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ON HYPERSTABILITY OF ADDITIVE MAPPINGS ONTO BANACH SPACES

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Abstract

Let $(X, +)$ be an Abelian group and E be a Banach space. Suppose that $f : X \to E$ is a surjective map satisfying the inequality

$$|\,\|f(x) - f(y)\| - \|f(x - y)\|\,| \le \varepsilon \min\{\|f(x) - f(y)\|^p, \|f(x - y)\|^p\}$$

for some $\varepsilon > 0$, $p > 1$ and for all $x, y \in X$. We prove that f is an additive map. However, this result does not hold for $0 < p \le 1$. As an application, we show that if f is a surjective map from a Banach space E onto a Banach space F so that for some $\epsilon > 0$ and $p > 1$

$$|\,\|f(x) - f(y)\| - \|f(u) - f(v)\|\,| \le \epsilon \min\{\|f(x) - f(y)\|^p, \|f(u) - f(v)\|^p\}$$

whenever $\|x - y\| = \|u - v\|$, then f preserves equality of distance. Moreover, if $\dim E \ge 2$, there exists a constant $K \ne 0$ such that Kf is an affine isometry. This improves a result of Vogt ['Maps which preserve equality of distance', *Studia Math.* **45** (1973) 43–48].

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1. Introduction

The well-known Ulam stability problem for a functional equation asks whether, for a map satisfying the functional equation approximately, there exists another map close to the original one which satisfies the equation exactly. The first answer to this question was given by Hyers [12], who proved the following celebrated result.

THEOREM 1.1 (Hyers). *Let f be a map from a Banach space E into a Banach space F, and assume that there is an $\varepsilon > 0$ so that*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in E$. Then there is a unique additive map $g : E \to F$ satisfying $\|f(x) - g(x)\| \le \varepsilon$ for all $x \in E$.

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This result is now called the Hyers–Ulam stability theorem for the functional equation $f(x + y) = f(x) + f(y)$. Rassias [17] introduced a new notion by using a function depending on $\|x\|$ and $\|y\|$ instead of the constant ε. Indeed, he showed that if $f : E \to F$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for some $\theta > 0$, $0 \le p < 1$ and for all $x, y \in E$, then there is a unique additive map $g : E \to F$ which satisfies

$$\|f(x) - g(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in E$. This stability phenomenon is called Hyers–Ulam–Rassias stability for functional equations. We refer to the survey [18] and the celebrated book [14] for more details and some discussions.

The strong stability phenomenon, that each map satisfying a functional equation approximately in some sense actually satisfies the functional equation, is called hyperstability. It seems that the first hyperstability result appeared in [3], which was about ring homomorphisms. Some of the latest developments in the hyperstability of functional equations can be found in the papers [1, 4–6, 8, 16]. The term *hyperstability* is often confused with *superstability* and we refer to the survey [7] for an explanation of the difference.

The first aim of this paper is to establish a hyperstability result for the functional equation

$$\|f(x - y)\| = \|f(x) - f(y)\|, \tag{1.1}$$

where f is a map from an Abelian group X onto a Banach space E. The functional equation (1.1) has been studied by several authors (see, for instance, [9, 10, 19, 20]). In particular, Sikorska [19] proved the Hyers–Ulam stability of (1.1).

The second aim of this paper is to study the hyperstability of maps which preserve the equality of distance. We say that a map f from a Banach space E into a Banach space F *preserves the equality of distance* if $\|f(x) - f(y)\| = \|f(u) - f(v)\|$ whenever $x, y, u, v \in E$ and $\|x - y\| = \|u - v\|$. Such maps were first studied by Vogt [21], who proved the following result.

THEOREM 1.2 (Vogt [21]). *Let $f : E \to F$ with $f(0) = 0$ be a surjective map which preserves the equality of distance. If $\dim E \ge 2$, there exists a constant $K \ne 0$ such that Kf is a linear isometry.*

Other results on such maps preserving the equality of distance can be found in [11, 15, 19]. In this paper, using the hyperstability of (1.1), we also show an analogous result for maps preserving the equality of distance.

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2. Main results

We start this section by recalling the following useful result of John [13] (see also [2, Corollary 14.8]).

Lemma 2.1. *Let f be a local homeomorphism from a connected open set Ω in a Banach space E onto an open subset of a Banach space F satisfying*

$$\lim_{y\to x} \frac{\|f(y)-f(x)\|}{\|y-x\|} = 1$$

for all $x \in \Omega$. Then f is the restriction of an affine isometry from E onto F.

