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ON THE GRADUATION OF 'AMOUNTS'

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ABSTRACT

The provision of graduated mortality rates, for the United Kingdom pensioners' experience, based on the so-called 'amounts' data sets is addressed. Specifically a methodology is investigated, building on the existing methods practiced by the CMI Bureau, which takes a more detailed account of the underlying structure of the data involved. The method is applied to the U.K. pensioners' experience and recent mortality trends in this experience revealed.

KEYWORDS

Graduation; Pensioners; Mortality; Trends

1. INTRODUCTION

The graduation of mortality data for the pensioners' experience under United Kingdom life office pension schemes gives rise to some distinctive technical problems. These have their origin in the work of the Continuous Mortality Investigation (CMI) Bureau, which exercises a collective responsibility for the collation and graduation of these data. Typically, for any such experience based on a fixed observation window consisting of one or more consecutive calendar years, two sets of data are available for analysis. These comprise the number of pension policies ceasing through death together with the associated exposures, the so-called 'lives' data, and the total amounts of pension ceasing through death together with the associated exposures, the so-called 'amounts' data. Both types of data sets are graduated as separate exercises by the CMI Bureau. We emphasise a key feature of this work, namely that it is a well accepted result that 'amounts' based experiences (for pensioners, annuitants, etc.) show a lower set of mortality rates than the corresponding 'lives' based experiences, except possibly at very high ages. Also, the average amount of pension per policy ceasing is less than the average amount of pension per policy in force in the exposed to risk. For a more detailed description of these and other issues involved the reader is referred to Section 5 of the CMI (1974) paper, to Section 3 of CMI Report No. 2 (1976) and the accompanying tables, and to Section 16.4 of the comprehensive paper on U.K. actuarial parametric graduation practice by Forfar, McCutcheon & Wilkie (1988) (referred to as FMW).

This paper is primarily concerned with the practice of graduating 'amounts', and aims to contribute to the discussion on the methodology for conducting these graduations instigated and practiced by the CMI Bureau in the papers cited above. For data based on 'amounts', the Bureau resolved to divide both the deaths and exposures at each age by a constant factor before graduation proceeds in the manner outlined in FMW for data based on 'lives'. The factor is calculated by dividing the total exposure across all ages based on 'amounts' by the total exposure across all ages based on 'lives'. The reasons given for applying this simple transformation to the data before graduation proceeds are not compelling other than to accommodate the application of certain well tried tests of a graduation, while possibly hinting at the need for further research into the practice of graduating 'amounts'. In Section 2 of this paper we introduce a number of assumptions designed to take a more detailed account of structure in the 'amounts' data sets than is allowed for by the simple data transformation currently applied. The assumptions, which are then discussed in Section 3, are formulated in the spirit of the graduation methods advocated by FMW and extended by Renshaw (1991, 1992). In Section 4 the alternative approach developed here is applied to the graduation of the 'amounts' for the male pensioners' 1979–82 experience, and the results compared with the FMW graduation of these data. In Section 5 the methods are used to investigate recent mortality trends in the U.K. male pensioners' experience.

In essence, the proposed method for graduating 'amounts' gives a different graduation to the CMI because variation in claim size is modelled, whereas the CMI methodology intrinsically assumes that all claim sizes are the same.

2. Distribution Assumptions

2.1 *Preliminaries*

For notational convenience, focus on a set of cells or units $\{u\}$. Either $u \equiv x$ when mortality is modelled over a set of ages $\{x\}$ for a fixed observation window in real time, or $u \equiv x, t$ when mortality is modelled over a rectangular grid defined by a set of ages $\{x\}$ and a set of calendar years $\{t\}$. Define the following (for each cell):

N_u = random variable representing the number of pension policies ceasing through deaths
r_u = exposure to risk based on 'lives'
A_u = random variable representing the amounts of pension ceasing through deaths
e_u = exposure to risk based on 'amounts'
$X_u^{(i)}$ = random variable representing the distribution of amounts per pension policy ceasing through death

such that:

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$$A_u = \sum_{i=1}^{N_u} X_u^{(i)}. \tag{2.1}$$

The data available for analysis comprise the realisations $(n_u, r_u; a_u, e_u)$ defined for the set of cells $\{u\}$, while the primary purpose of this analysis is to predict either the force of mortality or the probability of death based on the 'amounts' experience (a_u, e_u).

2.2 *Targeting the Force of Mortality*

Focus first on the prediction of the force of mortality and define the following:

μ_u = the force of mortality based on 'lives'
μ_u^* = the force of mortality based on 'amounts'.

For this case, both sets of exposures $\{r_u\}$ and $\{e_u\}$ are central exposures.

Begin with the prediction of μ_u based on the recorded mortality experience of 'lives' (n_u, r_u). To facilitate this make:

Assumption Ia. Model the N_us as independent Poisson response variables of a generalised linear (or non-linear) model (GLM) with possible overdispersion so that:

$$\mathrm{E}(N_u) = m_u = \mu_u r_u \text{ and } \mathrm{Var}(N_u) = \tau m_u.$$

The scale parameter, $\tau > 1$, which induces overdispersion in the Poisson response variable, is needed when the data (n_u, r_u) are based on the number of pension policies ceasing through deaths rather than on the number of pensioners dying, as the same pensioner may have more than one policy. A more detailed account of this effect is to be found in Renshaw (1992). Here the scale parameter is assumed constant across all cells, but this assumption can be relaxed, if deemed necessary, by making τ a function of the cells u. It is also possible to pre-set $\tau = 1$.

Focus next on $X_u^{(i)}$, the amount associated with death i and make:

Assumption II. The $X_u^{(i)}$s are modelled as independent, identically distributed non-negative random variables for all i, fixed u, independent also of N_u.

Denoting the generic form of the $X_u^{(i)}$s by X_u, the following standard results from collective risk theory are based on this assumption in combination with the identity (2.1), namely:

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$$E(A_u)=E(X_u)E(N_u)$$

and

$$\text{Var}(A_u)=\text{Var}(X_u)E(N_u)+\{E(X_u)\}^2\,\text{Var}(N_u). \tag{2.2}$$

Under Assumption Ia, $\text{Var}(N_u)=\tau E(N_u)$, so that expression (2.2), giving the variance of A_u, can be rewritten as:

$$\text{Var}(A_u)=\phi_u E(A_u) \tag{2.3}$$

where:

$$\phi_u=\frac{\text{Var}(X_u)}{E(X_u)}+\tau E(X_u). \tag{2.4}$$

Write $E(X_u)=\rho_u$ and define:

$$\bar{X}_u=\frac{A_u}{n_u}$$

to be the average amount associated with cell u, conditional on knowing the value n_u of N_u. Then make:

Assumption III. Model the cell averages $\bar{X}_u$, as independent gamma response variables of a GLM for which:

$$E(\bar{X}_u)=\rho_u \text{ and } \text{Var}(\bar{X}_u)=\frac{\psi\rho_u^2}{n_u}$$

with weights n_u, scale parameter ψ, and variance function $V(\rho_u)=\rho_u^2$.

