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# AN EXPONENTIAL NONUNIFORM BERRY–ESSEEN BOUND OF THE MAXIMUM LIKELIHOOD ESTIMATOR IN A JACOBI PROCESS

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**Abstract**

We establish the exponential nonuniform Berry–Esseen bound for the maximum likelihood estimator of unknown drift parameter in an ultraspherical Jacobi process using the change of measure method and precise asymptotic analysis techniques. As applications, the optimal uniform Berry–Esseen bound and optimal Cramér-type moderate deviation for the corresponding maximum likelihood estimator are obtained.

*Keywords:* Berry–Esseen bound; maximum likelihood estimator; Jacobi process; Cramér-type moderate deviation

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## 1. Introduction and main results

### 1.1. Introduction

Consider the following ultraspherical Jacobi process

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sqrt{1 - X_t^2}\,\mathrm{d}W_t, \qquad X_0 = 0,\ t \geq 0, \tag{1.1}$$

where $W = \{W_t\colon t \geq 0\}$ is a standard Brownian motion with unknown drift parameters $b \in (-\infty, -1)$. This process is well defined under the assumption $b < -1$, i.e. we have $|X_t| < 1$ for all $t \geq 0$. The Jacobi process, also called Wright–Fisher diffusion, was originally used to model gene frequencies [9, 16]. More recently, it has also been applied to describe financial factors. For example, [7] models interest rates by the Jacobi process and studies moment-based techniques for pricing bonds. Moreover, this process has also been applied to model stochastic correlation matrices [2] and credit default swap indexes [5]. For the multivariate case, see [1, 13].

For $b \in (-\infty, -1)$, the Jacobi process (1.1) has stationary distribution Beta$(-b, -b)$, i.e. the Beta distribution with shape parameter $-b$ and scale parameter $-b$. Here, $b$ also represents the mean-reverting parameter. For practical applications, it is crucial to construct the estimator for $b$ and study the corresponding asymptotic properties. Let $\mathbb{P}_b$ be the probability distribution

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of the solution of (1.1) on $C(\mathbb{R}_+, \mathbb{R})$. Under $\mathbb{P}_b$, the maximum likelihood estimator of $b$ is given by

$$\widehat{b}_T = \frac{\int_0^T X_t/(1-X_t^2)\,\mathrm{d}X_t}{\int_0^T X_t^2/(1-X_t^2)\,\mathrm{d}t}. \tag{1.2}$$

The Jacobi process was subordinated in [8] by the method of random time change, and the corresponding semi-group density was obtained. Together with the Girsanov formula technique [4, 12], they gave the large deviations for $\widehat{b}_T$. For the Girsanov formula technique, see also [20]. Moreover, [14] studied the moderate deviations for $\widehat{b}_T$.

From [17], it follows that

$$\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \xrightarrow{\mathrm{d}} N(0, 1),$$

which immediately implies that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x) \right| \to 0 \tag{1.3}$$

as $T \to \infty$. The Berry–Esseen bound for (1.3) bounds the global convergence speed which is uniform in all $x \in \mathbb{R}$. In this paper, our motivation is to study the nonuniform convergence rate of the difference

$$\mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x),$$

which will depend on $x$ as well as $T$. This local bound provides more accurate information than the Berry–Esseen bound; see Corollaries 1.1 and 1.2 and Remark 1.1. Our approach is based on the change of measure method [19] and precise asymptotic analysis techniques [15]. The explicit calculations and estimations for the Laplace functionals of $\int_0^T X_t/(1-X_t^2)\,\mathrm{d}X_t$ and $\int_0^T X_t^2/(1-X_t^2)\,\mathrm{d}t$ play crucial roles. Moreover, the techniques used here could potentially be used for a wider scope of diffusions that have a structure similar to our model.

### 1.2. Main results

We now state the main results of this paper, i.e. the exponential nonuniform Berry–Esseen bound and its application for $\widehat{b}_T$.

**Theorem 1.1.** *There exists some positive constant $C > 0$ depending only on $b$ such that, for $T$ large enough and any $\rho > 0$ with $|x| < \rho T^{1/6}$,*

$$\left| \mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x) \right| \le \frac{C}{\sqrt{T}}(x^2+1)\mathrm{e}^{-x^2/2}, \tag{1.4}$$

$$\frac{\mathbb{P}_b(\sqrt{-T/(2(1+b))}(\widehat{b}_T - b) > x)}{1-\Phi(x)} = \exp\left\{ O(1)\frac{x^3}{\sqrt{T}} \right\}, \tag{1.5}$$

*where the $O(1)$ term only depends on $b$.*

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As applications of the above nonuniform Berry–Esseen bound, we can obtain the following optimal uniform Berry–Esseen bound and optimal Cramér-type moderate deviations for $\widehat{b}_T$.

**Corollary 1.1.** *There exists some positive constant $C > 0$ depending only on b such that, for T large enough,*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x) \right| \le \frac{C}{\sqrt{T}}.$$

**Corollary 1.2.** *Let $\{\lambda_T, T > 0\}$ be a family of positive numbers satisfying $\lambda_T/T^{1/6} \to 0$, $T \to \infty$. Then, for any $\rho \ge 0$,*

$$\sup_{0\le x\le \rho\lambda_T} \left| \frac{\mathbb{P}_b(\sqrt{-T/(2(1+b))}(\widehat{b}_T - b) > x)}{1-\Phi(x)} - 1 \right| \to 0, \qquad T \to \infty.$$

**Remark 1.1.** Note that, for any $x > 0$,

$$\frac{x}{\sqrt{2\pi}(1+x^2)} \exp\left\{-\frac{x^2}{2}\right\} \le 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{x^2}{2}\right\}.$$

Hence, when $x \ge (\log T)^{1/2}$, $1 - \Phi(x) = o\big(T^{-1/2}\big)$. Moreover, using [14, Theorem 1.1], for $x \ge (\log T)^{1/2}$,

$$\mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) > x \right) = o\big(T^{-1/2}\big).$$

Then,

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| \mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x) \right| \\ &\quad = \sup_{|x|\le(\log T)^{1/2}} \left| \mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x) \right|. \end{aligned}$$

Therefore, the uniform Berry–Esseen bound cannot characterize the error

$$\mathbb{P}_b\left( \sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x \right) - \Phi(x)$$

for large $x$ depending on $T$. Our nonuniform Berry–Esseen bound (1.4) obtained in Theorem 1.1) can fill this gap.

