Glasgow Math. J. **58** (2016) 1–15. © Glasgow Mathematical Journal Trust 2015.
doi:10.1017/S0017089515000038.

A LOCALISABLE CLASS OF PRIMITIVE IDEALS OF UNIFORM NILPOTENT IWASAWA ALGEBRAS

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(Received 12 February 2013; revised 4 March 2014; accepted 4 March 2014; first published online 21 July 2015)

Abstract. We study the injective hulls of faithful characteristic zero finite dimensional irreducible representations of uniform nilpotent pro-p groups, seen as modules over their corresponding Iwasawa algebras. Using this we prove that the kernels of these representations are classically localisable.

2010 *Mathematics Subject Classification.* 16L30, 16P50, 16S99, 20E18.

1. Introduction. The study of Iwasawa algebras is one that has gained some prominence in recent years. Notably, of late attention has turned towards the noncommutative angle, with efforts in number theory to understand more about their modules (e.g. [**6**, **19**]), in particular with respect to finding a noncommutative analogue of the Iwasawa main conjecture [**5**, **12**].

Although noncommutative Iwasawa algebras were first considered by Lazard in a groundbreaking paper [**13**] nearly 50 years ago, many of their algebraic properties are still yet to be understood. It has been shown that they form a class of noetherian algebras with pleasant properties, and some similarities to the theory of Lie algebras – see [**3**] for more details. However, there are only a few known examples of prime ideals for any given Iwasawa algebra, and properties of such ideals are only just coming to light (e.g. [**2**]). The purpose of this paper is to study the simplest examples of noncommutative Iwasawa algebras, those of nilpotent groups, with respect to localisability conditions (an important tool used for example in [**5**]). Perhaps surprisingly, we find this situation to be very similar to the commutative case.

Let G be a compact p-adic Lie group, and K a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We define the *Iwasawa algebra* with coefficients in $\mathcal{O}$ to be

$$\mathcal{O}G := \varprojlim_{U \triangleleft_o G} \mathcal{O}[G/U],$$

where the inverse limit runs over the open normal subgroups of G. Write KG for the tensor product $K \otimes_{\mathcal{O}} \mathcal{O}G$.

For a K-algebra A, write $\mathrm{Prim}^{\mathrm{fd}}(A)$ for the set of primitive ideals P such that A/P is finite dimensional over K – we may think of these as kernels of faithful finite-dimensional irreducible representations of G. The aim of this paper is to prove the following:

THEOREM 1.1. *Let G be a uniform nilpotent pro-p group and K a finite extension of $\mathbb{Q}_p$. Then any completely prime ideal $P \in \mathrm{Prim}^{\mathrm{fd}}(KG)$ is classically localisable.*

Corollary 1.2. *If $\mathcal{O}$ is the ring of integers of K, then $P \cap \mathcal{O}G$ is also classically localisable.*

The theorem is trivially true in the case G is abelian, as then KG is a commutative ring. More generally, Ardakov [**1**] proved that Corollary 1.2 was true in the special case $P = I_G$, the augmentation ideal, which corresponds to the trivial module K. We shall take a different approach.

A prime ideal P is localisable only if it does not link to any other prime (see Section 2 for definitions). Jategaonkar's main Lemma [**11**] indicates that links are determined by the injective hull $E_{KG}(KG/P)$; further, the same object determines whether the prime ideal satisfies the second layer condition, which is a necessary criterion for localisation. Thus, we begin by determining this hull.

The case where $KG/P \cong K$ has already been well studied. Since $\mathcal{O}$ is essential in K as an $\mathcal{O}$-module, we see that the injective hulls for KG and $\mathcal{O}G$ coincide. When G is abelian, $\mathcal{O}G$ is just a power series ring in d variables X_i (e.g. [**18**]). Northcott [**16**] computed the injective hull of the trivial module K in this case, and showed it was the polynomial ring in X_i^{-1} with the canonical action. Musson [**14**] subsequently proved a similar statement for group rings of torsionfree nilpotent groups. It is therefore unsurprising that when we consider the case $P = I_G$ (where G is a uniform nilpotent pro-p group), we find that $E_{KG}(K)$ is a form of polynomial ring upon which I_G acts by lowering degree (Theorem 3.12). Our method of proving this is moderately technical, but one which allows an explicit determination of the action on the module, which will be useful to us in determining the clique.

To handle the general case, we observe (Lemma 4.1) that injectivity is preserved under tensoring by locally finite modules. Thus, $E_{KG}(K)$ may in fact be used as a base object for the injective hull of a certain class of finite-dimensional modules M, twisted by M at each point (Theorem 4.4). This explicit structure of the hull enables us to prove Theorem 1.1 in Section 5.

Aside from applying to all uniform nilpotent pro-p groups, many of the results in this paper are in fact fundamental to studying the larger class of uniform soluble pro-p groups. In a forthcoming publication, the author will build upon this work to answer similar questions on the injective hulls and localisability of $\mathrm{Prim}^{\mathrm{fd}}(KG)$ in case where G is in this class.

2. Preliminaries.

2.1. The second layer condition. We recall some relations on the prime spectrum of a noetherian ring R. For two prime ideals P, Q in R, we say that there is a *link* $P \rightsquigarrow Q$ (or that P *links to* Q) if there exists an ideal A of R such that $PQ \leq A < P \cap Q$, where $(P \cap Q)/A$ is torsionfree on both sides as an $(R/P, R/Q)$-bimodule. We define the *clique* of P to be the connected component of the graph of links formed on the spectrum of R by putting an edge between any two linked primes. It is well known that a (two-sided) denominator set may be formed within the regular elements mod P only if it is contained in the regular elements mod Q for all primes Q in the clique (see [**8**, 14.17]).

