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AN ORDERING OF MEASURES OF THE WELFARE COST OF INFLATION IN ECONOMIES WITH INTEREST-BEARING DEPOSITS

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This paper builds on Lucas [*Econometrica* 68 (2000), 247–274] and on Cysne [*Journal of Money, Credit and Banking* 35 (2003), 221–238] to derive and order six alternative measures of the welfare costs of inflation (five of them already existing in the literature) for any vector of opportunity costs. We provide examples and closed-form solutions for each welfare measure based both on log–log and on semilog money demands, whenever possible in terms of elementary functions. Estimates of the maximum relative error a researcher can incur when using any of these measures are given. Everything is done for economies with or without interest-bearing deposits.

Keywords: Inflation, Welfare, Interest-Bearing Assets, Money Demand, Divisia Index

1. INTRODUCTION

Measurement of the welfare cost of inflation, even when restricted to a money-demand approach [e.g., Bailey (1956), Lucas (2000), Simonsen and Cysne (2001), Cysne (2003), Jones et al. (2004), and Serletis and Yavari (2004)], presents the researcher with at least two types of problems.

First is that of choosing among the several welfare measures existing in the literature and being aware of the size of the resulting bias with regard to the other measures. For instance, Lucas (2000) derives a measure of the welfare cost of inflation based on the Sidrauski (1967) model and another one based on the McCallum–Goodfriend (1987) model. He shows, using numerical simulations, that both measures are very close to Bailey's (1956). However, he does not provide, either analytically or numerically, an ordering among these three functions of the nominal interest rate. An empirical investigation [like the one carried out in Ireland (2009), where Bailey's unidimensional measure is employed] might profit from knowing how such measures compare to each other (if they can be ordered) and, moreover, what would be the consequences (the maximum relative error, for instance) for the welfare figures of using any particular measure vis-à-vis the others.[1]

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A second problem the researcher might face is the necessity of taking into consideration the existence in the economy of interest-bearing deposits performing monetary functions. This leads us, as conjectured by Lucas (2000, p. 270) and later shown by Cysne (2003), to welfare measures based on Divisia indices of monetary services.[2] Using the "unidimensional" welfare formulas (i.e., disregarding other types of monies) in this case can be misleading, particularly because the demand for non-interest-bearing money can be very sensitive to variations in the opportunity costs of other assets providing monetary services.

Just as different unidimensional formulas can be defined, different Divisia indices can be used to calculate welfare losses in the multidimensional setting (dimension here referring to the number of assets performing monetary services in the economy). Once again, the question of how such welfare measures relate to each other emerges, but now in a more complicated fashion. Knowing in advance that some measures always lead to welfare figures that are higher (or lower) than others, for any vector of opportunity costs, is again valuable information for those interested in investigating the losses generated by inflation.

Answers to such questions have been provided in the literature by Simonsen and Cysne (2001) and by Cysne (2003), but only in relation to four measures of the welfare cost of inflation: Lucas's (2000) shopping-time measure, two (easier-to-calculate) approximations of Lucas's measure introduced by Simonsen and Cysne (2001), and Bailey's measure.

This paper extends such contributions, both in the unidimensional and in the multidimensional case, by reconsidering the previous ordering with respect to two additional measures: Lucas's (2000) measure based on Sidrauski's model, which he derives taking as a reference the GDP prevailing at the current interest rate; and a measure that is new in the literature, the one emerging from Sidrauski's model when the reference for income compensation is the GDP associated with an interest rate equal to zero.

The remainder of the paper is organized as follows. In Section 2 our results concerning the unidimensional case worked out by Lucas (2000) are presented. Section 3 extends those results to an economy where different assets perform monetary functions. Section 4 is devoted to analytically and numerically exemplifying our results with the bilogarithmic and semilogarithmic money-demand specifications, as well as presenting the resulting maximum relative error. Section 5 concludes.

2. SIX ALTERNATIVE MEASURES OF THE WELFARE COST OF INFLATION

In this section the case of an economy with only one type of (non-interest-bearing) money is analyzed. We present six alternative measures of the welfare cost of inflation and show how they can be ordered. By this it is meant that all of these six functions can be pairwise compared, i.e., do not cross each other.

2.1. The Shopping-Time Measure and Its Approximations

Lucas (2000, p. 265) shows that in McCallum and Goodfriend's (1987) shopping-time model the welfare cost of inflation (equivalently, the portion of time dedicated to shopping) s is given as the solution to the initial value problem

$$\begin{cases} s'(r) = -\dfrac{-r\eta'(r)}{1 - s(r) + r\eta(r)}(1 - s(r)) \\ s(0) = 0 \end{cases}, \tag{1}$$

where $r \in \mathbf{R}_+$ stands for the nominal interest rate and $\eta : \mathbf{R}_+ \to (0, +\infty]$ is the equilibrium strictly decreasing and differentiable money-demand function.

Since the nonseparable differential equation in (1) does not have any obvious solution in the general case, it is natural to look for approximations.[3] Simonsen and Cysne (2001) have shown that reasonable approximations to s are the upper bound $A : \mathbf{R}_+ \to [0, +\infty]$ given by

$$A(r) := \int_0^r -\frac{\rho\eta'(\rho)}{1 + \rho\eta(\rho)}\, d\rho = \int_{\eta(r)}^{\eta(0)} \frac{\psi(\mu)}{1 + \mu\psi(\mu)}\, d\mu, \tag{2}$$

where $\psi := \eta^{-1}$, and the lower bound $1 - e^{-A}$.

Let $B : \mathbf{R}_+ \to [0, +\infty]$ stand for Bailey's (1956) measure

$$B(r) := \int_0^r -\rho\eta'(\rho)\, d\rho = \int_{\eta(r)}^{\eta(0)} \psi(\mu)\, d\mu, \tag{3}$$

the area under the inverted money-demand curve. Simonsen and Cysne (2001) have also shown that $1 - e^{-A} < s < A < B$, where we write "$f < g$" for "$f(r) < g(r), \forall r \in \mathbf{R}_{++}$," since all these measures coincide only at 0. This notation for the ordering of unidimensional welfare measures will be used throughout the paper.

2.2. The Sidrauski Model

In the next subsection, two measures of the welfare cost of inflation that emerge from the Sidrauski (1967) model will be introduced. Both are based on Lucas's version of this model, and since one of them is new in the literature, the model is presented here in detail. Another reason for doing so is that later on we will want to generalize this model, so that it also accounts for the possibility of existence of other types of monies in the economy.

We shall assume a forever-living perfectly foresighted representative agent maximizing a time-separable constant relative risk aversion utility function, the arguments of which are the flows of real consumption of a single nonmonetary nonstorable good and the holdings of real cash balances.