**Remark 1.2.** When $T$ takes values in positive integers, letting

$$\xi_i = \int_{i-1}^{i} \frac{X_t}{\sqrt{1-X_t^2}}\, \mathrm{d}W_t,$$

we have

$$\widehat{b}_T - b = \frac{\sum_{i=1}^{T} \xi_i}{\sum_{i=1}^{T} E(\xi_i^2 \mid \mathcal{F}_{i-1})},$$

where $\mathcal{F}_i = \sigma(W_t, t \le i)$. Then $\{(\xi_i, \mathcal{F}_i), i \in \mathbb{N}\}$ is a sequence of martingale differences, and $\{\widehat{b}_T - b, T > 0\}$ can be viewed as self-normalized martingales. The exponential nonuniform Berry–Esseen bound in Theorem 1.1 is a parallel result to the self-normalized sum of independent variables [19].

Moreover, it is worth noting that [11] obtained an exponential nonuniform Berry–Esseen bound and Cramér-type moderate deviation for self-normalized martingales. However, for the martingale differences $\{(\xi_i, \mathcal{F}_i), i \in \mathbb{N}\}$ it is difficult to verify the Bernstein condition in [11, (A1)]. So, our results cannot be covered or deduced directly by [11]. For more details on this topic, see [10] and the references therein.

The rest of this paper is organized as follows. In Section 2, we first give the explicit calculation of Laplace functionals related to a Jacobi process, which plays an important role in our asymptotic analysis. The proofs of the main results and their corollaries then follow in Section 3 and Appendix A.

## 2. Explicit calculation of Laplace functionals related to a Jacobi process

Recall from (1.2) that we can write

$$\widehat{b}_T - b = \frac{\int_0^T X_t/(1-X_t^2)\,\mathrm{d}X_t - b\int_0^T X_t^2/(1-X_t^2)\,\mathrm{d}t}{\int_0^T X_t^2/(1-X_t^2)\,\mathrm{d}t}.$$

Observe that, for $x > 0$,

$$\begin{aligned}\mathbb{P}_b&\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) > x\right)\\ &= \mathbb{P}_b\left(\int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t - \left(b + x\sqrt{-\frac{2(1+b)}{T}}\right)\int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t > 0\right),\end{aligned}$$

and similarly for $x < 0$:

$$\begin{aligned}\mathbb{P}_b&\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T - b) \le x\right)\\ &= \mathbb{P}_b\left(\int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t - \left(b + x\sqrt{-\frac{2(1+b)}{T}}\right)\int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t \le 0\right).\end{aligned}$$

Set

$$G_{T,x} = \int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t - \left(b + x\sqrt{-\frac{2(1+b)}{T}}\right)\int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t + x\sqrt{-\frac{T}{2(1+b)}}; \tag{2.1}$$

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then,

$$\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)>x\right)=\mathbb{P}_b\left(G_{T,x}>x\sqrt{-\frac{T}{2(1+b)}}\right), \qquad x>0,$$

$$\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)\le x\right)=\mathbb{P}_b\left(G_{T,x}\le x\sqrt{-\frac{T}{2(1+b)}}\right), \qquad x<0.$$

Write

$$h_{T,x}=x\sqrt{-\frac{2(b+1)}{T}}, \tag{2.2}$$

and define a new probability measure $\mathbb{Q}$ which is absolutely continuous with respect to $\mathbb{P}_b$, with Radon–Nikodym density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_b}=\frac{\exp\{h_{T,x}G_{T,x}\}}{\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})}. \tag{2.3}$$

Now, we can obtain, for $x>0$,

$$\begin{aligned}\mathbb{P}_b&\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)>x\right)\\&=\mathbb{E}_{\mathbb{Q}}\left(\frac{\mathrm{d}\mathbb{P}_b}{\mathrm{d}\mathbb{Q}}\mathbf{1}_{\{G_{T,x}>x\sqrt{-T/(2(1+b))}\}}\right)\\&=\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})\mathbb{E}_{\mathbb{Q}}\left(\exp\{-h_{T,x}G_{T,x}\}\mathbf{1}_{\{G_{T,x}>x\sqrt{-T/(2(1+b))}\}}\right),\end{aligned} \tag{2.4}$$

and, for $x<0$,

$$\begin{aligned}\mathbb{P}_b&\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)\le x\right)\\&=\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})\mathbb{E}_{\mathbb{Q}}\left(\exp\{-h_{T,x}G_{T,x}\}\mathbf{1}_{\{G_{T,x}\le x\sqrt{-T/(2(1+b))}\}}\right).\end{aligned} \tag{2.5}$$

To prove Theorem 1.1, we need to analyze the Laplace functionals of $G_{T,x}$ under $\mathbb{P}_b$ and $\mathbb{Q}$ explicitly. Recall that if $\{X_t,\ t\ge 0\}$ is defined by (1.1), then its transition semi-group density is given by [8, p. 526]

$$P_t(y)=\sqrt{2\pi}K_\alpha\frac{\mathrm{e}^{\gamma_0^2t/2}}{\sqrt{t}}\sum_{n=0}^{\infty}\frac{\Gamma(2n+\alpha+\frac{3}{2})}{4^n n!\Gamma(n+\alpha+1)}(1-y^2)^{n+\alpha}f_{T_1}*f_{C_{2n+\gamma_0}}(1/t), \quad y\in[-1,1], \tag{2.6}$$

where $*$ is the convolution of two functions and

$$\alpha=-b-1, \quad K_\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})2^{\alpha+1/2}}, \quad \gamma_0=\alpha+\frac{1}{2}, \quad f_{T_1}(s)=\sum_{k\ge 0}\mathrm{e}^{-(\pi^2/2)(k+1/2)^2s}\mathbf{1}_{\{s>0\}},$$

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with

$$f_{C_{2n+\gamma_0}}(s)=\frac{2^{2n+\gamma_0}}{\sqrt{2\pi s^3}\Gamma(2n+\gamma_0)}\sum_{k\geq 0}(-1)^k\frac{(2k+2n+\gamma_0)\Gamma(k+2n+\gamma_0)}{k!}\mathrm{e}^{-((2k+2n+\gamma_0)^2)/2s}\mathbf{1}_{\{s>0\}}.$$

The following lemma plays an important role in our analysis.