This pre-supposes that, in addition to Assumption II, the individual amounts $X_u^{(i)}$ have the gamma density:

$$f(x|\rho_u,\alpha)=\frac{1}{\Gamma(\alpha)}\left(\frac{\alpha x}{\rho_u}\right)^{\alpha}\frac{1}{x}\exp\left(-\frac{\alpha x}{\rho_u}\right)$$

for which:

$$E(X_u)=\rho_u \text{ and } \text{Var}(X_u)=\frac{\rho_u^2}{\alpha}=\psi\rho_u^2 \tag{2.5}$$

so that $\psi=\alpha^{-1}$. For this distribution the expected value ρ_u is a function of the cells u, and the scale parameter ψ is constant across all cells. The implications of equations (2.5) for the variance of A_u follow from equation (2.3) on adjusting equation (2.4) accordingly. In order to facilitate the prediction of the μ_u^*s, this leads finally to:

Assumption IVa. Model the amounts A_u as the independent overdispersed Poisson responses of a GLM such that:

$$E(A_u) = h_u = \mu_u^* e_u \text{ and } \mathrm{Var}(A_u) = \phi_u h_u$$

where, using equations (2.4) and (2.5):

$$\phi_u = (\psi + \tau)\rho_u. \tag{2.6}$$

2.3 *Targeting the Probability of Death*

Focus next on the prediction of the probability of death. For this case it is necessary to modify Assumption Ia and Assumption IVa while retaining Assumption II and Assumption III intact. Define:

q_x or q_{xt} = the probability that a life, age x, dies before age $x+1$ based on 'lives'

q_x^* or q_{xt}^* = the probability that a life, age x, dies before age $x+1$ based on 'amounts'

and let $u \equiv x$ or $u \equiv x,t$ as the case may be. In addition, both sets of exposures $\{r_u\}$ and $\{e_u\}$ are initial exposures.

For the prediction of q_u it is necessary to replace Assumption Ia with:

Assumption Ib. Model the N_us as independent binomial response variables of a generalised linear (or non-linear) model (GLM) with possible overdispersion so that:

$$E(N_u) = m_u = q_u r_u \text{ and } \mathrm{Var}(N_u) = \tau m_u\left(1 - \frac{m_u}{r_u}\right).$$

Assumption II and Assumption III are retained, but this time, under Assumption Ib:

$$\mathrm{Var}(N_u) = \tau E(N_u)\left\{1 - \frac{E(N_u)}{r_u}\right\}$$

so that expression (2.2), giving the variance of A_u, becomes:

$$\mathrm{Var}(A_u) = \left\{\frac{\mathrm{Var}(X_u)}{E(X_u)} + \tau E(X_u)\right\} E(A_u) - \frac{\tau}{r_u}\{E(A_u)\}^2.$$

This reduces to:

$$\mathrm{Var}(A_u) = (\psi + \tau)\rho_u \mathrm{E}(A_u) - \frac{\tau}{r_u}\{\mathrm{E}(A_u)\}^2 \tag{2.7}$$

on using equation (2.5). Thus finally, for the prediction of q_u^*, it becomes necessary to replace Assumption IVa with:

Assumption IVb. Model the amounts A_u as the independent responses of a GLM with:

$$\mathrm{E}(A_u) = h_u = q_u^* e_u \text{ and } \mathrm{Var}(A_u) = (\psi + \tau)\rho_u \mathrm{E}(A_u) - \frac{\tau}{r_u}\{\mathrm{E}(A_u)\}^2.$$

3. Discussion of the Distribution Assumptions and their Implications

3.1 *Graduating 'Lives'*

Focus first on Assumptions Ia and Ib and the prediction of the μ_us or q_us as the case may be. When the set of cells $\{u\}$ is synonymous with a set of sequential ages $\{x\}$, the assumptions are identical to those underpinning modern actuarial parametric graduation methodology as practised and developed by the CMI Committee in the U.K. See e.g. FMW for a comprehensive description of this practice, while Renshaw (1991) has highlighted the connection with GLMs. When used in conjunction with an appropriate predictor-link relationship, this process supplies graduated values for the μ_xs or q_xs and an estimate for the overdispersion or scale parameter τ. For example, typically when graduating μ_x, FMW focus on Gompertz-Makeham graduation formulae of the type:

$$\mu_x = \eta_x = GM_x(r,s) = \sum_{j=0}^{r-1} \alpha_j x^j + \exp\left(\sum_{j=0}^{s-1} \beta_j x^j\right)$$

for non-negative integers r and s subject to the convention that $r=0$ implies the absence of the polynomial term and $s=0$ implies the absence of the exponentiated polynomial term. For $r>0$ and $s>0$ the right hand side $\eta_x = GM_x(r,s)$ is a non-linear predictor in the unknown parameters α_i and β_j, and is linked to μ_x by the identity function. When $u \equiv x,t$ the assumption may be viewed as an extension of current actuarial graduation practice to encompass trends in mortality. It is relevant to comment on the role played by the scale parameter in the fitting procedure. Here τ is taken to be constant across all units or ages x, and as such it does not contribute to the values of the predicted responses and graduated values, only to their standard errors. The estimation of τ is described within the context of the two applications discussed in Sections 4 and 5.

3.2 *Analysis of 'Amounts'*

Focus next on Assumption III which builds on Assumption II. This stage of the process is needed to provide estimates for the mean response amounts ρ_u and the scale parameter ψ, for use later under Assumptions IVa and IVb. It is used in conjunction with predictor-link relationships of the type:

$$g(\rho_u) = \eta_u = \sum_{j=1}^{p} z_{uj}\beta_j$$

with known covariate structure z_{uj}, unknown parameters β_j, and where typically the link function g is either the reciprocal function or the log function. Applications are presented in Sections 4 and 5. As an extreme case, by selecting the saturated model structure under this assumption so that the crude rates a_u/n_u estimate the ρ_us and by setting the scale parameter ψ to zero, it is then possible to bypass this part of the modelling process.