For every $t \in [0, +\infty)$, let $O_t \in [0, +\infty]$, $M_t \in [0, +\infty]$, $H_t \in \mathbf{R}$, $Y_t \in \mathbf{R}_{++}$, and $C_t \in \mathbf{R}_+$ represent the nominal values of, respectively, holdings of government bonds and of cash, a lump-sum tax (if negative, a transfer from the government to

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the individual), nominal output and consumption at time t. The budget constraint faced by our representative agent is $\dot{O}_t + \dot{M}_t = Y_t - H_t - C_t + r_t O_t$, where the dots mean time derivatives and $r_t \in \mathbf{R}_+$ stands for the nominal interest rate bonds yield at time t (by definition, cash is a monetary asset always yielding a nominal interest rate of 0). Let $P_t \in \mathbf{R}_{++}$ be the (both expected and realized) price level, $\pi_t := \dot{P}_t / P_t$ the inflation rate at time t, and $\gamma := \dot{y}_t / y_t$ the constant rate of growth of real output $y := Y/P$.

Other than y, lowercase variables will represent the real counterparts to the above nominal variables as fractions of output (that is, $o := O/Y$, etc.). As in Lucas (2000), the utility function U is assumed to be homogeneous of degree $1-\sigma$, whence $U\,(C_t/P_t, M_t/P_t) = y_0^{1-\sigma} e^{(1-\sigma)\gamma t} U\,(c_t, m_t)$, and our agent's problem (P) can be written as

$$\max_{c,o,m\geq 0} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t, m_t)\, dt \tag{P}$$

subject to

$$\dot{o}_t + \dot{m}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma) o_t - (\pi_t + \gamma) m_t$$
$$o_0 > 0 \text{ and } m_0 > 0 \text{ given.}$$

Following Lucas (2000), we use an instantaneous utility function $U : (0, +\infty] \times [0, +\infty] \to [-\infty, +\infty]$ with the functional form

$$U(c, m) = \frac{1}{1-\sigma} \left(c\varphi \left(\frac{m}{c} \right) \right)^{1-\sigma},$$

extended by continuity to the ray $\{0\} \times [0, +\infty]$. Here, $\sigma \in (0, 1) \cup (1, +\infty)$ and $\varphi : [0, +\infty] \to [0, +\infty]$ is a twice-differentiable function satisfying the property: there exists $\overline{m} \in (0, +\infty]$ such that $\varphi\,|_{[0,\overline{m}]}$ is strictly increasing and strictly concave and $\varphi\,|_{[\overline{m},+\infty]}$ is constant.[4]

Using the budget constraint to substitute for c, one gets a standard problem of the calculus of variations in the variables o and m. Assuming (P) has an interior solution, it must satisfy the Euler equation

$$r = \frac{U_m}{U_c}. \tag{4}$$

Due to the concavity of U, this condition really corresponds to a maximum.[5]

In equilibrium, because c is taken as a fraction of output, $c = 1$. Equation (4) then becomes

$$r = \frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)}, \tag{5}$$

which corresponds to equation 3.7 in Lucas (2000) (there obtained using Bellman's Optimality Principle instead). This equation gives r as a nonnegative differentiable function of m, for which we write $r = \psi(m)$, where $\psi : (0, \overline{m}] \to \mathbf{R}_+$.[6] Because, for any $m \in (0, \overline{m})$,

$$\psi'(m) = \frac{\varphi(m)\varphi''(m)}{(\varphi(m) - m\varphi'(m))^2} < 0, \tag{6}$$

ψ is strictly decreasing, and therefore one-to-one. Since $\varphi'(\overline{m}+) = 0$ and $\varphi \in C^1$ (for being twice-differentiable), one has $\varphi'(\overline{m}) = 0$, so that $\psi(\overline{m}) = 0$. Let $\overline{r} := \psi(0+)$ (possibly $\overline{r} = +\infty$). We shall call ψ's inverse function $\eta : [0, \overline{r}) \to (0, \overline{m}]$ a "money-demand function." This money demand is strictly decreasing $[\eta'(r) = 1/\psi'(\eta(r)) < 0]$ and surjective by construction.

As a practical matter, because the economist does not know the function φ, he or she ends up using a money-demand function η estimated by the econometric practice, implying a specific functional form for φ. This can be done by noting that equation (5) can be rewritten as

$$\varphi'(m) = \frac{\psi(m)}{1 + m\psi(m)}\varphi(m), \tag{7}$$

a separable equation yielding the general solution

$$\varphi(m) = De^{\int_{m^*}^{m} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu}, \tag{8}$$

where m^* is any finite number belonging to the image of η, and $D > 0$ due to the nonnegativity and strict increasingness of φ over $[0, \overline{m}]$. It is interesting to note that (8) can also be put in terms of A, if this welfare measure does not explode for finite m:

$$\varphi(m) = De^{\int_{m^*}^{\eta(0)} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu} e^{\int_{\eta(0)}^{\eta(\psi(m))} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu} = Ee^{-A(\psi(m))}, \tag{9}$$

where $E := De^{A(\psi(m^*))}$ is a positive constant. That is, A can also be used in a task completely different from its original one in Simonsen and Cysne (2001): that of finding a φ in the Sidrauski–Lucas framework that rationalizes a given money demand.

A final observation is that in steady state ($\dot{c} = \dot{m} = 0$) the Euler equation relative to o gives the modified Fisher equation

$$r = \rho + \pi + \sigma\gamma, \tag{10}$$

which is what justifies taking the welfare cost of inflation as a function of the nominal interest rate instead of inflation itself.

2.3. The Welfare Cost of Inflation in the Sidrauski Model: Two Different Approaches

In Sidrauski's framework, Lucas (2000, p. 257) "define[s] the welfare cost $w(r)$ of a nominal rate r to be the percentage income compensation needed to leave the household indifferent between r and 0." There are two diametrically distinct

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ways of interpreting this definition. The first, employed by Lucas [see also Serletis and Yavari (2004)], uses the initial interest rate as a reference and measures the percentage rise in income necessary to make people as well off as they would be if the nominal interest rate were to fall to zero. Given the first-degree homogeneity of U, one can simply write

$$U\,(1+\overline{w}(r),\eta(r)) = U(1,\overline{m}), \tag{11}$$

provided $\overline{w}(r) \geq 0$.

It may be noted that $\overline{w}$ defined in this way will always exist over some maximal interval closed at 0.[7] This interval may be a proper subset of $[0,\overline{r})$, as will be illustrated in Section 4 with the log–log case. All the math below, including Proposition 1, is done within the domain of existence of $\overline{w}$.

Let $\overline{\varphi} := \sup_{m>0}\varphi(m) = \varphi(\overline{m})$. In our framework, (11) implies

$$(1+\overline{w}(r))\varphi\left(\frac{\eta(r)}{1+\overline{w}(r)}\right) = \varphi\left(\overline{m}\right) = \overline{\varphi}.$$

Differentiating with respect to r, dividing through by $\varphi'\left(\eta(r)/\left[1+\overline{w}(r)\right]\right)$, and using (7),

$$\overline{w}'(r) = -\psi\left(\frac{\eta(r)}{1+\overline{w}(r)}\right)\eta'(r). \tag{12}$$

This is equation 3.11 in Lucas's paper, which, together with the condition $\overline{w}(0) = 0$, enables us to find $\overline{w}$.

We now turn our attention to another natural way of interpreting Lucas's definition for the welfare cost of inflation in Sidrauski's model: the one that takes as reference an interest rate equal to zero. That is, the welfare cost of inflation will be understood as the percentage fall in the representative agent's income that would make him or her as well off as he or she would have been, had no increase in the nominal interest rate taken place:

$$U\,(1,\eta(r)) = U(1-\underline{w}(r),\overline{m}), \tag{13}$$

provided $0 \leq \underline{w}(r) \leq 1$.