**Lemma 2.1.** *For any $u\in\mathbb{R}$ and $h_{T,x}$ as defined in (2.2),*

$$\mathbb{E}_b\exp\left\{\left(h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}}\right)G_{T,x}\right\}$$
$$=\exp\left\{\left(h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}}\right)\left(x\sqrt{-\frac{T}{2(1+b)}}-\frac{T}{2}\right)-(b-c_{T,x}(u))\frac{T}{2}+\mathcal{R}_{T,x}(u)\right\}\quad(2.7)$$

*holds for all $T>0$. Here, $\mathrm{i}=\sqrt{-1}$ and*

$$c_{T,x}(u)=-1-\sqrt{(b+1)^2+2\left(h_{T,x}+\frac{\mathrm{i}u}{\sqrt{T}}\right)\left(b+1+x\sqrt{-\frac{2(1+b)}{T}}\right)},$$

$$\varphi_{T,x}(u)=h_{T,x}+\frac{\mathrm{i}u}{\sqrt{T}}+b-c_{T,x}(u),$$

$$\mathcal{R}_{T,x}(u)=\log\left(\sqrt{2\pi}K_{\alpha_1}\frac{\mathrm{e}^{\gamma_1^2T/2}}{\sqrt{T}}\sum_{n=0}^{\infty}\frac{\Gamma(2n-c_{T,x}(u)+\frac{1}{2})}{4^n n!\Gamma(n-c_{T,x}(u))}f_{T_1}*f_{C_{2n+\gamma_1}}(1/T)\right.$$
$$\left.\times B\left(n-c_{T,x}(u)-\frac{1}{2}\varphi_{T,x}(u),\frac{1}{2}\right)\right).$$

*Proof.* From the definition of $G_{T,x}$ (2.1), it follows that

$$\mathbb{E}_b\exp\left\{\left(h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}}\right)G_{T,x}\right\}$$
$$=\exp\left\{\left(h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}}\right)\left(x\sqrt{-\frac{T}{2(1+b)}}\right)\right\}$$
$$\times\mathbb{E}_b\exp\left\{\lambda_{T,x}(u)\int_0^T\frac{X_t}{1-X_t^2}\,\mathrm{d}X_t-\mu_{T,x}(u)\int_0^T\frac{X_t^2}{1-X_t^2}\,\mathrm{d}t\right\},$$

where $\lambda_{T,x}(u)$ and $\mu_{T,x}(u)$ are defined by

$$\lambda_{T,x}(u)=h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}},\qquad \mu_{T,x}(u)=\left(h_{T,x}+\frac{u\mathrm{i}}{\sqrt{T}}\right)\left(b+x\sqrt{-\frac{2(1+b)}{T}}\right).$$

Let us define the joint log-Laplace transformation

$$g_T(\lambda_{T,x}(u),\mu_{T,x}(u))=\log\mathbb{E}_b\exp\left\{\lambda_{T,x}(u)\int_0^T\frac{X_t}{1-X_t^2}\,\mathrm{d}X_t-\mu_{T,x}(u)\int_0^T\frac{X_t^2}{1-X_t^2}\,\mathrm{d}t\right\}.$$

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Denote the Brownian filtration by $\mathcal{F}_t = \sigma(W_s, s \le t)$; then, by virtue of the Girsanov theorem, we have, for any $\mathrm{Re}(b_0) < -1$,

$$\frac{\mathrm{d}\mathbb{P}_b}{\mathrm{d}\mathbb{P}_{b_0}}\bigg|_{\mathcal{F}_t} = \exp\bigg\{(b-b_0)\int_0^t \frac{X_s}{1-X_s^2}\,\mathrm{d}X_s - \frac{1}{2}(b^2-b_0^2)\int_0^t \frac{X_s^2}{1-X_s^2}\,\mathrm{d}s\bigg\}, \qquad t \ge 0.$$

Now, choose $b_0 = -1 - \sqrt{(b+1)^2 + 2(\lambda_{T,x}(u) + \mu_{T,x}(u))} := c_{T,x}(u) := c$; then

$$\begin{aligned}
&\exp\{g_T(\lambda_{T,x}(u), \mu_{T,x}(u))\}\\
&\quad = \mathbb{E}_c\bigg(\frac{\mathrm{d}\mathbb{P}_b}{\mathrm{d}\mathbb{P}_c}\exp\bigg\{\lambda_{T,x}(u)\int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t - \mu_{T,x}(u)\int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t\bigg\}\bigg)\\
&\quad = \mathbb{E}_c\exp\bigg\{(\lambda_{T,x}(u) + b - c_{T,x}(u))\int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t\\
&\qquad\qquad - \bigg(\mu_{T,x}(u) + \frac{1}{2}(b^2 - c_{T,x}^2(u))\bigg)\int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t\bigg\}\\
&\quad = \mathbb{E}_c\exp\bigg\{(\lambda_{T,x}(u) + b - c_{T,x}(u))\bigg(\int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t + \int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t\bigg)\bigg\}.
\end{aligned}$$

Applying Itô's formula,

$$-\frac{1}{2}\log(1-X_T^2) - \frac{T}{2} = \int_0^T \frac{X_t}{1-X_t^2}\,\mathrm{d}X_t + \int_0^T \frac{X_t^2}{1-X_t^2}\,\mathrm{d}t$$

and, as a consequence,

$$\exp\{g_T(\lambda_{T,x}(u), \mu_{T,x}(u))\} = \exp\bigg\{-\frac{T}{2}\varphi_{T,x}(u)\bigg\}\mathbb{E}_c(1-X_T^2)^{-\varphi_{T,x}(u)/2}.$$

Using the transition density function of $X_T$ [8], see also (2.6), we obtain

$$\begin{aligned}
&\mathbb{E}_c(1-X_T^2)^{-\varphi_{T,x}(u)/2}\\
&= \sqrt{2\pi}K_{\alpha_1}\frac{\mathrm{e}^{\gamma_1^2 T/2}}{\sqrt{T}}\sum_{n=0}^{\infty}\frac{\Gamma(2n - c_{T,x}(u) + \frac{1}{2})}{4^n n!\,\Gamma(n - c_{T,x}(u))} f_{T_1} * f_{C_{2n+\gamma_1}}(1/T)\,B\bigg(n - c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg),
\end{aligned}$$

where $B$ stands for the Beta function and

$$\alpha_1 = -c_{T,x}(u) - 1, \qquad K_{\alpha_1} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+\frac{3}{2})2^{\alpha_1+1/2}}, \qquad \gamma_1 = \alpha_1 + \frac{1}{2}. \qquad \square$$

Recalling the definition of $\mathcal{R}_{T,x}(u)$ in the above lemma we can write

$$\begin{aligned}
&f_{T_1} * f_{C_{2n+\gamma_1}}(1/T)\\
&= \sum_{k,\ell\ge 0}\frac{(-1)^k}{\sqrt{2\pi}}U_{k,n}(u)\int_T^{\infty}\exp\bigg\{-\frac{1}{2}\bigg[(2n+2k+\gamma_1)^2 s + \pi^2\bigg(\ell+\frac{1}{2}\bigg)^2\bigg(\frac{s-T}{Ts}\bigg)\bigg]\bigg\}\frac{\mathrm{d}s}{\sqrt{s}},
\end{aligned}$$

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where

$$U_{k,n} = \frac{\Gamma(2n+k+\gamma_1)2^{2n+\gamma_1}(2n+2k+\gamma_1)}{k!\Gamma(2n+\gamma_1)}.$$

In order to derive the precise estimation of $\mathcal{R}_{T,x}(u)$, the following expansion of Gamma functions plays a crucial role.