3.3 *Graduating 'Amounts'*

Turning next to Assumption IVa and the prediction of the μ_u^*s. When $u \equiv x$ we get the actuarial graduation methodology associated with Assumption Ia, but with one notable difference. Here the scale parameters ϕ_u are functions of the cells or ages x, and their reciprocals are needed to form the weights in the model fitting procedure. As such, and unlike the impact of constant scale parameters, they have a direct bearing on the resulting graduated values. The ϕ_us are computed using equation (2.6) once the estimates for τ, ψ and the ρ_us become available through fitting the other two GLMs.

An equivalent situation exists under Assumption IVb for the prediction of the q_u^*, except that here the variance of A_u, given by expression (2.7), although similar in form, does not partition into that of an overdispersed binomial variate in general. Assumption IVb is implemented by declaring:

$$Y_u = \frac{A_u}{e_u}, \text{ for which } \mathrm{E}(Y_u) = q_u^* \text{ and } \mathrm{Var}\,(Y_u) = \frac{1}{\omega_u}\left\{\mathrm{E}(Y_u) - \frac{\{\mathrm{E}(Y_u)\}^2}{\kappa_u}\right\}$$

as the responses with weights ω_u, where:

$$\omega_u = \frac{e_u}{(\psi + \tau)\rho_u} \text{ and } \kappa_u = \frac{r_u}{\tau\omega_u}.$$

The corresponding expression for the deviance, which is also needed to implement the model, is:

$$d(\underline{y};\ \underline{\hat{y}})=2\sum_u \omega_u\left\{y_u\log\left(\frac{y_u}{\hat{y}_u}\right)+(\kappa_u-y_u)\log\left(\frac{\kappa_u-y_u}{\kappa_u-\hat{y}_u}\right)\right\}$$

where $\hat{y}_u=\hat{q}_u^*$ denote the fitted or predicted values under the model structure.

Identical predictor-link, and hence graduation, formulae to those applied under Assumption Ia and Assumption Ib also apply under Assumption IVa and Assumption IVb respectively. The predictor structure finally adopted under each GLM is based on a thorough analysis of the associated residuals backed up by formal statistical tests of the graduation where appropriate.

3.4 *Discussion*

It is informative to contrast this methodology of graduating 'amounts' with that used previously by the CMI Bureau, albeit with certain reservations, as explained in Section 5 of the CMI (1974) paper. There no attempt is made to model the distribution of amounts, and, effectively, a constant scale parameter is applied across all ages for both μ_x graduations and q_x graduations. Here it is suggested that the data, as currently available, can provide some insight into the patterns of the claims amounts involved, and this has been modelled through the introduction of Assumptions II & III, leading to the methodology encapsulated in Assumptions IV(a & b). It transpires that the points at issue involve the second moment properties of the modelling assumptions and, as such, represent a refinement to the graduation process; a point which is demonstrated next by the application in Section 4. There is one further difference between the graduation methods used in general by the CMI Bureau as described in FMW, and the GLM formulation of these methods as described in Renshaw (1991, 1992) concerning such refinements to the graduation process. Under the GLM formulation, any such refinement is incorporated into the structure of the model and the raw data are not transformed before graduation proceeds. Under current CMI practice, the refinement is not built into the structure of the model, but rather the data are transformed, typically by dividing both the number of deaths and exposures by so-called variance ratios, before graduation proceeds.

4. An Application

4.1 *The Data*

An example, drawn from Section 6 of FMW, is presented next to illustrate the method. A comparison of the results obtained by the two approaches is also possible. The data relate to the male pensioners, normal or late, 1979–82 experience. The 'lives' data are taken from columns two and four of Table

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16.5 and the 'amounts' data reconstituted on multiplying columns two and four of Table 16.8 by £324.416, the unit amount used by FMW in their analysis of these data. The ages range from 19 to 108 years, although the data are either non-existent or extremely sparse outside the range 56 to 101 years.

4.2 *CMI Graduations*

Both sets of data are graduated by FMW using the Gompertz-Makeham formula $\mu_x = GM_x(1,s)$ with $s=3$, where:

$$\mu_x = GM_x(1,s) = \alpha + \exp\left(\sum_{j=0}^{s-1} \beta_j x^j\right). \qquad (4.1)$$

The right hand side, $GM_x(1,s)$, is a non-linear predictor in the unknown parameters α and the β_js and is linked to μ_x by the identity function. The parameters are estimated by maximising the Poisson log-likelihood expression:

$$\sum_x (n_x \log(m_x) - m_x) + \text{constant}$$

where $m_x = \mu_{x+1/2} r_x$ and n_x and r_x denote the 'actual' deaths and central exposures respectively at age x. This is identical to Assumption Ia with $\tau = 1$. Working within the GLM framework, it is possible to linearise graduation formulae of the type $\mu_x = GM_x(1,s)$ by rewriting equation (4.1) as:

$$\log(\mu_x - \alpha) = \eta_x = \sum_{j=0}^{s-1} \beta_j x^j$$

comprising a polynomial predictor η_x, which is linear in the unknown parameters β_j; while treating the α as an integral part of a parameterised log-link function. For fixed s, the optimum value of α is determined by constructing the deviance profile through the repeated fitting of this formula, subject to incremental changes in α. Since this involves a parameterised version of the log-link rather than the log-link, it is not feasible to utilise the offset facility in GLIM when fitting this model, and it becomes necessary to rewrite Assumption Ia in a slightly different form and hence declare:

$$Y_x = \frac{N_x}{r_x}, \text{ for which } E(Y_x) = \mu_{x+1/2} \text{ and } \mathrm{Var}(Y_x) = \tau \frac{\mu_{x+1/2}}{r_x}$$

as the responses with weights r_x. The deviance corresponding to the graduated values $\hat{\mu}_{x+1/2}$ is:

