 3.5$, attain their limiting values, and the electron and impurity thermal forces increase linearly with $\langle Z_j\rangle$.

4.1.2. Stationary processes in a moving reference frame. Here we consider the existence of solutions of the system (80)–(86) that depend only on η:

$$\eta = x - v_0 t, \qquad v_0 = \text{const}. \tag{94}$$

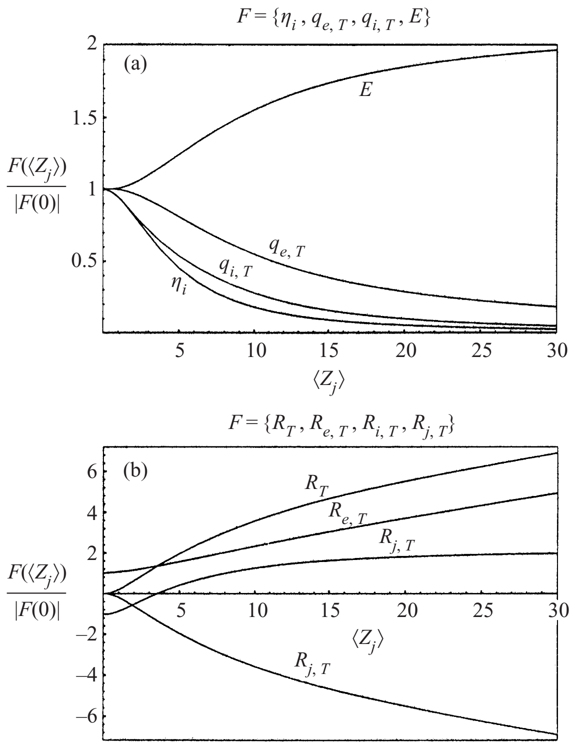

Figure 1. (*a*) Normalized electric field E, thermal fluxes $q_{i,T}$ and $q_{e,T}$, and ion viscosity η_i as functions of the mean charge $\langle Z_j\rangle$ for $\xi_j = 0.05$. (*b*) Thermal forces $R_{i,T}, R_{e,T}, R_{j,T} = \sum_{Z_j} R_{Z_j,T}$ and $R_T = R_{e,T} + R_{i,T}$ as functions of $\langle Z_j\rangle$ for $\xi_j = 0.05$

The continuity equations (80) and (81) allow us to express the velocity of the ions of type α as a function of their density:

$$v_\alpha = v_0 + \frac{\Gamma_{\alpha 0}}{n_\alpha}, \qquad \alpha = i, j, \tag{95}$$

where $\Gamma_{\alpha 0}$ are constant particle fluxes. From (93), we get the first integral

$$m_i\Gamma_{i0}\left(v_0 + \frac{\Gamma_{i0}}{n_i}\right) + m_j\Gamma_{j0}\left(v_0 + \frac{\Gamma_{j0}}{n_j}\right) + T[2n_i + (1 + \langle Z_j\rangle)n_j] = c_0, \tag{96}$$

with the total energy c_0 in the moving coordinate system. Equation (96) describes the relation between n_i, n_j and T:

$$n_i(T, n_j) = \frac{1}{4T}[a_0 + (a_0^2 - a_1 T)^{1/2}], \tag{97}$$

with

$$a_0 = c_0 - (m_i\Gamma_{i0} + m_j\Gamma_{j0})v_0 - m_j\frac{\Gamma_{j0}^2}{n_j} - n_jT(1 + \langle Z_j\rangle), \tag{98a}$$

$$a_1 = 8m_i\Gamma_{i0}^2. \tag{98b}$$

For the case of small impurity fluxes, a_0 is constant and n_i depends on T only.

From the impurity momentum balance equation (83), we obtain

$$-m_j \frac{\Gamma_{j0}^2}{n_j^2}\frac{dn_j}{d\eta} + \frac{d}{d\eta}(n_j T) = R_j^*, \tag{99}$$

where

$$\begin{aligned} R_j^* = n_j \frac{dT}{d\eta}&\left[C_4(Z_{\text{eff}})(\langle Z_j^2\rangle - \langle Z_j\rangle Z_{\text{eff}}) + \langle Z_j^2\rangle C_4(Z_0) - \langle Z_j\rangle\right] \\ &-n_j\left[\langle Z_j\rangle T\frac{d}{d\eta}\ln n_e + \langle Z_j\rangle m_e C_3(Z_{\text{eff}})\frac{\nu_{ei}}{n_e n_i}(n_i\Gamma_e - n_e\Gamma_{i0})\right] \\ &-m_j\langle Z_j^2\rangle\left[C_3(Z_{\text{eff}})\frac{\nu_{je}}{n_e}\left(1-\langle Z_j\rangle\frac{n_j}{n_e}\right)(n_e\Gamma_{jo} - n_j\Gamma_e)\right. \\ &\left.+C_3(Z_0)\frac{\nu_{ji}}{n_i}(n_i\Gamma_{j0} - n_j\Gamma_{i0})\right], \end{aligned} \tag{100}$$

with

$$\Gamma_e = \Gamma_{i0} + \langle Z_j\rangle\Gamma_{j0} - \frac{J}{e}. \tag{101}$$