Our first goal in this section is to show the hyperstability of (1.1).

Theorem 2.2. *Let $(X, +)$ be an Abelian group and E be a Banach space. Assume that $g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a function with the following properties:*

(1) $g(s, 0) = g(0, t) = 0$ *for all* $s, t \in [0, +\infty)$;
(2) *for any* $\varepsilon > 0$, *there is a* $\delta(\varepsilon) > 0$ *such that*

$$g(s,t) \le s\varepsilon \quad \textit{for all } 0 \le s < \delta(\varepsilon)$$

and

$$g(s,t) \le t\varepsilon \quad \textit{for all } 0 \le t < \delta(\varepsilon).$$

If $f : X \to E$ is a surjective map satisfying the inequality

$$\big|\, \|f(x)-f(y)\| - \|f(x-y)\| \,\big| \le g(\|f(x)-f(y)\|, \|f(x-y)\|) \tag{2.1}$$

for all $x, y \in X$, then f is an additive map.

Proof. We may assume that f is injective. Indeed, we can put $M = \{x \in X : f(x) = 0\}$ and consider X/M. Clearly, it follows from (2.1) that

$$f(x) = f(y) \Leftrightarrow f(x-y) = 0. \tag{2.2}$$

Hence, M is a subgroup of X. If $y \in x + M$, then (2.2) implies that $f(x) = f(y)$. Hence, we may define a map $f_1 : X/M \to E$ by $f_1(x + M) = f(x)$. Now, (2.1) can be written as

$$\begin{aligned} &\big|\, \|f_1(x+M) - f_1(y+M)\| - \|f_1(x-y+M)\| \,\big| \\ &\qquad \le g(\|f_1(x+M) - f_1(y+M)\|, \|f_1(x-y+M)\|) \end{aligned}$$

for all $x, y \in X$. It is clear that f_1 is injective. We can consider f_1 instead of f, since additivity of f follows from that of f_1.

We may thus assume that f is both injective and surjective. Fix $x \in X$ and define a map $h_x : E \to E$ by $h_x(u) = f(f^{-1}(u) + x) - f(x)$ for all $u \in E$. Clearly, (2.1) implies that $f(0) = 0$ and hence $h_0 = I$ (the identity on E) and $h_x(0) = 0$.

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Assume that $0 < \varepsilon < 1$. If $\|f(x) - f(y)\| < \delta(\varepsilon)/2 < \delta(\varepsilon)$, then

$$\begin{aligned}\|f(x-y)\| &\le \big|\,\|f(x-y)\| - \|f(x) - f(y)\|\,\big| + \|f(x) - f(y)\| \\ &\le g(\|f(x) - f(y)\|, \|f(x-y)\|) + \|f(x) - f(y)\| \\ &\le (\varepsilon + 1)\|f(x) - f(y)\| \le 2\|f(x) - f(y)\|. \end{aligned} \tag{2.3}$$

If $\|u - v\| < \delta(\varepsilon)/2$, (2.3) implies that

$$\|f(f^{-1}(u) - f^{-1}(v))\| \le 2\|f(f^{-1}(u)) - f(f^{-1}(v))\| < \delta(\varepsilon)$$

and hence

$$\begin{aligned}&\big|\,\|h_x(u) - h_x(v)\| - \|u - v\|\,\big| \\ &\quad = \big|\,\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| - \|f(f^{-1}(u)) - f(f^{-1}(v))\|\,\big| \\ &\quad \le \big|\,\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| - \|f(f^{-1}(u) - f^{-1}(v))\|\,\big| \\ &\qquad + \big|\,\|f(f^{-1}(u) - f^{-1}(v))\| - \|f(f^{-1}(u)) - f(f^{-1}(v))\|\,\big| \\ &\quad \le g(\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\|, \|f(f^{-1}(u) - f^{-1}(v))\|) \\ &\qquad + g(\|f(f^{-1}(u) - f^{-1}(v))\|, \|f(f^{-1}(u)) - f(f^{-1}(v))\|) \\ &\quad \le \varepsilon(\|f(f^{-1}(u) - f^{-1}(v))\| + \|f(f^{-1}(u)) - f(f^{-1}(v))\|) \\ &\quad \le 3\varepsilon\|f(f^{-1}(u)) - f(f^{-1}(v))\| \\ &\quad = 3\varepsilon\|u - v\|. \end{aligned} \tag{2.4}$$

