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HARDY AND RELLICH INEQUALITIES ON THE COMPLEMENT OF CONVEX SETS

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Abstract

We establish existence of weighted Hardy and Rellich inequalities on the spaces $L_p(\Omega)$, where $\Omega = \mathbf{R}^d \backslash K$ with K a closed convex subset of $\mathbf{R}^d$. Let $\Gamma = \partial\Omega$ denote the boundary of Ω and d_Γ the Euclidean distance to Γ. We consider weighting functions $c_\Omega = c \circ d_\Gamma$ with $c(s) = s^\delta(1+s)^{\delta'-\delta}$ and $\delta, \delta' \geq 0$. Then the Hardy inequalities take the form

$$\int_\Omega c_\Omega |\nabla\varphi|^p \geq b_p \int_\Omega c_\Omega\, d_\Gamma^{-p} |\varphi|^p$$

and the Rellich inequalities are given by

$$\int_\Omega |H\varphi|^p \geq d_p \int_\Omega |c_\Omega\, d_\Gamma^{-2} \varphi|^p$$

with $H = -\mathrm{div}(c_\Omega \nabla)$. The constants b_p, d_p depend on the weighting parameters $\delta, \delta' \geq 0$ and the Hausdorff dimension of the boundary. We compute the optimal constants in a broad range of situations.

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1. Introduction

The classical Hardy and Rellich inequalities are estimates for second-order elliptic operators on the spaces $L_p(\mathbf{R}^d \backslash \{0\})$, $p \in \langle 1, \infty\rangle$. These operators describe diffusion around a point obstacle at the origin and the existence of the inequalities is related to uniqueness properties of the diffusion. Both inequalities have been studied for operators with coefficients proportional to $|x|^\delta$, where $x \in \mathbf{R}^d \backslash \{0\}$ and $\delta \geq 0$. Our intention is to derive similar estimates on $\Omega = \mathbf{R}^d \backslash K$ where K is a closed, nonempty, convex subset of $\mathbf{R}^d$ with $K \neq \mathbf{R}^d$ and for operators with coefficients exhibiting different power behaviours at the boundary of Ω and at infinity. This dichotomy is natural in applications to diffusion and in a recent paper [Rob18] we derived existence results for strong forms of both inequalities on $L_2(\mathbf{R}^d \backslash \{0\})$. Background information on the classical inequalities and references to the extensive literature can be found in the recent monograph [BEL15].

The existence of both types of inequality depends on conditions involving the Hausdorff dimension d_H of the boundary $\Gamma(=\partial\Omega)$ of Ω. If the dimension $\dim(K)$ of K, that is, the dimension of the affine closure A_K of K, takes one of the values $0, 1, \ldots, d-1$, then $d_H = \dim(K)$ but if $\dim(K) = d$, then $d_H = d-1$. Moreover, the general inequalities depend on the Euclidean distance d_Γ to the boundary, that is, $d_\Gamma(x) = \inf_{y\in\Omega^c}|x-y|$ for $x \in \Omega$. We begin by establishing the existence of weighted Hardy inequalities.

THEOREM 1.1. *Let $\Omega = \mathbf{R}^d\backslash K$, where K is a closed, nonempty, convex subset of $\mathbf{R}^d$ with $K \neq \mathbf{R}^d$. Denote the Hausdorff dimension of the boundary Γ of Ω by d_H. Further, let $c_\Omega = c \circ d_\Gamma$, where $c(s) = s^\delta(1+s)^{\delta'-\delta}$ with $\delta, \delta' \geq 0$. If $d - d_H + (\delta\wedge\delta') - p > 0$ with $p \in [1, \infty\rangle$, then*

$$\int_\Omega c_\Omega|\nabla\varphi|^p \geq \int_\Omega c_\Omega|(\nabla d_\Gamma)\cdot(\nabla\varphi)|^p \geq a_p^p \int_\Omega c_\Omega\, d_\Gamma^{-p}|\varphi|^p \tag{1.1}$$

for all $\varphi \in C_c^1(\Omega)$, where $a_p = (d - d_H + (\delta\wedge\delta') - p)/p$.