The equation for the temperature reads

$$\begin{aligned} \frac{3}{2}\;&(\Gamma_e + \Gamma_{i0} + \Gamma_{j0})\frac{dT}{d\eta} + \frac{5}{2}Tn_j\nu_Z \\ &-T\left(\Gamma_e\frac{d}{d\eta}\ln n_e + \Gamma_{i0}\frac{d}{d\eta}\ln n_i + \Gamma_{j0}\frac{d}{d\eta}\ln n_j\right) \\ &-\frac{d}{d\eta}\kappa_e\frac{dT}{d\eta} \\ &= H_{\text{ext}} + H_j^* - Q_R, \end{aligned} \tag{102}$$

with

$$\begin{aligned} H_j^* = C_5(Z_{\text{eff}})&\frac{1}{n_i}(\Gamma_e n_i - \Gamma_{i0}n_e)\frac{dT}{d\eta} \\ &+m_e C_3(Z_{\text{eff}})\left[\frac{\nu_{ei}}{n_e n_i^2}(\Gamma_e n_i - \Gamma_{i0}n_e)^2 + \frac{\nu_{ej}}{n_e n_j^2}Z_0(\Gamma_e n_j - \Gamma_{j0}n_e)^2\right] \\ &+m_i C_3(Z_0)\frac{\nu_{ij}}{n_i n_j^2}Z_0(\Gamma_{i0}n_j - \Gamma_{j0}n_i)^2 \\ &-\frac{d}{d\eta}\left[TC_5(Z_0)\frac{1}{n_j}(\Gamma_{i0}n_j - \Gamma_{j0}n_i) + TC_4(Z_{\text{eff}})\frac{1}{n_e}(\Gamma_e n_i - \Gamma_{i0}n_e)\right. \\ &\left.+TZ_0C_4(Z_{\text{eff}})\frac{n_i}{n_e n_j}(\Gamma_e n_j - \Gamma_{j0}n_e)\right]. \end{aligned} \tag{103}$$

Together with the equation

$$\frac{d}{\partial\eta}\langle Z_j\rangle = \frac{n_j}{\Gamma_{j0}}\nu_Z^{(*)}, \tag{104}$$

we obtain a closed, strongly nonlinear system of four ordinary first-order differential equations for T, $dT/d\eta$, $\langle Z_j\rangle$ and n_j. Solutions of the system for the case

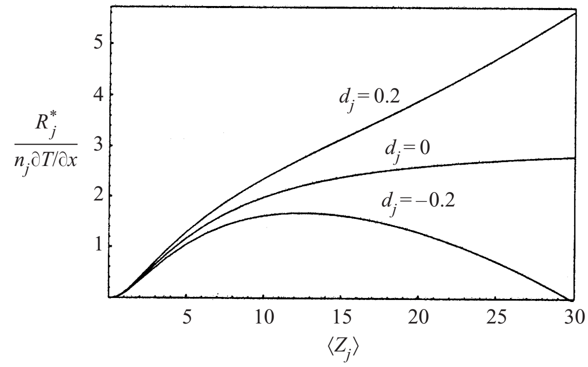

Figure 2. $R_j^*/(n_j \partial T/\partial x)$ as a function of $\langle Z_j \rangle$ for $\xi_j = 0.05$ and different values of the parameter $d_j = m_j(v_i - v_j)L_T \nu_{ji}/T$.

$v_e = v_i$, $\langle Z_j \rangle \sim T^{1/2}$ and $\langle Z_j \rangle n_j \ll n_i$ have been analysed by Morozov and Herrera (1996).

4.2. Currentless plasma models

4.2.1. Plasma with different particle velocities. The condition for both charge neutrality and a currentless plasma,

$$J = e[n_i(v_i - v_e) + \langle Z_j \rangle n_j (v_j - v_e)] = 0, \tag{105}$$

leads to

$$v_e = \frac{n_i v_i + \langle Z_j \rangle n_j v_j}{n_i + \langle Z_j \rangle n_j}. \tag{106}$$

We obtain, assuming $v_i \neq v_j$, the system of equations for $n_i, n_j, v_i, v_j, \langle Z_j \rangle$ and T derived in Sec. 4.1, where we have to take into account $J = 0$. The role of the different ion velocities is demonstrated in Fig. 2, where R_j^* is plotted as a function of the mean charge $\langle Z_j \rangle$ for $\xi_i = 0.05$ for various values of the parameter $d_j = m_j(v_i - v_j)L_T \nu_{ji}/T$ that characterizes the relation of the friction to the thermal force, where L_T is the temperature scale length. It can be seen that for the case of equal ion velocities ($v_i = v_j, d_j = 0$), R_j^* attains a constant value for large $\langle Z_j \rangle$.