We write $E_R(M)$ for the injective hull of the R-module M – we will drop the subscript if R is clear. Following [**11**], we say that a prime ideal Q satisfies the *second* *layer condition* if every associated prime P of the right R-module

$$E_R(R/Q)/\mathrm{ann}_{E_R(R/Q)}(Q),$$

both links to Q, and the image of $\mathrm{ann}_{E_R(R/Q)}(P)$ in the above quotient has no fully faithful submodule. By Jategaonkar's main Lemma ([**8**, 12.1] – see Section 3 for the statement), this is automatically satisfied if no such associated prime is contained in Q. We say a clique X (resp. ring R) satisfies the second layer condition if every element of X (resp. Spec R) does. We recall [**11**, 7.2.5]:

THEOREM 2.1 (Jategaonkar). *Any finite clique is classically localisable if and only if it satisfies the second layer condition.*

The definition of classical localisability can be found in [**11**, 7.1]/[**8**, p251] – we shall repeat it here for convenience. For a ring R and prime ideal Q, write $\mathcal{C}(Q)$ for the set of elements of R which are regular modulo Q. A set of primes X in R is said to be *right localisable* if $\mathcal{C}(X) = \bigcap_{Q\in X}\mathcal{C}(Q)$ is an Ore set, the ideals QR_X (for $Q \in X$) of the corresponding localisation R_X are its only primitive right ideals, and all such quotient rings R_X/QR_X are simple artinian. We further call the localisation *classical* if for all $Q \in X$, $E_{R_X}(R_X/QR_X)$ is the union of its socle series. Left localisation is defined analogously; we drop the side if X is both right and left (classically) localisable.

2.2. Iwasawa algebras. The following conventions will be used throughout this paper. Recall [**7**] a pro-p group is uniform if it is finitely generated, torsionfree and powerful, that is, $G/\overline{G^p}$ is abelian (or $G/\overline{G^4}$ in case $p = 2$). Unless otherwise stated, G will be a uniform nilpotent group, and KG its Iwasawa algebra, where K is a finite extension of $\mathbb{Q}_p$. In contrast, the group algebra of G over K will be denoted $K[G]$.

Let $\nu : KG \to KG^{op}$ be the K-linear continuous algebra homomorphism determined by $\nu(g) := g^{-1}$ for all $g \in G$. Using this map, if M is a right KG-module under the action $\rho : KG \to \mathrm{End}(M)$, then M is a right KG^{op}-module under the action $m \cdot r = \rho(\nu(r))(m)$. Thus, it may also be viewed as a left KG-module via this action. In this paper, we shall only deal with right modules; results for their left analogues may be derived from this observation.

2.3. Augmentation ideals. We recall the definition of the $\mathcal{O}$-*augmentation ideal* for a closed normal subgroup N of G:

$$J_{N,G} := \ker\left(\epsilon_N : \mathcal{O}G \to \mathcal{O}(G/N)\right),$$

where ϵ_N is the $\mathcal{O}$-algebra homomorphism satisfying $\epsilon_N(g) = gN$ for all $g \in G$. In the case $N = G$, we suppress the extra subscript and write J_G. In [**1**], Ardakov showed that $J_{N,G}$ was localisable if and only if N was finite-by-nilpotent. It is easy to show that any prime ideal satisfying the second layer condition has trivial clique if it also satisfies the AR-property (for example, via [**8**, 13.6]). This motivates the following result:

PROPOSITION 2.2. *J_G satisfies the AR-property whenever G is nilpotent.*

This follows as a direct corollary of the following lemma, using [**11**, 3.3.19].

LEMMA 2.3. *Let G be nilpotent; then J_G is a polycentral ideal of $\mathcal{O}G$.*

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Proof. We follow the line of proof of [**17**, 11.2.14]: Let $G = G_1 \supseteq G_2 \supseteq \dots$ be the lower central series of G. Since G is nilpotent, there is an integer m such that $G_m = 1$. Define $J := J_G$ and $J_i = J_{G_i,G}$; we claim that $J = J_1 \supseteq \dots \supseteq J_m = 0$ is a central series for J. Since G has finite rank, G_i/G_{i+1} is (topologically) finitely generated for each i, say $G_i = \overline{\langle G_{i+1}, x_1, \dots, x_{d_i}\rangle}$. In particular, if K is an open normal subgroup of G, then G_iK/K is a finite group generated by $\{G_{i+1}K, x_1K, \dots, x_{d_i}K\}$, and so

$$J_{G_iK/K,G/K} = J_{G_{i+1}K/K,G/K} + \sum_{j=1}^{d_i}(x_j - 1)\mathcal{O}[G/K].$$

Since, this is true for all open normal subgroups of G, and the above formulation is preserved under restrictions $G_iK/K \to G_iL/L$, where $L \geq K$ runs over the open normal subgroups of G, we deduce that

$$J_i = J_{i+1} + \sum_{j=1}^{d_i}(x_j - 1)\mathcal{O}G.$$

Moreover, by definition of G_{i+1}, we have that for all $g \in G$ and for all positive j,

$$[x_j - 1, g] = x_jg - gx_j = (x_jgx_j^{-1}g^{-1} - 1)gx_j \in J_{i+1},$$

i.e. J_i is centrally generated modulo J_{i+1}. □

Write $I_{N,G}$ and I_G for $K \otimes_{\mathcal{O}} J_{N,G}$ and $K \otimes_{\mathcal{O}} J_G$, respectively. We note that Proposition 2.2 may also be stated for I_G, as KG is just the localisation of $\mathcal{O}G$ at the uniformiser of $\mathcal{O}$.

3. The injective hull of the trivial module. Our main tools for determining cliques are two classical results by Arun Jategaonkar and Ken Brown (e.g. [**11**, Section 6.1] or [**8**, Chapter 12]). Here, $U <_e M$ denotes that U is an essential submodule of M; that is, $U \neq 0$ and has nonzero intersection with any other nonzero submodule of M.

THEOREM (Jategaonkar's main Lemma). *Let R be a noetherian ring, and M a right R-module with affiliated series $0 < U <_e M$, with corresponding affiliated primes Q and P. Let M' be a submodule of M properly containing U such that $A = ann_R(M')$ is maximal with respect to such submodules. Then one of the following cases holds:*

(1) (The undesirable case) *$P < Q$, $M'P = 0$ and both M' and M'/U are faithful torsion R/P-modules; or*

(2) (The desirable case) *$P \rightsquigarrow Q$ with linking bimodule $P \cap Q/A$, and if U is a torsionfree R/Q-module, then M/U is also a torsionfree R/P-module.*

THEOREM (Jategaonkar, Brown). *Let R be a noetherian ring with prime ideals P and Q. Then $P \rightsquigarrow Q$ if and only if there exists a finitely generated uniform right R-module M with affiliated series $0 < U < M$ such that U is isomorphic to a right ideal of R/Q, and M/U to a right ideal of R/P.*

By these results, links to a prime P are determined by essential extensions of the quotient ring viewed as a right module. Similarly, links from P are determined by the left module analogue, using the fact that $P \rightsquigarrow Q$ in a ring R if and only if $Q \rightsquigarrow P$ in R^{op}. We are thus interested in determining the injective hulls of KG/P.