Here and in the sequel all functions are real valued. Moreover, we use the standard notation $|\nabla\varphi| = (\sum_{k=1}^d |\partial_k\varphi|^2)^{1/2}$. Then the left-hand inequality in (1.1) follows since d_Γ is a Lipschitz function with $|\nabla d_\Gamma| \leq 1$. The choice of the weight c is governed by the asymptotic properties $c(s)/s^\delta \to 1$ as $s \to 0$ and $c(s)/s^{\delta'} \to 1$ as $s \to \infty$. Although this theorem and the subsequent one are stated for the particular coefficient c, the general conclusions are valid for a large class of c with similar asymptotic properties. Note that if $\delta = \delta'$, then $c(s) = s^\delta$, which is the conventional weight function used in the discussion of Hardy and Rellich inequalities.

The original Rellich inequality for the Laplacian $\Delta = -\sum_{k=1}^d \partial_k^2 = -\nabla^2$ was established on $L_2(\mathbf{R}^d\backslash\{0\})$ but was subsequently extended by Davies and Hinz [DH98] to the spaces $L_p(\mathbf{R}^d\backslash\{0\})$ with $p \in \langle 1, \infty\rangle$ and to weighted operators $c_\Omega\Delta$. In particular, optimal estimates were derived for the weights $c(s) = s^\delta$. These developments are described in [BEL15, Sections 6.1–6.3] or in greater detail in [MSS15]. In Section 3 we establish the existence of Rellich inequalities for weighted operators $H = -\sum_{k=1}^d \partial_k c_\Omega \partial_k = -\mathrm{div}(c_\Omega\nabla)$ on the spaces $L_p(\mathbf{R}^d\backslash K)$ by extension of the method of Davies and Hinz. These operators are defined on the universal domain $C_c^2(\Omega)$ and all estimates are on this domain.

THEOREM 1.2. *Let $\Omega = \mathbf{R}^d\backslash K$, where K is a closed, nonempty, convex subset of $\mathbf{R}^d$ with $K \neq \mathbf{R}^d$. Denote the Hausdorff dimension of the boundary Γ of Ω by d_H. Further, let $c_\Omega = c \circ d_\Gamma$, where $c(s) = s^\delta(1+s)^{\delta'-\delta}$ with $\delta, \delta' \in [0, 2\rangle$. If $d - d_H + p(\delta\wedge\delta') - 2p \geq 2p|\delta - \delta'|(2 - \delta\vee\delta')^{-1}$ with $p \in \langle 1, \infty\rangle$, then there is a $c_p \in \langle 0, C_p]$, where $C_p = (p-1)(d-d_H)(d - d_H + p(\delta\wedge\delta') - 2p)p^{-2}$, such that*

$$\int_\Omega |H\varphi|^p \geq c_p^p \int_\Omega |c_\Omega\, d_\Gamma^{-2}\varphi|^p \tag{1.2}$$

for all $\varphi \in C_c^2(\Omega)$. Moreover, if $\delta = \delta'$, then $c_p = C_p$.

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The proof of the theorem allows one to identify c_p as a function of d_H, δ and δ' but the result is significantly more complicated than the expression for C_p.