4.2.2. Streaming plasma. If we assume that all velocities are equal, $v_e = v_i = v_j \equiv v$, we obtain the following system of equations describing the dynamics of a currentless, with one velocity, streaming plasma:

$$\frac{dn_i}{dt} + n_i \frac{\partial v}{\partial x} = 0, \tag{107}$$

$$\frac{dn_j}{dt} + n_j \frac{\partial v}{\partial x} = 0, \tag{108}$$

$$\frac{d}{dt}\langle Z_j \rangle = \nu_Z^{(*)}, \tag{109}$$

$$(m_i n_i + m_j n_j)\frac{dv}{dt} + \frac{\partial}{\partial x}\{[2n_i + (1 + \langle Z_j \rangle) n_j]T\} = 0, \tag{110}$$

$$m_j n_j \frac{dv}{dt} + \frac{\partial}{\partial x}(n_j T) = R_j^*, \tag{111}$$

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$$\frac{3}{2}\frac{d}{dt}[2n_i + (1 + \langle Z_j \rangle n_j)]T + \frac{5}{2}[2n_i + (1 + \langle Z_j \rangle n_j)]T\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left(\kappa_e \frac{\partial T}{\partial x}\right) = H_{\text{ext}} - Q_R \tag{112}$$

with $d/dt = \partial/\partial t + v\partial/\partial x$ and

$$R_j^* = n_j\left[-\langle Z_j \rangle T\frac{\partial}{\partial x}\ln(n_i + \langle Z_j \rangle n_j) + G_j(T, n_j/n_i)\frac{\partial T}{\partial x}\right], \tag{113a}$$

$$G_j = \alpha - \beta - \langle Z_j \rangle, \tag{113b}$$

$$\alpha = [C_4(Z_{\text{eff}}) + C_4(Z_0)]\langle Z_j^2 \rangle, \tag{113c}$$

$$\beta = C_4(Z_{\text{eff}})Z_{\text{eff}}\langle Z_j \rangle. \tag{113d}$$

We arrive at six equations for the five unknowns $n_i, n_j, v, \langle Z_j \rangle$ and T because both the equation of continuity and momentum of the impurities govern n_j.

4.2.3. Wave-front solutions. Here we consider again the existence of solutions depending only on $\eta = x - v_0 t$, $v_0 = \text{const}$ (cf. Sec. 4.1.2). In this case of so-called wave-front-like solutions ($d/dt = (v - v_0)d/d\eta$) we obtain from the continuity equations the relations

$$v = v_0 + \frac{\Gamma_{i0}}{n_i} = v_0 + \frac{\Gamma_{j0}}{n_j} \Rightarrow \xi_j = \frac{n_j}{n_i} = \frac{\Gamma_{j0}}{\Gamma_{i0}}, \tag{114}$$

where the impurity density is a constant fraction of the ion density. The density n_i as a function of the temperature reads

$$n_i(T) = \frac{1}{2Tg_i}[d_0 + (d_0^2 - d_1 T)^{1/2}], \tag{115}$$

with

$$d_0 = c_0 - (m_i + m_j\xi_j)\Gamma_{i0}v_0, \tag{116a}$$

$$d_1 = 4(m_i + m_j\xi_j)g_i\Gamma_{i0}^2, \tag{116b}$$

$$g_i = 2 + (1 + \langle Z_j \rangle)\xi_j. \tag{116c}$$

The equation for the temperature is

$$\left(\Gamma_{i0}g_i\frac{dT}{d\eta} + \nu_Z^{(*)}n_i\xi_j T\right)\left(\frac{5}{2} + M\right) - \frac{d}{d\eta}\left[\kappa_e(T)\frac{dT}{d\eta}\right] = H_{\text{ext}} - n_i^2\xi_j(1 + \langle Z_j \rangle \xi_j)L_{\text{rad}} \tag{117}$$

with

$$M = \frac{d_1 T}{2[d_0 + (d_0^2 - d_1 T)^{1/2}](d_0^2 - d_1 T)^{1/2}}. \tag{118}$$

In the case of $d_1 T \ll d_0^2$, $\nu_Z^{(*)} = 0$ the left-hand side of (117) has the form of the usual reaction–diffusion equation in the travelling wave approximatiom (Fife 1979; Grindrod 1996).