As one could guess, the welfare cost of inflation that takes as reference an interest rate equal to zero ($\underline{w}$) is lower than the one that takes as reference the prevailing (supposedly positive) interest rate ($\overline{w}$), with Bailey's measure lying somewhere in between.[8] This will indeed follow from Proposition 1 below.

In our model, definition (13) implies that

$$\frac{\varphi(\eta(r))}{1-\underline{w}(r)} = \varphi\left(\frac{\overline{m}}{1-\underline{w}(r)}\right). \tag{14}$$

Since $0 \leq \underline{w}(r) \leq 1$ by construction, one has $\overline{m}/[1-\underline{w}(r)] \geq \overline{m}$, so that $\varphi(\overline{m}/[1-\underline{w}(r)]) = \overline{\varphi}$, and (14) ends up yielding the very simple formula for the

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welfare cost of inflation $\underline{w}$

$$\underline{w}(r) = \frac{\overline{\varphi} - \varphi(\eta(r))}{\overline{\varphi}}. \tag{15}$$

From (8),

$$\frac{\varphi(\eta(r))}{\overline{\varphi}} = \frac{De^{\int_{m^*}^{\eta(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu}}{De^{\int_{m^*}^{\overline{m}} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu}} = e^{\int_{\overline{m}}^{\eta(r)} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu} = e^{-A(r)}.$$

Taking (15) into account, one obtains

$$\underline{w}(r) = 1 - e^{-A(r)}. \tag{16}$$

In Simonsen and Cysne (2001) and in Cysne (2003), the measure $1 - e^{-A}$ has been derived in a completely different context (the shopping-time one) as an easier-to-calculate approximation to s. Here it has been shown that a different model, the money-in-the-utility-function model, can provide a sensible explanation for this measure.

The domain of existence of $\underline{w}$, unlike that of $\overline{w}$, is necessarily the whole range of interest rates, $[0, \overline{r})$, because A is defined over this set, and expression (16) yields $0 \leq \underline{w}(r) \leq 1$. Moreover, comparing (16) with (12), $\underline{w}$ can be seen to have at least one advantage over $\overline{w}$: computational ease. It is no longer necessary to solve a nonseparable nonlinear differential equation—although one still does need to deal with a possibly very difficult integral, perhaps one with no elementary primitive whatsoever.

The reader may suspect that $\underline{w}$ will always be lower than $\overline{w}$, simply because $\underline{w}$ can be interpreted as a percentage decrease in income, whereas $\overline{w}$, as a compensation through a percentage increase in income. This indeed follows from Proposition 1, which gives the main result of this section. All the proofs are collected in the Appendix.

PROPOSITION 1. *Let $\eta : [0, \overline{r}) \to (0, \overline{m}]$ be a differentiable and strictly decreasing money-demand function, rationalizable by the Sidrauski and the shopping-time models, so that it can be used to calculate s, A, B, $\overline{w}$, and $\underline{w}$ [by simply plugging it into (1), (2), (3), (12), and (16)].*[9] *Then the following inequality chain is true: $\underline{w} = 1 - e^{-A} < s < A < B < \overline{w}$.*

3. THE PRESENCE OF INTEREST-BEARING MONIES IN THE ECONOMY

The results obtained in the previous section are extended here to a framework in which n types of monies are available. This is important because, as argued in the Introduction, welfare formulas based on only one type of money can be misleading when there are different assets in the economy performing monetary functions. We begin by fixing some notation.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in [0, +\infty]^n$ represent the vector of real quantities demanded of all types of money as a fraction of nominal GDP (where m_1 is chosen to be m, currency per output). Each m_i yields a nominal interest rate of $r_i \geq 0$, with $r_1 = 0$ by definition. We shall write $\mathbf{u} := (r - r_1, r - r_2, \ldots, r - r_n) \in \mathbf{R}_+^n$ for the vector of opportunity costs of holding each type of money instead of government bonds.[10]

The reader may note that if $\mathbf{u} = \mathbf{0}$, then $r = r - r_1 = 0$, whereas if $r = 0$, then $\mathbf{u} = -(r_1, r_2, \ldots, r_n) \in \mathbf{R}_-^n$, so that necessarily $\mathbf{u} = \mathbf{0}$. That is,

$$r = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}. \tag{17}$$

The steady-state interest rate r in this extended model is determined by (10), as in the unidimensional case. Expressions (17) and (10) show how each of the opportunity costs u_i relates to the rate of inflation. When the nominal interest rate is equal to zero, all the opportunity costs are also equal to zero and inflation generates no welfare loss at all. The converse is also true.

All our results (including (17)) are valid for any given vector of opportunity costs $\mathbf{u}$, regardless of how $\mathbf{u}$ is determined. Here, though, is an optional way of thinking about how $\mathbf{u}$ can be determined and linked to inflation. Think of all interest rates other than r_1 as being determined by a competitive and costless banking system subject to $n - 1$ exogenous non-interest-bearing reserve requirements $k_2, \ldots, k_n \in (0, 1)$. In this case, $r_2 = (1 - k_2)r, \ldots, r_n = (1 - k_n)r$, and, using (10), $\mathbf{u} = (\rho + \pi + \sigma\gamma)(1, k_2, \ldots, k_n)$.

The first-order conditions of the money-in-the-utility-function model, as well as of the shopping-time model, will imply a function $\boldsymbol{\psi}$ taking $\mathbf{m}$ into $\mathbf{u}$, analogous to the function ψ from Section 2. But since this $\boldsymbol{\psi}$ is not necessarily invertible anymore, here the welfare measures will be evaluated at $\mathbf{m}$ rather than the more natural option $\mathbf{u}$. To keep the notation uniform, the same symbols used so far to denote each of the six welfare measures will be maintained, except for the addition of the subscript M (for "multidimensional").

3.1. The Shopping-Time Measure and Its Approximations in the Multidimensional Case

The multidimensional shopping-time model is introduced and solved in Cysne (2003). It yields a generalized "inverted money-demand function" $\boldsymbol{\psi} : (0, +\infty]^n \to \mathbf{R}_+^n$ taking $\mathbf{m}$ into $\mathbf{u}$. The multidimensional analog of (1) [Cysne (2003, eq. 14)] will be

$$(s_M)_{x_i}(\mathbf{m}) = -\frac{\psi_i(\mathbf{m})}{1 - s_M(\mathbf{m}) + \boldsymbol{\psi}(\mathbf{m}) \cdot \mathbf{m}}(1 - s_M(\mathbf{m})), \forall i \in \{1, \ldots, n\}, \tag{18}$$

where the ψ_i's are the component functions of $\boldsymbol{\psi}$ and $\cdot$ is the canonical inner product of $\mathbf{R}^n$.