Theorems 1.1 and 1.2 establish criteria for existence of the Hardy and Rellich inequalities (1.1) and (1.2), respectively, but they give no information about optimality of the constants a_p^p and c_p^p. This problem is addressed in Section 4. It is straightforward to prove by local estimates that a_p^p is optimal for the Hardy inequality if $K = \{0\}$ but the situation is more complicated if $k = \dim(K) \geq 1$. Then the geometry plays a significant role. If $k \in \{1, \ldots, d-1\}$ and $\delta \leq \delta'$, then a_p^p is still optimal because local properties are the most significant in the corresponding variational problem. If, however, $\delta' < \delta$, then global properties are important. Nevertheless, we establish that a_p^p is still optimal if $k \in \{1, \ldots, d-1\}$ and the 'dimension at infinity' k_∞ of K is equal to k. The latter dimension, which is defined in Section 4, is a measure of the size of K at infinity; for example, if K is bounded, then $k_\infty = 0$ and, if K is affine, then $k_\infty = k$. Our optimality results for the Rellich inequality are limited to the case that $\delta = \delta' \in [0, 2\rangle$. Then $c_p = C_p$ and c_p^p is the optimal constant for the Rellich inequality if $K = \{0\}$ or if $k \in \{1, \ldots, d-1\}$ and $k_\infty = k$. But these results leave open room for improvement. In particular, if $p = 2$ and $K = \{0\}$, then it follows from [Rob18] that C_2^2 is the optimal constant for all $\delta, \delta' \geq 0$ such that $\delta + \delta' \leq 4$ with the sole restriction $C_2 > 0$.

The inequality (1.2) is a natural 'weighted operator' generalization of the original Rellich inequality for the Laplacian, which differs from the weighted Rellich inequality that is usually studied, that is, the inequality similar to (1.2) but with H replaced by $c_\Omega \Delta$. The latter can be viewed as a 'weighted measure' version of the original inequality for the Laplacian, since the replacement of Δ by $c_\Omega \Delta$ corresponds to replacing Lebesgue measure dx by $c_\Omega^p \, dx$. We briefly discuss weighted measure inequalities at the end of the paper.

2. Hardy inequalities

In this section we prove Theorem 1.1. As a preliminary to the proof, we need to establish local convexity of the distance function d_Γ. Since K is a closed convex subset, it follows from Motzkin's theorem (see, for example, [Hör94, Theorem 2.1.20] or [BEL15, Theorem 2.2.9]) that each point $x \in \Omega$ has a unique nearest point $n(x) \in K$, that is, there is a unique $n(x) \in K$ such that $d_\Gamma(x) = |x - n(x)|$. Moreover, d_Γ is differentiable at each point $x \in \Omega$ and $(\nabla d_\Gamma)(x) = (x - n(x))/|x - n(x)|$. Thus, $|\nabla d_\Gamma| = 1$ and $(\nabla d_\Gamma^2)(x) = 2(x - n(x))$. Note that in the degenerate case $K = \{0\}$ one has $d_\Gamma(x) = |x|$ and consequently $\nabla^2 d_\Gamma^2 = 2d$. In the nondegenerate case it is not, however, clear that d_Γ is even twice differentiable. But this follows from local convexity.

Proposition 2.1. *The distance function d_Γ is convex on all open convex subsets of Ω. In particular, it is twice differentiable almost everywhere in Ω and the corresponding Hessian $(\partial_k \partial_l d_\Gamma)(x)$ is positive definite for almost all $x \in \Omega$.*

Proof. First, we prove the convexity in an open neighbourhood of an arbitrarily chosen point $x \in \Omega$.

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Let $n(x) \in \Gamma$ be the unique near point of $x \in \Omega$. Then there is a unique tangent hyperplane T_x at the point $n(x)$ which is orthogonal to $x - n(x)$. The hyperplane separates $\mathbf{R}^d$ into two open half spaces, $\Gamma_x^{(+)} \subset \Omega$ and $\Gamma_x^{(-)} \supset \mathrm{Int}(\Omega^c)$. Moreover, $\Omega = \bigcup_{x \in \Omega} \Gamma_x^{(+)}$ and $\mathrm{Int}(\Omega^c) = \bigcap_{x \in \Omega} \Gamma_x^{(-)}$. Now fix a point $x_0 \in \Omega$ and an $r > 0$ such that the open Euclidean ball $B_{x_0}(r)$ with centre x_0 and radius r is contained in Ω. Next, choose r sufficiently small that $B_{x_0}(r) \subset \bigcap_{x \in B_{x_0}(r)} \Gamma_x^{(+)}$. This is possible since if $x_k \in \Omega$ converges pointwise to $x \in \Omega$, then $n(x_k) \to n(x)$ (see [BEL15, Lemma 2.2.1]). Therefore, the family of open subsets $s > 0 \mapsto \Lambda_{x_0}(s) = \bigcap_{x \in B_{x_0}(s)} \Gamma_x^{(+)}$ increases to $\Gamma_{x_0}^{(+)} \supset B_{x_0}(r)$ as s decreases to zero. But the balls $B_{x_0}(s)$ decrease as $s \to 0$. Therefore, there is an r_0 such that $B_{x_0}(r) \subset \bigcap_{x \in B_{x_0}(r_0)} \Gamma_x^{(+)}$ for all $r \in \langle 0, r_0 \rangle$.