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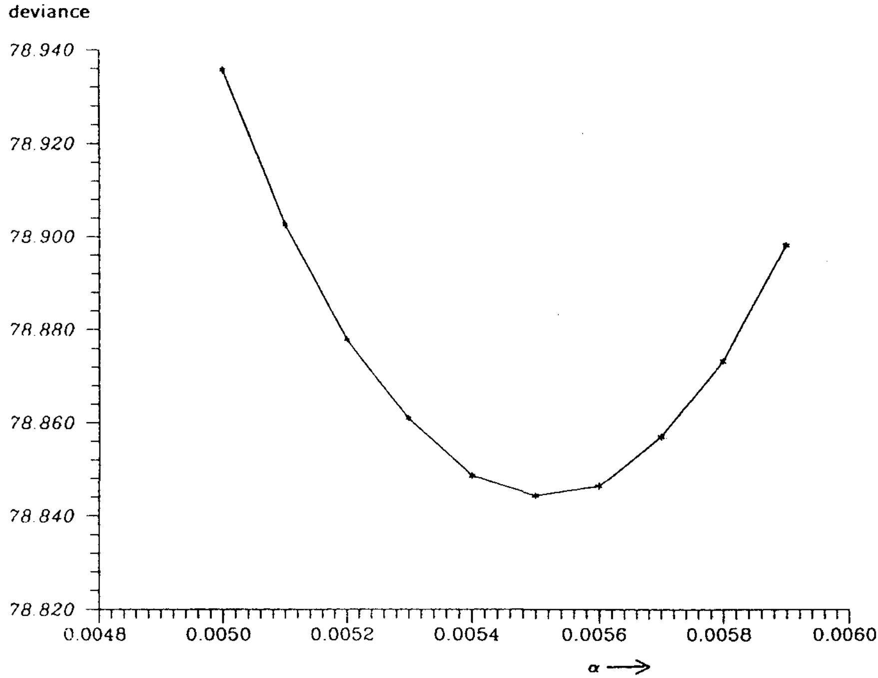

Figure. 4.1. Deviance profile, parameterised log-link, quadratic predictor in age effects

$$d(\underline{n};\underline{\hat{m}}) = 2\sum_x \left(n_x \log\left(\frac{n_x}{\hat{m}_x}\right) - (n_x - \hat{m}_x) \right)$$

where $\hat{m}_x = \hat{\mu}_{x+1/2} r_x$.

By way of illustration, we merely reproduce the deviance profile associated with fitting the 'lives' data in Figure 4.1, which has a minimum at $\alpha = 0.00551$. This differs slightly from the value $\alpha = 0.00557(29)$ quoted in Table 16.3 of FMW, a feature which would appear to be inherent in the different approaches adopted to determine the optimum value of α. In any event, this issue is of no consequence in the current context. Using the value $\alpha = 0.00557(29)$ it is possible to regraduate the data by this method and verify the detail of Table 16.5 and Table 16.8 of FMW, while the graduated values differ slightly from those quoted when based on the value $\alpha = 0.00551$. For either value of α, the corresponding optimum value of the deviance is 78.845 on 75 degrees of freedom, providing an estimate $\hat{\tau} = 1.051$ (based on the value of the optimum deviance divided by the degrees of freedom). Such a value, slightly in excess of one, is consistent with the very few known duplicates in the CMI pensioners' experience.

4.3 *Analysis of 'Amounts'*

We turn next to the application of Assumption III and the estimation of ψ and the ρ_xs. Recall that this is a necessary intermediate stage on the way to graduating 'amounts' under this formulation. Since the data are, at best, extremely sparse outside the ages 56 to 101 years, it is necessary to confine the modelling to this age range. The data and the results of the analysis are presented in Table 4.1. The responses are the average pension amounts (per annum) presented in the second column of Table 4.1. These are computed by dividing the reconstituted total amounts of pension for the policies ceasing (through death of the pensioner) by the number of policies ceasing. The latter also form the weights in this analysis and are reproduced in column three of Table 4.1. Because of the irregular pattern in the responses in the early part of the age range in the vicinity of 56 to 65 years, it was decided to model the responses with a break-point predictor of the type:

$$\eta_x = \sum_{j=0}^{J} \beta_{0j} x^j + \sum_{k=1}^{K} \sum_{j=1}^{J} \beta_{kj} (x - x_k)_+^j$$

with knots x_k, where $(x-x_k)_+^j = (x-x_k)^j$ if $x > x_k$ and $(x-x_k)_+^j = 0$ otherwise. This is then linked to the mean response ρ_x, through the log function:

$$\log(\rho_x) = \eta_x.$$

The fitted, or estimated, values $\hat{\rho}_x$ presented in column four of Table 4.1 are based on the line segment formula with $J=1$ and $K=8$ knots, positioned at ages 60, 61, 63, 65, 70, 75, 82 and 93 years. Details of the fit are presented in Table 4.2 including the estimated value of the scale parameter ψ. The knots are positioned by trial and error and the significance of the parameter estimates may be judged by referring to Table 4.2. The deviance for the current model under Assumption III is given by:

$$d(\underline{n}; \underline{\hat{m}}) = \sum_x d_x = 2 \sum_x n_x \left\{ -\log\left(\frac{a_x}{n_x \hat{\rho}_x} \right) + \left(\frac{a_x - n_x \hat{\rho}_x}{n_x \hat{\rho}_x} \right) \right\}$$

and the deviance residuals defined by:

$$\text{sign}\left\{ \frac{a_x}{n_x} - \hat{\rho}_x \right\} \sqrt{d_x}.$$

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Table 4.1. Analysis of amounts

Age x	Average a_x/n_x	Weight n_x	Fitted $\hat{\rho}_x$	Deviance residual	Estimated ϕ_x^{-1}
<56		6	281.2		0.0007735
56	1009.6	5	860.8	0.366	0.0002527
57	701.1	10	692.4	0.039	0.0003141
58	402.3	11	557.0	−1.024	0.0003904
59	594.1	7	448.1	0.783	0.0004853
60	357.4	25	360.5	−0.043	0.0006033
61	1073.7	51	1099.1	−0.166	0.0001979
62	1231.1	43	1167.2	0.352	0.0001863
63	1268.4	58	1239.5	0.176	0.0001755
64	672.6	99	708.8	−0.518	0.0003068
65	425.4	937	405.3	1.497	0.0005366
66	383.7	2408	374.8	1.153	0.0005803
67	339.4	3008	346.5	−1.135	0.0006276
68	314.1	3556	320.4	−1.189	0.0006787
69	287.1	3945	296.3	−1.961	0.0007340
70	280.7	4209	274.0	1.581	0.0007938
71	256.1	4448	259.5	−0.861	0.0008381
72	254.9	4806	245.7	2.553	0.0008850
73	229.0	4808	232.7	−1.114	0.0009345
74	224.4	5149	220.4	1.294	0.0009868
75	216.2	5047	208.7	2.508	0.0010420
76	194.8	5037	205.5	−3.770	0.0010584
77	201.2	4867	202.3	−0.375	0.0010750
78	198.0	4727	199.2	−0.409	0.0010919
79	197.2	4456	196.1	0.381	0.0011090
80	188.3	4049	193.1	−1.567	0.0011265
81	193.2	3509	190.1	0.960	0.0011442
82	196.4	3016	187.1	2.689	0.0011622
83	179.2	2448	186.8	−2.057	0.0011640
84	177.0	2126	186.5	−2.405	0.0011659
85	189.6	1775	186.2	0.760	0.0011677
86	185.3	1467	185.9	−0.139	0.0011696
87	211.6	1234	185.6	4.707	0.0011715
88	175.0	1021	185.3	−1.813	0.0011734
89	181.4	842	185.0	−0.581	0.0011752
90	194.1	627	184.8	1.247	0.0011771
91	191.5	482	184.5	0.831	0.0011790
92	184.3	365	184.2	0.015	0.0011809
93	173.2	233	183.9	−0.904	0.0011828
94	144.5	165	187.2	−3.186	0.0011617
95	144.4	134	190.6	−3.072	0.0011410
96	242.6	76	194.1	2.020	0.0011206
97	230.6	57	197.6	1.198	0.0011006
98	193.0	28	201.2	−0.217	0.0010810
99	271.3	25	204.8	1.475	0.0010617
100	103.8	11	208.6	−2.073	0.0010428
101	223.0	8	212.3	0.140	0.0010242
>101		13	151.2		0.0014381