4.2.4. Reaction-diffusion equation (RDE). Introducing the mass density ρ_m, the total density N and the total pressure p,

$$\rho_m = m_i n_i + m_j n_j, \tag{119a}$$

$$N = n_e + n_i + n_j = 2n_i + (1 + \langle Z_j \rangle) n_j, \tag{119b}$$

$$p = NT, \tag{119c}$$

one obtains the following reduced system of equations:

$$\frac{dN}{dt} + N\frac{\partial v}{\partial x} = n_j \nu_Z^{(*)}, \tag{120}$$

$$\frac{d\rho_m}{dt} + \rho_m \frac{\partial v}{\partial x} = 0, \tag{121}$$

$$\frac{d}{dt}\langle Z_j \rangle = \nu_Z^{(*)}, \tag{122}$$

$$\rho_m \frac{dv}{dt} + \frac{\partial p}{\partial x} = 0, \tag{123}$$

$$m_j n_j \frac{dv}{dt} + \frac{\partial}{\partial x}(n_j T) = R_j^*, \tag{124}$$

$$\frac{3}{2}\frac{dp}{dt} + \frac{5}{2}p\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}\left(\kappa_e \frac{\partial T}{\partial x}\right) = H_{\rm ext} - (n_i + \langle Z_j \rangle n_j) n_j L_{\rm rad}. \tag{125}$$

Using the continuity equation, the energy balance equations can be rewritten in the form

$$NT\frac{d}{dt}\ln(T^{3/2}N^{-1}) - \frac{\partial}{\partial x}\left(\kappa_e \frac{\partial T}{\partial x}\right) = H_{\rm ext} - (n_i + \langle Z_j \rangle n_j) n_j L_{\rm rad} - \frac{5}{2}T n_j \nu_Z^{(*)}. \tag{126}$$

This system of equations represents a fluid description by means of the model functions N, v, T, ρ_m and $\langle Z_j \rangle$. Of course, $\nu_Z^{(*)}$, κ_e and the impurity radiation depend on the densities n_i and n_j. According to (119b), n_i can be eliminated by

$$n_i = \tfrac{1}{2}[N - (1 + \langle Z_j \rangle) n_j]. \tag{127}$$

It is our aim in this paper to describe the effect of impurities within the simplest framework by a single RDE for the temperature. The idea now is to reduce the above equations down to one RDE by

(i) using the ADPAK data (Post et al. 1994) for $\langle Z_j \rangle$, where the main charge is considered only as a function of T that corresponds approximately to the relation $\nu_Z^{(*)} = 0$;

(ii) introducing Lagrangian coordinates; and

(iii) applying the equation of state

$$p = NT = p(T) \tag{128}$$

where $p(T)$ is to be understood as a given function.

Lagrangian coordinates. With the ansatz of a travelling-wave solution $\eta = x - v_0 t$, we are looking for steady-state solutions in a reference frame that moves with the constant velocity v_0 (which has to be determined) relative to the laboratory system. This suggest that we introduce Lagrangian coordinates according to (cf. Spatschek 1990)

$$\tau = t, \qquad y = x - \int_0^\tau d\tau'\, v(y, \tau'), \tag{129}$$

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$$\frac{d}{dt} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \frac{1}{s}\frac{\partial}{\partial y}, \qquad s = 1 + \int_0^\tau d\tau' \, \frac{\partial v(y,\tau')}{\partial y}. \tag{130}$$

It follows from the continuity equation (120) that

$$s = \frac{N_0(y)}{N(y,\tau)}, \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial \tau}\left(\frac{N_0}{N}\right), \tag{131}$$

where $N_0(y) = N(y,0)$ is the initial density distribution.

Applying the equation of state (128) and eliminating N, we obtain a single nonlinear reaction–diffusion equation that describes the temperature evolution in Lagrangian coordinates (τ, y):

$$\frac{\partial T}{\partial \tau} - \frac{2}{5N_0}\xi_p \frac{\partial}{\partial y}\left(\kappa_e \frac{p}{TN_0}\frac{\partial T}{\partial y}\right) = \frac{2}{5}\xi_p \frac{T}{p}\left\{H_{\text{ext}}(y,\tau) - \frac{1}{2}\left[\frac{p}{T} + (\langle Z_j\rangle - 1)n_j\right] n_j L_{\text{rad}}\right\}, \tag{132}$$

with

$$\xi_p^{-1} = 1 - \frac{2}{5}\frac{\partial \ln p}{\partial \ln T}.$$

The solutions of this equation are $N_0(y)$-dependent. This dependence is removed by introducing new Lagrangian coordinates (τ, z) according to (cf. Meerson 1989)

$$\tau = t, \qquad z(x,t) = \int_{x_1(t)}^{x} dx' N(x',t), \tag{133a}$$

$$\frac{d}{dt} = \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = N(x,t)\frac{\partial}{\partial z}, \tag{133b}$$

$x_1(t) = \{x | v(x,t) = 0\}$, which leads, including the equation of state (128), to a second Lagrangian RDE for the temperature:

$$\frac{\partial T}{\partial \tau} - \frac{2}{5}\xi_p \frac{\partial}{\partial y}\left(\kappa_e \frac{p}{T}\frac{\partial T}{\partial y}\right) = \frac{2}{5}\xi_p \frac{T}{p}\left\{H_{\text{ext}}(z,\tau) - \frac{1}{2}\left[\frac{p}{T} + (\langle Z_j\rangle - 1)n_j\right] n_j L_{\text{rad}}\right\}. \tag{134}$$

If one is interest in temperature solutions that are independent of N_0 one has to prefer the RDE (134) rather than (132).