Secondly, we argue that if $r < r_0$, then d_Γ is convex on $B_{x_0}(r)$. To this end, choose three points $x, y, z \in B_{x_0}(r)$ such that $x = \lambda y + (1 - \lambda) z$ with $\lambda \in \langle 0, 1 \rangle$. Since $r < r_0$, it follows that $B_{x_0}(r) \subset \Gamma_x^{(+)}$. Thus, the tangent plane T_x separates $B_{x_0}(r)$ and Ω^c. Next, let $\tilde{x}, \tilde{y}, \tilde{z}$ denote the orthogonal projections of x, y, z onto T_x. Then $\tilde{x} = n(x)$, by definition, and $d_\Gamma(x) = |x - \tilde{x}|$. But

$$|y - \tilde{y}| = \inf_{y_0 \in \Gamma_x^{(-)}} |y - y_0| \le \inf_{y_0 \in \Omega^c} |y - y_0| = d_\Gamma(y).$$

Similarly, $|z - \tilde{z}| \le d_\Gamma(z)$. Moreover, $\tilde{x} = \lambda \tilde{y} + (1 - \lambda) \tilde{z}$ and

$$|x - \tilde{x}| = \lambda |y - \tilde{y}| + (1 - \lambda) |z - \tilde{z}|.$$

Therefore, $d_\Gamma(x) \le \lambda\, d_\Gamma(y) + (1 - \lambda)\, d_\Gamma(z)$. Since this is valid for all choices of $x, y, z \in B_{x_0}(r)$ and $\lambda \in \langle 0, 1 \rangle$ with $x = \lambda y + (1 - \lambda) z$, it follows that d_Γ is convex on $B_{x_0}(r)$.

Thirdly, it follows from Motzkin's theorem that d_Γ is once differentiable at each $x \in \Omega$. But since d_Γ is convex on $B_{x_0}(r)$, it follows from Alexandrov's theorem (see [EG92, Section 6.4]) that d_Γ is twice differentiable almost everywhere on $B_{x_0}(r)$. Since this is valid for each $x_0 \in \Omega$ for some $r > 0$, it then follows that d_Γ is twice differentiable almost everywhere on Ω. The Hessian of a convex function is automatically positive definite. Hence, the Hessian of d_Γ is positive definite almost everywhere on Ω.

Finally, let $d_\Gamma^{(\varepsilon)}$, $\varepsilon > 0$, denote a family of local mollifications/regularizations of d_Γ (see [EG92, Section 4.2.1]). Then the $d_\Gamma^{(\varepsilon)}$ are C^2-functions and their Hessians are positive definite. In fact, the proof of Alexandrov's theorem relies on proving the positive definiteness of the regularizations. Next, it follows by a standard consequence of convexity (see [Sim11, Theorem 1.5]) that $d_\Gamma^{(\varepsilon)}$ is convex on all open convex subsets suitably distant from the boundary. But $d_\Gamma^{(\varepsilon)} \to d_\Gamma$ as $\varepsilon \to 0$. Therefore, in the limit d_Γ is convex on all open convex subsets of Ω. □

The subsequent proof of the Hardy inequalities of Theorem 1.1 depends on control of the second derivatives of d_Γ.

