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# Degenerating sequences of conformal classes and the conformal Steklov spectrum

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*Abstract.* Let $\Sigma$ be a compact surface with boundary. For a given conformal class $c$ on $\Sigma$ the functional $\sigma_k^*(\Sigma, c)$ is defined as the supremum of the $k$th normalized Steklov eigenvalue over all metrics in $c$. We consider the behavior of this functional on the moduli space of conformal classes on $\Sigma$. A precise formula for the limit of $\sigma_k^*(\Sigma, c_n)$ when the sequence $\{c_n\}$ degenerates is obtained. We apply this formula to the study of natural analogs of the Friedlander–Nadirashvili invariants of closed manifolds defined as $\inf_c \sigma_k^*(\Sigma, c)$, where the infimum is taken over all conformal classes $c$ on $\Sigma$. We show that these quantities are equal to $2\pi k$ for any surface with boundary. As an application of our techniques we obtain new estimates on the $k$th normalized Steklov eigenvalue of a nonorientable surface in terms of its genus and the number of boundary components.

## 1 Introduction and main results

Let $(\Sigma, g)$ be a compact Riemannian surface with boundary. In this paper, we always assume that $\Sigma$ is connected and the boundary of $\Sigma$ is nonempty and smooth. Consider *the Steklov problem* defined in the following way

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Sigma, \end{cases}$$

where $\Delta = -\operatorname{div}_g \circ \operatorname{grad}_g$ is the Laplace–Beltrami operator and $\frac{\partial}{\partial n}$ is the outward unit normal vector field along the boundary. The collection of all numbers $\sigma$ for which the Steklov problem admits a solution is called the *Steklov spectrum* of the surface $\Sigma$. The Steklov spectrum is a discrete set of real numbers called Steklov eigenvalues with finite multiplicities satisfying the following condition (see e.g., [GP17])

$$0 = \sigma_0(g) < \sigma_1(g) \le \sigma_2(g) \le \cdots \nearrow +\infty.$$

The Steklov spectrum enables us to define the following homothety-invariant functional on the set $\mathcal{R}(\Sigma)$ of Riemannian metrics on $\Sigma$

$$\overline{\sigma}_k(\Sigma, g) := \sigma_k(g) L_g(\partial\Sigma),$$

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where $L_g(\partial\Sigma)$ stands for the length of the boundary of $\Sigma$ in the metric $g$. The functional $\overline{\sigma}_k(\Sigma, g)$ is called the *k-th normalized Steklov eigenvalue*. It was shown in [CSG11] (see also [Has11, Kok14]) that if $\Sigma$ is an orientable surface, then the functional $\overline{\sigma}_k(\Sigma, g)$ is bounded from above. Moreover, the following theorem holds

***Theorem 1.1*** [GP] *Let $(\Sigma, g)$ be a compact orientable surface of genus $\gamma$ with $l$ boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma, g) \leq 2\pi k(\gamma + l).$$

In this paper, we prove that a similar estimate holds for nonorientable surfaces.

***Theorem 1.2*** *Let $\Sigma$ be a compact nonorientable surface of genus $\gamma$ with $l$ boundary components. Then one has*

$$\overline{\sigma}_k(\Sigma, g) \leq 4\pi k(\gamma + 2l).$$

Here, the genus of a nonorientable surface is defined as the genus of its orientable cover.

***Remark 1.1*** The estimate in Theorem 1.1 has been improved in [Kar17] by a bound which is linear in $k + \gamma + l$ instead of $k(\gamma + l)$. However, the proof of this result uses orientability in an essential way, see [Kar17, Section 6]. It would be interesting to obtain a similar improvement in Theorem 1.2.

Theorems 1.1 and 1.2 enable us to define the following functionals

$$\sigma_k^*(\Sigma) := \sup_{\mathcal{R}(\Sigma)} \overline{\sigma}_k(\Sigma, g),$$

and

$$\sigma_k^*(\Sigma, [g]) := \sup_{[g]} \overline{\sigma}_k(\Sigma, g).$$

***Remark 1.2*** Note that we cannot define the functionals $\sigma_k^*(\Sigma)$ and $\sigma_k^*(\Sigma, [g])$ in higher dimensions. Indeed, it was proved in the paper [CSG19] that if $n = \dim M \geq 3$ then the functional $\overline{\sigma}_k(M, g) := \sigma_k(g) Vol(\partial M, g)^{1/(n-1)}$, where $Vol(\partial M, g)$ denotes the volume of the boundary with respect to the metric $g$, is not bounded from above on the set of Riemannian metrics $\mathcal{R}(M)$. Moreover, it is not even bounded from above in the conformal class $[g]$.

The functional $\sigma_k^*(\Sigma)$ is an object of intensive research during the last decade (see e.g., [FS11, FS16, CGR18, Pet19, GL20, MP20a]).

The functional $\sigma_k^*(\Sigma, [g])$ which is called the *kth conformal Steklov eigenvalue* is less studied. Let us mention some results concerning $\sigma_k^*(\Sigma, [g])$. First, since the disc admits the unique conformal structure one can conclude that $\sigma_k^*(\mathbb{D}^2, [g_{can}]) = \sigma_k^*(\mathbb{D}^2)$, where $g_{can}$ stands for the Euclidean metric on $\mathbb{D}^2$ with unit boundary length. The value of $\sigma_k^*(\mathbb{D}^2)$ is known: $\sigma_k^*(\mathbb{D}^2) = 2\pi k$ (see [Wei54] for $k = 1$ and [GP10] for all $k \geq 1$). Let us also mention the resent paper [FS20], where the authors particularly obtain new results about the functional $\sigma_k^*(\mathbb{D}^2)$.

The functional $\sigma_k^*(\Sigma, [g])$ is the main research object of the paper [Pet19].

***Theorem 1.3*** [Pet19] *For every Riemannian metric $g$ on a compact surface $\Sigma$ with boundary one has*

$$\sigma_k^*(\Sigma,[g]) \geq \sigma_{k-1}^*(\Sigma,[g]) + \sigma_1^*(\mathbb{D}^2,[g_{can}]), \tag{1.1}$$

*particularly*

$$\sigma_k^*(\Sigma,[g]) \geq 2\pi k. \tag{1.2}$$

*Moreover, if the inequality (1.1) is strict then there exists a Riemannian metric $\tilde{g} \in [g]$ such that $\overline{\sigma}_k(\Sigma,\tilde{g}) = \sigma_k^*(\Sigma,[g])$.*

New interesting results about the functional $\sigma_k^*(\Sigma,[g])$ were recently obtained in the paper [KS20].

***Remark 1.3*** The result analogous to Theorem 1.3 for the conformal spectrum of the Laplace–Beltrami operator on closed surfaces also holds (see [NS15a, NS15b, Pet14, Pet18, KNPP20]). For further information concerning the spectrum of the Laplace–Beltrami operator on closed surfaces see the surveys [Pen13, Pen19] and references therein.

It is easy to see that the connection between the functionals $\sigma_k^*(\Sigma)$ and $\sigma_k^*(\Sigma,[g])$ is expressed by the formula

$$\sigma_k^*(\Sigma) = \sup_{[g]} \sigma_k^*(\Sigma,[g]).$$

One can ask what do we get if we replace $\sup_{[g]}$ by $\inf_{[g]}$ in this formula? In this case, we get the following quantity

$$I_k^\sigma(\Sigma) := \inf_{[g]} \sigma_k^*(\Sigma,[g]),$$

It is an analog of the Friedlander–Nadirashvili invariant of closed manifolds. The first Friedlander–Nadirashvili invariant of a closed manifold was introduced in the paper [FN99] in 1999. The $k$th Nadirashvili–Friedlander invariant of a closed surface has been recently studied in the paper [KM20].

In the study of functionals like $\sigma_k^*(\Sigma)$ and $I_k^\sigma(\Sigma)$, one considers maximizing and minimizing sequences of conformal classes $\{c_n\}$ on the *moduli space of conformal classes on* $\Sigma$, i.e., $\sigma_k^*(\Sigma,c_n) \to \sigma_k^*(\Sigma)$ or $\sigma_k^*(\Sigma,c_n) \to I_k^\sigma(\Sigma)$ as $n \to \infty$. Due to the Uniformization theorem conformal classes on $\Sigma$ are in one-to-one correspondence (up to an isometry) with metrics on $\Sigma$ of constant Gauss curvature and geodesic boundary. Therefore, any sequence of conformal classes $\{c_n\}$ on $\Sigma$ corresponds to a sequence of Riemannian surfaces of constant Gauss curvature and geodesic boundary $\{(\Sigma,h_n)\}$, $h_n \in c_n$ and we can consider the moduli space of conformal classes on $\Sigma$ as the set of all $(\Sigma,h)$, where $h$ is a metric of constant Gauss curvature and geodesic boundary, endowed with $C^\infty$-topology (see Section 4). Note that the moduli space of conformal structures is a noncompact topological space. For any sequence $\{c_n\}$ there are two possible scenarios: either this sequence remains in a compact part of the moduli space or it escapes to infinity. Let $(\Sigma_\infty,c_\infty)$ denote the *limiting space*, i.e., $(\Sigma_\infty,c_\infty) = \lim_{n\to\infty}(\Sigma,c_n)$. We compactify $\Sigma_\infty$ if necessary. Let $\widehat{\Sigma_\infty}$ denote the compactified limiting space. It turns out that if the first scenario realizes, then we get $\widehat{\Sigma_\infty} = \Sigma$ and $c_\infty$ is a genuine conformal class on $\Sigma$ for which the value $\sigma_k^*(\Sigma)$ or $I_k^\sigma(\Sigma)$ is attained. If the second scenario realizes, then we say that the sequence $\{c_n\}$ *degenerates*. It turns out that in this case there exists a finite collection of pairwise disjoint geodesics for the metrics $h_n$ whose lengths in $h_n$ tend to 0 as $n$ tends to $\infty$. We refer to these geodesics as *pinching* or *collapsing*. They can be of the following three types: the collapsing boundary components, the collapsing geodesics with no self-intersection crossing the boundary $\partial\Sigma$ at two points and the collapsing geodesics with no self-intersection which do not cross $\partial\Sigma$. Note that in this case, the topology of $\Sigma$ necessarily changes when we pass to the limit as $n \to \infty$, i.e., the compact surfaces $\widehat{\Sigma}_\infty$ and $\Sigma$ are of different topological types. In particular, the surface $\widehat{\Sigma}_\infty$ can be disconnected (see Figure 1). We refer to Section 4 for more details.

The following theorem establishes the correspondence between $\sigma_k^*(\widehat{\Sigma}_\infty, c_\infty)$ and the limit of $\sigma_k^*(\Sigma, c_n)$ when the sequence of conformal classes $c_n$ degenerates (see Section 4 for the definition). It is an analog of [KM20, Theorem 2.8] for the Steklov setting.

***Theorem 1.4*** *Let $\Sigma$ be a compact surface of genus $\gamma$ with $l > 0$ boundary components and let $c_n \to c_\infty$ be a degenerating sequence of conformal classes. Consider the corresponding sequence $\{h_n\}$ of metrics of constant Gauss curvature and geodesic boundary. Suppose that there exist $s_1$ collapsing boundary components and $s_2$ collapsing geodesics with no self-intersection which cross the boundary at two points. Moreover, suppose that $\widehat{\Sigma_\infty}$ has $m$ connected components $\Sigma_{\gamma_i,l_i}$ of genus $\gamma_i$ with $l_i > 0$ boundary components, $\gamma_i + l_i < \gamma + l$, $i = 1, \dots, m$. Then one has*

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = \max\left(\sum_{i=1}^{m} \sigma_{k_i}^*(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2)\right),$$

*where the maximum is taken over all possible combinations of indices such that*

$$\sum_{i=1}^{m} k_i + \sum_{i=1}^{s_1+s_2} r_i = k.$$

***Remark 1.4*** Let $\Sigma$ denote either cylinder or the Möbius band. Theorem 1.4 particularly implies that if the sequence of conformal classes $\{c_n\}$ on $\Sigma$ degenerates then we necessarily have:

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k.$$

***Remark 1.5*** In Theorem 1.4 the sequence $\{h_n\}$ can also have collapsing geodesics not crossing the boundary of $\Sigma$. Moreover, it can happen that the limiting space $\widehat{\Sigma_\infty}$ has *closed* components (see Figure 2). Anyway, in Theorem 1.4 we take only components of $\widehat{\Sigma_\infty}$ which have nonempty boundary.

The main tool that we use in the proof of Theorem 1.4 is the *Steklov–Neumann boundary problem* also known as the *sloshing problem*. Let $\Omega$ be a Lipschitz domain in $(\Sigma, g)$ such that $\overline{\Omega} \cap \partial\Sigma = \partial^S\Omega \neq \emptyset$. Let $\partial^N\Omega = \partial\Omega \backslash \partial\Sigma$. Then the Steklov–Neumann

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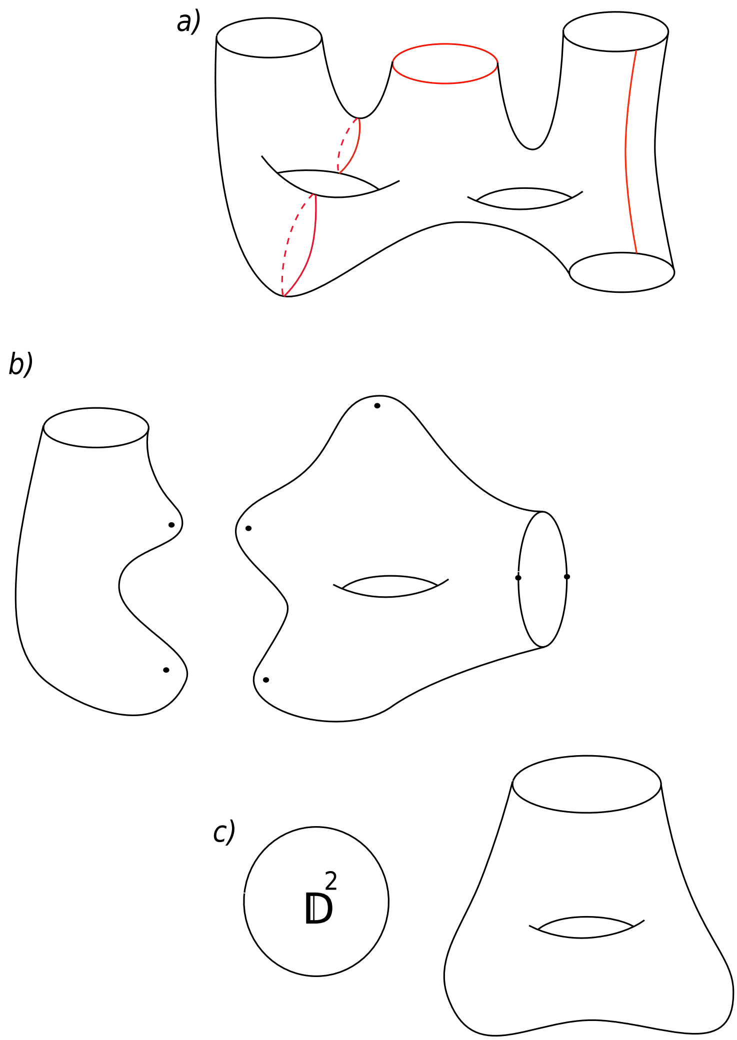


*Figure 1*: An example of a degenerating sequence of conformal classes $\{c_n\}$ on a surface $\Sigma$ of genus 2 with 4 boundary components. (a) The *red* curves correspond to collapsing geodesics for the sequence of metrics of constant Gauss curvature and geodesic boundary $\{h_n\}$, $h_n \in c_n$ corresponding to the degenerating sequence of conformal classes $\{c_n\}$. (b) The compactified limiting space $\widehat{\Sigma_\infty}$ (see Section 4). The black points correspond to the points of compactification. (c) The surface $\widehat{\Sigma_\infty}$ is homeomorphic to the disjoint union of a disc and a surface of genus 1 with 1 boundary component.

problem is defined as:

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial^N \Omega, \\ \frac{\partial u}{\partial n} = \sigma^N u & \text{on } \partial^S \Omega. \end{cases}$$

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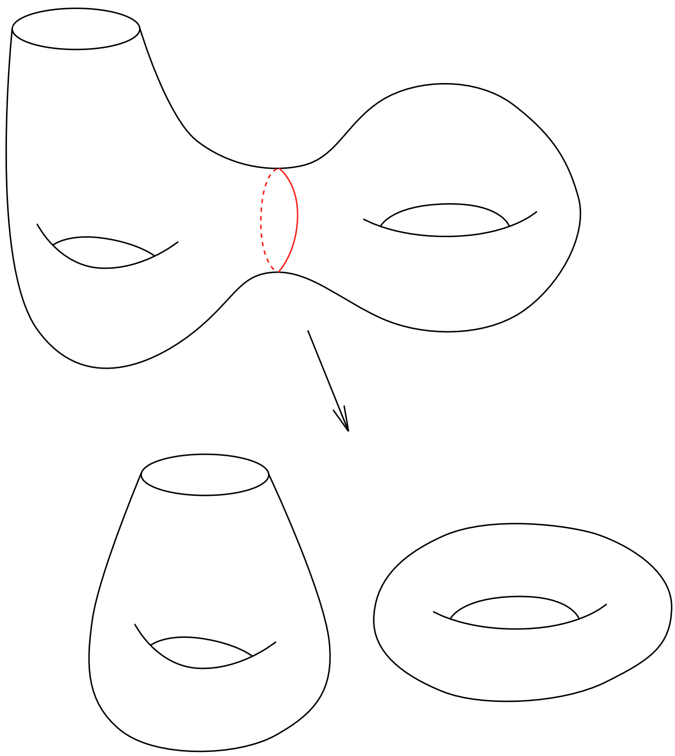

*Figure 2*: An example of a degenerating sequence of conformal classes $\{c_n\}$ on a surface of genus 2 with 1 boundary components such that the limiting space contains a closed component. In Theorem 1.4, we take only the component on the left which has nonempty boundary. Note that in this case $s_1 = s_2 = 0$.

The numbers $\sigma^N$ for which the Steklov–Neumann problem admits a solution are called *Steklov–Neumann eigenvalues*. It is known (see [BKPS10] and references therein) that the set of Steklov–Neumann eigenvalues is not empty and discrete

$$0 = \sigma_0^N(g) < \sigma_1^N(g) \le \sigma_2^N(g) \le \cdots \nearrow +\infty.$$

Every Steklov–Neumann eigenvalue admits the following variational characterization:

$$\sigma_k^N(g) = \inf_{V_k \subset \mathcal{H}^1(\Omega)} \sup_{0 \neq u \in V_k} \frac{\int_\Omega |\nabla u|^2 dv_g}{\int_{\partial^S \Omega} u^2 ds_g},$$

where the infimum is taken over all $k$-dimensional subspaces of the space $\mathcal{H}^1(\Omega) = \{u \in H^1(\Omega, g) \mid \int_{\partial^S \Omega} u ds_g = 0\}$.

Similarly to the case of the Steklov problem we define normalized Steklov–Neumann eigenvalues as

$$\overline{\sigma}_k^N(\Omega, \partial^S \Omega, g) := \sigma_k^N(g) L_g(\partial^S \Omega).$$

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In this notation, we always indicate the Steklov part of the boundary at the second place. Sometimes, we also use the notation $\sigma_k^N(\Omega, \partial^S\Omega, g)$ for $\sigma_k^N(\Omega, g)$ to emphasize that the Steklov boundary condition is imposed on $\partial^S\Omega$.

***Remark 1.6*** Consider $\Omega$ as a surface with Lipschitz boundary. It also follows from [Kok14, Theorem $A_k$] that the quantity $\overline{\sigma}_k^N(\Omega, \partial^S\Omega, g)$ is bounded from above on $[g]$ and we can define the invariant $\sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$ in the same way as the invariant $\sigma_k^*(\Sigma, [g])$.

Theorem 1.4 enables us to establish the value of $I_k^\sigma$.

***Theorem 1.5*** *Let $\Sigma$ be a compact surface with boundary. Then one has $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2) = 2\pi k$.*

## 1.1 Discussion

Let us discuss the estimate obtained in Theorem 1.2. The first estimate on $\overline{\sigma}_1(\Sigma, g)$ where $\Sigma$ is a nonorientable surface of genus $\gamma$ with boundary was obtained in the paper [Sch13]. It reads

$$\overline{\sigma}_1(\Sigma, g) \leq 24\pi(\gamma + 1),$$

if $\gamma \geq 1$ and

$$\overline{\sigma}_1(\Sigma, g) \leq 12\pi,$$

if $\gamma = 0$. Moreover, it follows from the papers [Kok14, Kar16] that

$$\overline{\sigma}_1(\Sigma, g) \leq 16\pi\Big[\frac{\gamma + 3}{2}\Big], \tag{1.3}$$

where $[x]$ stands for the integer part of the number $x$.