The unknown function n_j can be expressed by (i) a simple approximation $n_j = c_j n_i$ (c_j = const, cf. Sec. 5.3.2) or (ii) as a function of T: $n_j = n_j(T)$, which is outlined in the Sec. 4.2.5 (see Bachmann and Sünder 1998b).

Having solved the RDE (134), $T = T(z,\tau)$, the remaining quantities can be determined by

$$N(z,\tau) = \frac{p[T(z,\tau)]}{T(z,\tau)}, \tag{135}$$

$$v(z,\tau) = \int_0^z dz' \left\{1 - \frac{d \ln p[T(z',\tau)]}{d \ln T(z',\tau)}\right\} \frac{1}{p[T(z',\tau)]}\frac{\partial}{\partial \tau}T(z',\tau). \tag{136}$$

The relationship between the Eulerian and Lagrangian coordinates is given by the nonlinear relation

$$x(z,\tau) = x_1(\tau) + \int_0^z dz' \frac{T(z',\tau)}{p[T(z',\tau)]}. \tag{137}$$

4.2.5. The effect of the impurities on the RDE. The impurity density n_j affects (i) the radiation loss term of (134) and (ii) the electron heat conduction coefficient κ_e,

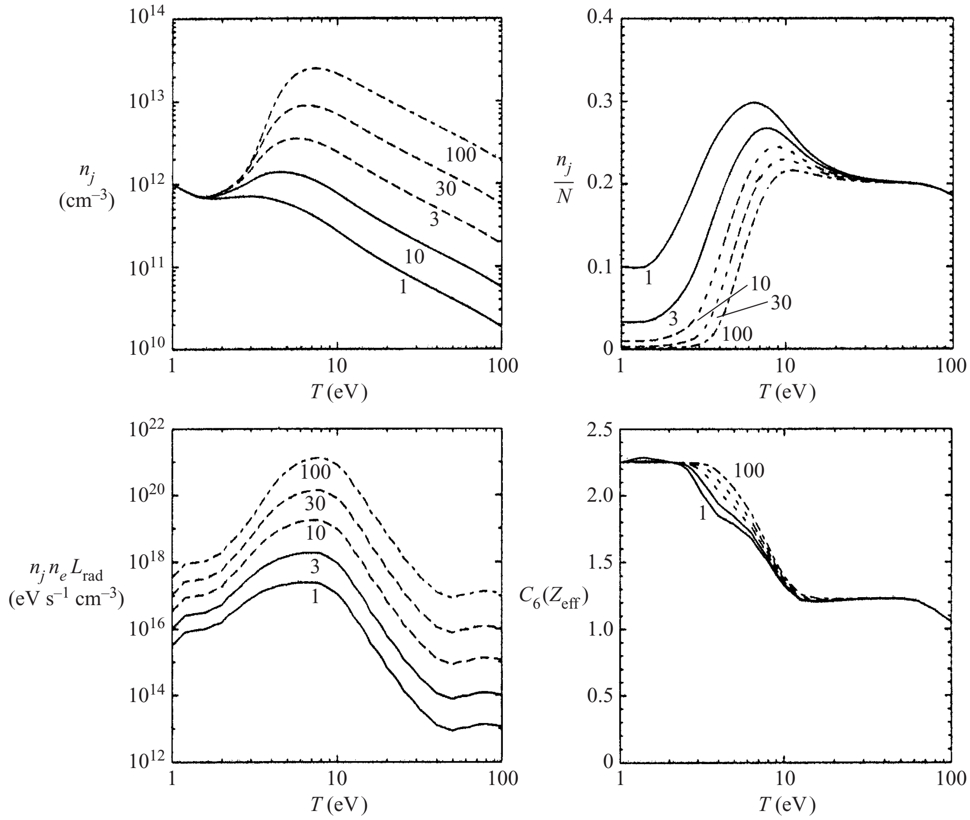

Figure 3. n_j, n_j/N, $n_j n_e L_{\rm rad}$ and $C_6(Z_{\rm eff})$ as functions of T for carbon and the parameter set $P1$; the numbers next to each curve indicate $p_0/(10^{13}\ {\rm cm}^{-3}\ {\rm eV})$.

which depends strongly on n_j via $C_6(Z_{\rm eff})$ (90). $\langle Z_j\rangle$ is a function of T, and $n_j(T)$ has to be calculated from the following first-order ordinary differential equation:

$$\begin{aligned}\frac{1}{n_j}\frac{dn_j}{dT} = \Bigg\{ & \left[\frac{p}{T} + (\langle Z_j\rangle - 1)n_j\right]\frac{G_j - 1}{T} \\ & + \langle Z_j\rangle\left(\frac{p}{T^2} - \frac{1}{T}\frac{dP}{dT} - n_j\frac{d\langle Z_j\rangle}{dT}\right)\Bigg\} \\ & \times\left[\frac{p}{T} + (\langle Z_j\rangle - 1)(\langle Z_j\rangle + 1)n_j\right]^{1/2}, \end{aligned} \tag{138}$$

derived from (111).