Corollary 2.2. *If $\Omega = \mathbf{R}^d \backslash K$, where K is a closed convex subset, then $\nabla^2 d_\Gamma^2 \ge 2(d - d_H)$, where d_H is the Hausdorff (Minkowski) dimension of Γ.*

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Proof. First, if K is a singleton, then one can assume that $K = \{0\}$. Hence, $d_\Gamma^2(x) = |x|^2$ and $\nabla^2 d_\Gamma^2 = 2d$.

Secondly, if $\dim(K) = k$ with $k \in \{1, \ldots, d-1\}$, one can factor $\mathbf{R}^d$ as a direct product $\mathbf{R}^k \times \mathbf{R}^{d-k}$, where $\mathbf{R}^k$ is identified with A_K, the affine hull of K. Thus, if $x = (y, z) \in \mathbf{R}^d$ with $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{d-k}$, one has $d_\Gamma^2(x) = d_K^2(y) + |z|^2$, where $d_K(y) = \inf_{y' \in K} |y - y'|$. In particular, if $y \in K$, then $d_\Gamma^2(x) = |z|^2$ and $\nabla^2 d_\Gamma^2 = \nabla_z^2 d_\Gamma^2 = 2(d-k) = 2(d - d_H)$ because $d_H = \dim(K)$. But if $y \notin K$, then $(\nabla_x^2 d_\Gamma^2)(x) = (\nabla_y^2 d_K^2)(y) + \nabla_z^2 |z|^2 > 2(d-k)$. Hence, one now has $\nabla^2 d_\Gamma^2 \geq 2(d - d_H)$ for all $k \in \{1, \ldots, d-1\}$.

Thirdly, if $\dim(K) = d$ and $K \neq \mathbf{R}^d$, then $\Gamma = \partial K$ and the Hausdorff dimension d_H of Γ is $d-1$. Then one can argue as in [BEL15, Sections 3.4.2 and 3.4.3] that $\nabla^2 d_\Gamma^2 \geq 2$. Specifically, if $x \in \Omega$, one can choose coordinates $x = (y_1, z)$ with $y_1 > 0$, $z \in \mathbf{R}^{d-1}$ and such that the near point of $(y_1, 0)$ is the origin. Then

$$(\nabla_x^2 d_\Gamma^2)(x) = \partial_{y_1}^2 y_1^2 + (\nabla_z^2 d_\Gamma^2)(x) \geq 2$$

since $(\nabla_z^2 d_\Gamma^2)(x) \geq 0$ by Proposition 2.1. In fact, the lower bound is attained if K has a proper face with dimension $d-1$. □

At this point we are prepared to establish the weighted Hardy inequalities (1.1) of Theorem 1.1.

Proof. Let $\chi_p = c_\Omega d_\Gamma^{-p} (\nabla d_\Gamma^2)$. Further, let $c'_\Omega = c' \circ d_\Gamma$. Then

$$\begin{aligned} \operatorname{div}\chi_p &= 2(c'_\Omega d_\Gamma / c_\Omega - p) c_\Omega d_\Gamma^{-p} |\nabla d_\Gamma|^2 + c_\Omega d_\Gamma^{-p} (\nabla^2 d_\Gamma^2) \\ &\geq 2 b_p c_\Omega d_\Gamma^{-p} \end{aligned}$$

with $b_p = (d - d_H + \delta \wedge \delta' - p)$, where we have used $|\nabla d_\Gamma|^2 = 1$, the estimate $\nabla^2 d_\Gamma^2 \geq 2(d - d_H)$ of Corollary 2.2 and the observation that $c'_\Omega d_\Gamma / c_\Omega \geq \delta \wedge \delta'$. (The last estimate follows since $s c'(s)/c(s) = (\delta + s\delta')/(1+s)$.)