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Table 4.2. Analysis of amounts, parameter estimates with standard errors

$\beta_{00}=$ 18.94 (10.37)	$\beta_{01}=-0.2176(0.1769)$	$\beta_{11}=1.332(0.5425)$
$\beta_{21}=-1.055(0.5205)$	$\beta_{31}=-0.6190(0.2464)$	$\beta_{41}=0.4806(0.1020)$
β_{51} $=0.02389(0.01167)$	$\beta_{61}=$ 0.03883(0.007712)	$\beta_{71}=0.01399(0.006914)$
	$\beta_{81}=$ 0.01960(0.03047)	

Deviance = 127.70 on 36 degrees of freedom with scale parameter $\psi=3.547$
$[\tau=1.051,\ \phi_x=(\tau+\psi)\rho_x]$

The latter, which are reproduced in column five of Table 4.1, form the basis of the many diagnostic checks conducted on the fit of the model. These checks are found to be supportive of the model, but are not reproduced here. Outside the age range 56 to 101 years, estimates for ρ_x are provided by the crude rates based on the grouped data for either all ages less than 56 years or all ages greater than 101 years, as the case may be.

4.4 *Graduation of 'Amounts'*

We turn finally to the application of Assumption IVa and the graduation of μ_x^*. The weights ϕ_x^{-1}, based on equation (2.6) and determined by the foregoing analysis, are given in column six of Table 4.1. It is possible to work either exclusively in the restricted age range 56 to 101 years, where the great bulk of the data lies, or in the extended age range 19 to 108 years by augmenting the weights as described above. Either way, because of the sparse nature of the data outside the age range 56 to 101 years, the results are essentially the same. In common with FMW, we focus initially on the Gompertz-Makeham formula $\mu_x^*=GM_x(1,3)$ to facilitate a comparison of the results produced by the two different approaches. The deviance profile based on the parameterised log-link is similar in shape to that displayed in Figure 4.1, with this time a minimum at $\alpha=-0.000867$ with a standard error of 0.00483. As such it does not differ significantly from zero, with the implication that we should set α to zero and focus on the formula $\mu_x^*=GM_x(0,3)$ or:

$$\log(\mu_x^*)=\sum_{j=0}^{2}\beta_j x^j.$$

Any modification to Assumption IVa is not necessary when fitting formulae of this type, since the application of the log-link to $E(A_x)$ in Assumption IVa implies that:

$$\log(h_x)=\log(e_x)+\log(\mu_x^*).$$

Thus to fit the model, the A_xs are retained as responses and the $\log(e_x)$ declared as offsets. It is of interest to note that FMW, in their graduation of

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these data, also found α to be a non-significant parameter, in spite of its retention in their final analysis. One possible strong reason for this concerns the unsatisfactory results obtained when the graduations are extrapolated down to age 19 years, where the data are effectively non existent. In order to compare results, we have decided to present the graduations in the age range 56 to 101 years, based on the formula $\mu_x^* = GM_x(0,3)$ for the two approaches. The graduations $\mu_x^{*\prime}$ for the approach based on Assumption IVa, together with the corresponding graduations $\mu_x^{*\prime\prime}$ based on the CMI approach, are given in columns two and three of Table 4.3 respectively. As anticipated, both approaches produce graduated values of a similar order of magnitude, since we are here concerned with second moment refinements to the modelling distribution underpinning the graduation process. A detailed comparison between corresponding entries in columns two and three reveals that the new rates differ by less than 1% from the old rates in the age range 64 to 93 years, increasing to 2.5% at either extremity of the age range quoted. Also for purposes of comparison, we present two separate graduations for μ_x using the corresponding 'lives' data. The first of these, presented in column four of Table 4.3, is based on the same graduation formula as the two preceding columns in Table 4.3, namely $\mu_x' = GM_x(0,3)$. The second graduation, presented in column five, is based on the different graduation formula $\mu_x'' = GM_x(1,3)$ and is therefore identical to the graduation presented in Table 16.5 of FMW. The comparison of μ_x' and μ_x'' reveals near identical graduations in the age range 63 to 92 years, say, where the data are at their thickest. The effect of including the additional parameter α in the graduation formula (equation (4.1)) is clearly visible by comparing the tails of the graduations. As already suggested, it has clearly been inserted by the CMI committee for this very reason. A comparison between either of $\mu_x^{*\prime}$ or $\mu_x^{*\prime\prime}$ and μ_x' indicates that the lower mortality rates associated with the 'amounts' based experience compared with the corresponding 'lives' based experience is not preserved as ages increase into the 90s, for the particular graduation formula in question.

5. Modelling Recent Short-Term Trends in Male Pensioner Mortality

5.1 *Preliminaries*

As a second application of the approach we model recent short-term mortality trends in the U.K. male pensioner experience, which is of considerable intrinsic interest in any event. The data, made available by the CMI Bureau, comprise initial exposures and the associated number of 'deaths' for both 'lives' and 'amounts' for individual ages x ranging from 60 to 95 years inclusive and for individual calendar years t, ranging from 1983 to 1990 inclusive; a total of $36 \times 8 = 288$ cells $u \equiv x, t$. We target q_{xt} and q_{xt}^* of Section 2.3 respectively.