We first deal with the case where $P = I_G$. In particular, KG/P is the one-dimensional trivial module K.

The following concept will make matters simpler. Write $\langle\langle X\rangle\rangle$ for the group generated topologically by X – we shall use $\langle X\rangle$ to denote the abstract group generated by X. We say that a finite ordered set $S = \{g_1, \ldots, g_d\}$ of group elements of G is *orderly* if for all pairs $1 \leq i, j \leq d$, $[g_i, g_j] \in \langle\langle g_k, \ldots, g_d\rangle\rangle$, where $k = \max\{i, j\} + 1$ and $[\cdot, \cdot]$ is the group commutator. In general, say that an unordered set $S \subset G$ is orderly if there exists an ordering on S satisfying the above condition.

We say a (topological) group G is *orderly* if it has an orderly minimal (topological) generating set S. It is clear that abelian groups are orderly. Inducing, we may show that:

LEMMA 3.1. *Let G be a uniform nilpotent group. Then, G is orderly.*

Proof. For any normal subgroup N of G, define $\sqrt{N}$ to be the isolator of N, that is, $\{x \in G | x^n \in N \text{ for some } n \in \mathbb{N}\}$. This is a subgroup by [**9**, 4.5], and by definition is isolated i.e. $G/\sqrt{N}$ is torsionfree.

Since G is torsionfree nilpotent, we see that $G \neq \sqrt{[G, G]}$ (else for all k, there exists n_k such that $G^{n_k} < G^{(k)}$, the kth commutator subgroup, and the derived series does not terminate). Further, $\sqrt{[G, G]}$ is a uniform normal subgroup of G by [**7**, 4.31(ii)], which is generated by fewer elements than G by [**7**, 4.8]. So, setting $G_0 = G$, let $G_{i+1} = \sqrt{[G_i, G_i]}$ for all i to obtain a series

$$G > G_1 > \ldots > G_n > G_{n+1} = 0,$$

for some n, with G_i/G_{i+1} abelian. As G is of finite rank, we can take minimal topological generating sets $\{g_{i,1}G_{i+1}, \ldots, g_{i,d(i)}G_{i+1}\}$ for each G_i/G_{i+1}, and lift them to obtain a topological generating set for G which (ordering lexicographically) we see is orderly. Further, this is a minimal generating set by [**7**, 4.8], and the lemma is proved. □

We shall assume from now on that $\{g_1, \ldots, g_d\}$ is an orderly minimal (topological) generating set for G as constructed in the above proof. In particular, each element $g \in G$ may be written uniquely as $g = g_1^{\lambda_1} \ldots g_d^{\lambda_d}$, for appropriate $\lambda_i \in \mathbb{Z}_p$ [**7**, 4.9].

Write G^* for the right KG-module $\mathrm{Hom}_K^{\mathrm{cts}}(KG, K)$ of continuous K-linear maps $KG \to K$, where G acts via $(fg)(x) = f(xg^{-1})$ for $f \in G^*, g \in G$. We note that G^* is canonically isomorphic to the G-module of continuous maps from G to K, and that in this setting the constant maps give a submodule isomorphic to K. Further:

LEMMA 3.2. *G^* is injective as a right KG-module.*

Proof. This is fundamentally the argument of [**8**, 5.3]: by Baer's criterion, it suffices to show that for any ideal I of KG, any KG-linear map $I \to G^*$ extends to a KG-linear map $KG \to G^*$. Let $ev_1 : G^* \to K$ be the evaluation map at 1; then, noting that the topology on K is defined by the uniformiser π of $\mathcal{O}$, and that multiplication by π commutes with K-linear maps, we see that the composition $ev_1 \circ f : I \to K$ is continuous. Since K is discretely valued and KG is locally compact, this extends to a continuous (K-linear) map $\theta : KG \to K$ by [**10**, Theorem 3]. It therefore follows that f extends to the map

$$\begin{aligned} \hat{f} : & KG \mapsto G^* \\ & r \to (\theta \circ \nu) \cdot r, \end{aligned}$$

since ν defines the right action of KG on G^*. □

https://doi.org/10.1017/S0017089515000038 Published online by Cambridge University Press

We may give a commutative ring structure to G^* by pointwise addition and multiplication, i.e. $(f+f')(r) = f(r) + f'(r)$ and $ff'(r) = f(r)f'(r)$. Combining this with the G-module structure, we note that $(ff') \cdot g = (f \cdot g)(f' \cdot g)$ for all $f, f' \in G^*, g \in G$.

Now define $Y_i : G \to \mathbb{Z}_p \subset K$ taking $g = g_1^{\lambda_1} \dots g_d^{\lambda_d}$ to λ_i; this is well defined by the uniqueness of the expansion. We wish to study the subring of G^* generated by $\{Y_1, \dots, Y_d\}$; the following determines the structure completely.

LEMMA 3.3. *The set* $\{Y_1, \dots, Y_d\}$ *is algebraically independent over* K.

Proof. Induct on d. For $d = 1$, let $f = \sum_j^n c_j Y_d^j$ for some natural number n and some $c_j \in K$. Applying f to g_d^t gives a polynomial in $K[t]$, which is zero only when $c_j = 0$ for all j.

In general, we may take $c_j \in K[Y_1, \dots, Y_{d-1}] \backslash \{0\}$ by the induction hypothesis. As $c_0 \neq 0$, there exists some $g \in \langle g_1, \dots, g_{d-1} \rangle$ such that $c_0(g) = \lambda_0 \neq 0$. Thus, applying f to gg_d^t gives a nonzero polynomial in $K[t]$, and so $f \neq 0$. □

Thus, the subring in question is the polynomial ring $K[Y_1, \dots, Y_d]$. This amounts to calculating the G-action on itself by right multiplication, in terms of the generating set.