**Lemma 2.2.** *Let* $\Gamma(z)$ *be the Gamma function defined on the complex plane* $\mathbb{C}$. *Then*

$$\frac{\Gamma(z_2)}{\Gamma(z_1)} = \frac{z_1}{z_2}\exp\left\{(z_1-z_2)\left(\gamma - \sum_{k=1}^{\infty}\frac{z_2}{k(k+z_2)}\right) + O\left(\frac{|z_1-z_2|^2}{|z_2|^2}\right)\right\},$$

*where* $z_1, z_2 \in \mathbb{C}$, $\mathrm{Re}(z_1), \mathrm{Re}(z_2) > 0$, *and* $\gamma = 0.5772\ldots$ *is the Euler constant. Moreover, when* $z_1 - z_2 = c \in \mathbb{R}_+$ *and* $\mathrm{Re}(z_2) > 1$,

$$\left|\frac{\Gamma(z_2)}{\Gamma(z_1)}\right| \le \left|\frac{z_1}{z_2}\right| \exp\{c\gamma\}.$$

*Proof.* By Euler's formula [3, p. 199] we have, for all $z \in \mathbb{C}$,

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\mathrm{e}^{-z/k},$$

where $\gamma = 0.5772\ldots$ is the Euler constant. As a consequence,

$$\begin{aligned}\frac{\Gamma(z_2)}{\Gamma(z_1)} &= \frac{z_1}{z_2}\mathrm{e}^{\gamma(z_1-z_2)}\prod_{k=1}^{\infty}\frac{k+z_1}{k+z_2}\mathrm{e}^{-(z_1-z_2)/k}\\ &= \frac{z_1}{z_2}\exp\left\{\gamma(z_1-z_2) + \sum_{k=1}^{\infty} -\frac{z_1-z_2}{k} + \sum_{k=1}^{\infty}\log\left(1+\frac{z_1-z_2}{k+z_2}\right)\right\}\\ &= \frac{z_1}{z_2}\exp\left\{\gamma(z_1-z_2) + (z_1-z_2)\sum_{k=1}^{\infty}\frac{-z_2}{k(k+z_2)} + O\left(\sum_{k=1}^{\infty}\left|\frac{z_1-z_2}{k+z_2}\right|^2\right)\right\}\\ &= \frac{z_1}{z_2}\exp\left\{(z_1-z_2)\left(\gamma - \sum_{k=1}^{\infty}\frac{z_2}{k(k+z_2)}\right) + O\left(\frac{|z_1-z_2|^2}{|z_2|^2}\right)\right\}.\end{aligned}$$

Moreover, when $z_1 - z_2 = c \in \mathbb{R}_+$ and $\mathrm{Re}(z_2) > 0$, we have

$$\left|\prod_{k=1}^{\infty}\left(1+\frac{c}{k+z_2}\right)\mathrm{e}^{-c/k}\right| \le \prod_{k=1}^{\infty}\left(1+\frac{c}{k}\right)\mathrm{e}^{-c/k} \le \prod_{k=1}^{\infty}\mathrm{e}^{c/k}\mathrm{e}^{-c/k} \le 1.$$

Thus, we obtain the desired estimate,

$$\left|\frac{\Gamma(z_2)}{\Gamma(z_1)}\right| \le \left|\frac{z_1}{z_2}\right| \exp\{c\gamma\}.$$

□

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Define the Laplace functional of $G_{T,x}$ under $\mathbb{Q}$ as

$$\varphi_{\mathbb{Q},G_{T,x}}(u)=\mathbb{E}_{\mathbb{Q}}\exp\{\mathrm{i}uG_{T,x}\},\qquad u\in\mathbb{R}.$$

In the following propositions, we give explicit asymptotic expansions of $\varphi_{\mathbb{Q},G_{T,x}}(u/\sqrt{T})$ for different ranges of $u$.

**Proposition 2.1.** *For any constant* $\rho>0$, *and* $|u|\le\rho T^{1/6}$, $|x|\le\rho T^{1/6}$,

$$\mathbb{E}_b\exp\{h_{T,x}G_{T,x}\}=\exp\left\{\frac{x^2}{2}+O(x^3T^{-1/2})\right\},\tag{2.8}$$

$$\varphi_{\mathbb{Q},G_{T,x}}\left(\frac{u}{\sqrt{T}}\right)=\exp\{\mathcal{A}_{1,T,x}u\mathrm{i}+\mathcal{A}_{2,T,x}u^2+O(u^3T^{-1/2})\},\tag{2.9}$$

*where*

$$\mathcal{A}_{1,T,x}=x\sqrt{-\frac{1}{2(1+b)}}+O(x^2T^{-1/2}),\qquad \mathcal{A}_{2,T,x}=\frac{1}{4(b+1)}+O(xT^{-1/2}).$$

*Proof.* By using (2.3) and (2.7), we can write

$$\mathbb{E}_b\exp\{h_{T,x}G_{T,x}\}=\exp\left\{h_{T,x}\left(x\sqrt{-\frac{T}{2(1+b)}}-\frac{T}{2}\right)-(b-c_{T,x}(0))\frac{T}{2}+\mathcal{R}_{T,x}(0)\right\}\tag{2.10}$$

and

$$\begin{aligned}\varphi_{\mathbb{Q},G_{T,x}}\left(\frac{u}{\sqrt{T}}\right)&=\mathbb{E}_b\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_b}\exp\left\{\frac{\mathrm{i}u}{\sqrt{T}}G_{T,x}\right\}\right)\\&=\frac{\mathbb{E}_b\exp\{(h_{T,x}+\mathrm{i}u/\sqrt{T})G_{T,x}\}}{\mathbb{E}_b\exp\{h_{T,x}G_{T,x}\}}\\&=\exp\left\{\frac{u\mathrm{i}}{\sqrt{T}}\left(x\sqrt{-\frac{T}{2(1+b)}}-\frac{T}{2}\right)-(c_{T,x}(0)-c_{T,x}(u))\frac{T}{2}+\mathcal{R}_{T,x}(u)-\mathcal{R}_{T,x}(0)\right\}.\end{aligned}\tag{2.11}$$

On the one hand, by Taylor's expansion, straightforward but tedious calculations will give

$$\begin{aligned}c_{T,x}(u)&=-1+(b+1)\left(1+2\left(h_{T,x}+\frac{\mathrm{i}u}{\sqrt{T}}\right)\left(\frac{1}{b+1}+\frac{x}{(b+1)^2}\sqrt{-\frac{2(1+b)}{T}}\right)\right)^{-1/2}\\&=\left(b+x\sqrt{-\frac{2(b+1)}{T}}-\frac{x^2}{T}+O(x^3T^{-3/2})\right)+(T^{-1/2}+O(x^2T^{-3/2}))u\mathrm{i}\\&\quad+\left(\frac{1}{2T(b+1)}+O(xT^{-3/2})\right)u^2+O(u^3T^{-3/2}),\end{aligned}\tag{2.12}$$

$$c_{T,x}(0)=b+x\sqrt{-\frac{2(b+1)}{T}}-\frac{x^2}{T}+O(x^3T^{-3/2}).\tag{2.13}$$