Very recently, in the paper [KS20], estimate (1.3) has been improved and extended for $k = 2$: consider $\Sigma$ as a domain with smooth boundary on a closed surface $M$, then one has

$$\overline{\sigma}_k(\Sigma, g) \leq \Lambda_k(M), \ k = 1, 2. \tag{1.4}$$

In this estimate, $\Lambda_k(M) := \sup_{g \in \mathcal{R}(M)} \lambda_k(g)\,\mathrm{Vol}(M, g)$, where $\lambda_k(g)$ is the $k$th Laplace eigenvalue of the metric $g$, $\mathrm{Vol}(M, g)$ is the volume of $M$ in the metric $g$ and $\mathcal{R}(M)$ is the set of Riemannian metrics on $M$. Note that estimate (1.4) does not depend on the number of boundary components. Combining estimate (1.4) with our estimate we get

$$\overline{\sigma}_k(\Sigma, g) \leq \min\{\Lambda_k(M), 4\pi k(\gamma + 2l)\}, \ k = 1, 2.$$

Particularly, for the Möbius band one has

$$\overline{\sigma}_k(\mathbb{MB}, g) \leq \min\{\Lambda_k(\mathbb{RP}^2), 8\pi k\}, \ k = 1, 2,$$

since $\mathbb{MB} \subset \mathbb{RP}^2$. The value $\Lambda_k(\mathbb{RP}^2)$ is known for all $k$ (see [Kar20]): $\Lambda_k(\mathbb{RP}^2) = 4\pi(2k + 1)$. Hence,

$$\overline{\sigma}_k(\mathbb{MB}, g) \leq \min\{4\pi(2k + 1), 8\pi k\} = 8\pi k, \ k = 1, 2.$$

In the paper [FS16] it was shown that $\overline{\sigma}_1(\mathbb{MB}, g) \leq 2\pi\sqrt{3}$ which is obviously $\leq 8\pi$.

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We proceed with the discussion of the functional $I_k^\sigma$. Unlike Theorem 1.4 in [KM20], Theorem 1.5 says nothing about conformal classes on which the value $I_k^\sigma(\Sigma)$ is attained. We conjecture that

***Conjecture 1.6*** *The infimum $I_k^\sigma(\Sigma)$ is attained if and only if $\Sigma$ is diffeomorphic to the disc $\mathbb{D}^2$.*

Note that this conjecture would be a corollary of the following one

***Conjecture 1.7*** *Let $\Sigma$ be a compact surface nondiffeomorphic to the disc. Then for every conformal class $c$ on $\Sigma$ one has*

$$\sigma_1^*(\Sigma, c) > \sigma_1^*(\mathbb{D}^2) = 2\pi.$$

This conjecture is an analog of the Petrides rigidity theorem for the first conformal Laplace eigenvalue [Pet14, Theorem 1]. Recently this conjecture has been confirmed in the case of the cylinder and the Möbius band (see [MP20b]). We plan to tackle Conjectures 1.6 and 1.7 in the subsequent papers.

Let us discuss the analogy between the quantity $I_k^\sigma$ and the Friedlander–Nadirashvili invariant of closed surfaces $I_k$. In the paper [KM20], it was conjectured that $I_k$ are invariants of cobordisms of closed surfaces (see Conjecture 1.8). Similarly, one can see that $I_k^\sigma$ are invariants of cobordisms of compact surfaces with boundary. Let us recall that two compact surfaces with boundary $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$ are called cobordant if there exists a three-dimensional *manifold with corners* $\Omega$ whose boundary is $\Sigma_1 \cup_{\partial\Sigma_1} W \cup_{\partial\Sigma_2} \Sigma_2$, where $W$ is a cobordism of $\partial\Sigma_1$ and $\partial\Sigma_2$ (i.e., $W$ is a surface with boundary $\partial\Sigma_1 \sqcup \partial\Sigma_2$). Following [BNR16] we denote a cobordism of two surfaces $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$ by $(\Omega; \Sigma_1, \Sigma_2, W; \partial\Sigma_1, \partial\Sigma_2)$. One can easily see that the cobordisms of surfaces with boundary are trivial. Indeed, we can construct the following cobordism of a surface $(\Sigma, \partial\Sigma)$ and $(\varnothing, \varnothing)$: $(\Sigma \times [0,1]; \Sigma \times \{0\}, \varnothing, \partial\Sigma \times [0,1] \cup \Sigma \times \{1\}; \partial\Sigma, \varnothing)$. A fundamental fact about cobordisms of surfaces with boundary is *Theorem about splitting cobordisms* (see [BNR16, Theorem 4.18]) which says that every cobordism of compact surfaces with boundary can be split into a sequence of cobordisms given by a handle attachment and cobordisms given by a *half-handle* attachment. We refer to [BNR16] for definitions and further information about cobordisms of compact manifolds with boundary. Analysing the proof of Theorem 1.5 one can remark that the value of $I_k^\sigma$ does not change under handle and half-handle attachments. Since by this procedure any surface $\Sigma$ can be reduced to the disc, we get $I_k^\sigma(\Sigma) = I_k^\sigma(\mathbb{D}^2) = 2\pi k$.

## 1.2 Plan of the paper

The paper is organized in the following way. In Section 2, we collect all the analytic facts which are necessary for the proof of Theorem 1.4. The main result here is Proposition 2.6. In Section 3, we prove Theorem 1.2 using the techniques developed in the previous section. Section 4 represents the geometric part of the paper. Here, we describe convergence on the moduli space of conformal structures on a surface with boundary. Section 5 is devoted to the proof of Theorem 1.4. In Section 6, we deduce Theorem 1.5 from Theorem 1.4. Finally, Section 7 contains some auxiliary technical results.

## 2 Analytic background

Here, we provide a necessary analytic background that we will use in the proof of Theorem 1.4 in Section 5. The propositions in this section are analogs of the propositions in [KM20, Section 4]. We postpone the proof of a proposition to Section 7.2 every time when it follows the exactly same way as the proof of an analogous proposition in [KM20, Section 4].

### 2.1 Convergence of Steklov–Neumann spectrum

We start with the following convergence result.

**Lemma 2.1** *Let $(M, g)$ be a compact Riemannian manifold with boundary. Consider a finite collection $\{B_\varepsilon(p_i)\}_{i=1}^l$ of geodesic balls of radius $\varepsilon$ centred at some points $p_1, \dots, p_l \in M$. Then the spectrum of the Steklov–Neumann problem*

$$\begin{cases} \Delta_g u = 0 & \text{in } M \backslash \cup_{i=1}^l B_\varepsilon(p_i), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \cup_{i=1}^l \partial B_\varepsilon(p_i) \backslash \partial M, \\ \frac{\partial u}{\partial n} = \lambda_k^N(M \backslash \cup_{i=1}^l B_\varepsilon(p_i), g) u & \text{on } \partial M \backslash \cup_{i=1}^l \partial B_\varepsilon(p_i) \end{cases}$$

*converges to the Steklov spectrum of $(M, g)$ as $\varepsilon \to 0$.*

**Proof** For the sake of simplicity, we only consider the case of one ball that we denote by $B_\varepsilon$ centred at $p \in M$. First, we consider the case when$B_\varepsilon \cap \partial M \neq \varnothing$, i.e., $p \in \partial M$.

Let $\mathcal{E}(u)$ denote the extension of the function $u$ by the unique solution of the problem

$$\begin{cases} \Delta_g \mathcal{E}(u) = 0 & \text{in } B_\varepsilon, \\ \frac{\partial \mathcal{E}(u)}{\partial n} = 0 & \text{on } \partial M \cap \partial B_\varepsilon, \\ \mathcal{E}(u) = u & \text{on } \partial B_\varepsilon \backslash \partial M. \end{cases}$$

**Claim 1.** The operator $\mathcal{E}(u)$ is uniformly bounded.

**Proof** The proof is similar to the proof of uniform boundedness of the harmonic continuation operator into small geodesic balls [RT75, Example 1]. Fix $0 < r < \varepsilon$ and let $B_r$ denote a geodesic ball of radius $r$ with the same center as $B_\varepsilon$. One has

$$||\mathcal{E}(u)||^2_{L^2(B_r, g)} \leq C||u||^2_{L^2(M \backslash B_r, g)} + C||\nabla u||^2_{L^2(M \backslash B_r, g)} \tag{2.1}$$

and

$$||\nabla \mathcal{E}(u)||^2_{L^2(B_r, g)} \leq C||\nabla u||^2_{L^2(M \backslash B_r, g)}. \tag{2.2}$$

Inequality (2.1) follows from estimate (7.1) and the trace inequality

$$||\mathcal{E}(u)||^2_{L^2(B_r, g)} \leq ||\mathcal{E}(u)||^2_{H^1(B_r, g)} \leq C||u||^2_{H^{1/2}(\partial B_r \backslash \partial M, g)} \leq C||u||^2_{H^1(M \backslash B_r, g)}.$$

Suppose that inequality (2.2) was false. Then, there exists a sequence of functions $\{u_n\}$ in $H^1(M \backslash B_r, g)$ such that

$$||\nabla u_n||_{L^2(M \backslash B_r, g)} \leq 1/n$$

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and

$$||\mathcal{E}(u_n)||_{L^2(B_r,g)} \geq 1.$$

Consider $\alpha_n = \frac{1}{Vol(M\backslash B_r,g)} \int_{M\backslash B_r} u_n dv_g$. We show that

$$||u_n - \alpha_n||_{H^1(M\backslash B_r,g)} \leq C/n.$$

Indeed, by the generalized Poincaré inequality one has

$$||u_n - \alpha_n||_{L^2(M\backslash B_r,g)} \leq C||\nabla u_n||_{L^2(M\backslash B_r,g)} \leq C/n$$

moreover

$$||\nabla(u_n - \alpha_n)||_{L^2(M\backslash B_r,g)} = ||\nabla u_n||_{L^2(M\backslash B_r,g)} \leq 1/n.$$

Note that $\mathcal{E}(u_n - \alpha_n) = \mathcal{E}(u_n) - \alpha_n$. Then, we can prove inequality (2.2)

$$\begin{aligned} ||\nabla\mathcal{E}(u_n)||_{L^2(B_r,g)} &= ||\nabla\mathcal{E}(u_n - \alpha_n)||_{L^2(B_r,g)} \leq ||\mathcal{E}(u_n - \alpha_n)||_{H^1(B_r,g)} \\ &\leq ||u_n - \alpha_n||_{H^{1/2}(\partial B_r\backslash\partial M,g)} \leq C||u_n - \alpha_n||_{H^1(M\backslash B_r,g)} \leq C/n, \end{aligned}$$

where in the second and third inequalities, we have used in order estimate (7.1) and the trace inequality. We got a contradiction. Hence, inequality (2.2) is true.

Note that for any $\rho r < \varepsilon$ the first inequality scales as

$$||\mathcal{E}(u)||^2_{L^2(B_{\rho r},g)} \leq C||u||^2_{L^2(M\backslash B_{\rho r},g)} + C\rho^2||\nabla u||^2_{L^2(M\backslash B_{\rho r},g)},$$

while the second inequality scales as

$$||\nabla\mathcal{E}(u)||^2_{L^2(B_{\rho r},g)} \leq C||\nabla u||^2_{L^2(M\backslash B_{\rho r},g)}.$$

Therefore, $||\mathcal{E}(u)||^2_{H^1(B_{\rho r},g)} \leq C||u||^2_{L^2(M\backslash B_{\rho r},g)} + C||\nabla u||^2_{L^2(M\backslash B_{\rho r},g)}$ for $\varepsilon$ small enough. ∎

**Claim 2.** One has

$$\limsup_{\varepsilon\to 0} \sigma_k^N(M\backslash B_\varepsilon, g) \leq \sigma_k(M,g).$$

**Proof** We only consider the case of $B_\varepsilon \cap \partial M \neq \varnothing$. The case of $B_\varepsilon \cap \partial M = \varnothing$ is easier and follows the exactly same arguments. The proof is similar to the proof of [Bog17, Theorem 3.5].

Let $V_k$ be a $k$-dimensional subspace of $H^1(M,g)$ and $v \in V_k$ such that

$$\sigma_k(M,g) = \max_{u\in V_k\backslash\{0\}} \frac{\int_M |\nabla u|^2 dv_g}{\int_{\partial M} u^2 ds_g}.$$

Let $u_1,\dots,u_k$ be an orthonormal basis in $V_k$. We modify the functions $u_i$, $i = 1,\dots,k$ as

$$u_{i,\varepsilon} = u_i - \frac{1}{L(\partial M\backslash\partial B_\varepsilon)} \int_{\partial M\backslash\partial B_\varepsilon} u_i ds_g.$$

Then, $\int_{\partial M\setminus\partial B_\varepsilon} u_{i,\varepsilon} ds_g = 0$. Consider the space $V_{k,\varepsilon} := span(u_{1,\varepsilon},\dots,u_{k,\varepsilon})$. Since $\dim V_{k,\varepsilon} = k$ one has

$$\sigma_k^N(M\setminus B_\varepsilon, g) \le \max_{u_\varepsilon\in V_{k,\varepsilon}\setminus\{0\}} \frac{\int_{M\setminus B_\varepsilon} |\nabla u_\varepsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\varepsilon} u_\varepsilon^2 ds_g}.$$

Moreover, since the dimension of $V_{k,\varepsilon}$ is finite then there exists a function $v_\varepsilon \in V_{k,\varepsilon}$ such that

$$\sigma_k^N(M\setminus B_\varepsilon, g) \le \frac{\int_{M\setminus B_\varepsilon} |\nabla v_\varepsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\varepsilon} v_\varepsilon^2 ds_g}. \tag{2.3}$$

Let $v_\varepsilon = \sum_{i=1}^k c_i u_{i,\varepsilon}$. We build the following function $v = \sum_{i=1}^k c_i u_i \in V_k \subset H^1(M,g)$. Note that $\nabla v_\varepsilon = \sum_{i=1}^k c_i \nabla u_{i,\varepsilon} = \sum_{i=1}^k c_i \nabla u_i = \nabla v$ on $M\setminus B_\varepsilon$. Thus, $\int_{M\setminus B_\varepsilon} |\nabla v_\varepsilon|^2 dv_g = \int_{M\setminus B_\varepsilon} |\nabla v|^2 dv_g \to \int_M |\nabla v|^2 dv_g$ as $\varepsilon \to 0$. Moreover, it is easy to see that

$$\int_{\partial M\setminus\partial B_\varepsilon} v_\varepsilon^2 ds_g = \sum_i c_i^2 \left( \int_{\partial M\setminus\partial B_\varepsilon} u_i^2 dv_g - \frac{1}{L(\partial M\setminus\partial B_\varepsilon, g)} \left( \int_{\partial M\setminus\partial B_\varepsilon} u_i ds_g \right)^2 \right)$$
$$+ \sum_{i\neq j} 2c_i c_j \left( \int_{\partial M\setminus\partial B_\varepsilon} u_i u_j ds_g - \frac{1}{L(\partial M\setminus\partial B_\varepsilon, g)} \int_{\partial M\setminus\partial B_\varepsilon} u_i ds_g \int_{\partial M\setminus\partial B_\varepsilon} u_j ds_g \right),$$

which converges to $\int_{\partial M} v^2 ds_g$ as $\varepsilon \to 0$. Then (2.3) implies

$$\limsup_{\varepsilon\to 0} \sigma_k^N(M\setminus B_\varepsilon, g) \le \limsup_{\varepsilon\to 0} \frac{\int_{M\setminus B_\varepsilon} |\nabla v_\varepsilon|^2 dv_g}{\int_{\partial M\setminus\partial B_\varepsilon} v_\varepsilon^2 ds_g} = \frac{\int_M |\nabla v|^2 dv_g}{\int_{\partial M} v^2 ds_g} \le \sigma_k(M,g). \qquad \blacksquare$$

Now, we are ready to prove the Lemma. The proof is similar to the proof of [MS20, Lemma 3.2]. Let $u_\varepsilon$ be a normalized $\sigma_k^N$-eigenfunction. By Claim 2 $u_\varepsilon$ are uniformly bounded. If $B_\varepsilon \cap \partial M = \emptyset$, then we take the harmonic continuation into $B_\varepsilon$. It is known that the operators of harmonic continuation into $B_\varepsilon$ are uniformly bounded (see [RT75, Example 1]). Otherwise we extend $u_\varepsilon$ into $B_\varepsilon$ by $\mathcal{E}(u_\varepsilon)$. By Claim 1 these operators are also uniformly bounded. Therefore, we get a uniformly bounded in $H^1(M,g)$ sequence $\{\tilde u_\varepsilon\}$. Then there exists $\varepsilon_l \to 0$ such that $\tilde u_{\varepsilon_l} \rightharpoonup u$ in $H^1(M,g)$. Thus, $\tilde u_{\varepsilon_l} \to u$ in $L^2(M,g)$ by the Rellich–Kondrachov embedding theorem. The standard elliptic estimates imply $u_{\varepsilon_l} \to u$ in $C^\infty_{loc}(M\setminus\{p\})$. Consider a function $\varphi \in C^\infty_c(M\setminus\{p\})$ such that $supp(\varphi) \subset M\setminus B_R$ for a ball $B_R$ centered at $p$ with $R$ fixed. Extracting a subsequence by Claim 2 one can assume that $\sigma_k^N(M\setminus B_{\varepsilon_l}, g) \to \sigma$. Then we have

$$\int_M \langle \nabla u, \nabla\varphi\rangle dv_g = \lim_{l\to 0} \int_{M\setminus B_R} \langle \nabla u_{\varepsilon_l}, \nabla\varphi\rangle dv_g$$
$$= \lim_{l\to 0} \sigma_k^N(M\setminus B_{\varepsilon_l}, g) \int_{M\setminus B_R} u_{\varepsilon_l}\varphi dv_g = \sigma \int_M u\varphi dv_g.$$

Hence, $u$ is an eigenfunction with eigenvalue $\sigma$. Thus all accumulation points of $\{\sigma_k^N(M\setminus B_{\varepsilon_l}, g)\}$ are in the Steklov spectrum of $M$. Our aim now is to show that $\sigma = \sigma_k(M,g)$. We will do this by showing that the $u$ is orthogonal in $L^2(\partial M, g)$ to the first $k-1$ Steklov eigenfunctions of $(M,g)$. We use the proof by induction.

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Let $u_\varepsilon$ be a first Steklov–Neumann eigenfunction of $(M\backslash B_\varepsilon, g)$. We have already shown that $\tilde{u}_\varepsilon \rightharpoonup u$ in $H^1(M,g)$ then by the trace embedding theorem one has $\tilde{u}_\varepsilon \to u$ in $H^{1/2}(\partial M, g)$ and hence in $L^2(\partial M, g)$. In particular, one has $||u_\varepsilon - u||_{L^2(\partial M\backslash\partial B_\varepsilon, g)} \to 0$ as $\varepsilon \to 0$. Then

$$\left|\int_{\partial M\backslash\partial B_\varepsilon}(u_\varepsilon - u)ds_g\right| \le \int_{\partial M\backslash\partial B_\varepsilon}|u_\varepsilon - u|ds_g$$
$$\le L(\partial M\backslash\partial B_\varepsilon, g)^{1/2}||u_\varepsilon - u||^{1/2}_{L^2(\partial M\backslash\partial B_\varepsilon, g)},$$

which converges tp 0 as $\varepsilon \to 0$. Since $\int_{\partial M\backslash\partial B_\varepsilon} u_\varepsilon ds_g = 0$ one then has that $\lim_{\varepsilon\to 0}\int_{\partial M\backslash\partial B_\varepsilon} u ds_g = \int_{\partial M} u ds_g = 0$. Therefore, $u$ cannot be a constant and since by Claim 2 $\limsup_{\varepsilon\to 0}\sigma_1^N(M\backslash B_\varepsilon, g) = \sigma \le \sigma_1(M,g)$ and $\sigma$ belongs to the Steklov spectrum of $(M,g)$ we conclude that $u$ is a first Steklov eigenfunction of $(M,g)$ and $\sigma = \sigma_1(M,g)$.

Now suppose that $\limsup_{\varepsilon\to 0}\sigma_i^N(M\backslash B_\varepsilon, g) = \sigma_i(M,g)$ for any $i < k$. Let $u_\varepsilon$ be a $k$th Steklov–Neumann eigenfucntion of $(M\backslash B_\varepsilon, g)$. Since $\tilde{u}_\varepsilon \rightharpoonup u$ in $H^1(M,g)$, then the trace embedding theorem implies that $\tilde{u}_\varepsilon \to u$ in $H^{1/2}(\partial M, g)$ in particular $\tilde{u}_\varepsilon \to u$ in $L^2(\partial M, g)$ whence $||u_\varepsilon - u||_{L^2(\partial M\backslash\partial B_\varepsilon, g)} \to 0$. Let $v_\varepsilon$ be an $i$th Steklov–Neumann eigenfunction of $(M\backslash B_\varepsilon, g)$ with $i < k$. Then $\int_{\partial M\backslash\partial B_\varepsilon} u_\varepsilon v_\varepsilon ds_g = 0$; moreover, we have supposed that $v$ is an $i$th Steklov eigenfunction of $(M,g)$. One has

$$\left|\int_{\partial M\backslash\partial B_\varepsilon}(u_\varepsilon v_\varepsilon - uv)ds_g\right|$$
$$\le \int_{\partial M\backslash\partial B_\varepsilon}|u_\varepsilon v_\varepsilon - uv|ds_g = \int_{\partial M\backslash\partial B_\varepsilon}|u_\varepsilon v_\varepsilon - u_\varepsilon v + u_\varepsilon v - uv|ds_g$$
$$\le \int_{\partial M\backslash\partial B_\varepsilon}|u_\varepsilon(v_\varepsilon - v)|ds_g + \int_{\partial M\backslash\partial B_\varepsilon}|v(u_\varepsilon - u)|ds_g$$
$$\le \left(\int_{\partial M\backslash\partial B_\varepsilon}u_\varepsilon^2 ds_g\right)^{1/2}\left(\int_{\partial M\backslash\partial B_\varepsilon}(v_\varepsilon - v)^2 ds_g\right)^{1/2}$$
$$+\left(\int_{\partial M\backslash\partial B_\varepsilon}v_\varepsilon^2 ds_g\right)^{1/2}\left(\int_{\partial M\backslash\partial B_\varepsilon}(u_\varepsilon - u)^2 ds_g\right)^{1/2} \to 0 \text{ as } \varepsilon \to 0.$$

Hence, $\int_{\partial M\backslash\partial B_\varepsilon} u_\varepsilon v_\varepsilon ds_g \to \int_{\partial M} uv ds_g$ as $\varepsilon \to 0$. But $\int_{\partial M\backslash\partial B_\varepsilon} u_\varepsilon v_\varepsilon ds_g = 0$ for all $\varepsilon$. Thus, $\int_{\partial M} uv ds_g = 0$. We conclude that $u$ is orthogonal in $L^2(\partial M, g)$ to the first $k-1$ Steklov eigenfunctions. Thus, $\sigma = \sigma_k^N(M,g)$. ■

We endow the set of Riemannian metrics on $\Sigma$ with the $C^\infty$–topology. Then the following "continuity" result holds.