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Table 4.3. Force of mortality, male pensioners experience 1979–82

For 'amounts' – $\mu_x^{*\prime} = GM_x(0,3)$ using new approach, $\mu_x^{*\prime\prime} = GM_x(0,3)$ using CMI approach.
For 'lives' – $\mu_x' = GM_x(0,3)$ using CMI approach, $\mu_x'' = GM_x(1,3)$ using CMI approach.

x	$\mu_x^{*\prime}$	$\mu_x^{*\prime\prime}$	μ_x'	μ_x''
56	0.00618	0.00634	0.00829	0.01091
57	0.00706	0.00722	0.00941	0.01179
58	0.00805	0.00822	0.01067	0.01281
59	0.00916	0.00934	0.01208	0.01397
60	0.01041	0.01059	0.01364	0.01528
61	0.01181	0.01199	0.01538	0.01677
62	0.01338	0.01356	0.01732	0.01846
63	0.01514	0.01531	0.01946	0.02037
64	0.01709	0.01725	0.02182	0.02250
65	0.01926	0.01941	0.02443	0.02490
66	0.02167	0.02181	0.02730	0.02757
67	0.02435	0.02447	0.03046	0.03055
68	0.02730	0.02741	0.03392	0.03386
69	0.03057	0.03065	0.03770	0.03752
70	0.03416	0.03422	0.04183	0.04156
71	0.03812	0.03815	0.04632	0.04600
72	0.04246	0.04246	0.05121	0.05086
73	0.04722	0.04719	0.05651	0.05618
74	0.05242	0.05236	0.06225	0.06196
75	0.05809	0.05800	0.06845	0.06825
76	0.06428	0.06415	0.07513	0.07504
77	0.07100	0.07085	0.08231	0.08237
78	0.07829	0.07811	0.09002	0.09025
79	0.08619	0.08599	0.09827	0.09868
80	0.09472	0.09452	0.10708	0.10768
81	0.10392	0.10372	0.11648	0.11726
82	0.11383	0.11364	0.12647	0.12740
83	0.12447	0.12432	0.13707	0.13811
84	0.13587	0.13579	0.14892	0.14838
85	0.14807	0.14808	0.16014	0.16120
86	0.16110	0.16123	0.17262	0.17353
87	0.17497	0.17527	0.18575	0.18635
88	0.18973	0.19024	0.19951	0.19964
89	0.20538	0.20616	0.21391	0.21335
90	0.22194	0.22306	0.22893	0.22743
91	0.23944	0.24096	0.24457	0.24184
92	0.25789	0.25990	0.26080	0.25651
93	0.27729	0.27988	0.27761	0.27138
94	0.29765	0.30092	0.29597	0.28638
95	0.31896	0.32304	0.31286	0.30144
96	0.34123	0.34624	0.33123	0.31647
97	0.36443	0.37052	0.35004	0.33139
98	0.38856	0.39589	0.36927	0.34612
99	0.41360	0.42232	0.38884	0.36057
100	0.43950	0.44981	0.40871	0.37465
101	0.46624	0.47834	0.42883	0.38825

5.2 *Modelling based on 'Lives'*

We begin with the analysis of the data based on 'lives'. The binomial modelling distribution is selected, since we are dealing here with initial exposures. We use Assumption Ib in conjunction with the complementary log–log link function:

$$\log\left\{-\log\left(1-\frac{m_{xt}}{r_{xt}}\right)\right\}=\log\{-\log(1-q_{xt})\}=\eta_{xt}$$

and polynomial predictor formulae of the type:

$$\eta_{xt}=\beta_0+\sum_{j=1}^{s}\beta_j L_j(x')+\sum_{i=1}^{r}\alpha_i t'^i+\sum_{i=1}^{r}\sum_{j=1}^{s}\gamma_{ij}L_j(x')t'^i \quad (5.1)$$

in which some of the parameters may be pre-set to zero. Here both the age and calendar year ranges have been mapped onto the interval $[-1,1]$ by translating the origin to the centre of the range concerned and then using the semi-range for scaling. We write:

$$x'=\frac{x-c_x}{\omega_x},\ t'=\frac{t-c_t}{\omega_t}$$

to denote the transformed ages and transformed calendar years respectively, where:

$$c_x=\frac{x_{\min}+x_{\max}}{2},\ \omega_x=\frac{x_{\max}-x_{\min}}{2}$$

with equivalent expressions for c_t and ω_t in terms of the maximum and minimum calendar years. The Legendre polynomials $L_j(x)$, of degree j, are generated by:

$$L_0(x)=1\ \ L_1(x)=x,\ (n+1)L_{n+1}(x)=(2n+1)xL_n(x)-nL_{n-1}(x)\ \text{(integer } n\geq 1)$$

so that:

$$L_2(x)=\frac{3x^2-1}{2},\quad L_3(x)=\frac{5x^3-3x}{2},\ \ldots$$

Similar predictor based formulae have been used by Renshaw, Haberman & Hatzopoulos (1996) to investigate mortality trends in the U.K. male assured

Table 5.1. Predictions based on 'lives', parameter estimates with standard errors

	$\beta_0 = -2.657$ (0.006440)	
$\beta_1 =$ 1.642 (0.01432)	$\beta_2 = -0.1665$ (0.01396)	$\beta_3 = -0.03783$ (0.01624)'
$\alpha_1 = -0.04721$ (0.01268)	$\alpha_2 = -0.03314$ (0.009019)	$\alpha_3 = -0.03846$ (0.01622)
	$\gamma_{11} = -0.02405$ (0.01394)	

Deviance = 442.28 on 280 degrees of freedom with scale parameter $\tau = 1.580$

lives experience. If the α_is and γ_{ij}s are pre-set to zero so that the model excludes trend effects, the combined formulae reduce to Gompertz 'law' on setting $s = 1$, and represent a generalisation of Gompertz 'law' otherwise. Writing equation (5.1) as:

$$\eta_{xt} = \sum_{j=0}^{s} \beta_j L_j(x') + \sum_{i=1}^{r} \left(\alpha_i + \sum_{j=1}^{s} \gamma_{ij} L_j(x') \right) t'^i$$

the second batch of terms may be interpreted as trend adjustment terms which are age specific, provided not all the γ_{ij}s are pre-set to zero.