In what follows, numerical solutions to (138) will be given for the case of the *isobaric approximation*, $p = p_0$. In addition to the parameter p_0, the solution to this equation requires the impurity density n_j to be known for a given temperature T. We solve (138) for two parameter sets

$P1$: $n_{j0} = n_j(T = 1.0\ {\rm eV}) = 10^{12}\ {\rm cm}^{-3}$, $p_0 = (1, 3, 10, 30, 100) \times 10^{13}\ {\rm cm}^{-3}\ {\rm eV}$;
$P2$: $p_0 = 10^{15}\ {\rm cm}^{-3}\ {\rm eV}$, $n_{j0} = (1, 3, 10, 30, 100, 300, 1000) \times 10^{12}\ {\rm cm}^{-3}$;

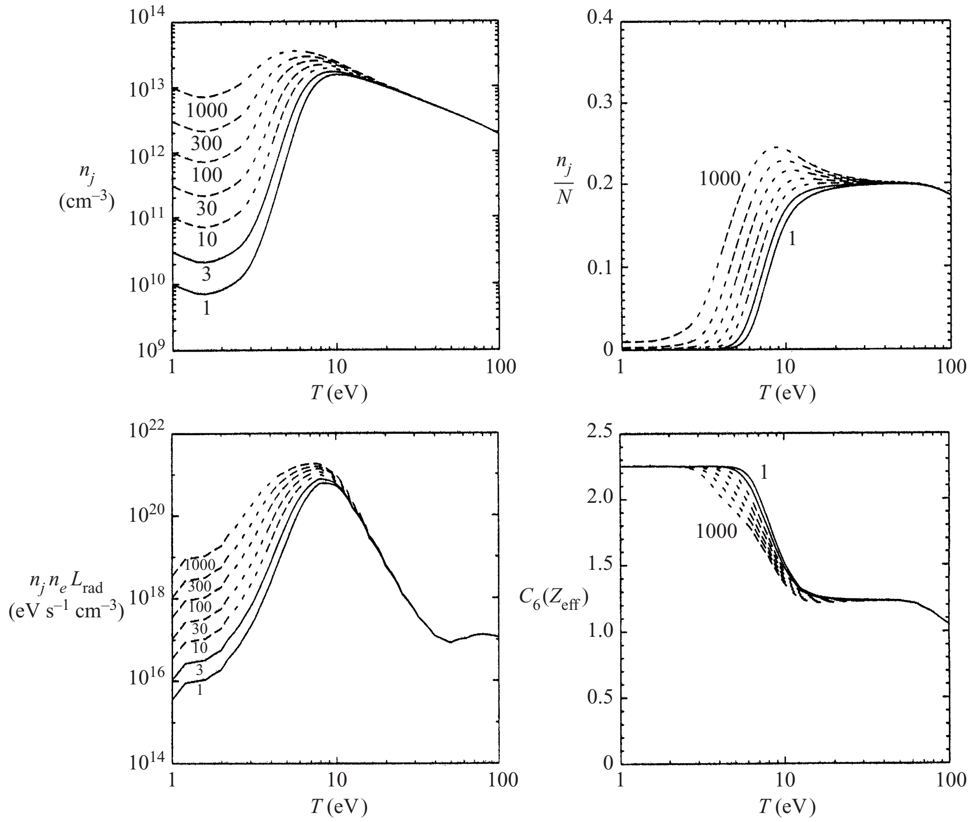

Figure 4. n_j, n_j/N, $n_jn_eL_{\rm rad}$ and $C_6(Z_{\rm eff})$ as functions of T for carbon and the parameter set $P2$; the numbers next to each curve indicate $n_{j0}/(10^{12}\ {\rm cm}^{-3})$.

and two reference cases, using ADPAK data (Post et al. 1994) for Z_f and the radiation loss function $L_{\rm rad}$ ($n_H/n_e = 10^{-7}$, $n_e\tau = 10^{13}$ s cm^{-3}):

(i) for carbon;

(ii) we assume that the mean charge number is given by

$$\langle Z_j \rangle = T^{1/2}, \qquad \langle Z_j^2 \rangle = T, \tag{139}$$

which is a reasonable approximation, especially for high-Z impurities (cf. Gervids et al. 1987).