Philip Hall has proved the following for abstract groups:

THEOREM 3.4 (Theorem 6.5 of [**9**]). *Let* $G = \langle x_1, \dots, x_d \rangle$ *be a nilpotent group with an orderly set of generators. Let* $x = x_1^{\lambda_1} \dots x_d^{\lambda_d}$ *and* $y = x_1^{\mu_1} \dots x_d^{\mu_d}$ *be elements of* G; *then for any* $\nu \in \mathbb{Z}$,

$$\begin{aligned} xy &= x_1^{f_1(\lambda_i, \mu_i)} \dots x_d^{f_d(\lambda_i, \mu_i)} \\ x^\nu &= x_1^{f'_1(\lambda_i, \nu)} \dots x_d^{f'_d(\lambda_i, \nu)}, \end{aligned}$$

where f_i *are polynomials in* $2d$ *variables and* f'_i *are polynomials in* $d+1$ *variables.*

The proof is rather technical but elementary. Following this result, Hall considers the idea of R-powered groups, defined as follows: call R a binomial ring if it is an integral domain which contains binomial coefficients $\binom{\lambda}{n}$, for all $\lambda \in R$ and $n \in \mathbb{Z}$. For any such R, an R-powered group is a nilpotent group G of class c such that x^λ is defined for all $x \in G$ and $\lambda \in R$, satisfying:

(i) $x^{\lambda+\mu} = x^\lambda x^\mu$; $x^{\lambda\mu} = (x^\lambda)^\mu$;
(ii) $(x^\lambda)^y = (x^y)^\lambda$; and
(iii) $x_1^\lambda \dots x_n^\lambda = t_1^\lambda t_2^{\binom{\lambda}{2}} \dots t_c^{\binom{\lambda}{c}}$,

for all $x, y \in G$, $n \in \mathbb{N}$ and $\lambda, \mu \in R$, where t_i are the Hall–Petresco words in $x_1, \dots, x_n$ (see [**9**, p23] for a definition). It is well known that any nilpotent group is a $\mathbb{Z}$-powered group; further:

LEMMA 3.5. *Any nilpotent pro-p group is a* $\mathbb{Z}_p$*-powered group* G.

Proof. Note that $\mathbb{Z}_p$ is clearly a binomial ring (as observed by Hall [**9**, p26]). It suffices to show (iii). For any $x_1, \dots, x_n \in G$ and $\lambda \in \mathbb{Z}_p$,

$$x_1^\lambda \dots x_n^\lambda = t_1^\lambda t_2^{\binom{\lambda}{2}} \dots t_c^{\binom{\lambda}{c}} y,$$

for some $y \in G$. However, we have $\lambda_k \in \mathbb{Z}$ such that $\lambda \equiv \lambda_k \bmod p^k$, and

$$x_1^{\lambda_k} \ldots x_n^{\lambda_k} \equiv t_1^{\lambda_k} t_2^{\binom{\lambda_k}{2}} \ldots t_c^{\binom{\lambda_k}{c}} \mod G^{p^k}.$$

Thus $y \equiv 1 \bmod G^{p^k}$, for all k. So $y = 1$ and the result follows. □

Hall observes after Theorem 3.4 that R-powered groups arise by extending the polynomials f' to R. More significantly for our purposes, it can be seen that the proof of Theorem 3.4 holds for all R-powered groups; in particular, it holds for pro-p groups by Lemma 3.5. Therefore, we have that:

Corollary 3.6. *$K[Y_1, \ldots Y_d]$ is a $K[G]$-submodule of G^*.*

We wish to extend this result to show that $K[Y_1, \ldots Y_d]$ is in fact a KG-module. By [**20**, 7.2.4], it suffices to show that the G-action is continuous. We shall do this by looking at the action in Theorem 3.4 in more detail, introducing a little extra terminology which will prove useful in further study of this object.

We can identify the monomials in $K[Y_1, \ldots, Y_d]$ with elements in $\mathbb{N}^d$ by the map

$$Y^{\mathbf{n}} = Y_1^{n_1} \ldots Y_d^{n_d} \leftrightarrow (n_1, \ldots, n_d) = \mathbf{n}.$$

We may thus give the monomials the colexicographic ordering, as follows: we will say that $Y_1^{m_1} \ldots Y_d^{m_d} <_{\text{colex}} Y_1^{n_1} \ldots Y_d^{n_d}$ if $m_d < n_d$ or $m_i = n_i$ for all $i = j+1, \ldots, d$, some $j \leq d$, and $m_j < n_j$. We extend this to a partial ordering on $K[Y_1, \ldots, Y_d]$: say that f has degree $\deg(f) = \mathbf{n}$ if $Y^{\mathbf{n}}$ appears as a term in f with nonzero coefficient, and $\mathbf{n}$ is maximal with respect to the colexicographic ordering amongst all such terms. Then, $K[Y_1, \ldots, Y_d]$ is partially ordered by degree with respect to the colexicographic ordering on $\mathbb{N}^d$.

We may now show the following:

Lemma 3.7. *For $l \leq d$, let ϵ_l be the natural projection map of $K[Y_1, \ldots, Y_d]$ onto the subspace $K[Y_1, \ldots, Y_l]$. For all $1 \leq i, j, l \leq d$,*

(i) $Y_i x = Y_i$ *for all* $x \in \langle\langle g_{i+1}, \ldots, g_d \rangle\rangle$;
(ii) $Y_i g_i = Y_i - 1$;
(iii) *Let* $Y^{\mathbf{n}} \in K[Y_1, \ldots, Y_d]$ *be a monomial. Then*

$$\epsilon_l(Y^{\mathbf{n}} \cdot (g_j - 1)) <_{\text{colex}} \epsilon_l(Y^{\mathbf{n}}).$$

Proof. We prove (iii); (i) and (ii) are proved similarly. Firstly, note that as g_j acts as an algebra homomorphism on $K[Y_1, \ldots Y_d]$, it suffices to prove (iii) for $\mathbf{n} = \delta_i$, where δ_i has value 1 in its only nonzero component i.

Consider $Y_i g_j$ restricted to $\langle\langle g_1, \ldots, g_l \rangle\rangle$, for given $l \leq d$. This is defined by its action on a generic element:

$$(Y_i g_j)\left(g_1^{\lambda_1} \ldots g_l^{\lambda_l}\right) = Y_i\left(g_1^{\lambda_1} \ldots g_l^{\lambda_l} g_j^{-1}\right).$$

If $l \leq j$, then $Y_i g_j = Y_i$ or $Y_i g_j = Y_i - 1$ and the result is clear.