Then

$$\|h_x(u) - h_x(v)\| \le (1 + 3\varepsilon)\|u - v\|. \tag{2.5}$$

On the other hand, if $\|h_x(u) - h_x(v)\| < \delta(\varepsilon)/2$, then (2.3) implies that

$$\|f(f^{-1}(u) - f^{-1}(v))\| \le 2\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| < \delta(\varepsilon)$$

and, corresponding to (2.4),

$$\begin{aligned}&\big|\,\|h_x(u) - h_x(v)\| - \|u - v\|\,\big| \\ &\quad \le \big|\,\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| - \|f(f^{-1}(u) - f^{-1}(v))\|\,\big| \\ &\qquad + \big|\,\|f(f^{-1}(u) - f^{-1}(v))\| - \|f(f^{-1}(u)) - f(f^{-1}(v))\|\,\big| \\ &\quad \le \varepsilon(\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| + \|f(f^{-1}(u) - f^{-1}(v))\|) \\ &\quad \le 3\varepsilon\|f(f^{-1}(u) + x) - f(f^{-1}(v) + x)\| = 3\varepsilon\|h_x(u) - h_x(v)\|. \end{aligned}$$

Then

$$\|u - v\| \le (1 + 3\varepsilon)\|h_x(u) - h_x(v)\|. \tag{2.6}$$

Since f is injective and surjective, h_x is also injective and surjective. Thus, combining (2.5) with (2.6) implies that h_x is a homeomorphism. Let $0 < \|u - v\| < \delta(\varepsilon)/2$. Then (2.4) implies that

$$\left|\frac{\|h_x(u) - h_x(v)\|}{\|u - v\|} - 1\right| = \frac{\big|\,\|h_x(u) - h_x(v)\| - \|u - v\|\,\big|}{\|u - v\|} \le 3\varepsilon.$$

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Hence,

$$\lim_{u\to v}\frac{\|h_x(u)-h_x(v)\|}{\|u-v\|}=1.$$

Now, by applying Lemma 2.1 and noting that $h_x(0)=0$, we obtain that $h_x : E \to E$ is a linear isometry.

Now fix $z \in X$ with $\|f(z)\| \le \delta(1)$. By letting $y = f^{-1}(u)$ for any $u \in E$,

$$h_z(f(y)) = f(z+y) - f(z) \tag{2.7}$$

for all $y \in X$. Hence, (2.1) implies that

$$\begin{aligned}\|h_z(u)-u\| &= \|h_z(f(y))-f(y)\|\\ &= \|f(z+y)-f(z)-f(y)\|\\ &\le \|f(z+y)-f(y)\|+\|f(z)\|\\ &\le |\,\|f(z+y)-f(y)\|-\|f(z+y-y)\|\,|+2\|f(z)\|\\ &\le g(\|f(z+y)-f(y)\|,\|f(z)\|)+2\|f(z)\|\\ &\le 3\|f(z)\|.\end{aligned}$$

It follows from the linearity of the isometry h_z that $h_z(u)=u$ for all $u \in E$. Then (2.7) can be written as $f(z+y)=f(z)+f(y)$ for all $y \in X$. Hence,

$$f(nz) = nf(z) \tag{2.8}$$

and

$$f(nz+y) = f(nz)+f(y) \tag{2.9}$$

for all $y \in X$ and $n \in \mathbb{N}$.

For any $x \in X$, there exists an $n_0 \in \mathbb{N}$ such that $\|f(x)/n_0\| \le \delta(1)$. Since f is surjective, there exists a $z \in X$ such that $f(z) = f(x)/n_0$. Now (2.8) and injectivity of f imply that $x = n_0 z$. Since x is arbitrary, the result follows from (2.9). □

Since the map g from $[0,+\infty)\times[0,+\infty)$ to $[0,+\infty)$ given by $g(s,t)=\varepsilon\min\{s^p,t^p\}$, where $p>1$ and $\varepsilon>0$, satisfies the conditions in Theorem 2.2, we have the following result.

THEOREM 2.3. *Let $(X,+)$ be an Abelian group and E be a Banach space. Assume that $f : X \to E$ is a surjective map. If there exist $\varepsilon > 0$ and $p > 1$ so that*

$$|\,\|f(x)-f(y)\|-\|f(x-y)\|\,| \le \varepsilon\min\{\|f(x)-f(y)\|^p,\|f(x-y)\|^p\}$$

for all $x, y \in X$, then f is an additive map.