Next, for $\varepsilon > 0$, set $\varphi_\varepsilon = (\varphi^2 + \varepsilon^2)^{1/2} - \varepsilon$. Then $\varphi_\varepsilon \geq 0$ is a regularized approximation to $|\varphi|$ with the same support as φ. But $\varphi^2 + \varepsilon^2 = (\varphi_\varepsilon + \varepsilon)^2 \geq \varphi_\varepsilon^2 + \varepsilon^2$, so $\varphi_\varepsilon \leq |\varphi|$. In addition, $\nabla \varphi_\varepsilon = (\varphi / (\varphi^2 + \varepsilon^2)^{1/2}) \nabla \varphi$. Now assume that $p \in \langle 1, \infty \rangle$ and $b_p > 0$. Then

$$\begin{aligned} 0 < 2 b_p \int_\Omega c_\Omega \, d_\Gamma^{-p} \varphi_\varepsilon^p &\leq \int_\Omega (\operatorname{div}\chi_p) \varphi_\varepsilon^p \\ &= -2p \int_\Omega c_\Omega \, d_\Gamma^{-p+1} (\nabla d_\Gamma) \cdot (\nabla \varphi_\varepsilon) \varphi_\varepsilon^{p-1} \\ &\leq 2p \Big(\int_\Omega (c_\Omega \, d_\Gamma^{-p+1})^p |(\nabla d_\Gamma) \cdot (\nabla \varphi)|^p \psi^p \Big)^{1/p} \cdot \Big(\int_\Omega \varphi_\varepsilon^p \psi^{-q} \Big)^{1/q}, \end{aligned}$$

where q is the conjugate of p and ψ is a strictly positive function. The last step uses the Hölder inequality. Choosing $\psi = c_\Omega^{-1/q} d_\Gamma^{p-1}$,

$$0 < b_p \int_\Omega c_\Omega \, d_\Gamma^{-p} \varphi_\varepsilon^p \leq p \Big(\int_\Omega c_\Omega |(\nabla d_\Gamma) \cdot (\nabla \varphi)|^p \Big)^{1/p} \cdot \Big(\int_\Omega c_\Omega \, d_\Gamma^{-p} \varphi_\varepsilon^p \Big)^{1/q}.$$

Dividing by the last factor and raising the inequality to the power p,

$$\int_\Omega c_\Omega|\nabla\varphi|^p \ge \int_\Omega c_\Omega|(\nabla d_\Gamma)\cdot(\nabla\varphi)|^p \ge a_p^p \int_\Omega c_\Omega\, d_\Gamma^{-p}\varphi_\varepsilon^p$$

for all $\varphi \in C_c^1(\Omega)$. Then the Hardy inequality of the theorem follows in the limit $\varepsilon \to 0$ by dominated convergence.

The proof for $p = 1$ is similar but simpler. The Hölder inequality is not necessary. □

The existence of a weighted Hardy inequality of the form (1.1) in the situation where $\delta = \delta'$ and with $d_H < d - 1$ follows from [LV16, Theorem 4.2]. This paper also indicates a number of interesting directions to extend the current results.

Remark 2.3. The foregoing proof only uses some general features of the weight function c. The estimates (1.1) follow for any strictly positive differentiable c on $\langle 0, \infty\rangle$ with $c'(s)s/c(s) \ge \delta \wedge \delta'$. If one makes the replacement $c(s) \to c(s) = s^\delta(a + bs)^{\delta'-\delta}$ with $a, b > 0$, then $c'(s)s/c(s) = (a\delta + b\delta' s)/(a + bs) \ge \delta \wedge \delta'$ and the theorem remains valid. Moreover, the constant a_p in the Hardy inequality (1.1) is unchanged but now $c(s)/s^\delta \to a^{\delta'-\delta}$ as $s \to 0$ and $c(s)/s^{\delta'} \to b^{\delta'-\delta}$ as $s \to \infty$.