An examination of the deviance profile induced by changes in the structure of the linear predictor formula (5.1), coupled with copious graphical tests of the corresponding deviance residuals, leads to the adoption of the model formula:

$$\log\{-\log(1 - q_{xt})\} = \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + \sum_{i=1}^{3} \alpha_i t'^i + \gamma_{11} L_1(x') t'.$$

The left hand side coupled with the first four terms of the right hand side may be interpreted as a natural extension to Gompertz 'law' as suggested by Renshaw (1991), while the four remaining terms on the right hand side, the last of which is age specific, may be collectively interpreted as a trend perturbation effect. Specific details of the fit are presented in Table 5.1, but details of the copious tests, which are highly supportive of the model structure, are withheld to save space. The resulting predicted values of q_{xt} are presented in Table 5.2. It is of particular interest to monitor the percentage improvement per year in mortality predicted by the model as a function of age. This is done for the period 1983 to 1990 by computing values of the statistic:

$$\iota_x = 100 \times \left\{ 1 - \sqrt[7]{\frac{q_{x,1990}}{q_{x,1983}}} \right\}.$$

Such values, based on the appropriate entries taken from Table 5.2, for

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Table 5.2. Predicted q_{xt} probabilities based on 'lives' (x = age, t = calendar year)

Age	1983	1984	1985	1986	1987	1988	1989	1990
60	0.01280	0.01245	0.01219	0.01198	0.01174	0.01141	0.01094	0.01029
61	0.01426	0.01386	0.01359	0.01335	0.01309	0.01273	0.01221	0.01150
62	0.01587	0.01544	0.01514	0.01489	0.01460	0.01420	0.01363	0.01284
63	0.01767	0.01720	0.01687	0.01660	0.01628	0.01584	0.01521	0.01433
64	0.01967	0.01915	0.01880	0.01850	0.01815	0.01767	0.01697	0.01600
65	0.02189	0.02132	0.02093	0.02061	0.02023	0.01970	0.01893	0.01786
66	0.02435	0.02373	0.02330	0.02295	0.02254	0.02196	0.02111	0.01992
67	0.02707	0.02639	0.02593	0.02554	0.02510	0.02446	0.02353	0.02221
68	0.03007	0.02933	0.02883	0.02841	0.02793	0.02723	0.02620	0.02475
69	0.03338	0.03257	0.03203	0.03158	0.03105	0.03029	0.02916	0.02756
70	0.03702	0.03613	0.03555	0.03507	0.03450	0.03367	0.03243	0.03065
71	0.04101	0.04005	0.03942	0.03890	0.03828	0.03738	0.03602	0.03407
72	0.04539	0.04434	0.04366	0.04310	0.04244	0.04146	0.03996	0.03782
73	0.05016	0.04903	0.04830	0.04770	0.04699	0.04592	0.04428	0.04193
74	0.05537	0.05414	0.05335	0.05272	0.05195	0.05079	0.04900	0.04642
75	0.06102	0.05969	0.05885	0.05817	0.05735	0.05609	0.05415	0.05132
76	0.06713	0.06570	0.06481	0.06408	0.06320	0.06185	0.05973	0.05664
77	0.07374	0.07220	0.07124	0.07048	0.06954	0.06808	0.06578	0.06241
78	0.08084	0.07919	0.07817	0.07736	0.07637	0.07479	0.07231	0.06864
79	0.08844	0.08668	0.08560	0.08475	0.08370	0.08201	0.07933	0.07535
80	0.09656	0.09468	0.09355	0.09266	0.09154	0.08974	0.08685	0.08254
81	0.10520	0.10319	0.10200	0.10108	0.09990	0.09798	0.09487	0.09022
82	0.11434	0.11221	0.11096	0.11000	0.10877	0.10673	0.10340	0.09839
83	0.12397	0.12172	0.12042	0.11943	0.11814	0.11598	0.11242	0.10704
84	0.13408	0.13171	0.13035	0.12933	0.12799	0.12571	0.12192	0.11616
85	0.14462	0.14214	0.14073	0.13969	0.13830	0.13590	0.13187	0.12572
86	0.15558	0.15298	0.15153	0.15047	0.14904	0.14652	0.14225	0.13571
87	0.16689	0.16418	0.16270	0.16162	0.16015	0.15752	0.15302	0.14607
88	0.17851	0.17569	0.17418	0.17310	0.17160	0.16886	0.16412	0.15678
89	0.19038	0.18746	0.18592	0.18484	0.18331	0.18047	0.17551	0.16777
90	0.20241	0.19940	0.19785	0.19678	0.19523	0.19230	0.18712	0.17898
91	0.21454	0.21145	0.20989	0.20883	0.20728	0.20426	0.19887	0.19035
92	0.22668	0.22352	0.22196	0.22093	0.21937	0.21627	0.21068	0.20180
93	0.23874	0.23552	0.23397	0.23297	0.23142	0.22826	0.22248	0.21324
94	0.25062	0.24735	0.24582	0.24486	0.24333	0.24011	0.23416	0.22459
95	0.26223	0.25892	0.25742	0.25651	0.25500	0.25174	0.24564	0.23575

Table 5.3. Measure of percentage improvement per year in mortality by age (ι_x—for 'lives', ι_x^*—for 'amounts', x—age)

Age	ι_x	ι_x^*
60	3.07	2.83
65	2.86	2.83
70	2.66	2.81
75	2.44	2.78
80	2.22	2.73
85	1.98	2.66
90	1.86	2.56
95	1.51	2.41

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Table 5.4. Predicted ρ_{xt} values (x = age, t = calendar year) with $\hat{\psi} = 6.134$

Age	1983	1984	1985	1986	1987	1988	1989	1990
60	630	5293	3849	2895	956	2962	1066	4164
61	904	3868	3064	3955	1258	3188	1741	4998
62	1297	2826	2438	5403	1655	3431	2844	5999
63	1860	2065	1941	7380	2178	3693	4646	7201
64	1030	1280	1298	2578	1436	2171	2694	3232
65	570	794	869	900	947	1276	1562	1451
66	524	714	788	825	879	1151	1419	1389
67	482	642	714	755	816	1039	1289	1330
68	443	577	648	692	758	938	1171	1274
69	407	519	588	633	704	847	1065	1220
70	374	467	533	580	653	764	968	1168
71	344	420	483	531	606	690	879	1119
72	316	378	438	487	563	623	799	1071
73	290	340	398	446	523	562	726	1026
74	267	306	361	408	485	507	660	982
75	245	275	327	373	450	458	600	941
76	232	295	287	360	396	432	537	638
77	246	271	279	349	378	416	493	600
78	262	248	271	338	359	401	453	564
79	279	228	263	327	342	386	416	530
80	261	231	256	301	319	350	396	460
81	244	233	250	278	297	318	377	399
82	229	236	244	256	277	288	359	346
83	223	233	238	249	272	287	342	336
84	217	230	233	243	267	286	327	327
85	212	228	227	236	262	284	312	319
86	207	225	222	230	257	283	298	310
87	201	222	217	223	252	282	284	302
88	196	220	211	217	248	280	271	294
89	191	217	207	212	243	279	259	286
90	187	215	202	206	239	278	247	279
91	192	217	214	209	231	250	262	262
92	198	220	226	212	223	226	278	247
93	203	223	239	215	215	203	295	232
94	209	225	253	218	208	183	313	219
95	215	228	268	221	201	165	332	206

values of x at five yearly intervals, are reported in the second column of Table 5.3. It is of interest to note that the pattern of these values with age is consistent with external evidence to the effect that the rates of improvement in mortality decrease monotonically with age.