***Proposition 2.2*** *Let $\Sigma$ be a surface with boundary and $\Omega \subset \Sigma$ be a Lipschitz domain. Let the sequence of Riemannian metrics $g_m$ on $\Sigma$ converge in $C^\infty$–topology to the metric $g$. Then $\sigma_k^*(\Sigma, [g_m]) \to \sigma_k^*(\Sigma, [g])$. Similarly, if $h_m|_{\overline{\Omega}}$ converge to $g|_{\overline{\Omega}}$ in $C^\infty$-topology, then $\sigma_k^{N*}(\Omega, \partial^S\Omega, [h_m|_{\overline{\Omega}}]) \to \sigma_k^{N*}(\Omega, \partial^S\Omega, [g|_{\overline{\Omega}}])$.*

**Proof** We provide a proof for the functional $\sigma_k^*(\Sigma, [g])$. The proof for the functional $\sigma_k^{N*}(\Omega, [g|_{\overline{\Omega}}])$ follows the exactly same arguments.

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Choose any $\varepsilon > 0$ and consider $m$ large enough. One has

$$\frac{1}{1+\varepsilon} f g_m(v,v) \le f g(v,v) \le (1+\varepsilon) f g_m(v,v), \quad \forall v \in \Gamma(TM \backslash \{0\}),$$

where $f$ is any positive smooth function on $\Sigma$. Then by [CGR18, Proposition 32] one has

$$\frac{1}{(1+\varepsilon)^6} \bar{\sigma}_k(\Sigma, f g_m) \le \bar{\sigma}_k(\Sigma, f g) \le (1+\varepsilon)^6 \bar{\sigma}_k(\Sigma, f g_m).$$

Taking the supremum over all $f$ yields

$$\frac{1}{(1+\varepsilon)^6} \sigma_k^*(\Sigma, [g_m]) \le \sigma_k^*(\Sigma, [g]) \le (1+\varepsilon)^6 \sigma_k^*(\Sigma, [g_m]),$$

which completes the proof since this inequality holds for any $\varepsilon > 0$. ∎

## 2.2 Discontinuous metrics

Let $\Sigma$ be a compact surface with boundary. Consider a set of pairwise disjoint Lipschitz domains $\{\Omega_i\}_{i=1}^s$ in $\Sigma$ such that $\Sigma = \bigcup_{i=1}^s \overline{\Omega}_i$. Let $C_+^\infty(\Sigma, \{\Omega_i\})$ denote a set of functions on $\bigcup_{i=1}^s \overline{\Omega}_i$ such that $\rho \in C_+^\infty(\Sigma, \{\Omega_i\})$ means that $\rho|_{\Omega_i} = \rho_i \in C^\infty(\overline{\Omega}_i)$ are positive for every $i$. Similarly, $C^\infty(\Sigma, \{\Omega_i\})$ denotes a set of "smooth" functions on $\bigcup_{i=1}^s \overline{\Omega}_i$. We introduce discontinuous metrics on $\Sigma$ defined as $\rho g \in [g]$, where $\rho \in C_+^\infty(\Sigma, \{\Omega_i\})$ and $g$ is a genuine Riemannian metric. The set $C^k(\Sigma, \{\Omega_i\})$ of functions which are of class $C^k$ in every $\overline{\Omega}_i$ is defined in a similar way. The Steklov spectrum of the metric $\rho g$ is defined as the set of critical values of the Rayleigh quotient

$$R_{\rho g}[\varphi] = \frac{\int_\Sigma |\nabla_g \varphi|_g^2 dv_g}{\int_{\partial\Sigma} \rho^{\frac{1}{2}} \varphi^2 ds_g}.$$

This is the Rayleigh quotient of the *Steklov problem with density* $\rho$. The Steklov spectrum with density $\rho$ is well-defined for any non-negative $\rho \in L^\infty(\Sigma, g)$ (see [Kok14, Proposition 1.3]). Elliptic regularity implies that the eigenfunctions are at least 1/2-Hölder continuous on $\partial\Sigma$. Therefore, Steklov eigenvalues of the metric $\rho g$ admit the following variational characterization

$$\sigma_k(\Sigma, \rho g) = \inf_{E_{k+1}} \sup_{\varphi \in E_{k+1}} R_{\rho g}[\varphi],$$

where $E_{k+1}$ ranges over all $(k+1)$-dimensional subspaces of $C^0(\Sigma)$.

We introduce the following notation

$$\sigma_k^*(\Sigma, \{\Omega_i\}, [g]) = \sup\{\bar{\sigma}_k(\rho g) \mid \rho \in C_+^\infty(\Sigma, \{\Omega_i\})\},$$

where $\bar{\sigma}_k(\rho g)$ is the normalized $k$th eigenvalue given by

$$\bar{\sigma}_k(\rho g) = \sigma_k(\rho g) L_{\rho g}(\partial\Sigma).$$

The following lemma particularly asserts that the quantity $\sigma_k^*(\Sigma, \{\Omega_i\}, [g])$ is well-defined.

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***Lemma 2.3*** *Let $(\Sigma, g)$ be a Riemannian surface with boundary. Consider a set of pairwise disjoint Lipschitz domains $\Omega_i$ such that $\Sigma = \bigcup_{i=1}^{s} \overline{\Omega}_i$. Then one has*

$$\sigma_k^*(\Sigma, \{\Omega_i\}, [g]) = \sigma_k^*(\Sigma, [g]).$$

**Proof** The proof follows the same steps as the proof of Lemma 2 in the paper [FN99]. We provide it here.

Since the set of discontinuous metrics is larger than the set of continuous ones, we have $\sigma_k^*(\Sigma, \{\Omega_i\}, [g])) \geq \sigma_k^*(\Sigma, [g])$. Therefore, we have to prove that

$$\sigma_k^*(\Sigma, \{\Omega_i\}, [g])) \leq \sigma_k^*(\Sigma, [g]),$$

which is equivalent to

$$\sigma_k(\Sigma, \rho g) \leq \sigma_k^*(\Sigma, [g]), \tag{2.4}$$

where $\rho \in C_+^\infty(\Sigma, \{\Omega_i\})$ and $\int_{\partial\Sigma} \rho^{1/2} ds_g = 1$.

Let $E_k$ be the eigenspace corresponding to the $k$th Steklov eigenvalue of the metric $\rho g$. We put

$$S = \left\{ u \in H^1(\Sigma, \rho g) \mid u \perp_{L^2(\partial\Sigma, \rho g)} E_0, \dots, E_{k-1}, \int_{\partial\Sigma} \rho^{1/2} u^2 ds_g = 1 \right\}$$

For any $\varepsilon > 0$ we consider the functional

$$\mathcal{F}_\rho[u] := \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma, \rho g) - \varepsilon) \int_{\partial\Sigma} \rho^{1/2} u^2 ds_g.$$

It immediately follows that $\mathcal{F}_\rho[u] \geq \varepsilon, \forall u \in S$.

Let $0 < a := \min_{\cup\{\Omega_i\}} \rho$ and $\max_{\cup\{\Omega_i\}} =: b < \infty$. We define a smooth nondecreasing function $\chi(t)$ on $\mathbb{R}_+$ that equals zero if $t < 1/2$ and equals 1 when $t > 1$ and define the following parametrized family of functions:

$$\rho_\delta(x) = \begin{cases} \rho(x) & \text{if } x \notin U, \\ \rho(x)\chi\left(\frac{d^2(x)}{\delta}\right) + b\left(1 - \chi\left(\frac{d^2(x)}{\delta}\right)\right) & \text{if } x \in U, \end{cases}$$

where $d$ is the distance function from a point $x \in \Sigma$ to $\cup\{\partial\Omega_i \cap \partial\Omega_j\},\ i \neq j$ and $U$ is a sufficiently small tubular neighborhood of $\cup\{\partial\Omega_i \cap \partial\Omega_j\},\ i \neq j$ where $d^2$ is smooth. We have:

(i) $\left(\frac{a}{b}\right)\rho \leq \rho_\delta \leq \left(\frac{b}{a}\right)\rho$;
(ii) $\lim_{\delta\to 0} \int_{\partial\Sigma} \rho_\delta^{1/2} ds_g = 1$; and
(iii) $\lim_{\delta\to 0} \int_{\partial\Sigma} |\rho_\delta^{1/2} - \rho^{1/2}|^q ds_g = 0, \forall q < \infty$.

We want to prove that $\mathcal{F}_{\rho_\delta}[u] \geq 0, \forall u \in S$.

Consider $T = (\sigma_k(\Sigma, \rho g) - \varepsilon)\sqrt{\frac{b}{a}}$ and divide the set $S$ into two parts $S_1$ and $S_2$:

$$S_1 := \left\{ u \in S \middle| \int_\Sigma |\nabla_g u|^2 dv_g \geq T \right\},$$

$$S_2 := S \backslash S_1 = \left\{ u \in S \middle| \int_\Sigma |\nabla_g u|^2 dv_g < T \right\}.$$

If $u \in S_1$ then

$$\mathcal{F}_{\rho_\delta}[u] = \int_\Sigma |\nabla_g u|^2 dv_g - (\sigma_k(\Sigma, \rho g) - \varepsilon) \int_{\partial\Sigma} \rho_\delta^{1/2} u^2 ds_g$$

$$\geq (\sigma_k(\Sigma,\rho g)-\varepsilon)\left(\sqrt{\frac{b}{a}}-\int_{\partial\Sigma}\rho_\delta^{1/2}u^2ds_g\right)$$
$$\geq (\sigma_k(\Sigma,\rho g)-\varepsilon)\sqrt{\frac{b}{a}}\left(1-\int_{\partial\Sigma}\rho^{1/2}u^2ds_g\right)=0.$$

Let us show that $||u||_{L^p(\partial\Sigma,g)}$ with $p\geq 2$ is bounded for any $u\in S_2$. We consider the following operator $L[u]:=\int_{\partial\Sigma}u\rho^{1/2}ds_g$. For this operator, one has

$$|L[u]|\leq C\int_{\partial\Sigma}|u|ds_g\leq C||u||_{L^2(\partial\Sigma,g)}\leq C||u||_{H^1(\Sigma,g)},$$

which implies that $L\in H^{-1}(\Sigma,g)$. Here, we used in order the boundedness of $\rho$, the Cauchy–Schwarz and the trace inequalities. We also used the convention that $C$ denotes any positive constant depending only on $\Sigma$. [AH96, Lemma 8.3.1] applied to the operator $L$ implies that there exists a constant $C>0$ depending only on $\Sigma$ such that

$$||u||^2_{L^2(\Sigma,g)}\leq C||\nabla u||^2_{L^2(\Sigma,g)}<CT,$$

where we used the fact that $L[u]=0\ \forall u\in S$. By the trace theorem one then has

$$||u||^2_{H^{1/2}(\partial\Sigma,g)}\leq C'||u||^2_{H^1(\Sigma,g)}<C'',$$

where $C''=C'(CT+T)$. Finally by the Sobolev embedding theorem (see for instance [DNPV12, Theorem 6.9]) we get

$$||u||_{L^p(\partial\Sigma,g)}\leq C'''||u||_{H^{1/2}(\partial\Sigma,g)}<\tilde{C}\ \forall 2\leq p<\infty,$$

where $\tilde{C}=C'''\sqrt{C''}$. Therefore, if $u\in S_2$ then

$$\begin{aligned}\mathcal{F}_{\rho_\delta}[u]&=\int_\Sigma|\nabla_g u|^2dv_g-(\sigma_k(\Sigma,\rho g)-\varepsilon)\int_{\partial\Sigma}\rho_\delta^{1/2}u^2ds_g\\&=\int_\Sigma|\nabla_g u|^2dv_g-(\sigma_k(\Sigma,\rho g)-\varepsilon)-(\sigma_k(\Sigma,\rho g)-\varepsilon)\int_{\partial\Sigma}(\rho_\delta^{1/2}-\rho^{1/2})u^2ds_g\\&\geq\varepsilon-(\sigma_k(\Sigma,\rho g)-\varepsilon)\left(\int_{\partial\Sigma}(\rho_\delta^{1/2}-\rho^{1/2})^q ds_g\right)^{1/q}\left(\int_{\partial\Sigma}|u|^p ds_g\right)^{2/p}\\&\geq\varepsilon-(\sigma_k(\Sigma,\rho g)-\varepsilon)\frac{\varepsilon}{\sigma_k(\Sigma,\rho g)-\varepsilon}=0.\end{aligned}$$

In the last inequality, we put

$$\left(\int_{\partial\Sigma}(\rho_\delta^{1/2}-\rho^{1/2})^q ds_g\right)^{1/q}\left(\int_{\partial\Sigma}|u|^p ds_g\right)^{2/p}=\frac{\varepsilon}{\sigma_k(\Sigma,\rho g)-\varepsilon}$$

since $\int_{\partial\Sigma}(\rho_\delta^{1/2}-\rho^{1/2})^q ds_g\to 0$ as $\delta\to 0$ and $\int_{\partial\Sigma}|u|^p ds_g<+\infty$.

Hence, $\mathcal{F}_{\rho_\delta}[u]\geq 0, \forall u\in S$ which implies $\sigma_k(\Sigma,\rho_\delta g)\geq\sigma_k(\Sigma,\rho g)-\varepsilon$. We then have

$$\bar{\sigma}_k(\Sigma,\rho_\delta g)=\sigma_k(\Sigma,\rho_\delta g)L_{\rho_\delta g}(\partial\Sigma)\geq\sigma_k(\Sigma,\rho g)L_{\rho_\delta g}(\partial\Sigma)-\varepsilon L_{\rho_\delta g}(\partial\Sigma).$$

Therefore, $\sigma_k^*(\Sigma,[g]) \geq \sigma_k(\Sigma,\rho g)L_{\rho_\delta g}(\partial\Sigma) - \varepsilon L_{\rho_\delta g}(\partial\Sigma)$. Letting $\delta \to 0$ one then gets $\sigma_k^*(\Sigma,[g]) \geq \sigma_k(\Sigma,\rho g) - \varepsilon$ that implies (2.4) since $\varepsilon$ is arbitrary small. ■

Lemma 2.3 implies the following lemma whose proof is postponed to Section 7.2.

***Lemma 2.4*** *Let $(\Sigma, g)$ be a Riemannian surface with boundary. Consider a set of pairwise disjoint domains $\Omega_i$ such that $\Sigma = \bigcup_{i=1}^s \overline{\Omega}_i$ and $\Omega_i \cap \partial\Sigma = \partial^S\Omega_i$. Let $(\Omega, h) = \sqcup(\overline{\Omega}_i, g|_{\overline{\Omega}_i})$ and $\partial^S\Omega = \sqcup\partial^S\Omega_i$. Then for all $k \geq 0$ one has*

$$\sigma_k^*(\Sigma,[g]) \geq \sigma_k^{N*}(\Omega, \partial^S\Omega, [h]).$$

## 2.3 Steklov–Neumann spectrum of a subdomain

This section is devoted to the following technical lemma

***Lemma 2.5*** *Let $\rho_\delta \in C_+^\infty(\Sigma, \{\Omega, \Sigma\backslash\Omega\})$ such that $\rho_\delta|_\Omega \equiv 1$ and $\rho_\delta|_{\Sigma\backslash\Omega} \equiv \delta$. Then one has*

$$\liminf_{\delta\to 0} \sigma_k(\rho_\delta g) \geq \sigma_k^N(\Omega, \partial^S\Omega, g),$$

*where $\sigma_k^{N*}(\Omega, \partial^S\Omega, g)$ is the kth Steklov–Neumann eigenvalue of the domain $(\Omega, g)$ and $\partial^S\Omega = \partial\Sigma \cap \Omega \neq \emptyset$.*

**Proof** The idea of the proof comes from the proof of [EPS15, Section 2, Step 2].

**Case I.** First, we consider the case when $\Omega^c \cap \partial\Sigma \neq \emptyset$. Let $\Omega^c$ denotes $int(\Sigma\backslash\Omega)$ and $\partial^S\Omega^c = \partial\Omega^c \cap \partial\Sigma$. Since by elliptic regularity eigenfunctions of the Steklov problem with bounded density are in $H^1$ on the boundary we can restrict ourselves to the space $H^1(\partial\Sigma, g)$. More precisely, let $\psi$ be an eigenfunction with eigenvalue $\sigma$ then by elliptic regularity:

$$||\psi||^2_{H^1(\partial\Sigma,\rho_\delta g)} \leq C(||\sigma\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)} + ||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}) \leq C(\sigma^2+1)||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}$$

for some positive constant $C$. This implies

$$\frac{||\nabla\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\psi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leq C(\sigma^2+1) - 1.$$

More generally we see that if $\varphi \in \mathrm{span}\langle\psi_0,\dots,\psi_k\rangle$, where $\psi_i$ is in the $i$th eigenspace of $(\Sigma, g_\delta)$ then there exists a constant $C_k > 0$ such that

$$\frac{||\nabla\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leq C_k.$$

Therefore, we set

$$\mathcal{H} := \{\varphi \in H^1(\partial\Sigma, g) \mid \frac{||\nabla\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}}{||\varphi||^2_{L^2(\partial\Sigma,\rho_\delta g)}} \leq C_k\},$$

$$\mathcal{H}_1 := \left\{\varphi \in \mathcal{H} \mid \frac{\partial\hat{\varphi}}{\partial n} = 0 \text{ on } \partial^S\Omega^c\right\},$$

where $\hat{\varphi}$ stands for the harmonic continuation of $\varphi$ into $\Sigma$ and

$$\mathcal{H}_2 := \{\varphi \in \mathcal{H} \mid \varphi \in H_0^1(\partial^S\Omega^c, g),\ \varphi_{|\Omega} = 0\}.$$

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**Claim 1.** One has

$$\int_\Sigma \langle \nabla \hat{\varphi}, \nabla \hat{\psi} \rangle_{\tilde{g}} dv_{\tilde{g}} = 0, \forall \varphi \in \mathcal{H}_1, \psi \in \mathcal{H}_2,$$

for any metric $\widetilde{g} \in [g]$.

**Proof**

$$\begin{aligned}\int_\Sigma \langle \nabla \hat{\varphi}, \nabla \hat{\psi} \rangle_{\tilde{g}} dv_{\tilde{g}} &= \int_\Sigma \Delta_{\tilde{g}} \hat{\varphi} \hat{\psi} dv_{\tilde{g}} + \int_{\partial \Sigma} \frac{\partial \hat{\varphi}}{\partial \tilde{n}} \psi ds_{\tilde{g}} \\ &= \int_{\partial^S \Omega^c} \frac{\partial \hat{\varphi}}{\partial \tilde{n}} \psi ds_{\tilde{g}} + \int_{\partial^S \Omega} \frac{\partial \hat{\varphi}}{\partial \tilde{n}} \psi ds_{\tilde{g}} = 0.\end{aligned}$$ ∎

For the sake of simplicity, we use the symbols $\sigma_k^\delta$ for $\sigma_k(\rho_\delta g)$, $g_\delta$ for $\rho_\delta g$ and $R_\delta$ for the Rayleigh quotient

$$R_\delta[\varphi] = \frac{\int_\Sigma |\nabla \hat{\varphi}|^2_{g_\delta} dv_{g_\delta}}{\int_{\partial \Sigma} \varphi^2 ds_{g_\delta}}.$$

**Claim 2.** There exists a constant that we also denote by $C_k > 0$ such that $\sigma_k^\delta \le C_k$.

**Proof** Theorem 1.1 implies that there exists a constant $C(k) > 0$ such that

$$\sigma_k^*(\Sigma, [g]) \le C(k).$$

By Lemma 2.3 for every $\delta$ one has

$$\sigma_k^\delta L_{g_\delta}(\partial \Sigma) \le \sigma_k^*(\Sigma, [g]) \le C(k).$$

Therefore,

$$\sigma_k^\delta \le \frac{C(k)}{L_{g_\delta}(\partial \Sigma)} = \frac{C(k)}{L_g(\partial^S \Omega) + \delta^{1/2} L_g(\partial^S \Omega^c)} \le \frac{C(k)}{L_g(\partial^S \Omega)} = C_k.$$ ∎

Let $W_k$ be the set of $k+1$-dimensional subspaces of $\mathcal{H}$ satisfying the condition that $R_\delta|_{W_k} \le C_k$. Claim 2 particularly implies that the space spanned by the first $k+1$ eigenfunctions is in $W_k$, i.e., $W_k \ne \emptyset$.