Now suppose that there exists some $j', l' \leq d$ such that (iii) holds for all choices of l whenever $j > j'$, and also holds for all choices of $l < l'$ whenever $j = j'$. The lemma will then follow by induction on showing that $Y_i \cdot (g_{j'} - 1)$ acts on $\langle\langle g_1, \ldots, g_{l'} \rangle\rangle$ as a lower degree element of $K[Y_1, \ldots, Y_d]$ than $Y^{\mathbf{n}}$.

By above, assume that $j' < l'$. Then

$$g_1^{\lambda_1} \dots g_{l'}^{\lambda_{l'}} g_{j'}^{-1} = g_1^{\lambda_1} \dots g_{l'-1}^{\lambda_{l'-1}} g_{j'}^{-1} g_{l'}^{\lambda_{l'}} z,$$

where $z = [g_{l'}^{\lambda_{l'}}, g_{j'}^{-1}] \in \langle\langle g_{l'+1}, \dots, g_d \rangle\rangle$. Thus

$$Y_i(g_1^{\lambda_1} \dots g_{l'}^{\lambda_{l'}} g_{j'}^{-1}) = (Y_i \cdot (z^{-1} g_{l'}^{\lambda_{l'}}))(g_1^{\lambda_1} \dots g_{l'-1}^{\lambda_{l'-1}} g_{j'}^{-1}).$$

Since $l' > j'$, (iii) also holds in this case by the hypotheses. □

In particular, (iii) tells us that for any $g \in G$ and $Y^{\mathbf{n}} \in K[Y_1, \dots, Y_d]$, $Y^{\mathbf{n}} \cdot g$ is the sum of $Y^{\mathbf{n}}$ and some lower order term (in the colex ordering); that is to say, any element of $K[Y_1, \dots, Y_d]$ has a leading term which remains present under the G-action. We may deduce from this that the valuation on terms of $K[Y_1, \dots, Y_d]$ are preserved by the G-action, and so G acts continuously on M as required. Thus, by [**20**, 7.2.4].

PROPOSITION 3.8. *$K[Y_1, \dots Y_d]$ is a KG-submodule of G^*.*

We can find an alternative characterisation of $K[Y_1, \dots, Y_d]$, which sheds more light on its relation to I_G. Firstly, we need the following combinatorial fact. The author would like to thank David Saxton for the stated argument.

LEMMA 3.9. *Let R be a K-algebra for some field K of characteristic zero, and $\psi(X) \in R[X]$ a polynomial of degree d. Then $\Psi(n) = \sum_{i=1}^n \psi(i)$ is a polynomial of degree $d+1$ in n.*

Proof. It suffices to prove the lemma for $\psi_d(X) = X^d$; write $\Psi_j(n)$ for $\sum_{i=1}^n i^j$. We induct on d. For $d = 0$, $\Psi_0(n) = n$, so the induction starts. For $d > 0$, suppose that the lemma holds for all $j < d$, and consider the sum

$$\sum_{i=1}^{n} \left((i+1)^{d+1} - i^{d+1}\right) = \sum_{j=0}^{d} \left(\sum_{i=1}^{n} \binom{d+1}{j} i^j \right) = \sum_{j=0}^{d} \binom{d+1}{j} \Psi_j(n).$$

Now, this sum telescopes on the left-hand side to $(n+1)^{d+1} - 1$. Thus, by induction we see that $\Psi_d(n)$ is a polynomial of degree $d+1$. □

Now consider I_G. Note that if $f \in G^*$ is annihilated by I_G, then $f(g) = f(1)$ for all $g \in G$. Therefore f is constant on G, that is, $f \in K$. This raises the question of whether functions in G^* annihilated by a certain power of I_G are united by a certain notion of degree. We shall use the following definition to approach this question. Say f has size $\leq m$ if no monomial has an exponent (of any Y_i) greater than m, with equality if it is not of size $\leq m-1$. Write P_m for the set of polynomials in the Y_i of size at most m; that is, the mth filtered part in the size filtration.

We have just seen that $\mathsf{P}_0 = K = \mathrm{ann}_{I_G}(G^*)$. Further:

LEMMA 3.10. *For all $m \in \mathbb{N}$,*

$$\{f \in G^* : f \cdot I_G^{m+1} = 0\} \subseteq \mathsf{P}_m \subset \{f \in G^* : f \cdot I_G^n = 0, \text{ some } n \in \mathbb{N}\}.$$

Proof. We first show the first inclusion. We note that it suffices to show that any f in the first set is polynomial on $\langle g_1, \dots, g_d \rangle$ of the correct size, as this group is dense in G. We proceed by induction: we have already covered the case $m = 0$. Now

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let $f \cdot I_G^{m+1} = 0$; then by our inductive hypothesis $fI_G \subseteq \mathsf{P}_{m-1}$. More precisely, for any $y \in G$, then the action of $y^{-1} - 1$ gives some $\alpha_y \in K[Y_1, \ldots, Y_d]$ of size $< m$ such that

$$f(xy) = f(x) + \alpha_y(x), \qquad \text{for all } x \in G. \tag{1}$$

For brevity, we write $\alpha_j = \alpha_{g_j}$ for all $1 \leq j \leq d$.

We shall use a secondary induction on j (defined as follows) to utilise the power of equation (1). Fix j, and assume that whenever $f \in G^*$ is annihilated by I_G^{m+1}, then there exists some element f' of P_m such that $f(x) = f'(x)$ for all $x \in \langle g_1, \ldots, g_j \rangle$. The first inclusion will then follow on showing that there exists $f'' \in \mathsf{P}_m$ such that $f(x) = f''(x)$ for all $x \in \langle g_1, \ldots, g_{j+1} \rangle$.