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Consequently, we find that

$$-(c_{T,x}(0)-c_{T,x}(u))\frac{T}{2}=\left(\frac{T^{1/2}}{2}+O(x^2T^{-1/2})\right)u\mathrm{i} + \left(\frac{1}{4(b+1)}+O(xT^{-1/2})\right)u^2+O(u^3T^{-1/2}), \tag{2.14}$$

$$-(b-c_{T,x}(0))\frac{T}{2}=\frac{x}{2}\sqrt{-2T(b+1)}-\frac{x^2}{2}+O(x^3T^{-1/2}). \tag{2.15}$$

On the other hand, by Proposition A.1,

$$\mathcal{R}_{T,x}(u)-\mathcal{R}_{T,x}(0)=O(T^{-1/2})u\mathrm{i}+O(u^2T^{-1})+O(u^3T^{-3/2}), \quad \mathcal{R}_{T,x}(0)=O(xT^{-1/2}).$$

Therefore, combining (2.10), (2.15), and (A.1), we get (2.8). Moreover, (2.9) can be obtained via (2.11), (2.14), and (A.1). □

**Proposition 2.2.** *For any constant* $\rho>0$, *and* $|u|\le(|b|/32)T^{1/2}$, $|x|\le\rho T^{1/6}$,

$$\left|\varphi_{\mathbb{Q},G_{T,x}}\left(\frac{u}{\sqrt{T}}\right)\right|\le 4(\sqrt{2}+1)\sqrt{\pi}\exp\left\{-\frac{u^2}{8\sqrt{2}|b|}+\frac{\gamma}{2}\right\},$$

*where* $\gamma$ *is the Euler constant.*

*Proof.* Recall the calculation in (2.11); it suffices to analyze the real part of $-(c_{T,x}(0)-c_{T,x}(u))T/2$ and give an upper bound for $|\exp(\mathcal{R}_{T,x}(u))|$. Using

$$\mathrm{Re}(\sqrt{1+z})=\sqrt{\frac{|1+z|+\mathrm{Re}(1+z)}{2}},\ c_{T,x}(u) \tag{2.16}$$

$$=c_{T,x}(0)\left(1+\frac{4u\mathrm{i}}{c_{T,x}^2(0)\sqrt{T}}\left(b+1+x\sqrt{-\frac{2(1+b)}{T}}\right)\right)^{1/2}, \tag{2.17}$$

we have

$$\mathrm{Re}(c_{T,x}(u))=\frac{c_{T,x}(0)}{\sqrt{2}}\left(1+\sqrt{1+\frac{16u^2}{c_{T,x}^4(0)T}\left(b+1+x\sqrt{-\frac{2(1+b)}{T}}\right)^2}\right)^{1/2}.$$

Now, for $T$ large enough and $|u|\le(|b|/32)T^{1/2}$, $|x|\le\rho T^{1/6}$,

$$-\frac{b}{2}\le -b-1-x\sqrt{-\frac{2(1+b)}{T}}\le -2b, \qquad \left|\frac{b}{2}\right|\le|c_{T,x}(0)|\le 2^{1/4}|b|.$$

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Consequently,

$$\begin{aligned}
&\frac{T}{2}|\mathrm{Re}(c_{T,x}(u)-c_{T,x}(0))| \\
&=\frac{|c_{T,x}(0)|T}{2\sqrt{2}}\Bigg(\sqrt{1+\bigg|1+\frac{4u\mathrm{i}}{c_{T,x}^2(0)\sqrt{T}}\bigg(b+1+x\sqrt{-\frac{2(1+b)}{T}}\bigg)\bigg|}-\sqrt{2}\Bigg) \\
&=\frac{4\sqrt{2}u^2(b+1+x\sqrt{-2(1+b)/T})^2}{|c_{T,x}^3(0)|}\Bigg(\bigg|1+\frac{4u\mathrm{i}}{c_{T,x}^2(0)\sqrt{T}}\bigg(b+1+x\sqrt{-\frac{2(1+b)}{T}}\bigg)\bigg|+1\Bigg)^{-1} \\
&\quad\times\Bigg(\sqrt{1+\bigg|1+\frac{4u\mathrm{i}}{c_{T,x}^2(0)\sqrt{T}}\bigg(b+1+x\sqrt{-\frac{2(1+b)}{T}}\bigg)\bigg|}+\sqrt{2}\Bigg)^{-1} \\
&\geq\frac{u^2(b+1+x\sqrt{-2(1+b)/T})^2}{|c_{T,x}^3(0)|}\Bigg(1+\frac{16u^2}{c_{T,x}^4(0)T}\bigg(b+1+x\sqrt{-\frac{2(1+b)}{T}}\bigg)^2\Bigg)^{-3/4} \\
&\geq\frac{u^2}{8\sqrt{2}|b|},
\end{aligned}$$

which immediately implies that

$$\mathrm{Re}\bigg(-(c_{T,x}(0)-c_{T,x}(u))\frac{T}{2}\bigg)\leq-\frac{u^2}{8\sqrt{2}|b|}. \tag{2.18}$$

Next, we turn to estimating $|\exp(\mathcal{R}_{T,x}(u))|$. From (2.17),

$$\frac{|b|}{2}\leq|c_{T,x}(u)|=|c_{T,x}(0)|\bigg|1+\frac{4u\mathrm{i}}{c_{T,x}^2(0)\sqrt{T}}\bigg(b+1+x\sqrt{-\frac{2(1+b)}{T}}\bigg)\bigg|^{1/2}\leq\sqrt{2}|b|.$$

Together with (2.12) and (A.2)–(A.4), we have

$$\bigg|-c_{T,x}(u)-\frac{1}{2}\varphi_{T,x}(u)+\frac{1}{2}\bigg|\leq(\sqrt{2}+1)|b|,\qquad\bigg|-c_{T,x}(u)-\frac{1}{2}\varphi_{T,x}(u)\bigg|\geq\frac{1}{2}|b|.$$

By Lemma 2.2, we have

$$\bigg|\frac{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u))}{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)+1/2)}\bigg|\leq\bigg|\frac{-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)+1/2}{-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)}\bigg|\mathrm{e}^{\gamma/2}\leq2(\sqrt{2}+1)\mathrm{e}^{\gamma/2},$$

which, together with (A.8), implies that

$$|\exp\{\mathcal{R}_{T,x}(u)\}|=\bigg|B\bigg(-c_{T,x}(u)-\frac{1}{2}\varphi_{T,x}(u),\ \frac{1}{2}\bigg)(1+O(\mathrm{e}^{-2T}))\bigg|\leq4(\sqrt{2}+1)\sqrt{\pi}\mathrm{e}^{\gamma/2}. \tag{2.19}$$