Consider a C^1 path $\chi : [0, 1] \to (0, +\infty]^n$ such that $\chi(0) \in (0, +\infty]^n \setminus \mathbf{R}^n_{++}$ and $\chi(1) = \mathbf{m}$, and the following 1-forms in $(0, +\infty]^n$:[11]

$$dA_M := -\frac{1}{1 + \psi(\boldsymbol{\mu}) \cdot \boldsymbol{\mu}} \psi(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu},$$
$$dB_M := -\psi(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}.$$

The line integrals

$$A_M(\mathbf{m}) := \int_\chi dA_M \qquad \mathbf{(19)}$$

and

$$B_M(\mathbf{m}) := \int_\chi dB_M \qquad \mathbf{(20)}$$

extend Simonsen and Cysne's (2001) proxy measure A and Bailey's measure B to the present framework.[12] In Cysne (2003), it is shown that when ψ arises from the shopping-time model, these integrals are path-independent, so that A_M and B_M are indeed well-defined (Proposition 1 there), and that, as in the unidimensional case, $1 - e^{-A_M} < s_M < A_M < B_M$ (Remark 2).

As shown in Cysne and Turchick (2009, Lemma 2), each ψ_i is strictly decreasing along rays starting at the origin. From this property and (17) it is clear that the initial condition a welfare-cost function W_M (such as s_M, A_M, or B_M) has to satisfy in this multidimensional framework is $W_M(\mathbf{d}) = 0, \forall \mathbf{d} \in (0, +\infty]^n \setminus \mathbf{R}^n_{++}$, so that inequality chains such as the one above necessarily exclude these points. We present next the model to which measures $\overline{w}_M$ and $\underline{w}_M$ will be associated.

3.2. The Extended Sidrauski Model

Our representative agent's instantaneous utility will now have the form $U_M(c, \mathbf{m}) = U(c, G(\mathbf{m}))$, where U is the same as in Section 2 (to which a φ and an $\overline{m}$, with the same properties as before, are attached) and the money aggregator $G : [0, +\infty]^n \to [0, +\infty]$ is a twice-differentiable first-degree homogeneous concave function such that $G_{x_i} > 0$, $\lim_{m_i \to 0} G_{x_i}(\mathbf{m}) = +\infty$, $\lim_{m_i \to +\infty} G_{x_i}(\mathbf{m}) = 0$, and $G_{x_i x_i} < 0$ for all $i \in \{1, \ldots, n\}$.[13] This utility function U_M, similar to the one used in the money-in-the-utility-function model of Jones et al. (2004, Sect. 3.1), naturally generalizes the one in the unidimensional Sidrauski model. The maximization problem will be

$$\max_{c, o, \mathbf{m} \geq 0} \int_0^{+\infty} e^{(-\rho + (1-\sigma)\gamma)t} U_M(c_t, \mathbf{m}_t)\, dt \qquad (\mathbf{P}_M)$$

subject to

$$\dot{o}_t + \mathbf{1} \cdot \dot{\mathbf{m}}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma) o_t$$
$$+ (r_{1t} - \pi_t - \gamma, \ldots, r_{nt} - \pi_t - \gamma) \cdot \mathbf{m}_t$$
$$o_0 > 0 \text{ and } \mathbf{m}_0 > 0 \text{ given},$$

where all the nonbold letters have the same meaning as in the model of Section 2.2.

Considering only regular solutions and substituting for c as in the unidimensional setting, (P_M) becomes a standard variational problem. Its Euler equations with respect to o and each m_i yield

$$r - r_i = \frac{(U_M)_{x_i}}{(U_M)_c} = \frac{U_m G_{x_i}}{U_c}, \forall i \in \{1, \ldots, n\}. \quad (21)$$

This really corresponds to the optimum, by the concavity of U_M (which is a consequence of the concavity of G and of U, as well as of U being increasing in both of its coordinates).

In equilibrium $c = 1$, so that (21) gives

$$u_i = \psi_i(\mathbf{m}) = \frac{\varphi'(G(\mathbf{m}))}{\varphi(G(\mathbf{m})) - G(\mathbf{m})\,\varphi'(G(\mathbf{m}))} G_{x_i}(\mathbf{m}), \forall i \in \{1, \ldots, n\}, \quad (22)$$

where now $\psi : G^{-1}((0, \overline{m}]) \to \mathbf{R}^n_+$, G^{-1} denoting G's inverse image function. Equation (22) is analogous to (5), giving us a differentiable function $\boldsymbol{\psi}$ taking $\mathbf{m}$ into $\mathbf{u}$. It can be rewritten as

$$\mathbf{u} = \boldsymbol{\psi}(\mathbf{m}) = F(G(\mathbf{m}))\nabla G(\mathbf{m}), \quad (23)$$

where $F : (0, \overline{m}] \to \mathbf{R}_+$ is a differentiable and strictly decreasing function [as already calculated in (6)]. As noted in Cysne and Turchick (2009), this general form of multidimensional money demands also encompasses those originating from the extended shopping-time model.

In order for the measures A_M and B_M introduced in the last subsection to apply to $\boldsymbol{\psi}$, the path $\boldsymbol{\chi}$ should be taken so that $\boldsymbol{\chi}(0) \in G^{-1}(\{\overline{m}\})$, and the boundary condition for a generic welfare measure W_M is that $W_M(\mathbf{d}) = 0, \forall \mathbf{d} \in G^{-1}(\{\overline{m}\})$. It must also be checked that these measures are still well-defined, that is, that the line integrals in (19) and (20) are path-independent. This is done next.

LEMMA 1. *Let* $W_M(\mathbf{m}) := \int_{\boldsymbol{\chi}} J(G(\boldsymbol{\mu}))\,\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}$, *where* $J : (0, \overline{m}] \to \mathbf{R}$ *is any differentiable function, and* $\boldsymbol{\chi}$ *and* $\boldsymbol{\psi}$ *are as described above. Then* W_M *is well-defined. In particular,* A_M *and* B_M *are well-defined.*

3.3. The Welfare Cost of Inflation in the Extended Sidrauski Model: Two Different Approaches

The measures $\overline{w}_M$ and $\underline{w}_M$ of the welfare cost of inflation are defined following the same ideas as in Section 2.3. For an arbitrary $\mathbf{d} \in G^{-1}(\{\overline{m}\})$,

$$U_M(1 + \overline{w}_M(\mathbf{m}), \mathbf{m}) = U_M(1, \mathbf{d}), \text{ i.e., } U(1 + \overline{w}_M(\mathbf{m}), G(\mathbf{m})) = U(1, \overline{m}) \quad (24)$$

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and

$$U_M(1, \mathbf{m}) = U_M(1 - \underline{w}_M(\mathbf{m}), \mathbf{d}), \text{ i.e., } U(1, G(\mathbf{m})) = U(1 - \underline{w}_M(\mathbf{m}), \overline{m}), \tag{25}$$

provided $\overline{w}_M(\mathbf{m}) \geq 0$ and $0 \leq \underline{w}_M(\mathbf{m}) \leq 1$. The same observations made in the last section concerning the issue of existence of $\overline{w}$ and $\underline{w}$ obviously extend to $\overline{w}_M$ and $\underline{w}_M$.

The next result aims at simplifying the task of calculating the welfare cost of inflation in an economy with more than one type of money.