Remark 2.4. The condition $d - d_H + \delta \wedge \delta' - p > 0$ in Theorem 1.1 restricts the result to sets whose boundaries have small codimension. For example, if $p = 2$ and $\delta = 0 = \delta'$, then it requires $d_H < d - 2$. In fact, Davies' analysis of sectors in $\mathbf{R}^2$ with angle $\beta \in \langle \pi, 2\pi]$ (see [Dav95, Section 4]) shows that the optimal Hardy constant depends on the angle, that is, it is dependent on the more detailed geometry of the boundary. In particular, there is a critical angle $\beta^c \in \langle \pi, 2\pi\rangle$ such that the optimal constant in the Hardy inequality is given by $a_2 = 1/2$ if $\beta \in \langle \pi, \beta^c]$ but $a_2 < 1/2$ if $\beta \in \langle \beta^c, 2\pi]$ and the value decreases as β increases. The Hardy inequality (1.1) is, however, valid for these sectors if $p = 2$ and $\delta \wedge \delta' > 1$. The latter condition implies that $\delta > 1$ and this is sufficient to ensure that the corresponding diffusion process does not reach the boundary [LR16]. Moreover, if d_H is small, the result of Theorem 1.1 is useful since it allows one to deduce Rellich inequalities on $L_2(\Omega)$ for a large range of $\delta, \delta' \ge 0$ (see [Rob18] or Section 5).

The foregoing arguments may also be used to obtain Hardy inequalities on convex Ω but there are some significant differences. First, the distance d_Γ is a concave function, which is a help. Secondly, the set G of points in Ω at which d_Γ is differentiable can be small. This is a hindrance since d_Γ is differentiable at x if and only if x has a unique near point (see [BEL15, Section 2.2]). Nevertheless, one can obtain an analogue of Theorem 1.1 for convex Ω as a corollary of well-known results for the unweighted Hardy inequality.

Proposition 2.5. *Assume that Ω is convex. Again let $c_\Omega = c \circ d_\Gamma$ with $c(s) = s^\delta(1 + s)^{\delta'-\delta}$, where $\delta, \delta' \in \mathbf{R}$. If $p - 1 - |\delta| \vee |\delta'| > 0$, then*

$$\int_\Omega c_\Omega|\nabla\varphi|^p \ge a_p^p \int_\Omega c_\Omega\, d_\Gamma^{-p}|\varphi|^p$$

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for all $\varphi \in C_c^1(\Omega)$ *with* $a_p = (p - 1 - |\delta| \vee |\delta'|)/p$. *Moreover, if* $\delta \geq |\delta'|$, *then the constant* $((p - 1 - \delta)/p)^p$ *is optimal.*

PROOF. First, one has the well-known Hardy inequality

$$\|\nabla\varphi\|_p \geq (1 - 1/p)\|d_\Omega^{-1}\varphi\|_p$$

for all $\varphi \in C_c^1(\Omega)$ if $p > 1$ (see [BEL15, Theorem 3.3.4]). Therefore,

$$(1 - 1/p)\|c_\Omega^{1/p} d_\Omega^{-1}\varphi\|_p \leq \|\nabla(c_\Omega^{1/p}\varphi)\|_p.$$

Then, however,

$$\begin{aligned}\|\nabla(c_\Omega^{1/p}\varphi)\|_p &\leq \|c_\Omega^{1/p}(\nabla\varphi)\|_p + (1/p)\|c'_\Omega c_\Omega^{-1} c_\Omega^{1/p}\varphi\|_p \\ &\leq \|c_\Omega^{1/p}(\nabla\varphi)\|_p + ((|\delta| \vee |\delta'|)/p)\|d_\Omega^{-1} c_\Omega^{1/p}\varphi\|_p\end{aligned}$$

since $s|c'(s)|/c(s) \leq |\delta| \vee |\delta'|$ and $|\nabla d_\Omega| \leq 1$. Combining these estimates gives the Hardy inequality of the proposition.

The proof of the optimality statement will be indicated in Section 4 as part of the general discussion of sharp estimates. □

Recent results on the weighted Hardy inequality on convex sets with $\delta = \delta' \in \mathbf{R}$ can be found in [Avk15a, Avk15b] and references therein.