5.3 *Modelling based on the 'Amounts'*

We turn next to Assumption III and the prediction of the scale parameter ψ and average claim amounts ρ_{xt}. Again, because of the characteristic pattern in the responses similar to those encountered in Section 4, we resort to using the same spline predictor in conjunction with the log-link. The fitted or

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Table 5.5. Predicted q^*_{xt} probabilities based on 'amounts' (x=age, t=calendar year)

Age	1983	1984	1985	1986	1987	1988	1989	1990
60	0.00919	0.00893	0.00867	0.00843	0.00819	0.00796	0.00773	0.00751
61	0.01040	0.01010	0.00981	0.00954	0.00927	0.00901	0.00875	0.00850
62	0.01176	0.01142	0.01110	0.01079	0.01048	0.01018	0.00990	0.00962
63	0.01328	0.01290	0.01254	0.01218	0.01184	0.01150	0.01118	0.01086
64	0.01497	0.01455	0.01414	0.01374	0.01335	0.01297	0.01261	0.01225
65	0.01687	0.01639	0.01593	0.01548	0.01504	0.01462	0.01420	0.01380
66	0.01898	0.01844	0.01792	0.01742	0.01692	0.01645	0.01598	0.01553
67	0.02132	0.02072	0.02014	0.01957	0.01902	0.01848	0.01796	0.01745
68	0.02393	0.02325	0.02260	0.02196	0.02134	0.02074	0.02016	0.01959
69	0.02681	0.02606	0.02533	0.02462	0.02392	0.02325	0.02260	0.02196
70	0.03001	0.02917	0.02835	0.02755	0.02678	0.02603	0.02530	0.02458
71	0.03354	0.03260	0.03169	0.03080	0.02994	0.02910	0.02828	0.02749
72	0.03744	0.03640	0.03538	0.03439	0.03343	0.03249	0.03158	0.03069
73	0.04174	0.04057	0.03944	0.03834	0.03727	0.03623	0.03522	0.03423
74	0.04646	0.04517	0.04391	0.04269	0.04150	0.04034	0.03922	0.03812
75	0.05165	0.05021	0.04882	0.04746	0.04614	0.04486	0.04361	0.04240
76	0.05733	0.05574	0.05420	0.05270	0.05124	0.04982	0.04843	0.04709
77	0.06355	0.06180	0.06009	0.05843	0.05681	0.05524	0.05371	0.05222
78	0.07033	0.06840	0.06652	0.06469	0.06291	0.06117	0.05948	0.05784
79	0.07773	0.07560	0.07353	0.07151	0.06955	0.06763	0.06578	0.06397
80	0.08577	0.08343	0.08116	0.07894	0.07678	0.07468	0.07263	0.07064
81	0.09450	0.09194	0.08944	0.08700	0.08464	0.08233	0.08008	0.07789
82	0.10395	0.10114	0.09841	0.09575	0.09315	0.09062	0.08816	0.08576
83	0.11416	0.11110	0.10811	0.10520	0.10236	0.09960	0.09691	0.09428
84	0.12517	0.12183	0.11857	0.11540	0.11231	0.10929	0.10635	0.10348
85	0.13700	0.13337	0.12983	0.12638	0.12302	0.11973	0.11653	0.11341
86	0.14970	0.14577	0.14193	0.13818	0.13452	0.13095	0.12747	0.12408
87	0.16329	0.15903	0.15487	0.15082	0.14686	0.14299	0.13921	0.13553
88	0.17779	0.17320	0.15871	0.16432	0.16004	0.15586	0.15178	0.14779
89	0.19323	0.18828	0.18345	0.17872	0.17411	0.16959	0.16519	0.16089
90	0.20962	0.20430	0.19911	0.19403	0.18906	0.18421	0.17947	0.17483
91	0.22695	0.22126	0.21570	0.21026	0.20493	0.19972	0.19463	0.18965
92	0.24525	0.23918	0.23323	0.22741	0.22172	0.21614	0.21069	0.20535
93	0.26449	0.25803	0.25170	0.24550	0.23942	0.23347	0.22764	0.22194
94	0.28466	0.27781	0.27108	0.26449	0.25803	0.25170	0.24550	0.23942
95	0.30574	0.29849	0.29137	0.28439	0.27754	0.27082	0.26423	0.25778

Table 5.6. Predictions based on 'amounts', parameter estimates with standard errors

$\beta_0 = -2.830$ (0.01136)	$\beta_1 = 1.839$ (0.01992)	$\beta_2 = -0.1174$ (0.02918)
	$\alpha_1 = -0.1011$ (0.01144)	

predicted values, $\hat{\psi}$ and $\hat{\rho}_{xt}$, presented in Table 5.4, were determined by fitting this formula with knots positioned at ages 63, 65, 75, 76, 79, 82 and 90 years for each calendar year separately.

Turning finally to the application of Assumption IVb and the prediction of

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q^*_{xt}. The results presented in Table 5.5 are based on the relatively simple formula:

$$\log\{-\log(1-q^*_{xt})\}=\beta_0+\beta_1 L_1(x')+\beta_2 L_2(x')+\alpha_1 t'$$

which is supported by the data, subject to a few outliers, and the parameter estimates associated with the fit are presented in Table 5.6. Unlike the previous fit based on 'lives', these data are not supportive of higher order terms in calendar year effects. The corresponding values in the mortality improvement factors are recorded in the third column of Table 5.3.

6. Summary

The approach developed here for the graduation of 'amounts' pays more attention to the intrinsic structure of the data than the approach used previously by the CMI Bureau. The methodology is strongly connected with earlier work by Renshaw (1992) on duplicate policies, whose effects on the graduation approach are modelled through over-dispersion. Thus, here a person with a multiple unit of amount of benefit b say, is equivalent to a person with b sets of 1 unit benefit amount, and the multiple stage modelling develops naturally from earlier work. The essential difference between the two approaches centres on the different specification of the second moment properties of the modelling distribution.

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