Finally, by (2.11), (2.18), and (2.19), we can complete the proof of this proposition. □

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## 3. Proofs of the main results

In this section we give the proof of our main result, Theorem 1.1, and its two corollaries. Recall that the definitions of $G_{T,x}$ and the probability measure $\mathbb{Q}$ are given in (2.1) and (2.3) respectively. Consider the normalized version of $G_{T,x}$,

$$\widetilde{G}_{T,x}=\bigg(-\frac{T}{2(b+1)}\bigg)^{-1/2}\bigg(G_{T,x}-x\sqrt{-\frac{T}{2(b+1)}}\bigg).$$

Using the simple identity $\int_0^\infty \exp\{-xu\}\,\mathrm{d}\Phi(u)=\mathrm{e}^{x^2/2}(1-\Phi(x))$ and (2.4), we can write, for $x>0$,

$$\begin{aligned}\mathbb{P}_b\bigg(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)>x\bigg)&=\mathrm{e}^{-x^2}\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})\int_0^\infty \exp\{-xt\}\,\mathrm{d}\mathbb{Q}(\widetilde{G}_{T,x}\le t)\\&=\mathrm{e}^{-x^2}\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})(\mathcal{J}_{T,x}+\mathrm{e}^{x^2/2}(1-\Phi(x))),\end{aligned}\tag{3.1}$$

and for $x<0$, using (2.5),

$$\mathbb{P}_b\bigg(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)\le x\bigg)=\mathbb{E}_b(\exp\{h_{T,x}G_{T,x}\})(\widehat{\mathcal{J}}_{T,x}+\mathrm{e}^{x^2/2}\Phi(x)),$$

where $h_{T,x}=x\sqrt{-2(b+1)/T}$ and

$$\mathcal{J}_{T,x}=\int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}(\mathbb{Q}(\widetilde{G}_{T,x}\le t)-\Phi(t)),\qquad \widehat{\mathcal{J}}_{T,x}=\int_{-\infty}^0 \mathrm{e}^{-xt}\,\mathrm{d}(\mathbb{Q}(\widetilde{G}_{T,x}\le t)-\Phi(t)).\tag{3.2}$$

Next, we need to give the estimation of $\mathcal{J}_{T,x}$ and $\widehat{\mathcal{J}}_{T,x}$. The following lemma [18, Theorem 2, p. 109] will play a crucial role.

**Lemma 3.1.** *Assume $F$ and $G$ are probability distribution functions, and that $G$ has bounded derivative. Define $\varphi(u)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}ux}\,\mathrm{d}F(x)$ and $\psi(u)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}ux}\,\mathrm{d}G(x)$. Then, for any $M>0$,*

$$\sup_{x\in\mathbb{R}}|F(x)-G(x)|\le\frac{1}{\pi}\int_{-M}^{M}\bigg|\frac{\varphi(u)-\psi(u)}{u}\bigg|\,\mathrm{d}u+\frac{24}{\pi M}\sup_{x\in\mathbb{R}}|G'(x)|.$$

*Proof of Theorem* 1.1. Without loss of generality, we assume that $x>0$; the proof for $x<0$ is similar. $\mathcal{J}_{T,x}$, as defined by (3.2), can be estimated as follows. By using Lemma 3.1, we have

$$\sup_{t\in\mathbb{R}}|\mathbb{Q}(\widetilde{G}_{T,x}\le t)-\Phi(t)|\le\frac{1}{\pi}\int_{-(|b|/32)T^{1/2}}^{(|b|/32)T^{1/2}}\bigg|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\bigg|\,\mathrm{d}u+\frac{768}{|b|\pi\sqrt{2\pi}}T^{-1/2}.$$

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Moreover, we have

$$\frac{1}{\pi}\int_{-(|b|/32)T^{1/2}}^{(|b|/32)T^{1/2}}\left|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\right|\mathrm{d}u$$
$$\leq\frac{1}{\pi}\int_{-T^{1/6}}^{T^{1/6}}\left|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\right|\mathrm{d}u+\frac{1}{\pi}\int_{-(|b|/32)T^{1/2}}^{-T^{1/6}}\left|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\right|\mathrm{d}u$$
$$+\frac{1}{\pi}\int_{T^{1/6}}^{(|b|/32)T^{1/2}}\left|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\right|\mathrm{d}u.$$

From Proposition 2.1, it follows that, for $|u|\leq T^{1/6}$,

$$\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)=\mathrm{e}^{-\mathrm{i}ux}\varphi_{\mathbb{Q},G_{T,x}}\left(u\sqrt{-\frac{2(b+1)}{T}}\right)$$
$$=\mathrm{e}^{-u^2/2}\exp\{ui O(x^2T^{-1/2})+u^2O(xT^{-1/2})+O(u^3T^{-1/2})\}.$$

Together with Proposition 2.2, we have

$$\frac{1}{\pi}\int_{-(|b|/32)T^{1/2}}^{(|b|/32)T^{1/2}}\left|\frac{\varphi_{\mathbb{Q},\widetilde{G}_{T,x}}(u)-\mathrm{e}^{-u^2/2}}{u}\right|\mathrm{d}u$$
$$\leq\frac{1}{\pi}\int_{-T^{1/6}}^{T^{1/6}}\mathrm{e}^{-u^2/2}(O(x^2T^{-1/2})+uO(xT^{-1/2})+O(u^2T^{-1/2}))\,\mathrm{d}u$$
$$+\frac{2}{\pi}\int_{T^{1/6}}^{(|b|/32)T^{1/2}}\frac{|\varphi_{\mathbb{Q},G_{T,x}}(u\sqrt{-2(b+1)/T})|+\mathrm{e}^{-u^2/2}}{u}\,\mathrm{d}u$$
$$=O((x^2+1)T^{-1/2}).$$

Therefore, we obtain

$$|\mathcal{J}_{T,x}|\leq\sup_{t\in\mathbb{R}}|\mathbb{Q}(\widetilde{G}_{T,x}\leq t)-\Phi(t)|=O((x^2+1)T^{-1/2}).\tag{3.3}$$

Applying (2.7), (2.8), (3.1), and (3.3),

$$\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)>x\right)=\mathbb{P}_b\left(G_{T,x}>x\sqrt{-\frac{T}{2(b+1)}}\right)$$
$$=(1-\Phi(x))(1+O(x(x^2+1)T^{-1/2})).$$

Therefore, we get

$$\left|\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)>x\right)-(1-\Phi(x))\right|=(1-\Phi(x))O(x(x^2+1)T^{-1/2})$$
$$=\mathrm{e}^{-x^2/2}O((x^2+1)T^{-1/2}),$$

which completes the proof of (1.4).