PROPOSITION 2. *Given a money-demand specification in the form (23), $\psi = (F \circ G)\nabla G$, let W be any of the unidimensional measures s, A, B, $\overline{w}$, or $\underline{w}$, evaluated using the inverse money demand F, and W_M the corresponding multidimensional measure [in the Sidrauski framework, comparison between (5) and (22) shows that this amounts to taking the same φ in (P) and in (P_M)]. Then $W_M = W \circ F \circ G$.*

We are now ready to state our main ordering result, extending Proposition 1 to an economy with many types of monies. It follows immediately from Propositions 1 and 2.

PROPOSITION 3. *Let $\psi : G^{-1}((0, \overline{m}]) \rightarrow \mathbf{R}^n_+$ be a money-demand specification taking the general form (23), rationalizable by the extended Sidrauski and the extended shopping-time model, so that it can be used to calculate s_M, A_M, B_M, $\overline{w}_M$, and $\underline{w}_M$. Then the following inequality chain is true: $\underline{w}_M = 1 - e^{-A_M} < s_M < A_M < B_M < \overline{w}_M$.*

4. CALCULATING THE WELFARE COST OF INFLATION

4.1. Formulas for the Unidimensional Measures

Assume that a unidimensional bilogarithmic money demand specification, $m = Kr^{-\alpha}$, with $K > 0$ and $\alpha \in (0, 1)$, has been estimated. How can we calculate the different measures of the welfare cost of inflation associated with a nominal interest rate of r?

Fortunately, s has already been calculated in the literature [see Cysne (2005)] for this particular case, being given in implicit form by

$$(1 - s(r))\left[1 - (1 - s(r))^{-\frac{1}{\alpha}}\right] + \frac{K}{1-\alpha} r^{1-\alpha} = 0.$$

Both Bailey's measure and the proxy measure A are straightforward:

$$B(r) = \int_0^r -\rho(-\alpha K \rho^{-\alpha-1}) d\rho = \frac{\alpha K}{1-\alpha} r^{1-\alpha},$$

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$$A(r) = \int_0^r \frac{-\rho(-\alpha K\rho^{-\alpha-1})}{1+\rho K\rho^{-\alpha}} d\rho$$
$$= \frac{\alpha}{1-\alpha}\int_1^{1+Kr^{1-\alpha}} \frac{du}{u} = \frac{\alpha}{1-\alpha}\log(1+Kr^{1-\alpha}).$$

Regarding $\overline{w}$, since $\psi(m) = (K/m)^{\frac{1}{\alpha}}$, (12) takes the form

$$\overline{w}'(r) = -\left[\frac{K(1+\overline{w}(r))}{Kr^{-\alpha}}\right]^{\frac{1}{\alpha}}(-\alpha Kr^{-\alpha-1}) = \alpha K(1+\overline{w}(r))^{\frac{1}{\alpha}}r^{-\alpha},$$

leading to

$$\overline{w}(r) = -1 + (1 - Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}. \quad (26)$$

The formula above requires $r \in [0, K^{\frac{1}{\alpha-1}}]$.[14] As explained in Section 2.3, this will not be an issue for $\underline{w}$, for which one simply has

$$\underline{w}(r) = 1 - e^{-A(r)} = 1 - (1 + Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}. \quad (27)$$

The expression for A can also be used to illustrate the point made in Section 2.2 regarding the rationalization of a money demand in the Sidrauski–Lucas framework. For the log–log case, equation (9) gives

$$\varphi(m) = E[1 + K\psi(m)^{1-\alpha}]^{-\frac{\alpha}{1-\alpha}} = E\left(1 + K^{\frac{1}{\alpha}}m^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}, \quad (28)$$

for a positive constant E (so that $\overline{m} = +\infty$, as expected).

If we had started out with a semilog money demand instead, $m = Ke^{-\beta r}$, with $K, \beta > 0$, then no measure other than Bailey's could be written out, in explicit or implicit form, through elementary functions. From (3),

$$B(r) = \int_0^r -\rho(-\beta Ke^{-\beta\rho})\, d\rho$$
$$= \beta K\left[-\frac{1+\beta\rho}{\beta^2}e^{-\beta\rho}\right]_0^r = \frac{K[1-(1+\beta r)\,e^{-\beta r}]}{\beta}.$$

The other four can be evaluated numerically with the help of a mathematics software.[15]

All the measures of the welfare cost of inflation considered in Proposition 1 are plotted for the estimated money-demand functions $\eta(r) = 0.05r^{-0.5}$ and $\eta(r) = 0.35e^{-7r}$.[16] Note how s and A are indistinguishable to the naked eye.[17]

A question that naturally arises from looking at Figure 1 is, *How much greater can the Sidrauski–Lucas upper bound $\overline{w}$ be relative to the lower bound $\underline{w}$ of our set of welfare measures?* For reasonable interest rates, not much greater indeed.

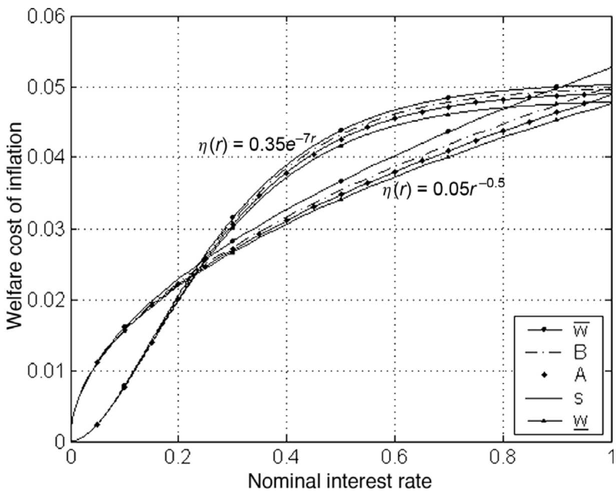

FIGURE 1. Welfare cost of inflation in the unidimensional case.

For example, for the log–log specification, (26) and (27) yield

$$\Delta(r) := \frac{\overline{w}(r) - \underline{w}(r)}{\underline{w}(r)} = \begin{cases} \dfrac{-1 + (1 - Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}}{1 - (1 + Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}} - 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

This Δ is the maximum relative error one can incur when choosing one of the measures s, A, B, $\overline{w}$, or $\underline{w}$ to work with, instead of any other of these. Figure 2 shows the behavior of Δ under the bilogarithmic (elasticity α) and the semilogarithmic (semielasticity β) specifications.

For instance, considering the log–log specification with $\alpha = 0.5$, one obtains $\Delta\,(0.15) \approx 3.9\%$. This value of the nominal interest rate could account for an inflation rate around 11% plus a long-term real interest rate around 4%. It could be understood as an upper bound for most industrialized economies. Thus, if one reports having found a welfare loss of, say, 1% of output associated with a money demand having a 0.5 elasticity and an 11% inflation rate, our results allow us to say that, regardless of which particular measure among the six was chosen, the estimate could vary at most between 0.96% and 1.04%, a very reasonable confidence interval. So one can feel secure about which measure to take, when considering low-inflation countries.