Consider the operator $\mathcal{E}$ defined in section 2.1 by

$$\begin{cases} \Delta_g \mathcal{E}(u) = 0 & \text{in } \Sigma, \\ \frac{\partial \mathcal{E}(u)}{\partial n} = 0 & \text{on } \partial^S \Omega^c, \\ \mathcal{E}(u) = u & \text{on } \partial^S \Omega. \end{cases}$$

For a function $\varphi \in H^1(\partial \Sigma, g)$, we fix its decomposition $\varphi_1 + \varphi_2$ with

$$\varphi_1 = \begin{cases} \varphi & \text{on } \partial^S \Omega, \\ \mathcal{E}(\varphi) & \text{on } \partial^S \Omega^c \end{cases}$$

and $\varphi_2 = \varphi_1 - \varphi$. Note that $\hat{\varphi}_1 = \mathcal{E}(\varphi_1)$.

**Claim 3.** For every $\varphi \in V \in W_k$ there exists a constant $C > 0$ such that

$$\int_{\partial^S \Omega^c} \varphi_2^2 \, ds_{g_\delta} \le C\sqrt{\delta} \int_{\partial \Sigma} \varphi^2 dv_{g_\delta}.$$

**Proof** By Claim 1, one has

$$\int_\Sigma \langle \nabla \hat{\varphi}_1, \nabla \hat{\varphi}_2 \rangle_g dv_g = 0.$$

Further, since $\varphi \in V \in W_k$, we have

$$\begin{aligned} C_k \geq R_\delta[\varphi] &= \frac{\int_\Sigma |\nabla \hat{\varphi}|_g^2 dv_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{\int_\Sigma |\nabla \hat{\varphi}_1|^2 dv_g + \int_\Sigma |\nabla \hat{\varphi}_2|_g^2 dv_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \\ &\geq \frac{\int_{\Omega^c} |\nabla \hat{\varphi}_2|_g^2 dv_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{1}{\delta^{1/2}} \frac{\int_{\Omega^c} |\nabla \hat{\varphi}_2|_g^2 dv_g}{\int_{\partial^s\Omega^c} \varphi_2^2 ds_g} \frac{\|\varphi_2\|^2_{L^2(\partial^s\Omega^c, g_\delta)}}{\|\varphi\|^2_{L^2(\partial\Sigma, g_\delta)}} \\ &\geq \frac{\sigma_1^D(\Omega^c, \partial^S\Omega^c, g)}{\sqrt{\delta}} \frac{\|\varphi_2\|^2_{L^2(\partial^s\Omega^c, g_\delta)}}{\|\varphi\|^2_{L^2(\partial\Sigma, g_\delta)}}, \end{aligned}$$

where $\sigma_1^D(\Omega^c, \partial^S\Omega^c, g)$ is the first nonzero Steklov–Dirichlet eigenvalue of $(\Omega^c, g)$ (see [BKPS10]). ■

**Claim 4.** For every $\varphi \in V \in W_k$ and for every sufficiently small $\delta$ there exists a constant $C > 0$ such that

$$\int_{\partial\Sigma} \varphi^2 \, ds_{g_\delta} \leq (1 + C\delta^{1/4}) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}.$$

**Proof** One has

$$\begin{aligned} \|\varphi\|^2_{L^2(\partial\Sigma, g_\delta)} &= \int_{\partial^s\Omega^c} (\varphi_1 + \varphi_2)^2 dv_{s_\delta} + \int_{\partial^s\Omega} \varphi_1^2 ds_{g_\delta} \\ &\leq \left(1 + \frac{1}{\varepsilon}\right) \int_{\partial\Sigma} \varphi_2^2 ds_{g_\delta} + (1 + \varepsilon) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}, \end{aligned}$$

for every $\varepsilon > 0$. Applying Claim 3, we obtain

$$\|\varphi\|^2_{L^2(\partial\Sigma, g_\delta)} \leq C\sqrt{\delta}\left(1 + \frac{1}{\varepsilon}\right) \int_{\partial\Sigma} \varphi^2 ds_{g_\delta} + (1 + \varepsilon) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta},$$

and hence,

$$\left(1 - C\sqrt{\delta}\left(1 + \frac{1}{\varepsilon}\right)\right) \|\varphi\|^2_{L^2(\partial\Sigma, g_\delta)} \leq (1 + \varepsilon)\|\varphi_1\|^2_{L^2(\partial\Sigma, g_\delta)}.$$

Choosing $\varepsilon = \delta^{1/4}$ completes the proof. ■

**Claim 5.** For every $\varphi \in V \in W_k$ and for every sufficiently small $\delta$, there exists a constant $C > 0$ such that

$$\int_{\partial^s\Omega^c} \varphi_1^2 \, ds_g \leq C \int_{\partial^s\Omega} \varphi_1^2 ds_g.$$

**Proof**

$$C_k \geq \frac{\int_{\partial\Sigma} |\nabla \varphi|^2_{g_\delta} dv_{g_\delta}}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geq \frac{\int_{\partial^s\Omega} |\nabla \varphi|_g^2 ds_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{\int_{\partial^s\Omega} |\nabla \varphi_1|_g^2 ds_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}},$$

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since $\varphi = \varphi_1$ on $\partial^S\Omega$. Then by Claim 4, one has

$$C_k \geq \frac{\int_{\partial^s\Omega} |\nabla\varphi_1|_g^2 ds_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \geq \frac{1}{1+C\delta^{1/4}} \frac{\int_{\partial^s\Omega} |\nabla\varphi_1|_g^2 ds_g}{\int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}},$$

which implies

$$\begin{aligned}\int_{\partial^s\Omega} |\nabla\varphi_1|_g^2 ds_g &\leq C_k(1+C\delta^{1/4}) \int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta} \\ &= C_k(1+C\delta^{1/4})\left(\int_{\partial^s\Omega} \varphi_1^2 ds_g + \delta^{1/2}\int_{\partial^s\Omega^c} \varphi_1^2 ds_g\right).\end{aligned} \tag{2.5}$$

For the rest of the proof $C$ stands for any positive constant depending possibly on $\Sigma$ and $g$ but not on $\delta$ or $\varphi$.

Note that $\partial^s\Omega$ has positive capacity (see [HP18, pp.102-105]). Applying in order the trace inequality, estimate (7.1), the Sobolev embedding and inequality (2.5) yield

$$\begin{aligned}\|\varphi_1\|^2_{L^2(\partial^s\Omega^c,g)} \leq C\|\hat\varphi_1\|^2_{H^1(\Sigma,g)} &\leq C\|\varphi_1\|^2_{H^{1/2}(\partial^s\Omega,g)} \\ &\leq C\|\varphi_1\|^2_{H^1(\partial^s\Omega,g)} = C(\|\varphi_1\|^2_{L^2(\partial^s\Omega,g)} + \|\nabla\varphi_1\|^2_{L^2(\partial^s\Omega,g)}) \\ &\leq C(1+C\delta^{1/4})\big(\|\varphi_1\|^2_{L^2(\partial^s\Omega,g)} + \delta^{1/2}\|\varphi_1\|^2_{L^2(\partial^s\Omega^c,g)}\big),\end{aligned}$$

which implies the required inequality for $\delta$ small enough. ∎

Further, by the fact that $\int_\Sigma \langle\nabla\hat\varphi_1, \nabla\hat\varphi_2\rangle_g dv_g = 0$ and by Claim 4 for every $\varphi \in V \in W_k$ and one has

$$\begin{aligned}R_\delta[\varphi] &= \frac{\int_\Sigma |\nabla\hat\varphi|_g^2 dv_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} = \frac{\int_\Sigma |\nabla\hat\varphi_1|_g^2 dv_g + \int_\Sigma |\nabla\hat\varphi_2|_g^2 dv_g}{\int_{\partial\Sigma} \varphi^2 ds_{g_\delta}} \\ &\geq \frac{1}{1+C\delta^{1/4}} \frac{\int_\Sigma |\nabla\hat\varphi_1|_g^2 dv_g + \int_\Sigma |\nabla\hat\varphi_2|_g^2 dv_g}{\int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}} \\ &\geq \frac{1}{1+C\delta^{1/4}} \frac{\int_\Sigma |\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial\Sigma} \varphi_1^2 ds_{g_\delta}} = \frac{1}{1+C\delta^{1/4}} \frac{\int_\Sigma |\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^s\Omega} \varphi_1^2 dv_g + \delta^{1/2}\int_{\partial^s\Omega^c} \varphi_1^2 dv_g}\end{aligned}$$

and by Claim 5, we get

$$\begin{aligned}R_\delta[\varphi] &\geq \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \frac{\int_\Sigma |\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^s\Omega} \varphi_1^2 ds_g} \\ &\geq \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \frac{\int_\Omega |\nabla\hat\varphi_1|_g^2 dv_g}{\int_{\partial^s\Omega} \varphi_1^2 ds_g} \\ &\geq \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} R^N_{(\Omega,\partial^s\Omega,g)}[\varphi_{|\Omega}],\end{aligned}$$

where $R^N_{(\Omega,\partial^s\Omega,g)}$ denotes the Rayleigh quotient for the Steklov–Neumann problem in the domain $(\Omega, g)$.

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Let $V = \text{span}\langle\psi_0,\ldots,\psi_k\rangle$, where $\psi_i$ is in the $i$th eigenspace of $(\Sigma, g_\delta)$. Then

$$\begin{aligned}\sigma_k^\delta = \max_{\varphi\in V} R_\delta[\varphi] &\geq \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \max_{\varphi\in V} R^N_{(\Omega,\partial^s\Omega,g)}[\varphi_{|\Omega}] \\ &\geq \frac{1}{(1+C\delta^{1/4})(1+\delta^{1/2}C)} \sigma_k^N(\Omega,\partial^S\Omega,g),\end{aligned} \tag{2.6}$$

since the restriction to $\Omega$ of the functions $\psi_i$ form the space of the same dimension by unique continuation. Finally, passing to the lim inf as $\delta\to 0$ in (2.6) yields the lemma.

**Case II.** The case when $\Omega^c\cap\partial\Sigma = \emptyset$ is trivial. Indeed, in this case, we have $\partial^S\Omega = \partial\Sigma$. Then for any function $\varphi$, one has

$$R_\delta[\varphi] = \frac{\int_\Sigma |\nabla\hat\varphi|_g^2 dv_g}{\int_{\partial\Sigma}\varphi^2 ds_{g_\delta}} \geq \frac{\int_\Omega |\nabla\hat\varphi|_g^2 dv_g}{\int_{\partial^s\Omega}\varphi^2 ds_g} = R^N_{(\Omega,\partial^s\Omega,g)}[\varphi_{|\Omega}].$$

Therefore, considering $V = \text{span}\langle\psi_0,\ldots,\psi_k\rangle$, where $\psi_i$ is in the $i$th eigenspace of $(\Sigma, g_\delta)$ yields

$$\sigma_k^\delta = \max_{\varphi\in V} R_\delta[\varphi] \geq \max_{\varphi\in V} R^N_{(\Omega,\partial^s\Omega,g)}[\varphi_{|\Omega}] \geq \sigma_k^N(\Omega,\partial^S\Omega,g).$$

Taking lim inf as $\delta\to 0$ completes the proof.

Lemma 2.5 is the key ingredient in the proof of the following proposition. We postpone the proof to Section 7.2.

***Proposition 2.6*** *Let $(\Sigma, g)$ be a Riemannian surface with boundary, $\Omega\subset\Sigma$ a Lipschitz domain and $\partial^S\Omega = \partial\Sigma\cap\Omega \neq \emptyset$. Then for all k one has*

$$\sigma_k^*(\Sigma,[g]) \geq \sigma_k^{N*}(\Omega,\partial^S\Omega,[g|_{\overline{\Omega}}]).$$

*Similarly, let $(\Sigma, g)$ be a Riemannian surface whose boundary. Let $\partial^S\Sigma$ denote all boundary components with the Steklov boundary condition and $\Omega\subset\Sigma$ be a Lipschitz domain such that $\partial^S\Omega\subset\partial^S\Sigma$. Then for all k one has*

$$\sigma_k^{N*}(\Sigma,\partial^S\Sigma,[g]) \geq \sigma_k^{N*}(\Omega,\partial^S\Omega,[g|_{\overline{\Omega}}]).$$

As a corollary of Proposition 2.6, we get

***Corollary 2.7*** *Let $(M, g)$ be a compact Riemannian surface with boundary. Consider a sequence $\{K_n\}$ of smooth domains $K_n\subset M$ such that*

- *$K_r\subset K_s$ $\forall r > s$ and*
- *$\cap_n K_n = \{p_1,\ldots,p_l\}$ for some points $p_1,\ldots,p_l\in M$.*

*Then one has*

$$\lim_{n\to\infty}\sigma_k^{N*}(M\backslash K_n,\partial M\backslash\partial K_n,[g]) = \sigma_k^*(M,[g]).$$

The proof is postponed to Section 7.2.

## 2.4 Disconnected surfaces

The proofs of two lemmas below follow the exactly same arguments as the proofs of Lemmas 4.9 and 4.10 in [KM20]. Their proofs are postponed to Section 7.2.

***Lemma 2.8*** *Let $(\Omega, g) = \sqcup_{i=1}^{s}(\Omega_i, g_i)$ be a disjoint union of Riemannian surfaces with Lipschitz boundary. Set $\partial^S\Omega = \sqcup_{i=1}^{s}\partial^S\Omega_i$. Then for all $k > 0$ one has*

$$\sigma_k^{N*}(\Omega, \partial^S\Omega, [g]) = \max_{\sum_{i=1}^{s} k_i = k,\ k_i > 0} \sum_{i=1}^{s} \sigma_{k_i}^{N*}(\Omega_i, \partial^S\Omega_i, [g_i]).$$

***Lemma 2.9*** *Let $(\Sigma, g)$ be a Riemannian surface with boundary. Consider a set of pairwise disjoint Lipschitz domains $\{\Omega_i\}_{i=1}^{s}$ in $\Sigma$ such that $\Sigma = \bigcup_{i=1}^{s}\overline{\Omega_i}$ and $\Omega_i \cap \partial\Sigma = \partial^S\Omega_i \neq \emptyset$ for $1 \le i \le s'$. Then one has*

$$\sigma_k^*(\Sigma, [g]) \ge \max_{\sum_{i=1}^{s'} k_i = k,\ k_i \ge 0} \sum_{i=1}^{s'} \sigma_{k_i}^{N*}(\Omega_i, \partial^S\Omega_i, [g]).$$

## 3 Proof of Theorem 1.2

The proof is inspired by the methods of the papers [YY80, GP, Kar16]. Let $\Sigma$ be a nonorientable compact surface of genus $\gamma$ and $l$ boundary components. We pass to its orientable cover $\pi\colon \widetilde{\Sigma} \to \Sigma$. Note that $\Sigma$ is of genus $\gamma$ and has $2l$ boundary components. Let $\tau$ denote the involution exchanging the sheets of $\pi$. If $h$ is a metric on $\Sigma$ then $g := \pi^* h$ is a metric on $\widetilde{\Sigma}$ invariant with respect to $\tau$, i.e., $\tau$ is an isometry of $g$. Let $\mathcal{D}_{\widetilde{\Sigma}}$ be the Dirichlet-to-Neumann map acting on functions on $\widetilde{\Sigma}$. Then $\tau \circ \mathcal{D}_{\widetilde{\Sigma}} = \mathcal{D}_{\widetilde{\Sigma}} \circ \tau$ and hence Steklov eigenfunctions are divided into $\tau$-odd and $\tau$-even ones. The corresponding Steklov eigenvalues are also divided into odd and even ones. Let $\sigma_k^\tau(\widetilde{\Sigma}, g)$ the $k$th $\tau$-even Steklov eigenvalue. Then $\sigma_k^\tau(\widetilde{\Sigma}, g) = \sigma_k(\Sigma, h)$.

By a well-known theorem of Ahlfors [Ahl50], there exists a proper conformal branched cover $\psi\colon (\widetilde{\Sigma}, g) \to (\mathbb{D}^2, g_{can})$. The word "proper" means $\psi(\partial\widetilde{\Sigma}) = \mathbb{S}^1$. Let $d$ be its degree. Define the following pushed-forward metric $g^*$ on $\mathbb{D}^2$: consider a neighborhood $U$ of a nonbranching point $p \in \mathbb{D}^2$. Its pre-image is a collection of $d$ neighborhoods $U_i, i = 1, \dots, d$ on $\widetilde{\Sigma}$. Moreover, $\psi_i := \psi_{|U_i}\colon U_i \to U$ is a diffeomorphism. Then the metric $g^*$ is defined on $U$ as $\sum(\psi_i^{-1})^* g$. The metric $g^*$ is a metric on $\mathbb{D}^2$ with isolated conical singularities at branching points of $\psi$. The following lemma is trivial

***Lemma 3.1*** *For any function $u \in C^\infty(\mathbb{D}^2)$ one has*

$$\int_{\mathbb{S}^1} u\, dv_{g^*} = \int_{\partial\widetilde{\Sigma}} (\psi^* u)\, dv_g$$

*and*

$$d\int_{\mathbb{D}^2} |\nabla_{g^*} u|^2 dv_{g^*} = \int_{\widetilde{\Sigma}} |\nabla_g(\psi^* u)|^2 dv_g.$$

Further, suppose that there exists an involution $\iota$ of $\mathbb{D}^2$ such that

$$\psi \circ \tau = \iota \circ \psi. \tag{3.1}$$

***Lemma 3.2*** *The involution $\iota$ is an isometry of $(\mathbb{D}^2, g^*)$.*

**Proof** Indeed, let the neighborhood $U \subset \mathbb{D}^2$ be small enough and do not contain branching points. Then $\psi^{-1}(U) = \sqcup_{i=1}^{d} U_i$ and applying $\tau$ one gets: $\tau(\psi^{-1}(U)) =$

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$\sqcup_{i=1}^{d}\tau(U_i)$. Note that condition (3.1) implies $\tau(\psi^{-1}(U)) = \psi^{-1}(\iota(U))$. Whence $\psi^{-1}(\iota(U)) = \sqcup_{i=1}^{d}\tau(U_i)$. Let $\widetilde{\psi}_i := \psi_{\tau(U_i)}$. Then on $U$, one has

$$g^* = \sum_{i=1}^{d}(\widetilde{\psi}_i^{-1})^* g = \sum_{i=1}^{d}(\widetilde{\psi}_i^{-1})^*\tau^* g = \sum_{i=1}^{d}(\widetilde{\psi}_i^{-1}\circ\tau)^* g$$
$$= \sum_{i=1}^{d}(\iota\circ\widetilde{\psi}_i^{-1})^* g = \sum_{i=1}^{d}\iota^*(\widetilde{\psi}_i^{-1})^* g = \iota^* g^*. \qquad \blacksquare$$

Consider a $j$th $\iota$-even eigenfunction $u_j$ on $(\mathbb{D}^2, g^*)$ with corresponding eigenvalue $\sigma_j^\iota(\mathbb{D}^2, g^*)$. Then the function $\psi^* u_j$ on $\widetilde{\Sigma}$ is $\tau$-even and hence it projects to a well-defined function $v_j$ on $\Sigma$. We can construct the following function $v = \sum_{j=0}^{k-1} c_j v_j$. Note that $\pi^* v = \sum_{j=0}^{k-1} c_j \psi^* u_j = \psi^* u$, where $u := \sum_{j=0}^{k-1} c_j u_j$. Further, let $w_i$ denote an $i$th eigenfunction on $\Sigma$ with eigenvalue $\sigma_i(\Sigma, h)$. It is easy to see that one can always find some coefficients $c_0, \dots, c_{k-1}$ such that $\int_{\partial\Sigma} v w_i dv_h = 0$, $i = 0, \dots, k-1$. Then, we can use $v$ as a test function for $\sigma_k(\Sigma, h)$:

$$\sigma_k(\Sigma, h) \le \frac{\int_\Sigma |\nabla_h v|^2 dv_h}{\int_{\partial\Sigma} v^2 dv_h} = \frac{\int_{\widetilde{\Sigma}} |\nabla_g \psi^* u|^2 dv_g}{\int_{\partial\widetilde{\Sigma}} (\psi^* u)^2 dv_g} = d\frac{\int_{\mathbb{D}^2} |\nabla_{g^*} u|^2 dv_{g^*}}{\int_{\mathbb{S}^1} u^2 dv_{g^*}} = d\sigma_k^\iota(\mathbb{D}^2, g^*),$$

where we used Lemma 3.1. Moreover, the second identity in Lemma 3.1 implies $L_{g^*}(\mathbb{S}^1) = L_g(\partial\widetilde{\Sigma}) = 2L_h(\partial\Sigma)$. Whence

$$\bar{\sigma}_k(\Sigma, h) \le \frac{d}{2}\sigma_k^\iota(\mathbb{D}^2, g^*) L_{g^*}(\mathbb{S}^1). \tag{3.2}$$

Consider a conformal map $\psi$ between surfaces with involution $\psi: (\widetilde{\Sigma}, \tau) \to (\mathbb{D}^2, \iota)$ of minimal degree $d$. The map $\psi$ is conformal, moreover, every involution exchanging the orientation on $\mathbb{D}^2$ is conjugate to the involution $\iota_0(z) := \bar{z}$, where we identify $\mathbb{D}^2$ with the unit disc on the complex plane. Therefore, without loss of generality, we can assume that $\iota = \iota_0$. The fixed point set of $\iota_0$ is the diameter $\{z \in \mathbb{D}^2 \mid Re(z) = 0\}$. Let $H\mathbb{D}^2$ denote a half-disc for example the right one and $\partial^S H\mathbb{D}^2$ is the right half-circle. Thus, $\sigma_k^{\iota_0}(\mathbb{D}^2, g^*) = \sigma_k^N(H\mathbb{D}^2, \partial^S H\mathbb{D}^2, g^*)$ and inequality (3.2) implies:

$$\begin{aligned}\bar{\sigma}_k(\Sigma, h) &\le \frac{d}{2}\sigma_k^\iota(\mathbb{D}^2, g^*) L_{g^*}(\mathbb{S}^1) = d\bar{\sigma}_k^N(H\mathbb{D}^2, \partial^S H\mathbb{D}^2, g^*)\\ &\le d\sigma_k^{N*}(H\mathbb{D}^2, \partial^S H\mathbb{D}^2, [g^*]) \le d\sigma_k^*(\mathbb{D}^2, [g_{can}]) = 2\pi k d,\end{aligned} \tag{3.3}$$

where in the last inequality, we used Lemma 2.6 and the fact that there exists a unique up to an isometry conformal class $[g_{can}]$ on $\mathbb{D}^2$. We want to estimate $d$ in formula (3.3). It is known that there exists a proper conformal branched cover $f: (\widetilde{\Sigma}, g) \to (\mathbb{D}^2, g_{can})$ of degree $d' \le \gamma + 2l$ (see [Gab06]). One can construct the following map $F(x) := f(x)\bar{f}(\tau(x))$. Note that $\bar{F}(x) = F(\tau(x)) = \iota(F(x))$ and hence $\iota = \iota_0$. Moreover, $F$ is proper and the degree of $F$ is not greater than $2d' = 2(\gamma + 2l)$. Hence, there exists a proper map between $(\widetilde{\Sigma}, \tau)$ and $(\mathbb{D}^2, \iota_0)$ of degree not exceeding $2d' = 2(\gamma + 2l)$ satisfying (3.1). Inequality (3.3) then implies

$$\bar{\sigma}_k(\Sigma, h) \le 4\pi k(\gamma + 2l).$$

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## 4 Geometric background

The aim of this section is the proof of Theorem 1.4. For this purpose, we provide a necessary background concerning the geometry of moduli space of conformal classes on a surface with boundary. We start with closed orientable surfaces.

