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PROBLEMS AND SOLUTIONS

PROBLEMS

04.2.1. A Range Equality for Block Matrices with Orthogonal Projectors

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When considering the general Gauss–Markov model $Y = X\beta + e$, where $E(e) = 0$, $\mathrm{Cov}(e) = V$, one often needs to know whether the range inclusion $\mathrm{range}(X) \subseteq \mathrm{range}(V)$ holds. Show that if V is the following tridiagonal block matrix:

$$V = \begin{bmatrix} P+Q & PQ & & \\ QP & P+Q & \ddots & \\ & \ddots & \ddots & PQ \\ & & QP & P+Q \end{bmatrix}_{n\times n},$$

where P and Q are two orthogonal projectors of the same size, then

$$\mathrm{range}(V) = \mathrm{range}\begin{bmatrix} P+Q & & \\ & \ddots & \\ & & P+Q \end{bmatrix}_{n\times n}.$$

04.2.2. Characterizations of Hermitian Projectors

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Let P in $\mathbb{C}^{n\times n}$ be a projector with P^+ its Moore–Penrose inverse and P^H its conjugate transpose. Werner (2002) provides a list of equivalent conditions for P to be Hermitian: (i) $P = PP^HP$, (ii) $P^+P^+ = P^+$, (iii) $P^+ = P$, and (iv) $P^+ = P^H$. Extend this list and show that also condition (a) (resp. (b)) is sufficient and necessary for a projector P to be Hermitian:

(a) the composition P^HP is a projector,
(b) the composition PP^H is a projector.

REFERENCE

Werner, H.J. (2002) Partial isometry and idempotent matrices. Solution 28-7.5. *IMAGE, The Bulletin of the International Linear Algebra Society* **29**, 31–32.

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SOLUTION

04.2.1. Fixed Effects Estimation of the Population-Averaged Slopes in a Panel Data Random Coefficient Model—Solution

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a. Just substitute $\mathbf{b}_i = \boldsymbol{\beta} + \mathbf{d}_i$ into (1) and rearrange:

$$y_{it} = a_i + \mathbf{x}_{it}(\boldsymbol{\beta} + \mathbf{d}_i) + u_{it} = a_i + \mathbf{x}_{it}\boldsymbol{\beta} + (\mathbf{x}_{it}\mathbf{d}_i + u_{it})$$
$$\equiv a_i + \mathbf{x}_{it}\boldsymbol{\beta} + v_{it}, \qquad t = 1,\dots,T,$$

where $v_{it} \equiv \mathbf{x}_{it}\mathbf{d}_i + u_{it}$.

b. The time-demeaned equation is simply

$$\ddot{y}_{it} = \ddot{\mathbf{x}}_{it}\boldsymbol{\beta} + (\ddot{\mathbf{x}}_{it}\mathbf{d}_i + \ddot{u}_{lt}) = \ddot{\mathbf{x}}_{it}\boldsymbol{\beta} + \ddot{v}_{it}, \qquad t = 1,\dots,T, \tag{5}$$

where $\ddot{v}_{it} \equiv \ddot{\mathbf{x}}_{it}\mathbf{d}_i + \ddot{u}_{it}, t = 1,\dots,T$.

c. The fixed effects estimator is simply the pooled ordinary least squares estimator applied to the time-demeaned equation (5). So we can write the fixed effects estimator as

$$\hat{\boldsymbol{\beta}} = \left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\ddot{\mathbf{x}}_{it}'\ddot{y}_{it}\right)$$
$$= \boldsymbol{\beta} + \left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\ddot{\mathbf{x}}_{it}'\ddot{v}_{it}\right)$$
$$= \boldsymbol{\beta} + \left(N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{X}}_{i}'\ddot{\mathbf{X}}_{i}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\ddot{\mathbf{x}}_{it}'\ddot{v}_{it}\right).$$

By the rank condition (3) and the weak law of large numbers (WLLN), $(N^{-1}\sum_{i-1}^{N}\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i)^{-1} \xrightarrow{p} \mathbf{A}^{-1}$, where $\mathbf{A} \equiv \mathrm{E}(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i)$ Consistency follows, again by the WLLN, if

$$\mathrm{E}(\ddot{\mathbf{x}}_{it}'\ddot{v}_{it}) = \mathbf{0}, \qquad t = 1,\dots,T.$$

Sufficient is $\mathrm{E}(\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}\mathbf{d}_i) = \mathbf{0}$ and $\mathrm{E}(\ddot{\mathbf{x}}_{it}'\ddot{u}_{it}) = \mathbf{0}$, $t = 1,\dots,T$. But $\mathrm{E}(\ddot{u}_{it}|\ddot{\mathbf{x}}_{it}) = 0$, $t = 1,\dots,T$, under (2), which implies $\mathrm{E}(\ddot{\mathbf{x}}_{it}'\ddot{u}_{it}) = \mathbf{0}, t = 1,\dots,T$. Further, (4) is equivalent to $\mathrm{E}(\mathbf{d}_i|\ddot{\mathbf{x}}_{it}) = \mathbf{0}, t = 1,\dots,T$; the law of iterated expectations then implies that $\mathrm{E}(\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}\mathbf{d}_i) = \mathbf{0}$.

Condition (4) only restricts the mean dependence between $\mathbf{b}_i$, and the time-demeaned variables, $\ddot{\mathbf{x}}_{it}$; it places no direct restriction on the dependence between $\mathbf{b}_i$, and $\bar{\mathbf{x}}_i$. Often, it is reasonable to think that although the time averages of some covariates would be correlated with unobserved heterogeneity—in particular, with $\mathbf{b}_i$—deviations about the time average might not be. So the fixed

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effects estimator, although not robust to arbitrary dependence between $\mathbf{b}_i$ and $(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{iT})$, does have some useful robustness properties in the general random coefficient panel data model.

d. Zero correlation does not suffice because we need $\mathbf{b}_i$ to be uncorrelated with $\ddot{\mathbf{x}}_{it}'\ddot{\mathbf{x}}_{it}, t = 1, 2, \ldots, T$.

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