However, the preceding calculations tell only one side of the story. Consider now a country where the annual inflation rate has reached 400% (in Brazil, for instance, yearly inflation reached 1783% in 1989). For the same parameters, the relative measuring difference Δ reaches 22.2% (assuming $r = 4$, since the long-term real interest rate becomes negligible). For example, estimates of the welfare costs reported as being 8% of output in high-inflation countries could actually be

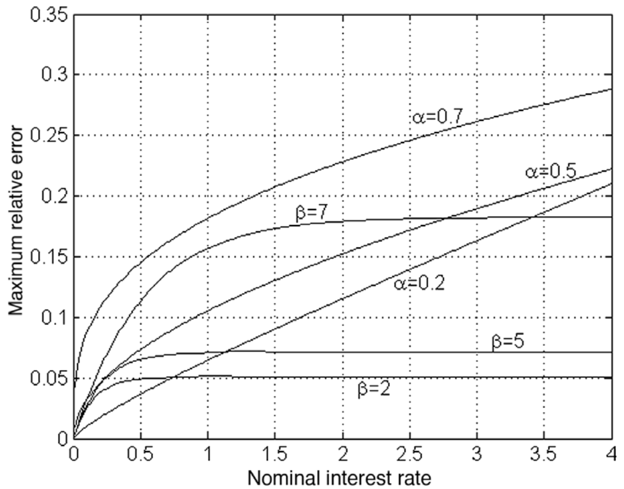

FIGURE 2. Maximum relative error.

as low as 6.5% [0.08/ $(1 + \Delta)$] or as high as 9.8% [$0.08 \times (1 + \Delta)$], depending on the formula being used. In such high-inflation cases, therefore, one has to be careful about which measuring strategy to pursue.[18]

Alternatively, the numerical computation leading to Figure 2 can be used to infer the relative error when semilog specifications of the money demand are used. For instance, using American data relative to the post-1980 period, Ireland (2009) estimates semielasticities ranging between 1.7944 and 1.9013. Applying Bailey's unidimensional formula, he reports welfare costs of inflation around 0.0136%, 0.0370%, and 0.2268% of income for yearly nominal interest rates of 3%, 5%, and 13%, respectively (corresponding to 0%, 2%, and 10% inflation rates, because the real interest rate is fixed at 3% there).[19]

Had Ireland chosen to use any of the other five unidimensional welfare measures displayed in this work, the maximum relative error he could incur would be negligible: $\Delta\,(0.03) \approx 0.50\%$, $\Delta\,(0.05) \approx 0.81\%$, and $\Delta\,(0.13) \approx 1.96\%$, thus implying the ranges [0.0135, 0.0137], [0.0367, 0.0373], and [0.2223, 0.2313] percent of income for his average welfare cost estimates. This is to say, these estimates are robust relative to the issue of choosing a particular unidimensional measure vis-à-vis the others.

It may still be noted that in the log–log case, for all practical purposes, the approximation $\Delta\,(r) \approx W\,(r)\,/\alpha$, where W is any of the welfare measures studied here, can be used. In fact, putting $z := Kr^{1-\alpha}$ and defining f, g_1, and g_2 such that $f\,(z) = \Delta\,(r)$, $g_1\,(z) = \underline{w}(r)/\alpha$, and $g_2\,(z) = \overline{w}(r)/\alpha$ (using (27) and (26)), it is easy to see that the first two terms in their Maclaurin series coincide (the series being of the form $0 + (1-\alpha)^{-1}\,z + O(z^2)$). Then $f\,(z) = g_i\,(z) + O(z^2)$, $\forall i \in \{1, 2\}$, and using Proposition 1 to extend this to all the other welfare measures lying in between $\underline{w}$ and $\overline{w}$, one gets $\Delta\,(r) = W\,(r)\,/\alpha + O(r^{2(1-\alpha)}) = W\,(r)\,/\alpha + o\,(1)$.

4.2. Formulas for the Multidimensional Measures

We now assume that a multidimensional log–log money demand specification,

$$\begin{cases} m_1 = L_1 \prod_{j=1}^{n} u_j^{\alpha_{1j}} \\ \vdots \\ m_n = L_n \prod_{j=1}^{n} u_j^{\alpha_{nj}} \end{cases}, \tag{29}$$

has been estimated. It can be checked that this is the type of demand that would emerge from taking (28) in (P_M), with G a weighted geometric mean, $G(m_1, \ldots, m_n) = \prod_{i=1}^{n} m_i^{\gamma_i}$ (where $\gamma_i \geq 0, \forall i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} \gamma_i = 1$). In fact, from (22) and the discussion in the previous section, we know that to this φ there corresponds a function F in (23) such that $F(G) = (K/G)^{\frac{1}{\alpha}}$. Therefore

$$\begin{cases} \psi_1(\mathbf{m}) = \dfrac{K^{\frac{1}{\alpha}}\gamma_1}{m_1} \prod_{j=1}^{n} m_j^{(1-\frac{1}{\alpha})\gamma_j} \\ \vdots \\ \psi_n(\mathbf{m}) = \dfrac{K^{\frac{1}{\alpha}}\gamma_n}{m_n} \prod_{j=1}^{n} m_j^{(1-\frac{1}{\alpha})\gamma_j} \end{cases},$$

which inverted gives (29).[20]

Proposition 2 makes it possible to obtain expressions for the multidimensional welfare measures by simply copying the formulae derived in Section 4.1. That is, at this point there is no need to solve line integrals or systems of partial differential equations. We get, in turn,

$$(1 - s_M(\mathbf{m}))\left[1 - (1 - s_M(\mathbf{m}))^{-\frac{1}{\alpha}}\right] + \frac{K^{\frac{1}{\alpha}}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}} = 0,$$

$$B_M(\mathbf{m}) = \frac{\alpha K^{\frac{1}{\alpha}}}{1-\alpha} G(\mathbf{m})^{1-\frac{1}{\alpha}},$$

$$A_M(\mathbf{m}) = \frac{\alpha}{1-\alpha} \log\left(1 + K^{\frac{1}{\alpha}} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right),$$

$$\overline{w}_M(\mathbf{m}) = -1 + \left(1 - K^{\frac{1}{\alpha}} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}$$

if $G(\mathbf{m}) \leq K^{\frac{1}{1-\alpha}}$, and

$$\underline{w}_M(\mathbf{m}) = 1 - \left(1 + K^{\frac{1}{\alpha}} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}.$$

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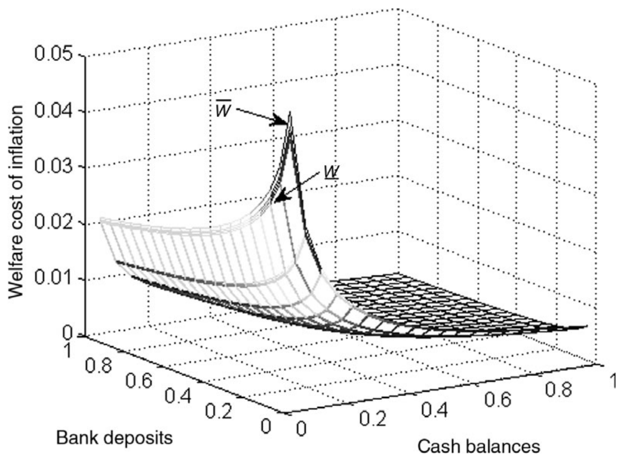

FIGURE 3. Welfare cost of inflation in the multidimensional case.