Fix an element $x \in \langle g_1, \ldots, g_j \rangle$. Setting $y = g_{j+1}$ and inductively applying equation (1) yields

$$f(xg_{j+1}^{n_j}) = f(x) + \sum_{i=0}^{n_j - 1} \alpha_j(xg_{j+1}^i).$$

By the induction hypothesis on j, $f(x) = f'(x) \in \mathsf{P}_m|_{\langle g_1, \ldots, g_j \rangle}$, and so may be replaced by a polynomial in $K[Y_1, \ldots, Y_j]$ of size $\leq m$. Similarly, α_j may be viewed as a polynomial of size $\leq m - 1$ in $(K[Y_1, \ldots, Y_j])[Y_{j+1}]$ by the inductive hypothesis on m. Again, as $Y_{j+1}(xg_{j+1}^i) = i$, and $Y_k(xg_{j+1}^i) = Y_k(x)$ for all $k \leq j$, $\alpha_j(xg_{j+1}^i)$ is a polynomial of degree $\leq m - 1$ in i over $K[Y_1, \ldots, Y_j]$. Applying Lemma 3.9, the sum over i gives a degree $\leq m$ polynomial β_j in n_j over the ring $K[Y_1, \ldots Y_j]$. Thus, we obtain that $f = f' + \beta_j(Y_{j+1})$ on $\langle g_1, \ldots, g_{j+1} \rangle$, where f' has no terms in Y_j. Thus, the first inclusion in the statement of the lemma follows on the degree conditions imposed upon f' and β_j in their construction.

For the second inclusion, we let $f \in K[Y_1, \ldots, Y_d]$. By Lemma 3.7(iii), we have that $f \cdot (g_j - 1) <_{\text{colex}} f$ for all $j \in \{1, \ldots, d\}$, and further therefore that $f \cdot r <_{\text{colex}} f$ for any $r \in I_G$. Since any decreasing sequence in the colexicographic order on $\mathbb{N}^d$ terminates, we must have that products of a sufficiently large number of terms in I_G annihilates f, and these products are of bounded length. So, there exists some $n \in \mathbb{N}$ such that $fI_G^n = 0$, as required. $\square$

Remark. Lemma 3.10 expresses $K[Y_1, \ldots, Y_d]$ as a submodule of G^* independent of the choice of the generating set. Thus, the choice of an orderly generating set to prove the results in this section is seen to be merely convenient rather than necessary.

This characterisation yields the following corollary.

Corollary 3.11. *The trivial KG-module K is essential in $K[Y_1, \ldots, Y_d]$.*

Proof. We already have that $K \leq K[Y_1, \ldots, Y_d]$. Let m be minimal such that f is right annihilated by I_G^m. There exists some element $r \in I_G^{m-1}$ such that fr is a nonzero map $G \to K$. However, fr is annihilated by I_G, and so must be constant; so $G^* \cap K$ is nontrivial for any submodule G^* of $K[Y_1, \ldots, Y_d]$ containing f. Since f is arbitrary, the statement follows. $\square$

Theorem 3.12. *Let G be a uniform nilpotent group; then $E(K) = K[Y_1, \ldots, Y_d]$.*

Proof. By Corollary 3.11, $E := E(K) \geq K[Y_1, \ldots, Y_d]$. Recall also that $K = \mathsf{P}_0$. Let M be a finitely generated submodule of E. By Proposition 2.2, I_G has the

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Artin–Rees property, and so there exists $k \in \mathbb{N}$ such that:

$$MI_G^k \cap \mathsf{P}_0 = MI_G^k \cap (M \cap \mathsf{P}_0) \leq (M \cap \mathsf{P}_0)I_G = 0.$$

Since $\mathsf{P}_0 = K$ is essential in E, $MI_G^k = 0$, and so $M \leq \mathsf{P}_{k-1}$ by Lemma 3.10. Since this holds for arbitrary finitely generated $M \leq E$, we deduce that $E = K[Y_1, \ldots, Y_d]$. □

REMARK. We say that a KG-module is *locally finite* if all its finitely generated submodules are finite dimensional as K-spaces. The above proof thus also shows that $E(K)$ is locally finite. In fact in a forthcoming paper [**15**], as mentioned in the introduction, we show that the injective hull of any locally finite KG-module is also locally finite.

Theorem 3.12 enables us to recover a result already known from [**1**, Theorem A]: that the augmentation ideal I_G of the Iwasawa algebra $\mathcal{O}G$ is localisable when G is nilpotent. Since any associated prime of $K[Y_1, \ldots, Y_d]/K$ must annihilate some cyclic submodule, we only need look at such annihilators. By Lemma 3.10, the preimage of a cyclic submodule is annihilated by some power of the augmentation ideal, and thus so is the image in the quotient; hence the associated prime must contain I_G. Therefore by Jategaonkar's main Lemma, I_G satisfies the second layer condition. Further, by maximality of I_G, we have that this associated prime is I_G, and so its clique is trivial. Therefore by Theorem 2.1, I_G is (classically) localisable.

4. Injective hulls for finite-dimensional modules. We wish to use the injective hull of the trivial module K, as found in the last section, as a template for hulls of finite-dimensional modules such as those formed as quotient rings of KG by elements of $\mathrm{Prim}^{\mathrm{fd}}(KG)$. In fact, this special case is stronger than that – we do not need repeat the above section, but merely patch our finite-dimensional module onto $E(K)$. The 'patching' in this case refers to taking the tensor product $\otimes_K$ over the base field K. More precisely, for KG-modules L and M, with L finite-dimensional, $L \otimes_K M$ has a KG-module structure on defining $(l \otimes_K m)g = lg \otimes_K mg$ for $g \in G$, $l \in L$ and $m \in M$. Note that this is well defined as L is finitely generated as a K-module. More generally, we may define a KG-module structure on $L \otimes_K M$ whenever L is a locally finite KG-module, as any element $\sum_{i=1}^n l_i \otimes_K m_i$ of this lies in $L' \otimes_K M$, where L' is a finitely generated, and thus finite dimensional, submodule of L.

We first establish an extension property of injective modules via this operation.

LEMMA 4.1. *Let G be any compact p-adic Lie group. Suppose that L is a locally finite KG-module and M an injective KG-module. Then $L \otimes_K M$ is injective.*

Proof. Firstly, we reduce to the case where L is finite dimensional over K. By Baer's criterion, we only need to show that for any right ideal I of KG, any KG-homomorphism $f : I \to L \otimes_K M$ extends to a KG-homomorphism $KG \to L \otimes_K M$. However, I is finitely generated since KG is noetherian, and so $f(I)$ is finitely generated in $L \otimes_K M$, and thus lives in $L' \otimes_K M$, for some finitely generated KG-module L'. Thus, L' is finite dimensional since L is locally finite.

We may therefore consider the case $L = L'$. Set $\{l_1, \ldots, l_n\}$ to be a K-basis for L, and $\{\partial_1, \ldots, \partial_n\}$ to be the dual basis for $L^* = \mathrm{Hom}_K(L, K)$.