The following example shows that Theorem 2.3 fails for $0 < p \le 1$.

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EXAMPLE 2.4. Define a surjective map $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0], \\ 2x & \text{if } x \in (0, 1/2), \\ x + 1/2 & \text{if } x \in [1/2, +\infty). \end{cases}$$

It is easy to prove that f satisfies

$$|\,\|f(x) - f(y)\| - \|f(x - y)\|\,| \le \min\{\|f(x) - f(y)\|^p, \|f(x - y)\|^p\}$$

for all $x, y \in X$ and $0 < p \le 1$. However, f is not an additive map.

We next prove a hyperstability result for maps preserving the equality of distance, which is a consequence of Theorems 1.2 and 2.2.

THEOREM 2.5. *Let g be as in Theorem 2.2. If f is a surjective map from a Banach space E onto a Banach space F so that*

$$|\,\|f(x) - f(y)\| - \|f(u) - f(v)\|\,| \le g(\|f(x) - f(y)\|, \|f(u) - f(v)\|)$$

whenever $\|x - y\| = \|u - v\|$, then f preserves equality of distance. Moreover, if $\dim E \ge 2$, there exists a constant $K \ne 0$ so that Kf is an affine isometry.

PROOF. Define $h(x) = f(x) - f(0)$ for every $x \in E$. Then $h : E \to F$ is a surjective map with $h(0) = 0$ and

$$|\,\|h(x) - h(y)\| - \|h(u) - h(v)\|\,| \le g(\|h(x) - h(y)\|, \|h(u) - h(v)\|) \tag{2.10}$$

whenever $\|x - y\| = \|u - v\|$.

By substituting $u = x - y$ and $v = 0$ in (2.10),

$$|\,\|h(x) - h(y)\| - \|h(x - y)\|\,| \le g(\|h(x) - h(y)\|, \|h(x - y)\|)$$

for all $x, y \in E$. Hence, Theorem 2.2 implies that h is additive.

Choose arbitrary $x, y, u, v \in E$ with $\|x - y\| = \|u - v\|$. For any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ so that $2^{-n}\|h(x) - h(y)\| < \delta(\varepsilon)$. Then the additivity of h and (2.10) give

$$\begin{aligned} |\,\|h(x) - h(y)\| - \|h(u) - h(v)\|\,| &= 2^n|\,\|h(2^{-n}x) - h(2^{-n}y)\| - \|h(2^{-n}u) - h(2^{-n}v)\|\,| \\ &\le 2^n g(\|h(2^{-n}x) - h(2^{-n}y)\|, \|h(2^{-n}u) - h(2^{-n}v)\|) \\ &\le 2^n g(2^{-n}\|h(x) - h(y)\|, 2^{-n}\|h(u) - h(v)\|) \\ &\le 2^n \varepsilon(2^{-n}\|h(x) - h(y)\|) \\ &= \varepsilon\|h(x) - h(y)\|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\|h(x) - h(y)\| = \|h(u) - h(v)\|$ and hence h preserves equality of distance. Since $f(x) = h(x) + f(0)$, f also preserves equality of distance.

Assume now that $\dim E \ge 2$. Then the result follows from Theorem 1.2. □

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An immediate consequence is the following theorem, which improves the theorem of Vogt mentioned in the Introduction.

THEOREM 2.6. *Let f be a surjective map from a Banach space E onto a Banach space F. If there exist an $\varepsilon > 0$ and a $p > 1$ so that*

$$|\,\|f(x) - f(y)\| - \|f(u) - f(v)\|\,| \leq \varepsilon \min\{\|f(x) - f(y)\|^p, \|f(u) - f(v)\|^p\} \tag{2.11}$$

whenever $\|x - y\| = \|u - v\|$, then f preserves equality of distance. Moreover, if $\dim E \geq 2$*, there exists a constant $K \neq 0$ so that Kf is an affine isometry.*

Theorem 2.6 also fails for $0 < p \leq 1$. In fact, using Example 2.4, we give the following counterexample between two-dimensional Banach spaces. Let f be as in Example 2.4. Define a map $g : \ell_\infty^2 \to \ell_\infty^2$ by $g(x, y) = (f(x), y)$ for all $(x, y) \in \ell_\infty^2$. It is not hard to show that g satisfies (2.11) for $\varepsilon = 1$ and for all $0 < p \leq 1$, but g does not preserve equality of distance.

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