The maximum relative error in this case is

$$\Delta_M(\mathbf{m}) := \frac{\overline{w}_M(\mathbf{m}) - \underline{w}_M(\mathbf{m})}{\underline{w}_M(\mathbf{m})}$$

$$= \begin{cases} \dfrac{-1 + \left(1 - K^{\frac{1}{\alpha}} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}}{1 - \left(1 + K^{\frac{1}{\alpha}} G(\mathbf{m})^{1-\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-1}}} - 1 & \text{if } \mathbf{m} \in \mathbf{R}^n_{++} \\ 0 & \text{if } \mathbf{m} \in (0, +\infty]^n \setminus \mathbf{R}^n_{++} \end{cases}.$$

The approximation for the log–log Δ derived in the preceding subsection, together with Proposition 2, give $\Delta_M(\mathbf{m}) = \Delta(F(G(\mathbf{m}))) = W(F(G(\mathbf{m})))/\alpha + O(F(G(\mathbf{m}))^{2(1-\alpha)}) = W_M(\mathbf{m})/\alpha + o(1)$, where here the relevant asymptotic behavior is at $\mathbf{m} \to \mathbf{d} \in (0, +\infty]^n \setminus \mathbf{R}^n_{++}$ (whence $G(\mathbf{m}) \to +\infty$), and the last equality (or pertinence relation) comes from $\lim_{G\to+\infty} F(G) = \lim_{G\to+\infty}(K/G)^{\frac{1}{\alpha}} = 0$.

Had we departed from a generalized semilog specification instead, i.e., one with $F(G) = \log(K/G)/\beta$ in (23), then

$$B_M(\mathbf{m}) = \frac{K\left[1 - \dfrac{G(\mathbf{m})}{K}\left(1 + \log\dfrac{K}{G(\mathbf{m})}\right)\right]}{\beta},$$

and the other measures, as well as the relative error, could only be evaluated numerically.

All five measures of the welfare cost of inflation in an economy with $n = 2$ (with m_1 standing for real balances and m_2 for interest-bearing bank deposits) are illustrated in Figure 3. The calculations use a log–log specification with $K = 0.05$ and $\alpha = 0.5$ (as in Figure 1), plus the money aggregation parameter $\gamma_1 = 0.7$

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(and $\gamma_2 = 1 - \gamma_1 = 0.3$). The ordering of the surfaces is the one implied by Proposition 3, with $\overline{w}_M$ on top and $\underline{w}_M$ on the bottom. As in the unidimensional case, A_M approximates s_M so well that it is practically impossible to visualize them separately.

5. CONCLUSION

The present work has extended the ordering of measures of the welfare costs of inflation provided by Cysne (2003) to include two new measures derived from Sidrauski's money-in-the-utility-function framework. The first measure is provided by Lucas (2000) (though only in relation to the unidimensional case), whereas the second is new in the literature (both for the unidimensional and the multidimensional case).

The main result of the paper, given in Proposition 3, is the ordering of six different measures of the welfare cost of inflation, both when there is only one type of money and when there are several interest-bearing deposits. Our results have been illustrated with the well-known bilogarithmic and semilogarithmic money demands, for which we have provided closed-form solutions to all welfare measures (whenever possible in terms of elementary functions), both in the unidimensional and in the multidimensional settings.

Our calculations for the unidimensional log–log case show that for parameter values such as those usually found in the literature, the maximum relative error a researcher can incur by deciding to use a particular welfare measure vis-à-vis the others is negligible in normal scenarios (only 4% of the measured welfare cost for annual nominal interest rates as high as 15%), being of relevance only in the case of hyperinflations (22% if interest rates are around 400%). Regarding the semilog case, applying our results to the estimates given in Ireland (2009) shows a maximum relative error around 2% of the measured welfare cost when annual nominal interest rates are as high as 13%.

NOTES

1. For a consideration of the error arising from the use of unidimensional rather than multidimensional measures of the welfare costs of inflation, see Cysne and Turchick (2010).

2. Barnett (1979, 1980) has applied aggregation theory in the construction of monetary aggregates, introducing the use of Divisia indices of monetary services in the analysis of monetary policy.

3. Cysne (2005) has provided a solution to it in the case of a log–log money demand. This solution will be helpful in the calculations to be carried out in this paper.

4. The satiation point $\overline{m}$ will turn out to be the maximum value the money-demand function arising from φ and (P) attains. For example, for the log–log money-demand specification $m = Kr^{-\alpha}$, we would have $\overline{m} = +\infty$, whereas for the semilog specification $m = Ke^{-\beta r}$, $\overline{m} = K$. Cf., for instance, the model of Cavalcanti and Villamil (2003), where a finite $\overline{m}$ is imposed.

5. Let $V(c, m) = c\varphi(m/c)$, so that U is a concave monotonic transformation of V, and U's concavity follows from that of V: for any $(m, c) \in [0, +\infty]^2$, $V_{mm}(c, m) = \varphi''(m/c)/c < 0$, and $V_{cc}(c, m)V_{mm}(c, m) - V_{cm}(c, m)^2 = 0$ from V's 1-homogeneity.

6. The fact that ψ is differentiable is a consequence from φ's twice-differentiability, whereas its nonnegativity results from φ's nonnegativity and strict concavity: for any $m \in (0, \overline{m}]$, $-\varphi(m) \leq \varphi(0) - \varphi(m) < \varphi'(m)(0 - m)$, whence $\varphi(m) - m\varphi'(m) > 0$.

7. In fact, obviously $\overline{w}(0) = 0 \geq 0$, and if $\overline{w}(r)$ exists and $\rho \in (0, r)$, since $U(1 + \overline{w}(r), \eta(\rho)) > U(1 + \overline{w}(r), \eta(r)) = U(1, \overline{m}) > U(1, \eta(\rho))$ (where η's strict decreasingness and $U_m > 0$ have been used), the Intermediate Value Theorem guarantees the existence of a $\overline{w}(\rho) \in (0, \overline{w}(r))$ (and therefore nonnegative) such that $U(1 + \overline{w}(\rho), \eta(\rho)) = U(1, \overline{m})$.

8. Note that the inequalities presented in this paper relate to measures of deadweight loss, rather than measures of welfare changes (as the consumers' surplus).

9. About the rationalizability issue, the reader may see Cysne and Turchick (2009).

10. That $\mathbf{u}$ is defined as a nonnegative vector is just an equilibrium-argument shortcut: if some r_i were larger than r in this risk-free economy, bonds would just cease to exist.

11. $(0, +\infty]^n \setminus \mathbf{R}^n_{++}$ is the set of vectors in $[0, +\infty]^n$ with strictly positive coordinates, at least one of which unbounded.

12. These are, in order, the additive inverses of the Divisia indices $DE(\chi)$ and $DG(\chi)$ presented in Cysne (2003).

13. If $n = 1$, the homogeneity of G would imply its linearity, whence $G'' = 0$. Therefore, our analysis in this section is restricted to the case $n > 1$. Even so, it yields exactly the same results as the $n = 1$ framework analyzed in Section 2.