### 4.1 Closed orientable surfaces

Let us recall the *Uniformization theorem*.

***Theorem 4.1*** *Let $\Sigma$ be a closed surface and $g$ be a Riemannian metric on it. Then in the conformal class $[g]$, there exists a unique (up to an isometry) metric $h$ of constant Gauss curvature and fixed area. The area assumption is unnecessary except in the case of the torus for which we fix the volume of $h$ to be equal to* 1*.*

***Remark 4.1*** It follows from the Gauss–Bonnet theorem that the metric $h$ in the Uniformization theorem is of Gauss curvature 1 in the case of the sphere, 0 in the case of the torus and −1 in the rest cases.

Recall that a Riemannian metric $h$ of constant Gaussian curvature −1 is called *hyperbolic* and a Riemannian surface $(\Sigma, h)$ endowed with a hyperbolic metric $h$ is called *a hyperbolic surface*. Note also that a hyperbolic surface is necessarily of negative Euler characteristic. We also say that the torus endowed with a metric of curvature $h = 0$ is a flat torus and the sphere endowed with the metric $h = 1$ is the standard (round) sphere.

### 4.2 Hyperbolic surfaces

We recall that a *pair of pants* is a compact surface of genus 0 with 3 boundary components. The following theorem plays an underlying role in the theory of hyperbolic surfaces.

***Theorem 4.2*** (Collar theorem (see e.g., [Bus92])) *Let $(\Sigma, h)$ be an orientable compact hyperbolic surface of genus $\gamma \geq 2$ and let $c_1, c_2, \dots, c_m$ be pairwise disjoint simple closed geodesics on $(\Sigma, h)$. Then the following holds*

(i) $m \leq 3\gamma - 3$.

(ii) *There exist simple closed geodesics $c_{m+1}, \dots, c_{3\gamma-3}$ which, together with $c_1, \dots, c_m$, decompose $\Sigma$ into pairs of pants.*

(iii) *The collars*

$$\mathcal{C}(c_i) = \left\{p \in \Sigma \mid dist(p, c_i) \leq w(c_i)\right\}$$

*of widths*

$$w(c_i) = \frac{\pi}{l(c_i)}\left(\pi - 2\arctan\left(\sinh\frac{l(c_i)}{2}\right)\right)$$

*are pairwise disjoint for $i = 1, \dots, 3\gamma - 3$.*

(iv) *Each $\mathcal{C}(c_i)$ is isometric to the cylinder*

$$\left\{(t, \theta) | -w(c_i) < t < w(c_i),\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}$$

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*with the Riemannian metric*

$$\left(\frac{l(c_i)}{2\pi\cos\left(\frac{l(c_i)}{2\pi}t\right)}\right)^2\left(dt^2+d\theta^2\right).$$

The decomposition of $(\Sigma, h)$ into pair of pants which we denote by $\mathcal{P}$ is called *the pants decomposition*. We also say that the geodesics $c_1,\dots,c_{3\gamma-3}$ form $\mathcal{P}$.

### 4.3 Convergence of hyperbolic metrics

We endow the set of hyperbolic metrics on a given surface $\Sigma$ with $C^\infty$–topology. In this section, we describe the convergence on this topological set which is called *the moduli space of conformal classes* on $\Sigma$. Essentially, two cases can happen: the injectivity radii of a sequence of hyperbolic metrics do not go to 0 or they do. The first case is described by *Mumford's compactness theorem* and the second one is treated by *the Deligne–Mumford compactification*.

***Proposition 4.3*** (Mumford's compactness theorem (see e.g., [Hum97])) *Let $\{h_n\}$ be a sequence of hyperbolic metrics on a surface $\Sigma$ of genus $\geq 2$. Assume that the injectivity radii* $\operatorname{inj}(\Sigma, h_n)$ *satisfy* $\limsup_{n\to\infty}\operatorname{inj}(\Sigma,h_n)>0$. *Then there exists a subsequence $\{h_{n_k}\}$, sequence $\{\Phi_k\}$ of smooth automorphisms of $\Sigma$ and a hyperbolic metric $h_\infty$ on $\Sigma$ such that the sequence of hyperbolic metrics $\{\Phi_k^* h_{n_k}\}$ converges in $C^\infty$-topology to $h_\infty$.*

If $\lim_{n\to\infty}\operatorname{inj}(\Sigma,h_n)=0$ then we say that the sequence $\{h_n\}$ *degenerates*. The thick-thin decomposition implies that if the sequence $\{h_n\}$ degenerates then for each $n$ there exists a collection $\{c_1^n,\dots,c_s^n\}$ of disjoint simple closed geodesics in $(\Sigma,h_n)$ whose lengths tend to 0 and the length of any geodesic in the complement $\Sigma_n=\Sigma\backslash(c_1^n\cup\cdots\cup c_s^n)$ is bounded from below by a constant independent of $n$. We call the geodesics $\{c_1^n,\dots,c_s^n\}$ "pinching" or "collapsing." The surface $(\Sigma_n,h_n)$ is possibly a disconnected hyperbolic surface with geodesic boundary. Let $\widehat{\Sigma_\infty}$ denote the surface having the same connected components as $\Sigma_n$, but with boundary component replaced by marked points. Note that each sequence $\{c_i^n\}$ corresponds to a pair of marked points $\{p_i,q_i\}$ on $\widehat{\Sigma_\infty}$, $i=1,\dots,s$. Then the punctured surface $\widehat{\Sigma_\infty}\backslash\{p_1,q_1,\dots,p_s,q_s\}$ that we denote by $\Sigma_\infty$ admits the unique hyperbolic metric $h_\infty$ with cusps at punctures. Now we are ready to formulate one of the underlying results in the theory of *moduli spaces of Riemann surfaces*.

***Proposition 4.4*** (Deligne–Mumford compactification (see e.g., [Hum97])) *Let $(\Sigma,h_n)$ be a sequence of hyperbolic surfaces such that* $\operatorname{inj}(\Sigma,h_n)\to 0$. *Then up to a choice of subsequence, there exists a sequence of diffeomorphisms $\Psi_n:\Sigma_\infty\to\Sigma_n$ such that the sequence $\{\Psi_n^* h_n\}$ of hyperbolic metrics converges in $C^\infty_{\mathrm{loc}}$-topology to the complete hyperbolic metric $h_\infty$ on $\Sigma_\infty$. Furthermore, there exists a metric of locally constant curvature $\widehat{h_\infty}$ on $\widehat{\Sigma_\infty}$ such that its restriction to $\Sigma_\infty$ is conformal to $h_\infty$.*

We call $(\widehat{\Sigma_\infty},\widehat{h_\infty})$ a *limiting space* of the sequence $(\Sigma,h_n)$. We also say that the limit of conformal classes $[h_n]$ is the conformal class $[\widehat{h_\infty}]$ on $\widehat{\Sigma_\infty}$.

***Remark 4.2*** We emphasize that $\widehat{h_\infty}$ has *locally* constant curvature, since $\widehat{\Sigma_\infty}$ is possibly disconnected and different connected components could have different signs of Euler characteristic.

## 4.4 Orientable surfaces with boundary of negative Euler characteristic

Our exposition of this topic essentially follows the book [Jos07].

Let $\Sigma$ be an orientable surface of genus $\gamma$ with $l$ boundary components. Consider its *Schottky double* $\Sigma^d$ defined in following way. We identify $\Sigma$ with another copy $\Sigma'$ of $\Sigma$ with opposite orientation along the common boundary. We get a closed oriented surface of genus $2\gamma + l - 1$. For example, the Schottky double of the disk is the sphere and the Schottky double of the cylinder is the torus. In the rest cases we always get a hyperbolic surface as the Schottky double. We endow the surface $\Sigma$ with a metric $g$. The next theorem plays a role of the Uniformization theorem for surfaces with boundary.

***Proposition 4.5*** [OPS88] *In the conformal class* $[g]$ *of a metric* $g$ *on the surface* $\Sigma$*, there exists a unique (up to an isometry) metric of constant Gauss curvature and geodesic boundary. More precisely, this metric is of curvature* $1$ *in the case of* $\mathbb{D}^2$*, of the curvature* $0$ *in the case of the cylinder and of curvature* $-1$ *in the rest cases.*

Denote the metric of constant Gauss curvature and geodesic boundary from Theorem 4.5 by $h$. Consider a Riemannian surface with boundary $(\Sigma, h)$. Its Schottky double admits the metric $h^d$ defined as $h^d_{|_\Sigma} = h$ and $h^d_{|_{\Sigma'}} = h$. It is a metric of constant curvature and the involution $\iota : \Sigma^d \to \Sigma^d$ that interchanges $\Sigma$ and $\Sigma'$ becomes an isometry with $\partial\Sigma$ as the fixed set. Moreover, $(\Sigma, h_n) = (\Sigma^d, h^d_n)/\iota$.

Theorem 4.5 also says that the set of conformal classes on the surface $\Sigma$ with boundary is in one-to-one correspondence with the set of metrics of constant Gauss curvature and geodesic boundary which is in the one-to-one correspondence with the set of “symmetric” metrics (metrics that go to themselves under the involution $\iota$) of constant curvature on the Schottky double. We endow the set of metrics of constant Gauss curvature and geodesic boundary with $C^\infty$−topology. Consider a sequence of conformal classes $\{c_n\}$ on $\Sigma$. It uniquely defines a sequence of “symmetric” metrics of constant curvature $\{h^d_n\}$ on $\Sigma^d$. For this sequence, we have the same dichotomy as we have seen in the previous sections. Precisely, either $\mathrm{inj}(\Sigma^d, h^d_n) \nrightarrow 0$ or $\mathrm{inj}(\Sigma^d, h^d_n) \to 0$. In the first case we get a genuine Riemannian metric on $\Sigma^d$ which is obviously “symmetric” and of constant curvature while in the second case one can find a set of simple closed geodesics $\{c^n_1, \dots, c^n_s\}$, where $s \le 6\gamma + 3l - 6$ whose lengths $l_{h^d_n}(c^n_i) \to 0$. For the geodesics $c^n_i$ there exist two possibilities: either $\iota(c^n_i) = c^n_i$ or $\iota(c^n_i) = c^n_j$ with $j \neq i$. The first possibility implies that the geodesic $c^n_i$ crosses $\partial\Sigma$ which corresponds to two situations as well: either $c^n_i$ has exactly two points of intersection with $\partial\Sigma$ or it belongs to $\partial\Sigma$, i.e., it is one of the boundary components. The second possibility implies that $c^n_i$ does not crosse $\partial\Sigma$. Taking quotient by $\iota$, we then get three types of pinching geodesics on $(\Sigma, h_n)$ with $\mathrm{inj}(\Sigma, h_n) \to 0$: pinching boundary components, pinching simple geodesics which have exactly two points of intersection with the boundary and pinching simple closed geodesics which do not cross the boundary.

### 4.5 Nonorientable surface with boundary of negative Euler characteristic

Let $\Sigma$ be a compact nonorientable surface with $l$ boundary components. Note that the Uniformization Theorem 4.5 also holds for nonorientable surfaces. Pick a metric $h$ of constant Gauss curvature and geodesic boundary. We pass to the orientable cover that we denote by $\widetilde{\Sigma}$. The surface $\widetilde{\Sigma}$ is a compact orientable surface with $2l$ boundary components. The pull-back of the metric $h$ that we denote by $\tilde{h}$ is a metric of constant Gauss curvature and with geodesic boundary. Moreover, this metric is invariant under the involution changing the orientation on $\widetilde{\Sigma}$. Consider a sequence $\{h_n\}$ on $\Sigma$ of metrics of constant Gauss curvature and geodesic boundary such that $\mathrm{inj}(\Sigma, h_n) \to 0$ as $n \to \infty$. This sequence corresponds to the sequence $\{\tilde{h}_n\}$ on $\widetilde{\Sigma}$ such that $\mathrm{inj}(\widetilde{\Sigma}, \tilde{h}_n) \to 0$ as $n \to \infty$. As we discussed in the previous section for the sequence $\{\tilde{h}_n\}$, one can find pinching geodesics of the following three types: pinching boundary components, pinching simple geodesics crossing the boundary at two points and pinching simple closed geodesics which do not cross the boundary. Note that for the geodesics of the second type the points of intersection with the boundary are not identified under the involution. Indeed, if the were identified then the corresponding pinching geodesic had fixed ends under the involution. Applying the involution to this geodesic we would get a pinching *closed* geodesic crossing the boundary at two points which is not one of the possible types of pinching geodesics. Consider now the geodesics of the third type. For every such geodesic there are two possible cases: either this geodesic maps to itself under the involution changing the orientation or it maps to another simple closed geodesic which does not cross the boundary. Then taking the quotient by the involution changing the orientation we get two types of simple closed geodesics on $\Sigma$ which do not crosse the boundary: *one-sided geodesics* which are the images of the geodesics described in the first case and *two-sided geodesics* which are the images of the geodesics described in the second case. The collars of one-sided geodesics are nothing but Möbius bands while the collars of two-sided geodesics are cylinders. Therefore, if $\mathrm{inj}(\Sigma, h_n) \to 0$ as $n \to \infty$, then one can find pinching geodesics of the following types: pinching boundary components, pinching simple geodesics which have exactly two points of intersection with the boundary, one-sided pinching simple closed geodesics not crossing the boundary and two-sided pinching simple closed geodesics not crossing the boundary.

### 4.6 Surfaces with boundary of non-negative Euler characteristic

Here we consider the cases of the disc, the cylinder $\mathbb{C}$ and the Möbius band $\mathbb{MB}$.

It is known that the disc has a unique conformal class (up to an isometry). We denote this conformal class as $[g_{can}]$ or $c_{can}$, where $g_{can}$ is the flat metric on the disc $\mathbb{D}^2$ with unit boundary length.

Accordingly to Theorem 4.5 in a conformal class on $\mathbb{C}$ there exists a flat metric with geodesic boundary, i.e., a metric on the right circular cylinder. This metric is unique if we fix the length of the boundary. The right circular cylinder is uniquely determined by its height. Therefore, conformal classes on $\mathbb{C}$ are in one-to-one correspondence with heights of right circular cylinders, i.e., the set of conformal classes is $\mathbb{R}_{>0}$. We will identify conformal classes on $\mathcal{C}$ with points of $\mathbb{R}_{>0}$. We say that the sequence $\{c_n\}$ of conformal classes degenerates if either $c_n \to 0$ or $c_n \to \infty$. The case $c_n \to 0$ corresponds to a pinching geodesic having intersection with two boundary components (i.e., the generatrix of the right circular cylinder). The case $c_n \to \infty$ corresponds to pinching boundary components.

In the case of the Möbius band we also use Theorem 4.5 which implies that in every conformal class on $\mathbb{MB}$ there exists a flat metric with geodesic boundary which is unique if we fix the length of the boundary. Passing to the orientable cover and pulling back the flat metric from $\mathbb{MB}$ we get a flat cylinder with geodesic boundary. Then the discussion in the previous paragraph implies that the conformal classes on $\mathbb{MB}$ are also encoded by $\mathbb{R}_{>0}$. Identifying again conformal classes on $\mathbb{MB}$ with points of $\mathbb{R}_{>0}$, we get two possible cases for a sequence of conformal classes $\{c_n\}$: either $c_n \to 0$ or $c_n \to \infty$. In both cases, we say that the sequence $\{c_n\}$ degenerates. The first case corresponds to a pinching geodesic having two points of intersection with boundary. The second case corresponds to the collapsing boundary.

## 5 Proof of Theorem 1.4

**Negative Euler characteristic.** Let $\Sigma$ be a surface with boundary and $c_n \to c_\infty$ a degenerating sequence of conformal classes. Consider the corresponding sequence of metrics $h_n$ of constant Gauss curvature and geodesic boundary. Then as we have noticed in Section 4.4, one can find $s = s_1 + s_2 + s_3$ pinching geodesics of the following three types: $s_1$ pinching boundary components, $s_2$ pinching geodesics that have two points of intersection with boundary and $s_3$ pinching simple closed geodesics that do not intersect the boundary.

We introduce the following notations

- $\gamma_i^n$ for collapsing geodesics, $i = 1, \ldots, s$. If we do not indicate the superscript then the symbol $\gamma_i$ stands for the genus;
- $\mathcal{C}_i^n$ for collars of collapsing geodesics, $i = 1, \ldots, s$. Their widths are denoted by $w_i^n$. Moreover, $\mathcal{C}_i^n := \{(t,\theta)\,|\, 0 \le t < w_i^n,\ 0 \le \theta \le 2\pi\}$ for $1 \le i \le s_1$ and $\mathcal{C}_i^n := \{(t,\theta)\,|\, -w_i^n < t < w_i^n,\ 0 \le \theta \le 2\pi\}$ for $s_1 + 1 \le i \le s$ (if the geodesic is one-sided then we consider $\mathcal{C}_i^n := \{(t,\theta)\,|\, -w_i^n < t < w_i^n,\ 0 \le \theta \le 2\pi\}/\sim$, where $\sim$ stands for $(t,\theta) \sim (-t, \pi + \theta)$). Note that geodesics correspond to the line $\{t = 0\}$, the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ are identified for $1 \le i \le s_1$ and for $s_1 + s_2 + 1 \le i \le s$ and they are not identified for $s_1 + 1 \le i \le s_1 + s_2$ and correspond to the segments of intersection with the boundary;
- for $0 < a < w_i^n$, we denote $\mathcal{C}_i^n(0,a)$ the subset $\{(t,\theta)\,|\, 0 \le t \le a, 0 \le \theta \le 2\pi\} \subset \mathcal{C}_i^n$ for $1 \le i \le s_1$ and for $-w_i^n < a < b < w_i^n$, we denote $\mathcal{C}_i^n(a,b)$ the subset $\{(t,\theta)\,|\, a \le t \le b, 0 \le \theta \le 2\pi\} \subset \mathcal{C}_i^n$ for $s_1 + 1 \le i \le s$;
- $\Gamma_i^n := \{(t,\theta) \in \mathcal{C}_i^n \,|\, \theta = 0 \text{ or } \theta = 2\pi\}$ for $s_1 + 1 \le i \le s_1 + s_2$;
- for $-w_i^n < a < b < w_i^n$, we set $\Gamma_i^n(a,b) := \{(t,\theta) \in \Gamma_i^n \,|\, a \le t \le b\}$ for $s_1 + 1 \le i \le s_1 + s_2$;
- $\Sigma_j^n$ for the $j$th connected component of $\Sigma \setminus \cup_{i=1}^s \mathcal{C}_i^n$. We enumerate $\Sigma_j^n$ by $1 \le j \le M$ such that $M$ denotes the number of $\Sigma_j^n$ and for all $1 \le j \le m$ one has $\Sigma_j^n \cap \partial\Sigma \ne \varnothing$;

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- let $\alpha^n = \cup_{i=1}^{s_1+s_2}\{\alpha^n_{i,-}, \alpha^n_{i,+}\}$, where $0 \le \alpha^n_{i,\pm} < w^n_i$. We denote by $\Sigma^n_j(\alpha^n)$ the connected component of

$$\Sigma \backslash \left( \bigcup_{i=1}^{s_1+s_2} \mathcal{C}^n_i(\alpha^n_{i,-}, \alpha^n_{i,+}) \cup \bigcup_{i=s_1+s_2+1}^{s} \gamma^n_i \right)$$

which contains $\Sigma^n_j$;
- for $\alpha^n = \cup_{i=1}^{s_1+s_2}\{\alpha^n_{i,-}, \alpha^n_{i,+}\}$, where $0 \le \alpha^n_{i,\pm} < w^n_i$ we set $I^n_j(\alpha^n) = \Sigma^n_j(\alpha^n) \cap \partial\Sigma$ and $I^n_j = \Sigma^n_j \cap \partial\Sigma$ where $1 \le j \le m$;
- we use the notation $a_n \ll b_n$ for two sequences $\{a_n\}$ and $\{b_n\}$ satisfying $a_n, b_n \to +\infty$ and $\frac{a_n}{b_n} \to 0$ as $n \to \infty$.