Having calculated the impurity density n_j, the total density is simply given by $N = p_0/T$. The electron and ion densities follow from (64) and (127). The results are shown in Figs 3–6 where in each case the impurity density n_j, the impurity fraction n_j/N, the radiation loss rate $n_jn_eL_{\rm rad}$ (Figs 3 and 4), the fraction n_jn_e (Figs 5 and 6) and the coefficient $C_6(Z_{\rm eff})$ are represented as functions of the temperature T. The last two quantities enter the description directly through the RDEs (132) and (134).

The solutions for the parameter set $P2$ show that at sufficiently high temperatures, each solution family approaches one curve that does not depend on the initial impurity density n_{j0}. The effect of the different constant-pressure values of

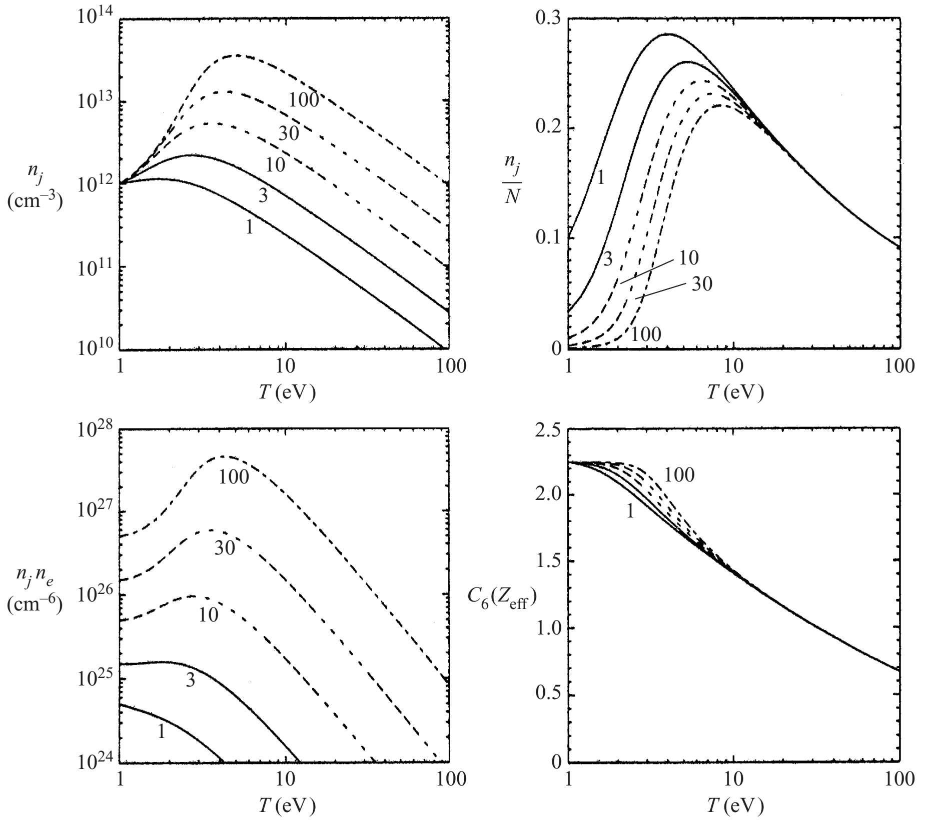

Figure 5. n_j, n_j/N, n_jn_e and $C_6(Z_{\rm eff})$ as functions of T for $\langle Z_j\rangle = T^{1/2}$ and the parameter set $P1$; the numbers next to each curve indicate $p_0/(10^{13}\ \text{cm}^{-3}\ \text{eV})$.

$P1$ is such that only the n_j/N and $C_6(Z_{\rm eff})$ families approach one curve in each case, which, of course, is identical with the one resulting from the $P2$ calculations. Thus the essential result of these investigations is that there are universal curves for n_j/N and $C_6(Z_{\rm eff})$ for temperatures higher than a characteristic temperature ($T_c \approx$ 10–20 eV for (i) and (ii)) in the sense that they are the same for both parameter sets $P1$ and $P2$. n_j can be estimated analytically:

$$n_j \approx \frac{N}{1+\langle Z_j\rangle}, \qquad 20\ \text{eV} < T < 100\ \text{eV}. \tag{140}$$

(i) For carbon (Figs 3 and 4), $n_j(T)$ behaves non-monotonically for T = 1–10 eV. For 10 eV $\leqslant T \leqslant$ 60 eV $n_j/N \approx$ const is a reasonable approximation for all parameter sets, and decreases for higher temperatures. For both parameter sets, $C_6(Z_{\rm eff})$ changes for T = 1–10 eV from 2.3 to 1, and is nearly constant for 10 eV $\leqslant T \leqslant$ 60 eV and then decreases again.

(ii) In the high Z_f approximation (Figs 5 and 6), the effect of high-Z impurities is much stronger. Each universal curve with a characteristic temperature $T_c \approx$ 10 eV tends to 0 as $T \to \infty$.