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Note that

$$\frac{x}{\sqrt{2\pi}(1+x^2)}\exp\left\{-\frac{x^2}{2}\right\} \le 1-\Phi(x) \le \frac{1}{\sqrt{2\pi}x}\exp\left\{-\frac{x^2}{2}\right\}, \qquad x>0. \tag{3.4}$$

Together with (1.4), we can get (1.5), i.e.

$$\frac{\mathbb{P}_b(\sqrt{-T/(2(1+b))}(\widehat{b}_T-b)\le x)}{\Phi(x)} = \exp\left\{O(1)\frac{x^3}{\sqrt{T}}\right\}.$$ □

*Proof of Corollary* 1.1. According to Theorem 1.1, for any $\rho>0$,

$$\sup_{x\le\rho T^{1/6}}\left|\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)\le x\right)-\Phi(x)\right| \le CT^{-1/2}. \tag{3.5}$$

Moreover, using [14, Theorem 1.1] and (3.4), we have

$$\begin{aligned}&\sup_{x\ge\rho T^{1/6}}\left|\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)\le x\right)-\Phi(x)\right|\\ &\quad\le \sup_{x\ge\rho T^{1/6}}\left|\mathbb{P}_b\left(\sqrt{-\frac{T}{2(1+b)}}(\widehat{b}_T-b)> x\right)\right| + \sup_{x\ge\rho T^{1/6}}(1-\Phi(x)) \le CT^{-1/2}. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we can complete the proof of Corollary 1.1. □

*Proof of Corollary* 1.2. Finally, Corollary 1.2 can be obtained immediately from (1.5) in Theorem 1.1. □

## Appendix A. Estimations of $\mathcal{R}_{T,x}(u)$ and $\mathcal{R}_{T,x}(0)$

In this appendix, we give precise estimates of $\mathcal{R}_{T,x}(u)$ and $\mathcal{R}_{T,x}(0)$, which play crucial roles in the proof of Proposition 2.1.

**Proposition A.1.** *For any constant $\rho>0$, and $|u|\le\rho T^{1/6}$, $|x|\le\rho T^{1/6}$,*

$$\mathcal{R}_{T,x}(u)-\mathcal{R}_{T,x}(0)=O(T^{-1/2})ui+O(u^2T^{-1})+O(u^3T^{-3/2}), \qquad \mathcal{R}_{T,x}(0)=O(xT^{-1/2}). \tag{A.1}$$

*Proof.* From (2.12) and (2.13), we have

$$\varphi_{T,x}(u)=O(x^2T^{-1})+O(x^2T^{-3/2})ui+O(u^2T^{-1})+O(u^3T^{-3/2}), \tag{A.2}$$

$$c_{T,x}(0)-b=O(xT^{-1/2}), \tag{A.3}$$

$$c_{T,x}(u)-c_{T,x}(0)=O(T^{-1/2})ui+O(u^2T^{-1})+O(u^3T^{-3/2}). \tag{A.4}$$

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By Lemma 2.2, we have the following asymptotic results:

$$\begin{aligned}\frac{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u))}{\Gamma(-c_{T,x}(u))} &= \frac{-c_{T,x}(u)}{-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)}\exp\{O(\varphi_{T,x}(u))\}\\ &= 1+O(\varphi_{T,x}(u)),\\ \frac{\Gamma(-c_{T,x}(u)+1/2)}{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)+1/2)} &= \frac{-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)+1/2}{-c_{T,x}(u)+1/2}\exp\{O(\varphi_{T,x}(u))\}\\ &= 1+O(\varphi_{T,x}(u)),\\ B\bigg(-c_{T,x}(u)-\frac{1}{2}\varphi_{T,x}(u),\,\frac{1}{2}\bigg) &= \frac{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u))\Gamma(1/2)}{\Gamma(-c_{T,x}(u)-(1/2)\varphi_{T,x}(u)+1/2)}\\ &= B\bigg(-c_{T,x}(u),\,\frac{1}{2}\bigg)(1+O(\varphi_{T,x}(u))).\end{aligned}\tag{A.5}$$

Similarly, we also have

$$\frac{B(-c_{T,x}(u),\,1/2)}{B(-c_{T,x}(0),\,1/2)} = 1+O(c_{T,x}(u)-c_{T,x}(0)),\qquad \frac{B(-c_{T,x}(0),\,1/2)}{B(-b,\,1/2)} = 1+O(c_{T,x}(0)-b).\tag{A.6}$$

Recall the Jacobi Theta function [6]:

$$\Theta(z) = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-\pi\ell^2 z} = 1+2\sum_{\ell=1}^{\infty}\mathrm{e}^{-\pi\ell^2 z}.$$

We have

$$\Theta(z) = \sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\pi(\ell+1)^2 z} \le \sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\pi(\ell+1/2)^2 z} \le \sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-\pi\ell^2 z} = \Theta(z).$$

Moreover, from the fact that

$$\sum_{\ell\ge 0}\exp\bigg\{-\frac{1}{2}\pi^2\bigg(\ell+\frac{1}{2}\bigg)^2\bigg(\frac{s}{T(s+T)}\bigg)\bigg\} = \frac{1}{2}\Theta\bigg(\frac{\pi s}{2T(s+T)}\bigg),$$

it follows that

$$\begin{aligned}&\sum_{\ell\ge 0}\int_T^{\infty}\exp\bigg\{-\frac{1}{2}\bigg[(2n+2k+\gamma_1)^2(s-T)+\pi^2\bigg(\ell+\frac{1}{2}\bigg)^2\bigg(\frac{s-T}{Ts}\bigg)\bigg]\bigg\}\frac{\mathrm{d}s}{\sqrt{Ts}}\\ &\quad= \sum_{l\ge 0}\int_0^{\infty}\exp\bigg\{-\frac{1}{2}\bigg[(2n+2k+\gamma_1)^2 s+\pi^2\bigg(\ell+\frac{1}{2}\bigg)^2\bigg(\frac{s}{T(s+T)}\bigg)\bigg]\bigg\}\frac{\mathrm{d}s}{\sqrt{T(s+T)}}\\ &\quad= \frac{1}{2}\int_0^{\infty}\exp\bigg\{-\frac{1}{2}(2n+2k+\gamma_1)^2 s\bigg\}\Theta\bigg(\frac{\pi s}{2T(s+T)}\bigg)\frac{\mathrm{d}s}{\sqrt{T(s+T)}}.\end{aligned}$$