## 5.1 Inequality $\geq$

We prove that

$$\liminf_{n\to\infty} \sigma^*_k(\Sigma, c_n) \ge \max\left( \sum_{i=1}^{m} \sigma^*_{k_i}(\Sigma_{\gamma_i, l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma^*_{r_i}(\mathbb{D}^2) \right), \tag{5.1}$$

For this aim we consider the domains $\mathcal{C}^n_i(0, \alpha^n_{i,+})$ for $1 \le i \le s_1$, $\mathcal{C}^n_i(\alpha^n_{i,-}, \alpha^n_{i,+})$ for $1 + s_1 \le i \le s_1 + s_2$, where $w^n_i - \alpha^n_{i,\pm} \ll w^n_i$, $\alpha^n_{i,\pm} \to \infty$ and the domains $\Sigma^n_j(\alpha^n)$ for $1 \le j \le m$. By Lemma 2.9, we have

(5.2)

$$\sigma^*_k(\Sigma, c_n) \ge \max\Bigg( \sum_{i=1}^{s_1} \sigma^{N*}_{r_i}(\mathcal{C}^n_i(0, \alpha^n_{i,+}), \gamma^n_i, c_n) + \sum_{i=1+s_1}^{s_1+s_2} \sigma^{N*}_{r_i}(\mathcal{C}^n_i(\alpha^n_{i,-}, \alpha^n_{i,+}), \Gamma^n_i(\alpha^n_{i,-}, \alpha^n_{i,+}), c_n) + \sum_{j=1}^{m} \sigma^{N*}_{k_j}(\Sigma^n_j(\alpha^n), I^n_j(\alpha^n), c_n) \Bigg).$$

For $1 \le i \le s_1$, we define the conformal maps $\Psi^n_i\colon (\mathcal{C}^n_i(0, \alpha^n_{i,+}), c_n) \to (\mathbb{D}^2, [g_{can}])$ as

$$\Psi^n_i(t, \theta) = e^{\sqrt{-1}(\theta + \sqrt{-1}t)}.$$

The images of $\Psi^n_i$ are the annuli $\mathbb{D}^2 \backslash \mathbb{D}^2_{e^{-\alpha^n_{i,+}}}$ exhausting $\mathbb{D}^2$ as $n \to \infty$. We also note that $\Psi^n_i(\gamma^n_i) = \mathbb{S}^1$.

For $s_1 + 1 \le i \le s_1 + s_2$, we define the conformal maps $\Psi^n_i\colon (\mathcal{C}^n_i(\alpha^n_{i,-}, \alpha^n_{i,+}), c_n) \to (\mathbb{D}^2, [g_{can}])$ as

$$\Psi^n_i(t, \theta) = \tan\left( \frac{\theta - \pi + \sqrt{-1}t}{4} \right).$$

The images of $\Psi^n_i$ that we denote by $\Omega^n_i$ exhaust $\mathbb{D}^2$ as $n \to \infty$. We also denote the image of $\Gamma^n_i(\alpha^n_{i,-}, \alpha^n_{i,+})$ by $\partial^S\Omega^n_i$. Note that $\partial^S\Omega^n_i$ exhaust $\mathbb{S}^1$ as $n \to \infty$.

Finally, we take restrictions of the diffeomorphisms $\Psi^{-1}_n$ given by Proposition 4.4 to obtain the conformal maps $\check{\Psi}^n_j\colon (\Sigma^n_j(\alpha^n), c_n) \to (\Sigma_\infty, \Psi^*_n c_n)$ where $1 \le j \le m$. Let $\check{\Omega}^n_j \subset \Sigma_\infty$ be the the image of $\check{\Psi}^n_j$ and $\partial^S\check{\Omega}^n_j := \check{\Psi}^n_j(I^n_j(\alpha^n))$. The following lemma holds

***Lemma 5.1*** *Let $\Sigma_j^\infty$ be the connected component $\check\Psi_j^n(\Sigma_j^n) \subset \Sigma_\infty$ where $1 \le j \le m$. Then the domains $\check\Omega_j^n$ exhaust $\Sigma_j^\infty$ and $\partial^S\check\Omega_j^n$ exhaust $\partial\Sigma_j^\infty$.*

**Proof** Passing to the Schottky double of the surface $\Sigma$, we immediately deduce this lemma from [KM20, Lemma 5.1]. ∎

Further, we apply the conformal transformations to (5.2) to get

$$\sigma_k^*(\Sigma, c_n) \ge \max\Bigg(\sum_{i=1}^{s_1}\sigma_{r_i}^{N*}(\mathbb{D}^2\backslash\mathbb{D}^2_{e^{-\alpha_{i,+}^n}}, \mathbb{S}^1, [g_{can}]) + \sum_{i=1+s_1}^{s_1+s_2}\sigma_{r_i}^{N*}(\Omega_i^n, \partial^S\Omega_i^n, [g_{can}]) + \sum_{j=1}^{m}\sigma_{k_j}^{N*}(\check\Omega_j^n, \partial^S\check\Omega_j^n, [(\Psi^n)^*h_n])\Bigg). \tag{5.3}$$

It follows from Corollary 2.7 that the first two terms on the right hand side converge to $\sigma_{r_i}(\mathbb{D}^2, [g_{can}])$. To complete the proof we will need the following lemma

***Lemma 5.2*** *Let $\widehat{\Sigma_j^\infty} \subset \widehat{\Sigma_\infty}$ be a closure of $\Sigma_j^\infty$, $1 \le j \le m$. Then for all r one has*

$$\liminf_{n\to\infty}\sigma_r^{N*}(\check\Omega_j^n, \partial^S\check\Omega_j^n, [(\Psi^n)^*h_n]) \ge \sigma_r^*(\widehat{\Sigma_j^\infty}, [\widehat{h_\infty}]).$$

We postpone the proof to Section 7.3.

Finally, taking $\liminf_{n\to\infty}$ in (5.3) completes the proof of (5.1).

## 5.2 Inequality ≤

We prove the inverse inequality,

$$\limsup_{n\to\infty}\sigma_k^*(\Sigma, c_n) \le \max\Bigg(\sum_{i=1}^{m}\sigma_{k_i}^*(\Sigma_{\gamma_i, l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2}\sigma_{r_i}^*(\mathbb{D}^2)\Bigg). \tag{5.4}$$

For this aim we choose a subsequence $c_{n_m}$ such that

$$\lim_{n_m\to\infty}\sigma_k^*(\Sigma, c_{n_m}) = \limsup_{n\to\infty}\sigma_k^*(\Sigma, c_n).$$

Then we relabel the subsequence and denote it by $\{c_n\}$. Therefore, one can choose subsequences without changing the value of lim sup.

**Case 1.** Suppose that up to a choice of a subsequence the following inequality holds

$$\sigma_k^*(\Sigma, c_n) > \sigma_{k-1}^*(\Sigma, c_n) + 2\pi.$$

Then by [Pet19, Theorem 2] in the conformal class $c_n$ there exists a metric $g_n$ of unit boundary length induced from a harmonic immersion with free boundary $\Phi_n$ to some $N(n)$-dimensional ball $\mathbb{B}^{N(n)}$, i.e.,

$$g_n = \frac{\langle\Phi_n, \partial_{\nu_n}\Phi_n\rangle_{h_n}}{\sigma_k^*(\Sigma, c_n)}h_n$$

and such that $\sigma_k(g_n) = \sigma_k^*(\Sigma, c_n)$. Here, the metric $h_n$ is the canonical representative in the conformal class $c_n$. It is known that for any compact surface the multiplicity of $\sigma_k(g_n)$ is bounded from above by a constant depending only on $k$ and the topology of

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$\Sigma$ (see for instance [FS12, KKP14]). Therefore, one can choose the number $N(n)$ large enough such that $N(n)$ does not depend on $n$.

Assume that for the sequence $\{c_n\}$ the following inequality holds

$$\limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n) > \max\left( \sum_{i=1}^{m} \sigma_{k_i}^*(\Sigma_{\gamma_i, l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma_{r_i}^*(\mathbb{D}^2) \right). \tag{5.5}$$

For $1 \le i \le s_1$ we consider the conformal map $\Psi_i^n : (\mathcal{C}_i^n, c_n) \to (\mathbb{D}^2, [g_{can}])$ defined as $\Psi_i^n(\theta, t) = e^{\sqrt{-1}(\theta + \sqrt{-1}t)}$. The image of this map is nothing but $\mathbb{D}^2 \backslash \mathbb{D}^2_{e^{-w_i^n}}$ which exhausts $\mathbb{D}^2$ as $n \to \infty$. The image of a pinching geodesic is $\mathbb{S}^1$. Then the map $\Phi_i^n := \Phi_n \circ (\Psi_i^n)^{-1} : \mathbb{D}^2 \backslash \mathbb{D}^2_{e^{-w_i^n}} \to \mathbb{B}^N$ satisfies the *bubble convergence theorem for harmonic maps with free boundary* [LP17, Theorem 1]. Hence, there exist a regular harmonic map with free boundary $\Phi_i : \mathbb{D}^2 \to \mathbb{B}^N$ and some harmonic extensions of nonconstant 1/2-harmonic maps $\omega_1^i, \dots, \omega_{t_i}^i : \mathbb{D}^2 \to \mathbb{B}^N$ such that

$$\int_{\mathbb{D}^2} |\nabla \Phi_i|^2 dv_{g_{can}} + \sum_{j=1}^{t_j} \int_{\mathbb{D}^2} |\nabla \omega_{t_i}^j|^2 dv_{g_{can}} = \lim_{n\to\infty} \int_{\gamma_i^n} ds_{g_n}.$$

We denote $\lim_{n\to\infty} \int_{\gamma_i^n} ds_{g_n}$ by $m_i$.

***Proposition 5.3*** *For $s_1 + 1 \le i \le s_1 + s_2$ there exist integers $t_i \ge 0$, non-negative sequences $\{a_{i,l}^n\}, \{b_{i,l}^n\}$ with $1 \le l \le t_i$ and a sequence $\{\alpha_i^n\}$ such that*

$$-w_i^n \ll \alpha_{i,-}^n = b_{i,0}^n \ll a_{i,1}^n \ll b_{i,1}^n \ll \cdots \ll a_{i,t_i}^n \ll b_{i,t_i+1}^n \ll a_{i,t_{i+1}}^n = \alpha_{i,+}^n \ll w_i^n$$

*and*

$$m_{i,l} = \lim_{n\to\infty} L_{g_n}(\Gamma_i^n(a_{i,l}^n, b_{i,l}^n)) > 0.$$

*Moreover, there exists a set $J \subset \{1, \dots, m\}$ such that for every $j \in J$ one has*

$$m_j = \lim_{n\to\infty} L_{g_n}(I_j^n(\alpha^n)) > 0$$

*satisfying*

$$\sum_{i=1}^{s_1} m_i + \sum_{i=1}^{s_1+s_2} \sum_{l=s_1+1}^{t_i} m_{i,l} + \sum_{j\in J} m_j = 1,$$

*with $s_1 + \sum_{i=s_1+1}^{s_1+s_2} t_i$ is maximal.*

**Proof** The proof follows the proofs of Claim 16, Claim 17 by [Pet19]. Precisely, denying the proposition one can construct $k+1$ test-functions such that $\sigma_k(g_n) \le o(1)$ which contradicts inequality (1.2). The construction of these functions is given in the proofs of Claim 16, Claim 17 by [Pet19]. Note that these functions equal 1 on $\Sigma_j^n$ for every $m + 1 \le j \le M$. ∎

We proceed with considering a sequence $\{d_{i,l}^n\}$ where $s_1 + 1 \le i \le s_1 + s_2$ and $1 \le l \le t_i$ such that

$$\lim_{n\to\infty} L_{g_n}(\Gamma_i^n(a_{i,l}^n, d_{i,l}^n)) = \lim_{n\to\infty} L_{g_n}(\Gamma_i^n(d_{i,l}^n, b_{i,l}^n)) = m_{i,l}/2.$$

Let $q_{i,l}^n \ll a_{i,l}^n$, $q_{i,l}^n \to +\infty$. Consider the conformal maps

$$\Psi_{i,l}^n \colon \left(\mathcal{C}_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n), c_n\right) \to (\mathbb{D}^2, [g_{can}])$$

defined as

$$\Psi_{i,l}^n(t,\theta) = \tan\left(\frac{\theta - \pi + \sqrt{-1}(t - t_{i,l}^n)}{4}\right)$$

Let

$$D_{i,j}^n = \Psi_{i,l}^n\left(\mathcal{C}_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n)\right)$$

and

$$S_{i,j}^n = \Psi_{i,l}^n\left(\Gamma_i^n(a_{i,l}^n - q_{i,l}^n, b_{i,l}^n + q_{i,l}^n)\right)$$

Then $D_{i,j}^n$ exhausts $\mathbb{D}^2$ and $S_{i,j}^n$ exhausts $\mathbb{S}^1$ as $n \to \infty$. We also set

$$\lim_{n\to\infty} L_{(\Psi_{i,l}^n)_* g_n}(S_{i,j}^n) = m_{i,l}.$$

Consider the map $\Phi_{i,l}^n = \Phi_n \circ (\Psi_{i,l}^n)^{-1} \colon (D_{i,j}^n, S_{i,j}^n) \to (\mathbb{B}^N, \mathbb{S}^{N-1})$. We endow $D_{i,j}^n$ with the metric $(\Psi_{i,l}^n)_* g_n$ and $\mathbb{B}^N$ with the Euclidean metric. Then the map $\Phi_{i,l}^n$ is harmonic with free boundary since $\Phi_n$ is harmonic with free boundary and $\Psi_{i,l}^n$ is conformal. Moreover, it is shown in [Pet19] that the measure $\mathbf{1}_{S_{i,j}^n}\langle \Phi_{i,l}^n, \partial_v \Phi_{i,l}^n\rangle_{g_{can}} ds_{g_{can}}$ does not concentrate at the poles $(0,1)$ and $(0,-1)$ of $\mathbb{D}^2$. Indeed, if the measure concentrated at the poles then one would obtain a contradiction with the maximality of $s_1 + \sum_{i=s_1+1}^{s_1+s_2} t_i$.

The exactly same procedure can be carried out for components $\Sigma_j^n(\alpha^n)$, $j \in J$. The only difference is that now we use restrictions of diffeomorphisms $\Psi^n$ given by Proposition 4.4 instead of the explicit harmonic map as above. As a result, one obtains domains $\check{\Omega}_j^n \subset \Sigma_\infty$ and harmonic maps with free boundary $\check{\Phi}_j^n \colon \check{\Omega}_j^n \to \mathbb{B}^N$ such that the measure $\mathbf{1}_{\partial\check{\Omega}_j^n}\langle \Phi_{i,l}^n, \partial_v \Phi_{i,l}^n\rangle_{g_{can}} ds_{g_{can}}$ does not concentrate at the marked points of $\widehat{\Sigma_\infty}$.

Now thanks to inequality (5.5), we can construct $k+1$ well-defined test-functions for the Rayleigh quotient of $\sigma_k$ using the limit functions of the sequences of maps $\hat{\Phi}_{i,l}^n$ and $\hat{\Phi}_i^n$ as it was shown in [Pet19]. Precisely, let $p_i$ be the maximal integers such that

$$\frac{\sigma_{p_i}^*(\mathbb{D}^2)}{m_i} < \limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n), \tag{5.6}$$

where $1 \le i \le s_1$, $p_{i,l}$ the maximal integers such that

$$\frac{\sigma_{p_{i,l}}^*(\mathbb{D}^2)}{m_{i,l}} < \limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n), \tag{5.7}$$

where $s_1 + 1 \le i \le s_1 + s_2$ and $p_j$ the maximal integers such that

$$\frac{\sigma_{p_j}^*(\widehat{\Sigma_j^\infty}, \widehat{c_\infty})}{m_j} < \limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n),\ j \in J. \tag{5.8}$$

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Then one has

$$\sigma^*_{p_i+1}(\mathbb{D}^2) \geq m_i \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ 1 \leq i \leq s_1,$$
$$\sigma^*_{p_{i,l}+1}(\mathbb{D}^2) \geq m_{i,l} \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ s_1+1 \leq i \leq s_1+s_2$$

and

$$\sigma^*_{p_j+1}(\widehat{\Sigma^\infty_j}, \widehat{c_\infty}) \geq m_j \limsup_{n\to\infty} \sigma^*_k(\Sigma, c_n),\ j \in J.$$

If $\sum_{i=1}^{s_1}(p_i+1) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i}(p_{i,l}+1) + \sum_{j\in J}(p_j+1) \leq k$ then by inequality (5.5) we have

$$\sum_{i=1}^{s_1}\sigma^*_{p_i+1}(\mathbb{D}^2) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i}\sigma^*_{p_{i,l}+1}(\mathbb{D}^2) + \sum_{j\in J}\sigma^*_{p_j+1}(\widehat{\Sigma^\infty_j}, \widehat{c_\infty}) < \limsup_{n\to\infty}\sigma^*_k(\Sigma, c_n),$$

which implies $\sum_{i=1}^{s_1} m_i + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i} m_{i,l} + \sum_{j\in J} m_j < 1$ and we arrive at a contradiction with Proposition 5.3. Hence, $\sum_{i=1}^{s_1}(p_i+1) + \sum_{i=s_1+1}^{s_1+s_2}\sum_{l=1}^{t_i}(p_{i,l}+1) + \sum_{j\in J}(p_j+1) \geq k+1$.