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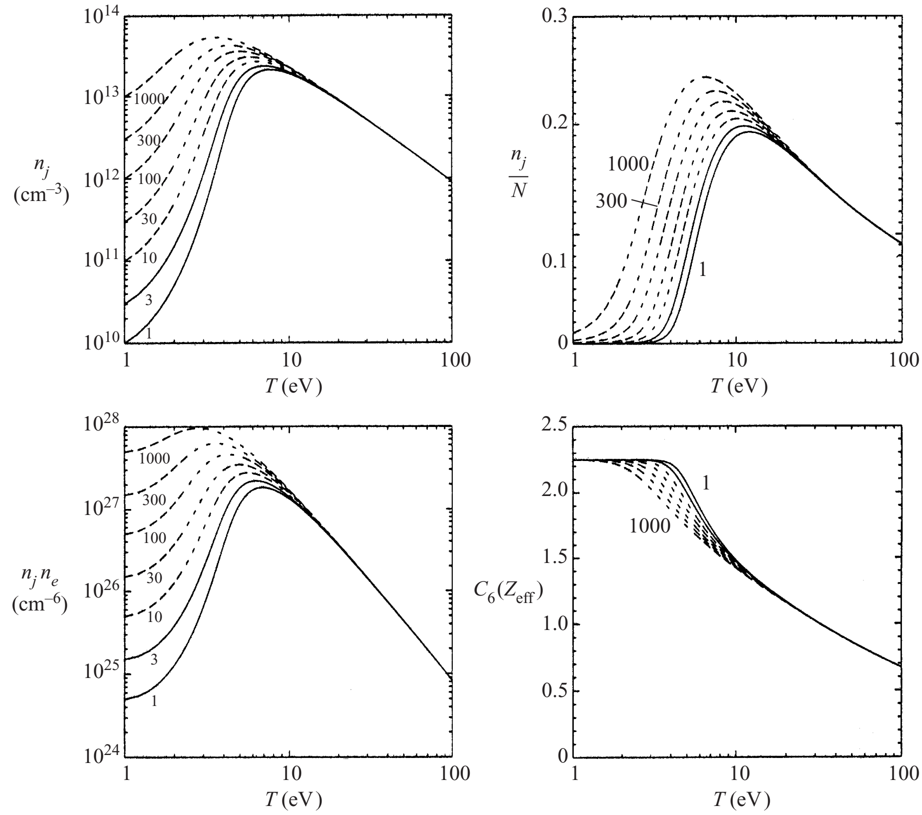

Figure 6. n_j, n_j/N, $n_j n_e$ and $C_6(Z_{\text{eff}})$ as functions of T for $\langle Z_j \rangle = T^{1/2}$ and the parameter set $P2$; the numbers next to each curve indicate $\eta_{j0}/(10^{12}\ \text{cm}^{-3})$.

5. Conclusions

This paper on 1D multifluid plasma models has been concerned with the plasma physical fundamentals by presenting a variety of simple 1D and time-dependent multifluid plasma models. The starting point is the system of multifluid MHD equations for hydrogen ions, impurity ions together with their ionization stages, and electrons. In order to simplify the treatment, especially for the case of high-Z materials, the impurity has been described as a single fluid.

By equalizing all temperatures and adopting the condition of quasineutrality, we have derived within the framework of a three-fluid description for a current-carrying plasma a system of equations whose ability to support travelling-wave solutions has been investigated. This system was reduced further to a currentless plasma description by assuming at first different flow velocities of the particles and then a currentless, streaming plasma model where all particles move with the same velocity.

By introducing Lagrangian coordinates and an equation of state, and deriving an equation that allows one to calculate the impurity density as a function of the temperature (through (138)) the reaction–diffusion equations (132) and (134) for the temperature were obtained.

The effect of the impurity ions on the reaction–diffusion process for the temperature is determined not only by the impurity radiation loss term but also by the electron heat conduction coefficient κ_e, which depends strongly on n_j via $C_6(Z_{\text{eff}})$. Both quantities can be calculated by solving (138) for carbon and high-Z impurities, which are of interest for present and future fusion devices (ITER). The essential result of these investigations is that there are universal curves for the impurity fraction n_j/N and $C_6(Z_{\text{eff}})$ for temperatures higher than a characteristic temperature ($T_c \approx$ 10–20 eV) in the sense that they nearly coincide for the parameters considered.

This paper may be seen as a contribution to the derivation of reduced multifluid plasma models with respect to their range of validity. It is the first of a series of papers that will apply the presented models to investigate effects related to impurity radiation phenomena. For instance, travelling-wave solutions will be considered, steady and time-dependent solutions to the RDE will be given, and problems arising from the nonlinear transformation from Eulerian to Lagrangian coordinates with respect to initial and boundary value problems will be discussed. First results can be found in Bachmann and Sünder (1998b).

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