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Recall the fact that $\Theta(x)=(1/\sqrt{x})\Theta(1/x)$, which yields

$$\frac{1}{2}\int_0^\infty \exp\left\{-\frac{1}{2}(2n+2k+\gamma_1)^2 s\right\}\Theta\left(\frac{\pi s}{2T(s+T)}\right)\frac{\mathrm{d}s}{\sqrt{T(s+T)}} = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left\{-\frac{1}{2}(2n+2k+\gamma_1)^2 s\right\}\Theta\left(\frac{2T(s+T)}{\pi s}\right)\frac{\mathrm{d}s}{\sqrt{s}}.$$

By the asymptotic expansion of the Jacobi Theta function,

$$\Theta\left(\frac{2T(s+T)}{\pi s}\right)=1+2\mathrm{e}^{-2T(s+T)/s}+o(\mathrm{e}^{-2T})\le 1+2\mathrm{e}^{-2T}+o(\mathrm{e}^{-2T}),$$

we obtain

$$\sum_{l\ge 0}\int_T^\infty \exp\left\{-\frac{1}{2}\left((2n+2k+\gamma_1)^2(s-T)+\pi^2\left(\ell+\frac{1}{2}\right)^2\left(\frac{s-T}{Ts}\right)\right)\right\}\frac{\mathrm{d}s}{\sqrt{Ts}} = \frac{1}{\sqrt{2\pi}}(1+O(\mathrm{e}^{-2T}))\int_0^\infty \exp\left\{-\frac{1}{2}((2n+2k+\gamma_1)^2 s)\right\}\frac{\mathrm{d}s}{\sqrt{s}}.$$

Consequently, we have

$$\begin{aligned}
&\frac{\mathrm{e}^{\gamma_1^2 T/2}}{\sqrt{T}} f_{T_1} * f_{C_{2n+\gamma_1}}\left(\frac{1}{T}\right)\\
&= \sum_{k,\ell\ge 0}\frac{(-1)^k}{\sqrt{2\pi}}U_{k,n}\mathrm{e}^{-((2n+2k)^2/2)T-(2n+2k)\gamma_1 T}\\
&\quad\times\int_T^\infty \exp\left\{-\frac{1}{2}\left((2n+2k+\gamma_1)^2(s-T)+\pi^2\left(\ell+\frac{1}{2}\right)^2\left(\frac{s-T}{Ts}\right)\right)\right\}\frac{\mathrm{d}s}{\sqrt{s}}\\
&= \frac{2^{2n+\gamma_1}}{\sqrt{2\pi}}\exp\{-2n^2T-2n\gamma_1 T\}\sum_{k\ge 0}\frac{(-1)^k\Gamma(2n+k+\gamma_1)}{k!\Gamma(2n+\gamma_1)}\mathrm{e}^{-2k^2T-4nkT-2k\gamma_1 T}(1+O(\mathrm{e}^{-2T})).
\end{aligned}$$

Using Lemma 2.2,

$$\frac{\Gamma(2n+k+\gamma_1)}{\Gamma(2n+\gamma_1)}=\frac{2n+\gamma_1}{2n+k+\gamma_1}\exp\left\{-k\gamma+\sum_{i\ge 1}\frac{k(2n+k+\gamma_1)}{i(i+2n+k+\gamma_1)}+O\left(\left|\frac{k}{2n+k+\gamma_1}\right|^2\right)\right\},$$

so we obtain

$$\sum_{k\ge 0}\frac{(-1)^k\Gamma(2n+k+\gamma_1)}{k!\Gamma(2n+\gamma_1)}\mathrm{e}^{-2k^2T-4nkT-2k\gamma_1 T}=1+o(\mathrm{e}^{-2T}),$$

and the small $o(\cdot)$ term is hold uniformly in $n$; this further implies that

$$\frac{\mathrm{e}^{\gamma_1^2 T/2}}{\sqrt{T}} f_{T_1} * f_{C_{2n+\gamma_1}}\left(\frac{1}{T}\right)=\frac{2^{2n+\gamma_1}}{\sqrt{2\pi}}\exp\{-2n^2T-2n\gamma_1 T\}(1+O(\mathrm{e}^{-2T})).$$

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Similarly, we have

$$
\begin{aligned}
&\frac{\mathrm{e}^{\gamma_1^2 T/2}}{\sqrt{T}} \sum_{n=0}^{\infty} \frac{\Gamma(2n - c_{T,x}(u) + 1/2)}{4^n n! \Gamma(n - c_{T,x}(u))} f_{T_1} * f_{C_{2n+\gamma_1}}\bigg(\frac{1}{T}\bigg) B\bigg(n - c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg) \\
&= \frac{2^{\gamma_1}\Gamma(-c_{T,x}(u)+1/2)}{\sqrt{2\pi}\,\Gamma(-c_{T,x}(u))} B\bigg(-c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg)(1 + O(\mathrm{e}^{-2T})) \\
&\quad \times \sum_{n=0}^{\infty} \frac{\Gamma(2n - c_{T,x}(u) + 1/2)\Gamma(-c_{T,x}(u)) B(n - c_{T,x}(u) - (1/2)\varphi_{T,x}(u), 1/2)}{n!\Gamma(-c_{T,x}(u)+1/2)\Gamma(n - c_{T,x}(u)) B(-c_{T,x}(u) - 1/2)} \mathrm{e}^{-2n^2 T - 2n\gamma_1 T} \\
&= \frac{2^{\gamma_1}\Gamma(-c_{T,x}(u)+1/2)}{\sqrt{2\pi}\,\Gamma(-c_{T,x}(u))} B\bigg(-c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg)(1 + O(\mathrm{e}^{-2T})). \qquad (\mathrm{A}.7)
\end{aligned}
$$

Applying (A.5) and (A.7), we get

$$
\begin{aligned}
&\sqrt{2\pi} K_{\alpha_1} \frac{\mathrm{e}^{\gamma_1^2 T/2}}{\sqrt{T}} \sum_{n=0}^{\infty} \frac{\Gamma(2n - c_{T,x}(u) + 1/2)}{4^n n! \Gamma(n - c_{T,x}(u))} f_{T_1} * f_{C_{2n+\gamma_1}}\bigg(\frac{1}{T}\bigg) B\bigg(n - c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg) \\
&\qquad = B\bigg(-c_{T,x}(u) - \frac{1}{2}\varphi_{T,x}(u), \frac{1}{2}\bigg)(1 + O(\mathrm{e}^{-2T})) \\
&\qquad = B\bigg(-c_{T,x}(u), \frac{1}{2}\bigg)(1 + O(\varphi_{T,x}(u)))(1 + O(\mathrm{e}^{-2T})). \qquad (\mathrm{A}.8)
\end{aligned}
$$

Combining this with (A.6) and (A.2)–(A.4), we have (A.1). □


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## Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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