Let X be a KG-module. Following [**4**, 4.4(c)], we define maps

$$\theta : \mathrm{Hom}_{KG}(X, L \otimes_K M) \to \mathrm{Hom}_{KG}(X \otimes_K L^*, M) \quad \text{and}$$
$$\phi : \mathrm{Hom}_{KG}(X \otimes_K L^*, M) \to \mathrm{Hom}_{KG}(X, L \otimes_K M)$$

by

$$\theta(f)(x \otimes \lambda) = \sum_j \lambda(y_j)m_j$$
$$\phi(\tilde{f})(x) = \sum_i l_i \otimes \tilde{f}(x \otimes \partial_i),$$

for any $f \in \mathrm{Hom}_{KG}(X, L \otimes_K M), \tilde{f} \in \mathrm{Hom}_{KG}(X \otimes_K L^*, M)$, $\lambda \in L^*$, and $x \in X$, such that $f(x) = \sum_j y_j \otimes m_j$. A simple check shows these to be mutually inverse K-maps, and so θ is an isomorphism of K-vector spaces.

Let I be a right ideal of KG. Using θ, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{KG}(KG, L \otimes_K M) \arrow[r, "\sim"] \arrow[d, "\iota^*"] & \mathrm{Hom}_{KG}(KG \otimes_K L^*, M) \arrow[d, "(\iota\otimes\mathrm{id})^*"] \\
\mathrm{Hom}_{KG}(I, L \otimes_K M) \arrow[r, "\sim"] & \mathrm{Hom}_{KG}(I \otimes_K L^*, M).
\end{tikzcd}$$

Since M is injective, $(\iota\otimes\mathrm{id})^*$ is surjective. Thus, so is ι^*. The lemma now follows by Baer's criterion. □

Thus, for any finite-dimensional KG-module M we have that M is isomorphic to a submodule of the injective module $M \otimes_K E(K)$. This gives us an 'upper bound' for $E(M)$. Conversely, we wish to determine the intersection of $M \cong M \otimes_K K$ in this injective module with the cyclic submodules. Fortunately, we have determined a detailed description of the KG-action in Section 3. This enables us to prove the following technical lemma, which will provide the analogue of Lemma 3.7(iii) in the case of modules over commutative quotients of KG (and thus act in place of Lemma 3.10). Note that elements of $M \otimes_K K[Y_1, \ldots, Y_d]$ may be written as $\sum_{\mathbf{n}\in\mathbb{N}^d} m_{\mathbf{n}} \otimes Y^{\mathbf{n}}$, where $m_{\mathbf{n}} \in M$; we may therefore extend our notion of degree from Section 3 to this setting. We set $\deg(0) = (-1, 0, \ldots, 0)$ for convenience.

LEMMA 4.2. *Let G be a uniform nilpotent group and M an irreducible finite-dimensional KG-module, which has completely prime annihilator. Let $x \in M \otimes_K K[Y_1, \ldots, Y_d]$. Then, there exists $r \in \mathrm{ann}_{K[G]}(M)$ such that r has lowers* $\deg(x)$ *by one; that is,* $\deg(xr)$ *is* $\deg(x)$ *with one subtracted from the final nonzero coefficient (or from the first coefficient if all are zero).*

Proof. Write $x = \sum_{\mathbf{n}\in\mathbb{N}^d} m_{\mathbf{n}} \otimes Y^{\mathbf{n}}$ as above. Let j be maximal such that the jth coefficient of $\deg(x)$ is nonzero. From Lemma 3.7(i), we see that $(Y_1^{n_1} \ldots Y_j^{n_j})g_j = Y_1^{n_1} \ldots Y_{j-1}^{n_{j-1}}(Y_j^{n_j} g_j)$. Thus, it suffices to show that there exists some $r \in K\langle\langle g_j\rangle\rangle$ (that is, the Iwasawa algebra of the group topologically generated by g_j) such that r reduces the degree of $y = \sum_i m_i \otimes Y_j^i$ by one, where $m_i \in M$.

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Let n be the degree of y. As M is finite dimensional, there exists some polynomial $\psi(g_j) = \sum_{k=0}^{l} \lambda_k g_j^k \in \mathrm{ann}_{KG}(M)$, where $\lambda_k \in K$ for all k, of minimal degree l. By Lemma 3.7(ii), $Y_j^i g_j^k = (Y_j - k)^i$; therefore using the diagonal action of G on the tensor product and expanding brackets, we obtain for all $i \in \mathbb{N}$,

$$\begin{aligned}\left(m_i \otimes Y_j^i\right) \psi(g_j) &= \sum_k \lambda_k \left(m_i g_j^k \otimes (Y_j - k)^i\right) \\ &= \sum_k m_i(\lambda_k g_j^k) \otimes Y_j^i + \varepsilon_i(Y_j) \\ &= m_i \psi(g_j) \otimes Y_j^i + \varepsilon_i(Y_j) = \varepsilon_i(Y_j),\end{aligned}$$

where $\varepsilon_i(Y_j) = \sum_{k=0}^{i-1} m_k^{(i)} \otimes Y_j^k$ for some $m_k^{(i)} \in M$. It thus remains to show that $m_{n-1}^{(n)} \neq 0$, as this is the coefficient of Y_j^{n-1} in $y\psi(g_j)$. We can calculate this explicitly from the above formulation:

$$m_{n-1}^{(n)} = -\sum_{k=0}^{l} \lambda_k k m_n g_j^k = m_n \varphi(g_j) g_j,$$

for some polynomial $\varphi(g_j)$ of degree $l-1$. Now, $KG/(\mathrm{ann}(M))$ is commutative by hypothesis; hence, if $m_n\varphi(g_j) = 0$, then for all $s \in KG$, $m_n s\varphi(g_j) = 0$ as $s\varphi(g_j) = \varphi(g_j)s + a$, for some $a \in \mathrm{ann}(M)$. However as M is irreducible, it is fully faithful over $KG/\mathrm{ann}(M)$. Hence, $\varphi(g_j) \in \mathrm{ann}(M)$, a contradiction of the minimality of l. Thus $m_n\varphi(g_j) \neq 0$. As g_j is invertible in KG, it acts faithfully on M, and so we deduce that $m_{n-1}^{(n)} \neq 0$, as required. □

PROPOSITION 4.3. *M is essential in $M \otimes_K E(K)$ under the hypotheses of Lemma 4.2.*

Proof. Let X be a submodule of $M \otimes_K E(K)$. For any $x \in X$, $\deg(x)$ is finite, and thus by repeated iteration of Lemma 4.2, there exists some $r \in KG$ such that $xr = m \in M\backslash\{0\}$, and so $X \cap M \neq 0$, as required. □

We can now deduce the main result of this section from Lemma 4.1 and Proposition 4.3.