14. This formula is equivalent to formula 6 in Serletis and Yavari (2004) (where the presence of a typographical error, the A's in the denominators, should be noted).

15. Even though $\overline{m} = K < +\infty$ in this case, the semilog money demand is also rationalizable through the shopping-time model, under a minor adjustment in the model [see Cysne and Turchick (2009)]. Thus s and its approximations can be sensibly evaluated.

16. These parameters were calibrated to fit the American economy in Lucas (2000).

17. In this figure, the 1 on the x-axis represents a 100% annual nominal interest rate, whereas the 0.01 on the y-axis means a welfare cost of 1%.

18. If one uses instead the value for α found by Serletis and Yavari (2004) for the post–WWII U.S. economy, 0.21, together with the value for K that can be inferred from their Figure 2, 0.112, then $\Delta\,(0.15) \approx 3.2\%$ and $\Delta\,(4) \approx 54.9\%$.

19. Such welfare-cost figures correspond to the averages of the numbers displayed in the three last columns of Table 6 in Ireland (2009).

20. Inversion of this system can be obtained by taking logs on both sides and then applying Cramer's rule to solve the resulting linear system in the variables $\log m_i$. Proceeding in this way yields the following expressions for the parameters in (29): $\alpha_{ij} = f_{ji}/\det(q)$ and $L_i = \prod_{j=1}^{n} (K^{\frac{1}{\alpha}} \gamma_j)^{-\alpha_{ij}}$, where $q_{ki} := (1 - 1/\alpha)\gamma_i$ if $i \neq k$ and $(1 - 1/\alpha)\gamma_i - 1$ otherwise, and f_{ki} is the cofactor of the element q_{ki} in the matrix q.

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APPENDIX

Proof of Proposition 1. The equality is demonstrated in the text. The first, second, and third inequalities have already been shown in Simonsen and Cysne (2001, Proposition 1), and the reader may note that their proof that $1 - e^{-A} < s < A < B$ draws on the strict decreasingness of η alone, and no other property enjoined by money-demand functions arising from the shopping-time model. As for the fourth inequality, (12) gives, for $\rho \in (0, r)$, $\overline{w}'(\rho) > -\psi(\eta(\rho))\eta'(\rho) = -\rho\eta'(\rho)$ (recalling that ψ is a strictly decreasing function and $\eta'(\rho) < 0$), so all that is left to do is integrate both sides of this inequality from 0 to r. ■

Proof of Lemma 1. In fact, it is well known that, given the simple connectedness of W_M's domain $G^{-1}((0, \overline{m}])$, it is only necessary to verify that the form $dW_M = J(G(\boldsymbol{\mu}))\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}$ is closed. Because

$$(J\psi_i)_{x_j} = J'G_{x_j}\psi_i + J\,(\psi_i)_{x_j} = J'G_{x_j}FG_{x_i} + J(F'G_{x_j}G_{x_i} + FG_{x_i x_j})$$

is symmetric in (i, j), we are done (in the above calculation, G and its derivatives are being evaluated at $\boldsymbol{\mu}$, whereas J, F, and their derivatives are being evaluated at $G(\boldsymbol{\mu})$).

Take $J = -1$. Then $W_M = B_M$, whence B_M is well-defined. Now take J such that $J\,(G) = -1/\,[1 + GF\,(G)]$. Applying Euler's formula for homogeneous functions to (23) yields $\boldsymbol{\psi}(\boldsymbol{\mu}) \cdot \boldsymbol{\mu} = F(G\,(\boldsymbol{\mu}))\,\nabla G\,(\boldsymbol{\mu}) \cdot \boldsymbol{\mu} = G(\boldsymbol{\mu})\,F(G(\boldsymbol{\mu}))$, so that $W_M = A_M$ and A_M is also well-defined. ■

Proof of Proposition 2. Let $m \in G^{-1}\,((0, \overline{m}])$ be given. If $W = \overline{w}$, then a simple comparison between (24) and (11), with $r = F\,(G\,(\mathbf{m}))$, yields

$$U(1 + \overline{w}_M(\mathbf{m}), G\,(\mathbf{m})) = U(1, \overline{m}) = U(1 + \overline{w}(F\,(G\,(\mathbf{m}))), G\,(\mathbf{m})),$$

and because $U_c > 0$, $\overline{w}_M\,(\mathbf{m}) = \overline{w}\,(F(G\,(\mathbf{m})))$. If $W = \underline{w}$, a similar argument comparing (25) with (13) yields

$$U(1 - \underline{w}_M(\mathbf{m}), \overline{m}) = U\,(1, G\,(\mathbf{m})) = U(1 - \underline{w}(F\,(G\,(\mathbf{m}))), \overline{m}),$$

and because $U_m > 0$, $\underline{w}_M(m) = \underline{w}(F\,(G\,(\mathbf{m})))$.

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If W can be written as $W(r) = \int_{\overline{m}}^{F^{-1}(r)} J(\mu)\, F(\mu)\, d\mu$, then

$$W_M(\mathbf{m}) := \int_{\chi} J(G(\boldsymbol{\mu}))\, \psi(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu} = \int_{\chi} J(G(\boldsymbol{\mu}))\, F(G(\boldsymbol{\mu}))\, \nabla G(\boldsymbol{\mu}) \cdot d\boldsymbol{\mu}$$

$$= \int_{\overline{m}}^{G(\mathbf{m})} J(\tilde{G}) F(\tilde{G}) d\tilde{G} = W(F(G(\mathbf{m}))),$$

where the third equality used Lemma 1. Putting J as in the proof of that lemma and comparing with (2) and (3) shows that both $W = A$ and $W = B$ are special cases of this argument.

Finally, if s solves (1), then $s_M := s \circ F \circ G$ can be seen to solve (18). In fact,

$$(s_M)_{x_i}(\mathbf{m}) = s'(F(G(\mathbf{m})))\, F'(G(\mathbf{m}))\, G_{x_i}(\mathbf{m})$$

$$= -\frac{F(G(\mathbf{m}))\,(F^{-1})'(F(G(\mathbf{m})))F'(G(\mathbf{m}))\, G_{x_i}(\mathbf{m})}{1 - s(F(G(\mathbf{m}))) + F(G(\mathbf{m}))\, G(\mathbf{m})}(1 - s(F(G(\mathbf{m}))))$$

$$= -\frac{F(G(\mathbf{m}))\, G_{x_i}(\mathbf{m})}{1 - s_M(\mathbf{m}) + F(G(\mathbf{m}))\, G(\mathbf{m})}(1 - s_M(\mathbf{m}))$$

$$= -\frac{\psi_i(\mathbf{m})}{1 - s_M(\mathbf{m}) + \psi(\mathbf{m}) \cdot \mathbf{m}}(1 - s_M(\mathbf{m})),$$

where in the last equality Euler's formula for homogeneous functions was applied to (23). Also, if $\mathbf{d} \in (0, +\infty]^n \setminus \mathbf{R}^n_{++}$, then $s_M(\mathbf{d}) = s(F(G(\mathbf{d}))) = s(F(+\infty)) = s(0) = 0$. ■

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