Further, let $dv_{g^i_\infty} = \lim_{n\to\infty}(\Psi^n_i)_* dv_{g_n}$, $dv_{g^{i,l}_\infty} = \lim_{n\to\infty}(\Psi^n_{i,l})_* dv_{g_n}$ and $dv_{g^j_\infty} = \lim_{n\to\infty}(\Psi^n_j)^* dv_{g_n}$. Denote by $\widehat{dv_{g^i_\infty}}$, $\widehat{dv_{g^{i,l}_\infty}}$ and $\widehat{dv_{g^j_\infty}}$ the measures induced by the compactification on $\mathbb{D}^2$ for $1 \leq i \leq s_1$ and $s_1+1 \leq i \leq s_1+s_2$ and on $\widehat{\Sigma^\infty_j}$, respectively. These measures are well-defined due to the nonconcentration argument explained above. Take orthonormal families of eigenfucntions $(\phi^0_i, \dots, \phi^{p_i}_i)$ in $L^2(\mathbb{D}^2, \widehat{dv_{g^i_\infty}})$ $1 \leq i \leq s_1$, $(\phi^0_i, \dots, \phi^{p_{i,l}}_i)$ in $L^2(\mathbb{D}^2, \widehat{dv_{g^{i,l}_\infty}})$ $s_1+1 \leq i \leq s_1+s_2$ and $(\psi^0_j, \dots, \psi^{p_j}_j)$ in $L^2(\widehat{\Sigma^\infty_j}, \widehat{dv_{g^j_\infty}})$ such that for $0 \leq e \leq p_i$ the function $\phi^e_i$ is an eigenfunction with eigenvalue $\sigma_e(\widehat{dv_{g^i_\infty}})$ on $\mathbb{D}^2$, for $0 \leq e \leq p_{i,l}$ the function $\phi^e_i$ is an eigenfunction with eigenvalue $\sigma_e(\widehat{dv_{g^{i,l}_\infty}})$ on $\mathbb{D}^2$ and for $0 \leq r \leq p_j$ the function $\psi^r_j$ is an eigenfunction with eigenvalue $\sigma_r(\widehat{dv_{g^j_\infty}})$ on $\widehat{\Sigma^\infty_j}$. The standard capacity computations (see for instance [Pet19, Claim 1]) imply the existence of smooth functions supported in a geodesic ball of a Riemannian manifold and having bounded Dirichlet energy. More precisely, there exist positive smooth functions $\eta_i$, $\eta_{i,l}$, and $\eta_j$ for $(\mathbb{D}^2, \widehat{dv_{g^i_\infty}})$, $(\mathbb{D}^2, \widehat{dv_{g^{i,l}_\infty}})$, and $(\widehat{\Sigma^\infty_j}, \widehat{dv_{g^j_\infty}})$, respectively supported in geodesic balls $B(x,r)$ centered at the compactification points $x$ of radius $r$ such that $\eta \in C^\infty_0(B(x,r))$ and $\eta = 1$ on $B(x, \rho_n r) \subset B(x,r)$, where $\rho_n \to 0$ as $n \to \infty$ and $\int_\Omega |\nabla\eta|^2_g dv_g \leq \frac{C}{\log\frac{1}{\rho_n}}$, where $\eta$ is one of the functions $\eta_i$, $\eta_{i,l}$ and $\eta_j$, $(\Omega, dv_g)$ is one of the corresponding manifolds $(\mathbb{D}^2, \widehat{dv_{g^i_\infty}})$, $(\mathbb{D}^2, \widehat{dv_{g^{i,l}_\infty}})$ and $(\widehat{\Sigma^\infty_j}, \widehat{dv_{g^j_\infty}})$. Moreover, if $(\Omega, dv_g) = (\mathbb{D}^2, \widehat{dv_{g^{i,l}_\infty}})$ then we additionally require $\rho_n$ to satisfy $\partial D^n_{i,l} \backslash S^n_{i,l} \subset B(x, \rho_n r)$. Then, we define the desired test-functions as

$$\xi^e_i = (\Psi^n_i)^{-1}\eta_i\phi^e_i,\ 1 \leq i \leq s_1$$

extended by 0 on $\Sigma$,

$$\xi^e_{i,l} = (\Psi^n_{i,l})^{-1}\eta_{i,l}\phi^e_i,\ s_1+1 \leq i \leq s_1+s_2$$

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extended by 0 on $\Sigma$ and

$$\xi_j^r = \Psi_j^n \eta_j \psi_j^r, \ j \in J$$

extended by 0 on $\Sigma$. Note that all these functions have pairwise disjoint supports. Then from the variational characterization of $\sigma_k(g_n)$ one gets

$$\sigma_k(g_n) \le \max\left\{ \max_{1\le i\le s_1} \frac{\int_\Sigma |\nabla \xi_i^e|^2_{g_n} dv_{g_n}}{\int_{\partial\Sigma}(\xi_i^e)^2 ds_{g_n}}, \max_{s_1+1\le i\le s_1+s_2} \frac{\int_\Sigma |\nabla \xi_{i,l}^e|^2_{g_n} dv_{g_n}}{\int_{\partial\Sigma}(\xi_{i,l}^e)^2 ds_{g_n}}, \max_{j\in J} \frac{\int_\Sigma |\nabla \xi_j^r|^2_{g_n} dv_{g_n}}{\int_{\partial\Sigma}(\xi_j^r)^2 ds_{g_n}} \right\},$$

and passing to lim sup as $n \to \infty$, we get

$$\limsup_{n\to\infty} \sigma_k^*(\Sigma, c_n) \le \max\left\{ \max_{1\le i\le s_1} \frac{\sigma_{p_i}^*(\mathbb{D}^2)}{m_i}, \max_{s_1+1\le i\le s_1+s_2} \frac{\sigma_{p_{i,l}}^*(\mathbb{D}^2)}{m_{i,l}}, \max_{j\in J} \frac{\sigma_{p_j}^*(\widehat{\Sigma_j^\infty}, \widehat{c_\infty})}{m_j} \right\}$$

which contradicts (5.6), (5.7), and (5.8). This means that if inequality (5.5) holds then the sequence $\{c_n\}$ cannot degenerate. We arrived at a contradiction and inequality (5.4) is proved.

***Remark 5.1*** Note that if $s_2 = 0$, i.e., there are no pinching geodesics having intersection with boundary components, then we take the set $J$ as $J = \{1, \dots, m\}$, i.e., we consider $\Sigma_j^n(\alpha^n)$, where $1 \le j \le m$. If all the boundary components are getting pinched then we set $J = \emptyset$ and we only have deal with the functions $\xi_i^e = (\Psi_i^n)^{-1}\eta_i \phi_i^e$ extended by 0 on $\Sigma$ and $\sigma_{p_i}^*(\mathbb{D}^2)$ where $1 \le i \le s_1$. If $s_1 = s_2 = 0$, i.e., only geodesics of the third type are getting pinched then we only have deal with functions $\xi_j^r = \Psi_j^n \eta_j \psi_j^r, \ j \in J$ extended by 0 on $\Sigma$ and $\sigma_{p_j}^*(\widehat{\Sigma_j^\infty}, \widehat{c_\infty})$ where $J = \{1, \dots, m\}$.

**Case 2.** Assume that up to a choice of a subsequence the following inequality holds

$$\sigma_k^*(\Sigma, c_n) \le \sigma_{k-1}^*(\Sigma, c_n) + 2\pi$$

then we prove inequality (5.4) by induction.

Consider the case $k = 1$ then by inequality (1.2) $\sigma_1^*(\Sigma, c_n) \ge 2\pi$. Suppose that up to a choice of a subsequence one has $\sigma_1^*(\Sigma, c_n) > 2\pi$. Then the case $k = 1$ falls under Case 1. Otherwise one has $\limsup_{n\to\infty} \sigma_1^*(\Sigma, c_n) = 2\pi$ and the inequality (5.4) reads as

$$2\pi = \limsup_{n\to\infty} \sigma_1^*(\Sigma, c_n) \le \max\{\sigma_1^*(\Sigma_{\gamma_i, l_i}, c_\infty); 2\pi\},$$

which is true. The base of induction is proved.

Suppose that the inequality holds for all numbers $k' \le k$. We show that it also holds for $k + 1$. Indeed, one has

$$\sigma_{k+1}^*(\Sigma, c_n) \le \sigma_k^*(\Sigma, c_n) + 2\pi = \sigma_k^*(\Sigma, c_n) + \sigma_1^*(\mathbb{D}^2)$$

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and we get

$$\limsup_{n\to\infty} \sigma^*_{k+1}(\Sigma, c_n) \le \max\left(\sum_{i=1}^{m} \sigma^*_{k_i}(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma^*_{r_i}(\mathbb{D}^2)\right) + \sigma^*_1(\mathbb{D}^2)$$
$$\le \max\left(\sum_{i=1}^{m} \sigma^*_{k_i}(\Sigma_{\gamma_i,l_i}, c_\infty) + \sum_{i=1}^{s_1+s_2} \sigma^*_{r_i}(\mathbb{D}^2)\right),$$

where the maximum is taken over all possible combinations of indices such that

$$\sum_{i=1}^{m} k_i + \sum_{i=1}^{s_1+s_2} r_i = k+1,$$

since the term $\sigma^*_1(\mathbb{D}^2)$ can be absorbed by one of the terms inside max using inequality (1.1). The proof is complete.

**Zero Euler characteristic.** The case of the cylinder was essentially considered in [Pet19, Section 7.1]. Indeed, it was proved that if the sequence of conformal classes $\{c_n\}$ degenerates then

$$\lim_{n\to\infty} \sigma^*_k(\mathbb{C}, c_n) \le \max_{i_1+\cdots+i_s=k} \sum_{q=1}^{s} \sigma^*_{i_q}(\mathbb{D}^2) = 2\pi k.$$

Applying then inequality (1.2), one immediately gets that $\lim_{n\to\infty} \sigma^*_k(\mathbb{C}, c_n) = 2\pi k$.

Consider the case of the Möbius band. If the sequence $\{c_n\}$ goes to 0 then it follows from [Pet19, Section 7.1] that

$$\lim_{n\to\infty} \sigma^*_k(\mathbb{MB}, c_n) \le \max_{i_1+\cdots+i_s=k} \sum_{q=1}^{s} \sigma^*_{i_q}(\mathbb{D}^2) = 2\pi k. \tag{5.9}$$

Indeed, we pass to the orientable cover which is a cylinder. Then inequality (5.9) follows from [Pet19, Section 7.1, the case $R_\alpha \to 1$as $\alpha \to +\infty$ in Petrides' notations].

If the sequence $\{c_n\}$ goes to $\infty$ then we prove that inequality (5.9) also holds. The proof follows the exactly same arguments as in the proof of inequality (5.4). The analog of the Case 1 for $\mathbb{MB}$ corresponds to the case of pinching boundary (see Remark (5.1)).

Therefore, in both cases inequality (5.9) holds. Applying inequality (1.2) once again we then get that $\lim_{n\to\infty} \sigma^*_k(\mathbb{MB}, c_n) = 2\pi k$.

## 6 Proof of Theorem 1.5

For the proof of Theorem 1.5, we will need to choose a "nice" degenerating sequence of conformal classes, i.e., a degenerating sequence of conformal classes such that the limiting space looks as simple as possible.

***Lemma 6.1*** *Let $\Sigma$ be a compact surface with boundary of negative Euler characteristic. Then there exists a degenerating sequence of conformal classes such that the limiting space is the disc.*

**Proof** The proof is purely topological.

Assume that $\Sigma$ is orientable. Then we consider collapsing geodesics shown in Figure 3. Passing to the limit when the lengths of all pinching geodesics tend to zero

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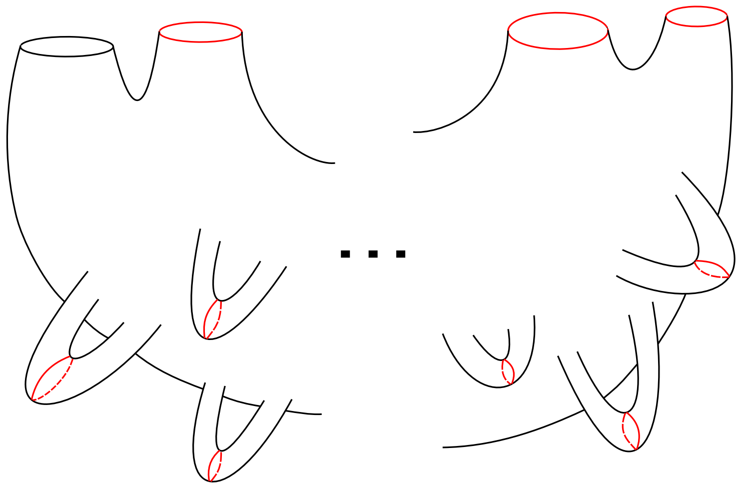

*Figure 3*: Orientable surface with boundary. The lengths of all *red* geodesics tend to zero.

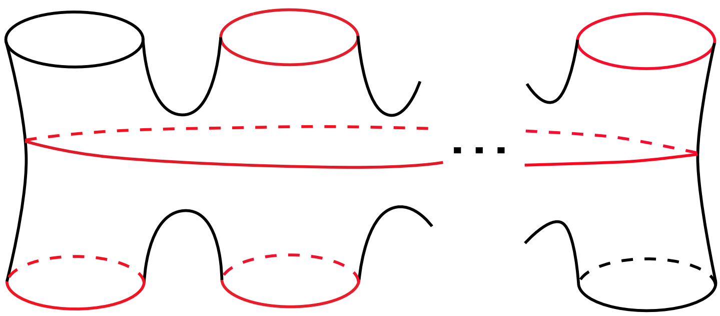

*Figure 4*: Orientable cover of a non-orientable surface of genus 0 with boundary. The lengths of all *red* geodesics tend to zero.

and using the one-point cusps compactification we get an orientable surface of genus 0 with one boundary component, i.e., the disc.

If $\Sigma$ is nonorientable then we pass to its orientable cover and we consider collapsing geodesics shown in Figure 4 for genus 0 and Figure 5 for genus $\neq 0$ (the pictures are symmetric with respect to the involution changing the orientation, "the antipodal map"). Passing to the limit when the lengths of all pinching geodesics tend to zero and using the one-point cusps compactification, we get a disconnected surface with two connected components which are topologically discs. The involution changing the orientation maps one component to another one and hence passing to the quotient by this involution we get just one disc. ■

Now we are ready to prove Theorem 1.5.

**Zero Euler characteristic.** Let $\Sigma$ be either the cylinder $\mathbb{C}$ or the Möbius band $\mathbb{MB}$. Then this case immediately follows from Theorem 1.4 by Remark 1.4. Indeed, if $\{c_n\}$

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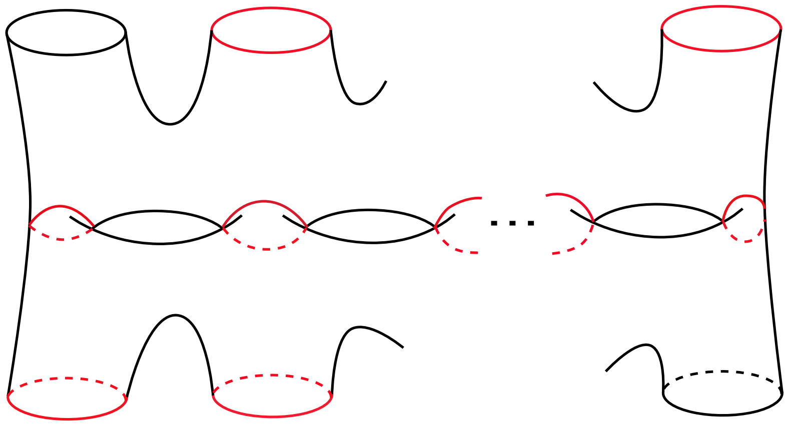

*Figure 5*: Orientable cover of a non-orientable surface of genus ≠ 0 with boundary. The lengths of all *red* geodesics tend to zero.

denotes a degenerating sequence of conformal classes on Σ then by Theorem 1.4:

$$I_k^\sigma(\Sigma) \le \lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k.$$

But $I_k^\sigma(\Sigma) \ge 2\pi k$ by (1.2). Thus, $I_k^\sigma(\Sigma) = \lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k$ and the degenerating sequence $\{c_n\}$ is minimizing.

**Negative Euler characteristic.** By Lemma 6.1, there exists a sequence of conformal classes $\{c_n\}$ such that the limiting space $\widehat{\Sigma_\infty}$ is the disc. Then by Theorem 1.4, we have

$$\lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = \max_{\sum k_j = k} \sum \sigma_{k_j}^*(\mathbb{D}^2).$$

Moreover, we know that $\sigma_k^*(\mathbb{D}^2) = 2\pi k$. Hence,

$$I_k^\sigma(\Sigma) \le \lim_{n\to\infty} \sigma_k^*(\Sigma, c_n) = 2\pi k.$$

Finally, by (1.2) one has $I_k^\sigma(\Sigma) \ge 2\pi k$ whence $I_k^\sigma(\Sigma) = 2\pi k$ which completes the proof.

# 7 Appendix

## 7.1 A well-posed problem

In this section, we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ u = g & \text{on } D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } N, \end{cases} \tag{7.1}$$

where $(M, h)$ is a Riemannian manifold with boundary such that $\overline{D} \cup \overline{N} = \partial M$ and $D$ has positive capacity.

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Let $G$ be a smooth function such that $G_{|_D} = g$ and consider the function $v = G - u$. Then substituting $u = G - v$ into (7.1) implies:

$$\begin{cases} \Delta v = \Delta G & \text{in } M, \\ v = 0 & \text{on } D, \\ \frac{\partial u}{\partial n} = \frac{\partial G}{\partial n} & \text{on } N. \end{cases} \tag{7.2}$$

We introduce the space $H^1_D(M,h)$ as the closure in $H^1$-norm of $C^\infty$-functions vanishing on $D$. For a function $u \in H^1_D(M,h)$, we have the following coercivity inequality:

$$||u||_{L^2(M,h)} \leq C||\nabla u||_{L^2(M,h)}, \tag{7.3}$$

with the best constant $C = \frac{1}{\sqrt{\lambda_1^{DN}(M,h)}}$, where $\lambda_1^{DN}(M,h)$ is the first non zero eigenvalue of the mixed problem

$$\begin{cases} \Delta u = \lambda u & \text{in } M, \\ u = 0 & \text{on } D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } N. \end{cases}$$

By the Lax–Milgram theorem and by virtue of the inequality (7.3) the problem (7.2) admits a unique solution on the space $H^1_D(M,h)$. Thus, problem (7.1) also has a solution. Moreover, it is easy to see that this solution is unique.

Our aim now is the following lemma.

***Lemma 7.1*** *Let $u$ satisfy the problem* (7.1). *Then one has*

$$||u||_{H^1(M,h)} \leq C||g||_{H^{1/2}(D,h)}.$$

**Proof** The weak formulation of (7.1) reads

$$\int_M \langle \nabla u, \nabla v \rangle dv_h = 0, \ \forall v \in H^1_D(M,h).$$

Let $G$ be any continuation of the function $g$ into $M$, i.e., $G \in H^1(M,h)$ is any function such that $G_{|_D} = g$. Then substituting $v = u - G$ in the previous identity yields

$$0 = \int_M \langle \nabla u, \nabla u - \nabla G \rangle dv_h = \int_M |\nabla u|^2 dv_h - \int_M \langle \nabla u, \nabla G \rangle dv_h,$$

whence

$$\int_M |\nabla u|^2 dv_h = \int_M \langle \nabla u, \nabla G \rangle dv_h \leq \frac{1}{2}\int_M |\nabla u|^2 dv_h + \frac{1}{2}\int_M |\nabla G|^2 dv_h. \tag{7.4}$$

Further, it is easy to see that

$$||u||_{L^2(M,h)} \leq ||u - G||_{L^2(M,h)} + ||G||_{L^2(M,h)}.$$

Moreover, since $u - G \in H^1_D(M,h)$ one has

$$||u - G||_{L^2(M,h)} \leq C||\nabla u - \nabla G||_{L^2(M,h)} \leq C(||\nabla u||_{L^2(M,h)} + ||\nabla G||_{L^2(M,h)}).$$

Substituting it in the previous inequality, we get

$$\|u\|_{L^2(M,h)} \leq C(\|\nabla u\|_{L^2(M,h)} + \|\nabla G\|_{L^2(M,h)}) + \|G\|_{L^2(M,h)}. \tag{7.5}$$

Plugging (7.4) in (7.5) yields

$$\|u\|_{L^2(M,h)} \leq C\|G\|_{H^1(M,h)}. \tag{7.6}$$

Finally (7.4) and (7.6) imply

$$\|u\|_{H^1(M,h)} \leq C\|G\|_{H^1(M,h)} \tag{7.7}$$

for any function $G \in H^1(M,h)$ such that $G_{|_D} = g$.

***Lemma 7.2*** *The norms*

$$\inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)} \text{ and } \|g\|_{H^{1/2}(D,h)}$$

*are equivalent.*

**Proof** By the trace inequality there exists a positive constant $C_1$ such that for every $G \in H^1(M,h)$ one has

$$\|g\|_{H^{1/2}(D,h)} \leq C_1\|G\|_{H^1(M,h)},$$

which implies:

$$\|g\|_{H^{1/2}(D,h)} \leq C_1 \inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)}; \tag{7.8}$$

Further, we construct a continuation $G' \in H^1(M,h)$ of $g$ with the property that there exists a positive constant $C_2$ such that for every $g \in H^{1/2}(D,h)$ one has:

$$\|G'\|_{H^1(M,h)} \leq C_2\|g\|_{H^{1/2}(D,h)}. \tag{7.9}$$

Let $\tilde{g}$ be any continuation of $g$ on $\partial M$ such that $\|\tilde{g}\|_{H^{1/2}(N,h)} \leq \|g\|_{H^{1/2}(D,h)}$. Therefore, $\|\tilde{g}\|_{H^{1/2}(\partial M,h)} \leq \sqrt{2}\|g\|_{H^{1/2}(D,h)} < \infty$ and $\tilde{g} \in H^{1/2}(\partial M,h)$. Then we take the harmonic continuation of $\tilde{g}$ into $M$ as $G'$. By [Tay11, Proposition 1.7] there exists a positive constant that $C_3$ such that:

$$\|G'\|_{H^1(M,h)} \leq C_3\|\tilde{g}\|_{H^{1/2}(\partial M,h)}.$$

Since $\|\tilde{g}\|_{H^{1/2}(\partial M,h)} \leq \sqrt{2}\|g\|_{H^{1/2}(D,h)}$ we get (7.9) with $C_2 = \sqrt{2}C_3$.

Therefore, (7.8) and (7.9) imply:

$$C_2^{-1}\|G'\|_{H^1(M,h)} \leq \|g\|_{H^{1/2}(D,h)} \leq C_1 \inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)},$$

whence

$$C_2^{-1} \inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)} \leq \|g\|_{H^{1/2}(D,h)} \leq$$
$$\leq C_1 \inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)},$$

since

$$\|G'\|_{H^1(M,h)} \geq \inf_{G\in H^1(M,h),\ G_{|_D}=g} \|G\|_{H^1(M,h)}.$$

And lemma follows. ∎

Finally, taking the infimum over all $G \in H^1(M,h)$ such that $G_{|_D} = g$ in (7.7) and using Lemma 7.2 complete the proof. ∎

## 7.2 Proofs of propositions of Section 2

This section contains the proofs of propositions in Section 2 analogous to propositions in [KM20, Section 4] whose adaptation to the Steklov setting is rather technical.

**Proof of Lemma 2.4** Let $h^m \in [h]$ be a maximizing sequence of metrics for $\sigma_k^{N*}(\Omega, \partial^S\Omega, [h])$ and $g^m \in [g]$ be a discontinuous metric on $\Sigma$ defined as $g|_{\Omega_i} = h_i$. By the variational characterization of eigenvalues for all $k$ one has $\sigma_k(\Sigma, g^m) \geq \sigma^N(\Omega, h^m)$ since the set of test functions for the Steklov–Neumann eigenvalues $C^0(\Sigma, \{\Omega_i\})$ is larger than the set $C^0(\Sigma)$ of test functions for $\sigma_k(\Sigma, g^m)$. Using the fact that $L_{g^m}(\partial\Sigma) = \sum_i L_{h^m}(\partial^S\Omega_i) \geq L_{g^m}(\partial^S\Omega_i)$ for any $i$ and taking the limit as $m \to \infty$ we get

$$\sigma_k^*(\Sigma, \{\Omega_i\}, [g]) \geq \sigma_k^{N*}(\Omega, \partial^S\Omega, [h]).$$

Finally by Lemma 2.3 one gets

$$\sigma_k^*(\Sigma, [g]) \geq \sigma_k^{N*}(\Omega, \partial^S\Omega, [h]).$$ ∎

**Proof of Proposition 2.6** The proof is similar for both cases. The obvious analog of Lemma 2.5 for the second case holds since its proof follows the exactly same arguments as the proof of Lemma 2.5. For that reason we only provide the proof of Proposition 2.6 for the first case.