THEOREM 4.4. *Let G be a uniform nilpotent group and M an irreducible finite-dimensional KG-module with completely prime annihilator. Then*

$$E(M) \cong M[Y_1, \ldots, Y_d] := M \otimes_K K[Y_1, \ldots, Y_d],$$

where $Y_1, \ldots, Y_d$ are defined as in Section 3.

Note that when $P \in \mathrm{Prim}^{\mathrm{fd}}(KG)$ is completely prime, KG/P satisfies the above criteria.

REMARK. The above results cannot be extended to the class of all modules, even when G is abelian. For example, let $G \cong \mathbb{Z}_p = \langle\langle g \rangle\rangle$ and $M = KG/I_G^2$, so $E(K) \cong K[Y]$. Let $x = \left(\overline{g-1}\right) Y + \overline{g} \in M[Y]$, where $\overline{y}$ is the image of $y \in KG$ in M. Then if $r \in KG$ satisfies $\deg(xr) = 0$, then $r \in I_G$; but $x \cdot (g-1) = 0$, and so there is no such r.

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5. Proof of Theorem 1.1.

LEMMA 5.1. *Let P be a prime ideal of KG. Then KG/P is a fully faithful module over itself.*

Proof. Suppose that (the right ideal) J annihilates some nonzero KG-submodule of KG/P. This submodule must be of the form I/P for some right ideal $I > P$ of KG. But then $IJ \subseteq P$, and so since P is prime, $J \leq P$. □

We now prove Theorem 1.1. Suppose from this point on that $P \in \mathrm{Prim}^{\mathrm{fd}}(KG)$, and is completely prime. Following Jategaonkar–Brown, as stated in Section 3, we wish to show that all of the associated primes of the second layer of $E(KG/P)$, as constructed in Theorem 4.4, are P itself. By Lemma 4.2, any polynomial annihilated by P must be constant. But any constant function is annihilated by P as it is an element of KG/P. Thus, we may write the second layer as

$$(K[Y_1, \dots, Y_d]/K) \otimes (KG/P),$$

where the K in the quotient is the constant functions.

Any associated prime must be a maximal ideal amongst annihilators of nonzero cyclic submodules. So, consider an element $f \in (KG/P)[Y_1, \dots, Y_d]$ (where G acts simultaneously on the monomials and KG/P as in Section 4). By Lemma 4.2, it suffices to only consider the images of linear polynomials.

Consider $m \otimes Y_j$ for any $j \geq 1$ and $m \in KG/P$. By Lemma 3.7(iii), $(m \otimes Y_j)g_1 = mg_1 \otimes (Y_j + \delta_j)$, for some $m \in M$ and some $\delta_j \in K[Y_1, \dots Y_d]$ of lower degree in the colex ordering. Suppose that $\delta_j \neq 0$; then similarly, $\delta_j(g_1 - 1)$ has lower degree in the colex ordering. So as $P \in \mathrm{Prim}^{\mathrm{fd}}(KG)$, we may consider $\psi(g_1) = \sum_{k=0}^{l} \lambda_k g_1^k \in P$ of minimal degree l as in Lemma 4.2. A similar calculation yields

$$\begin{aligned}\left(m \otimes Y_j\right) \psi(g_1) &= \sum_k \lambda_k m g_1^k \otimes (Y_j + k\delta_j + \varepsilon_k) \\ &= m\psi(g_1) \otimes Y_j + \sum_k \lambda_k k(mg_1^k) \otimes \delta_j + \varepsilon \\ &= mg_1\varphi(g_1) \otimes \delta_j + \varepsilon,\end{aligned}$$

where $\varphi(g_1)$ is a polynomial of degree $l - 1$ and $\varepsilon_k, \varepsilon \in M[Y_1, \dots, Y_d]$ are of strictly lower degree than δ_j in the colex ordering. Thus, this term is nonzero.

The above reduces the argument to Y_j such that $\delta_j = 0$. Now, by the construction of the orderly generating set $\{g_i : 1 \leq i \leq d\}$ in Lemma 3.1, we observe that $\delta_j = 0$ if and only if $i \leq d_0$, where

$$d_0 = \dim_{\mathbb{Z}_p}(G/\sqrt{[G, G]}) = \dim_{\mathbb{F}_p}(G_{ab}/\Phi(G_{ab})),$$

where $\Phi(G_{ab})$ is the Frattini subgroup of G_{ab} [**7**, 1.13]. So, it suffices to consider only modules generated by some element of the form $\sum_{i=1}^{d_0} m_i Y_i + KG/P$, where $m_i \in KG/P$.

By the choice of d_0, we have that G acts trivially on the Y_i in the expression above, and so for any $g \in G$ and $i \leq d_0$,

$$(m_i Y_i)g = (m_i g)(Y_i g) = (m_i g) Y_i.$$

Thus, for any $r \in KG$

$$\left(\sum_{i=1}^{d_0} m_i Y_i + KG/P\right) r = \sum_{i=1}^{d_0} (m_i r) Y_i + KG/P.$$

So, the associated primes of the second layer are precisely the associated primes of KG/P. However, by Lemma 5.1, there is only one such prime, P itself. Thus, we deduce that P is the only associated prime of $(KG/P) \otimes (K[Y_1, \ldots, Y_d]/K)$. So, using the result of Jategaonkar–Brown from Section 3, the clique of P is trivial, and is therefore (classically) localisable by Theorem 2.1.

Finally, we remark that Corollary 1.2 follows instantly since injective hulls over $\mathcal{O}G$ and KG are the same.

ACKNOWLEDGEMENTS. This work was undertaken whilst the author was a PhD student at the Department of Pure Mathematics and Mathematical Statistics in Cambridge University under Simon Wadsley. Further to these, he would also like to thank Rishi Vyas for several helpful conversations, and for proofreading previous versions of the manuscript; and also to the referee for their many helpful comments, and in particular for alerting the author to **[10]** in providing a detailed proof of Lemma 3.2.

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