Take a maximizing sequence of metrics $\{h_i \mid h_i \in [g|_\Omega]\}$ for the functional $\sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$, i.e.,

$$\lim_{i\to\infty} \bar\sigma_k^N(\Omega, \partial^S\Omega, h_i) = \sigma_k^{N*}(\Omega, \partial^S\Omega, [g])$$

Let $h_i = f_i g|_\Omega$, where $f_i \in C_+^\infty(\bar\Omega)$. We then define the metric $\widetilde{h_i} = \widetilde{f_i} g$ on $\Sigma$, where $\widetilde{f_i}$ is any positive continuation of the function $f_i$ into $\Omega^c$. It enables us to consider the metric $\rho_\delta \widetilde{h_i}$, where as before

$$\rho_\delta = \begin{cases} 1 & \text{in } \Omega, \\ \delta & \text{in } \Sigma\backslash\Omega. \end{cases}$$

Lemma 2.5 implies

$$\liminf_{\delta\to 0} \sigma_k(\rho_\delta \widetilde{h_i}) \geq \sigma_k^N(\Omega, \partial^S\Omega, h_i).$$

Moreover, $L_{\rho_\delta \widetilde{h_i}}(\partial\Sigma) \to L_{h_i}(\partial^S\Omega)$. By Lemma 2.3, we have

$$\sigma_k^*(\Sigma, [g]) = \sigma_k^*(\Sigma, \{\Omega, \Sigma\backslash\Omega\}, [g]) \geq \liminf_{\delta\to 0} \bar\sigma_k(\rho_\delta \widetilde{h_i}) \geq \bar\sigma_k^N(\Omega, \partial^S\Omega, h_i).$$

Therefore, passing to the limit as $i \to \infty$ one gets,

$$\sigma_k^*(\Sigma, [g]) \geq \sigma_k^{N*}(\Omega, \partial^S\Omega, [g]).$$ ∎

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**Proof of Corollary 2.7** We show that

$$\sigma_k^*(M,[g]) \le \liminf_{n\to\infty} \sigma_k^{N*}(M\backslash K_n, \partial M\backslash\partial K_n,[g]).$$

Let $g^m$ be a maximizing sequence for the functional $\sigma_k^*(M,[g])$. For a fixed $m$, we consider geodesic balls $B_{\varepsilon_n}(p_i)$ of radius $\varepsilon_n \to 0$ in metric $g^m$ centered at the points $p_1,\dots,p_l \in M$ such that $K_n \subset \cup_{i=1}^l B_{\varepsilon_n}(p_i)$. We see that $M\backslash \cup_{i=1}^l B_{\varepsilon_n}(p_i) \subset M\backslash K_n$. Then by Proposition 2.6 one has

$$\begin{aligned} &\sigma_k^{N*}(M\backslash K_n, \partial M\backslash\partial K_n,[g]) \\ &\ge \sigma_k^{N*}(M\backslash \cup_{i=1}^l B_{\varepsilon_n}(p_i), \partial M\backslash \cup_{i=1}^l \partial B_{\varepsilon_n}(p_i),[g]) \\ &\ge \bar\sigma_k^{N}(M\backslash \cup_{i=1}^l B_{\varepsilon_n}(p_i), \partial M\backslash \cup_{i=1}^l \partial B_{\varepsilon_n}(p_i),g^m). \end{aligned} \tag{7.10}$$

Note that $L(\partial M\backslash \cup_{i=1}^l \partial B_{\varepsilon_n}(p_i),g^m) \to L(\partial M, g^m)$ as $n\to\infty$ and by Lemma 2.1 one has $\sigma_k^{N}(M\backslash \cup_{i=1}^l B_{\varepsilon_n}(p_i), \partial M\backslash \cup_{i=1}^l \partial B_{\varepsilon_n}(p_i),g^m) \to \sigma_k(M,g^m)$. Hence, $\bar\sigma_k^{N}(M\backslash \cup_{i=1}^l B_{\varepsilon_n}(p_i), \partial M\backslash \cup_{i=1}^l \partial B_{\varepsilon_n}(p_i),g^m) \to \bar\sigma_k(M,g^m)$ as $n\to\infty$. Taking $\liminf_{n\to\infty}$ in (7.10) one then gets

$$\liminf_{n\to\infty} \sigma_k^{N*}(M\backslash K_n, \partial M\backslash\partial K_n,[g]) \ge \bar\sigma_k(M,g^m).$$

Passing to the limit as $m\to\infty$, we get the desired inequality.

The inequality

$$\limsup_{n\to\infty} \sigma_k^{N*}(M\backslash K_n, \partial M\backslash\partial K_n,[g]) \le \sigma_k^*(M,[g])$$

follows from Proposition 2.6. This completes the proof. ∎

**Proof of Lemma 2.8** Essentially the idea of the proof comes from the paper [WK94]. We denote by $\partial^S\Omega$ the part of the boundary with the Steklov boundary condition. We also call $\partial^S\Omega$ "Steklov boundary" and $L_g(\partial^S\Omega)$ "the length of Steklov boundary" in metric $g$.

**Inequality ≥.**

Fix the indices $k_i > 0$ satisfying $\sum k_i = k$ and consider a maximizing sequence of metrics $\{g_i^m\}$ such that $\bar\sigma_{k_i}^N(\Omega_i,\partial^S\Omega_i,g_i^m) \to \sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])$. One can assume that $\sigma_{k_i}^N(\Omega_i,\partial^S\Omega_i,g_i^m) = \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$. Then, one has

$$L_{g_i^m}(\partial^S\Omega_i) \to \frac{\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])}{\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])}.$$

Let $\{g^m\}$ be a sequence of metrics on $\Omega$ defined as $g^m|_{\Omega_i} = g_i^m$. Then for large enough $m$ one has that $\sigma_k^N(\Omega,\partial^S\Omega,g^m) = \sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$, since the spectrum of disjoint union is the union of spectra of each component. By definition of $\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$ we also have

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])L_{g^m}(\partial^S\Omega) = \sigma_k^{N}(\Omega,\partial^S\Omega,g^m)L_{g^m}(\partial^S\Omega) \le \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]),$$

i.e., $L_{g^m}(\partial^S\Omega) \le 1$. Thus, one has

$$1 \ge L_{g^m}(\partial^S\Omega) = \sum_i L_{g_i^m}(\partial^S\Omega_i) \to \frac{\sum_i \sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i])}{\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])}.$$

Passing to the limit $m\to\infty$ yields the inequality.

**Inequality ≤**

Assume the contrary, i.e.,

$$\sigma_k^{N*}(\Omega,\partial^S\Omega,[g]) > \max_{\sum_{i=1}^{s} k_i=k,\ k_i>0} \sum_{i=1}^{s} \sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g_i]). \tag{7.11}$$

Consider a maximizing sequence of metrics $\{g^m\}$ of unit total length of Steklov boundary such that $\sigma_k^N(\Omega,\partial^S\Omega,g^m)\to\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$. Let $g_i^m$ be a restriction of $g^m$ to $\Omega_i$ and $d_i^m$ be the largest number satisfying $\sigma_{d_i^m}^N(\Omega_i,\partial^S\Omega_i,g_i^m)<\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$ and $\limsup_{m\to\infty}\sigma_{d_i^m}^N(\Omega_i,\partial^S\Omega_i,g_i^m)<\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$. Let $L_i^m$ denote $L_{g_i^m}(\partial^S\Omega_i)$. Then, we have $d_i^m\le k$ and $L_i^m\le 1$. Therefore, up to a choice of a subsequence one can assume that $d_i^m=d_i$ does not depend on $m$ and $L_i^m\to L_i$ as $m\to\infty$.

We claim that $\sum_i(d_i+1)\ge k+1$. Otherwise, by (7.11) and definition of $d_i$ we have

$$\begin{aligned}\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])\sum_i L_i &\le \sum_i \limsup_{m\to\infty}\bar\sigma_{d_i+1}^N(\Omega_i,\partial^S\Omega_i,g_i^m)\\ &\le \sum_i \sigma_{d_i+1}^{N*}(\Omega_i,\partial^S\Omega_i,[g]) < \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]).\end{aligned}$$

Moreover, $\sum_i L_i=1$ since $g^m$ are of unit Steklov boundary length. Thus, we arrive at $\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])<\sigma_k^{N*}(\Omega,\partial^S\Omega,[g])$, which is a contradiction.

Therefore, the inequality $\sum(d_i+1)\ge k+1$ holds. Since the spectrum of a union is a union of spectra, we have

$$\sigma_k^N(\Omega,\partial^S\Omega,g^m)\in\bigcup_i\{\sigma_0(\Omega_i,g_i^m),\dots,\sigma_{d_i}(\Omega_i,g_i^m)\},$$

hence

$$\begin{aligned}\sigma_k^{N*}(\Omega,\partial^S\Omega,g) = \limsup_{m\to\infty}\sigma_k^N(\Omega,\partial^S\Omega,g^m) &\le \max_i \limsup_{m\to\infty}\sigma_{d_i}(\Omega_i,g_i^m)\\ &< \sigma_k^{N*}(\Omega,\partial^S\Omega,[g]).\end{aligned}$$

Since $g^m$ are of unit Steklov boundary length we arrive at a contradiction. ∎

**Proof of Lemma 2.9** Fix indices $k_i\ge 0$ such that $\sum_{i=1}^{s'}k_i=k$ and set $I=\{i\,|\,k_i>0\}$. Let $\Omega_1=\cup_{i\in I}\overline{\Omega}_i\subset\Sigma$, $\partial^S\Omega_1=\cup_{i\in I}\partial^S\Omega_i$, $(\Omega_2,h)=\sqcup_{i\in I}(\overline{\Omega}_i,g_{\overline{\Omega}_i})$ and $\partial^S\Omega_2=\sqcup_{i\in I}\partial^S\Omega_i$. One gets

$$\begin{aligned}\sigma_k^*(\Sigma,[g]) \ge \sigma_k^{N*}(\Omega_1,\partial^S\Omega_1,[g]) &\ge \sigma_k^{N*}(\Omega_2,\partial^S\Omega_2,[h])\\ &\ge \sum_{i\in I}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g]) = \sum_{i=1}^{s'}\sigma_{k_i}^{N*}(\Omega_i,\partial^S\Omega_i,[g]),\end{aligned}$$

where we used in order: Proposition 2.6, Lemmas 2.4 and 2.8 and the fact that $\sigma_0^{N*}(\Omega_j,\partial^S\Omega_j,[g])=0$ for any $j$ in the last equality. ∎

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### 7.3 Proof of Lemma 5.2

Fix $\varepsilon > 0$. An application of Corollary 2.7 to a compact exhaustion of $\Sigma_j^\infty$ yields the existence of a compact set $K \subset \Sigma_j^\infty \subset \widehat{\Sigma_j^\infty}$ such that

$$|\sigma_r^*(\widehat{\Sigma_j^\infty}, [\widehat{h_\infty}]) - \sigma_r^{N*}(K, \partial^S K, [\widehat{h_\infty}])| < \varepsilon,$$

where $\partial^S K = K \cap \partial\Sigma_j^\infty \neq \emptyset$. Since $\check{\Omega}_j^n$ exhaust $\Sigma_j^\infty$, then for all large enough $n$ one has $K \subset \check{\Omega}_j^n$. Then, by Proposition 2.6

$$\sigma_r^{N*}(\check{\Omega}_j^n, \partial^S \check{\Omega}_j^n, [(\Psi^n)^* h_n]) \geq \sigma_r^{N*}(K, \partial^S K, [(\Psi^n)^* h_n]).$$

Taking lim inf of both sides in the above inequality and using Proposition 2.2 yields

$$\liminf_{n\to\infty} \sigma_r^{N*}(\check{\Omega}_j^n, \partial^S \check{\Omega}_j^n, [(\Psi^n)^* h_n]) \geq \sigma_r^{N*}(K, \partial^S K, [\widehat{h_\infty}]) > \sigma_r^*(\widehat{\Sigma_j^\infty}, [\widehat{h_\infty}]) - \varepsilon.$$

Since $\varepsilon$ is arbitrary, this completes the proof.

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## References

[AH96] D. Adams and L. Hedberg, *Function spaces and potential theory*, Vol. 314. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1996.

[Ahl50] L. Ahlfors, *Open Riemann surfaces and extremal problems on compact subregions*. Comment. Math. Helv. **24**(1950), no. 1, 100–134.

[BKPS10] R. Banuelos, T. Kulczycki, I. Polterovich, and B. Siudeja, *Eigenvalue inequalities for mixed Steklov problems*. Oper. Theor. Appl. **231**(2010), 19–34.

[Bog17] B. Bogosel, *The Steklov spectrum on moving domains*. Appl. Math. Optim. **75**(2017), no. 1, 1–25.

[BNR16] M. Borodzik, A. Némethi, and A. Ranicki, *Morse theory for manifolds with boundary*. Algebr. Geometr. Topol. **16**(2016), no. 2, 971–1023.

[Bus92] P. Buser, *Geometry and spectra of compact Riemann surfaces*. Vol. 106. Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1992.

[CSG11] B. Colbois, A. El Soufi, and A. Girouard, *Isoperimetric control of the Steklov spectrum*. J. Funct. Anal. **261**(2011), no. 5, 1384–1399.

[CSG19] B. Colbois, A. El Soufi, and A. Girouard, *Compact manifolds with fixed boundary and large Steklov eigenvalues*. Proc. Am. Math. Soc. **147**(2019), no. 9, 3813–3827.

[CGR18] B. Colbois, A. Girouard, and B. Raveendran, *The Steklov spectrum and coarse discretizations of manifolds with boundary*. Pure Appl. Math. Quarter. **14**(2018), no. 2, 357–392.

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. Math. **136**(2012), no. 5, 521–573.

[EPS15] A. Enciso and D. Peralta-Salas, *Eigenfunctions with prescribed nodal sets*. J. Differ. Geometry **101**(2015), no. 2, 197–211.

[FS11] A. Fraser and R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*. Adv. Math. **226**(2011), no. 5, 4011–4030.

[FS12] A. Fraser and R. Schoen, *Minimal surfaces and eigenvalue problems*. Geometr. Anal. Math. Relat. Nonlinear Partial Differ. Equat. **599**(2012), 105–121.
[FS16] A. Fraser and R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*. Invent. Math. **203**(2016), no. 3, 823–890.
[FS20] A. Fraser and R. Schoen, *Some results on higher eigenvalue optimization*. Calculus Variat. Partial Differ. Equat. **59**(2020), no. 5, 1–22.
[FN99] L. Friedlander and N. Nadirashvili, *A differential invariant related to the first eigenvalue of the Laplacian*. Int. Math. Res. Notices **17**(1999), 939–952.
[Gab06] A. Gabard, *Sur la représentation conforme des surfaces de Riemann à bord et une caractérisation des courbes séparantes*. Comment. Math. Helv. **81**(2006), no. 4, 945–964.
[GL20] A. Girouard and J. Lagacé, *Large Steklov eigenvalues via homogenisation on manifolds*. Preprint, 2020. arXiv:2004.04044
[GP10] A. Girouard and I. Polterovich, *On the Hersch-Payne-Schiffer inequalities for Steklov eigenvalues*. Funct. Anal. Appl. **44**(2010), no. 2, 106–117.
[GP17] A. Girouard and I. Polterovich, *Spectral geometry of the Steklov problem (survey article)*. J. Spectr. Theory 7(2017), no. 2, 321–360.
[GP] A. Girouard and I. Polterovich, *Upper bounds for Steklov eigenvalues on surfaces*. Electronic Research Announcements, 19, 77.
[Has11] A. Hassannezhad, *Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem*. J. Funct. Anal. **261**(2011), no. 12, 3419–3436.
[HP18] A. Henrot and M. Pierre, *Shape variation and optimization*. Vol. 28. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2018. A geometrical analysis, English version of the French publication [MR2512810] with additions and updates.
[Hum97] C. Hummel, *Gromov's compactness theorem for pseudo-holomorphic curves*. Vol. 151. Progress in Mathematics, Birkhäuser Verlag, Basel, 1997.
[Jos07] J. Jost, *Bosonic Strings: A mathematical treatment*. Vol. 21. American Mathematical Society, Providence, RI, 2007.
[Kar16] M. Karpukhin, *Upper bounds for the first eigenvalue of the Laplacian on non-orientable surfaces*. IMRN **20**(2016), 6200–6209.
[Kar17] M. Karpukhin, *Bounds between Laplace and Steklov eigenvalues on nonnegatively curved manifolds*. Electron. Res. Announcements Math. Sci. **24**(2017), 100–109.
[Kar20] M. Karpukhin, *Index of minimal spheres and isoperimetric eigenvalue inequalities*. Invent. Math **223**(2021), 335–377.
[KKP14] M. Karpukhin, G. Kokarev, and I. Polterovich, *Multiplicity bounds for Steklov eigenvalues on Riemannian surfaces*. Ann. l'Institut Fourier **64**(2014), no. 6, 2481–2502.
[KM20] M. Karpukhin and V. Medvedev, *On the Friedlander–Nadirashvili invariants of surfaces*. Math. Ann. (2020), 1–39.
[KNPP20] M. Karpukhin, N. Nadirashvili, A. V. Penskoi, and I. Polterovich, *Conformally maximal metrics for Laplace eigenvalues on surfaces*. Preprint, 2020. arXiv:2003.02871
[KS20] M. Karpukhin and D. L. Stern, *Min-max harmonic maps and a new characterization of conformal eigenvalues*. Preprint, 2020. arXiv:2004.04086
[Kok14] G. Kokarev, *Variational aspects of Laplace eigenvalues on Riemannian surfaces*. Adv. Math. **258**(2014), 191–239.
[LP17] P. Laurain and R. Petrides, *Regularity and quantification for harmonic maps with free boundary*. Adv. Calculus Variat. **10**(2017), no. 1, 69–82.
[MP20a] H. Matthiesen and R. Petrides, *Free boundary minimal surfaces of any topological type in Euclidean balls via shape optimization*. Preprint, 2020. arXiv:2004.06051
[MP20b] H. Matthiesen and R. Petrides, *A remark on the rigidity of the first conformal Steklov eigenvalue*. Preprint, 2020. arXiv:2006.04364
[MS20] H. Matthiesen and A. Siffert, *Sharp asymptotics of the first eigenvalue on some degenerating surfaces*. Trans. Am. Math. Soc. **373**(2020), no. 8, 5903–5936.
[NS15a] N. Nadirashvili and Y. Sire, *Conformal spectrum and harmonic maps*. Moscow Math. J. **15**(2015), no. 1, 123–140.
[NS15b] N. Nadirashvili and Y. Sire, *Maximization of higher order eigenvalues and applications*. Moscow Math. J. **15**(2015), no. 4, 767–775.
[OPS88] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians*. J. Funct. Anal. **80**(1988), no. 1, 148–211.
[Pen13] A. V. Penskoĭ, *Extremal metrics for the eigenvalues of the Laplace-Beltrami operator on surfaces (Russian)*. Uspekhi Mat. Nauk **68**(2013), no. 414, 107–168.
[Pen19] A. V. Penskoĭ, Isoperimetric inequalities for higher eigenvalues of the Laplace-Beltrami operator on surfaces (Russian). *Trudy Matematicheskogo Instituta Imeni V. A. Steklova*,

305, Algebraicheskaya Topologiya Kombinatorika i Matematicheskaya Fizika, 2019, pp. 291–308.

[Pet14] R. Petrides, *Existence and regularity of maximal metrics for the first Laplace eigenvalue on surfaces*. Geometr. Funct. Anal. **24**(2014), no. 4, 1336–1376.

[Pet18] R. Petrides, *On the existence of metrics which maximize Laplace eigenvalues on surfaces*. IMRN **2018**(2018), no. 14, 4261–4355.

[Pet19] R. Petrides, *Maximizing Steklov eigenvalues on surfaces*. J. Differ. Geometry **113**(2019), no. 1, 95–188.

[RT75] J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*. J. Funct. Anal. **18**(1975), no. 1, 27–59.

[Sch13] R. Schoen, *Existence and geometric structure of metrics on surfaces which extremize eigenvalues*. Bull. Braz. Math. Soc. (N.S.) **44**(2013), no. 4, 777–807.

[Tay11] M. E. Taylor, *Partial differential equations I. Basic theory*. 2nd ed. Applied Mathematical Sciences, 117, Springer, 2011.

[Wei54] R. Weinstock, *Inequalities for a classical eigenvalue problem*. J. Rational Mech. Anal. **3**(1954), 745–753.

[WK94] S. A. Wolf and J. B. Keller, *Range of the first two eigenvalues of the Laplacian*. Proc. R. Soc. Lond. A. **447**(1994), no. 1930, 397–412.

[YY80] P. C. Yang and S. T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7(1980), no. 1, 55–63.

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