3. Rellich inequalities

In this section we establish the Rellich inequalities (1.2) of Theorem 1.2. Our proof is based on an extension of Theorem 4 in the paper of Davies and Hinz [DH98] (see [BEL15, Theorem 6.3.3]) from the Laplacian $\Delta = -\nabla^2$ to the weighted operator $H = -\mathrm{div}(c_\Omega\nabla)$.

PROPOSITION 3.1. *Let* Ω *be a general domain in* $\mathbf{R}^d$ *and fix* $p \in \langle 1, \infty\rangle$. *Define the closeable operator* $H = -\sum_{k=1}^d \partial_k c_\Omega \partial_k$ *on* $D(H) = C_c^\infty(\Omega)$. *If there is a* χ *in the domain of the* L_p*-closure* $\overline{H}$ *of* H *such that* $\chi > 0$, $\overline{H}\chi > 0$ *and* $\overline{H}\chi^{1+\gamma} \geq 0$ *for some* $\gamma > 0$, *then*

$$\int_\Omega |\overline{H}\chi||\varphi|^p \leq p^{2p}(p + \gamma(p-1))^{-p} \int_\Omega \chi^p |\overline{H}\chi|^{-p+1}|H\varphi|^p \tag{3.1}$$

for all $\varphi \in C_c^\infty(\Omega)$.

The proof of Proposition 3.1 is a direct repetition of the proof of [DH98, Theorem 4]. The latter proof only involves quadratic form arguments, for example, partial integration and Cauchy–Schwarz estimates, on the form associated with the Laplacian, and these are unchanged by replacement with the quadratic form of H. In fact, since the estimates are on $C_c^\infty(\Omega)$, it suffices that c_Ω is the operator of multiplication by a locally C_1-function. The proposition differs superficially from that of Davies–Hinz since we define the Laplacian as $\Delta = -\nabla^2$ instead of ∇^2. Similarly, we have introduced a minus sign in the definition of H. Moreover, the parameter δ in [DH98] is replaced by $1 + \gamma$ and this changes slightly the form of the constant in (3.1). For brevity we omit further details.

Proof of Theorem 1.2. Define χ on the open right half line by $\chi(s) = s^{-\alpha}(1+s)^{-\alpha'+\alpha}$ with $\alpha, \alpha' \geq 0$. Then set $\chi_\Omega = \chi \circ d_\Gamma$ and adopt the notation $\chi'_\Omega = \chi' \circ d_\Gamma$ etc. Our aim is to derive conditions on α and α' such that $H\chi_\Omega > 0$ with H (the closure of) the weighted operator of Theorem 1.2. In fact, one can obtain quite precise lower bounds on $H\chi_\Omega$. □

Lemma 3.2. *Let* $b_\alpha = (d - d_H + (\delta \wedge \delta'))(\alpha \wedge \alpha') - (\alpha \vee \alpha')(\alpha \vee \alpha' + 2)$.
It follows that $H\chi_\Omega \geq b_\alpha d_\Gamma^{-2} c_\Omega \chi_\Omega$. *Hence, if* $b_\alpha > 0$, *then* $H\chi_\Omega > 0$.

Proof. First, one has $\chi'(s) = -s^{-1}\chi(s)(\alpha + \alpha' s)(1+s)^{-1}$. Therefore,

$$-s^{-1}\chi(s)(\alpha \vee \alpha') \leq \chi'(s) \leq -s^{-1}\chi(s)(\alpha \wedge \alpha').$$

In addition,

$$\begin{aligned}\chi''(s) &= s^{-2}\chi(s)(1+s)^{-2}(\alpha(\alpha+1) + 2\alpha(\alpha'+1)s + \alpha'(\alpha'+1)s^2)\\ &\leq s^{-2}\chi(s)(\alpha \vee \alpha')(\alpha \vee \alpha' + 1).\end{aligned}$$

Secondly, one calculates that

$$H\chi_\Omega = -d_\Gamma^{-1} c_\Omega \chi'_\Omega (\nabla^2 d_\Gamma^2)/2 - (c'_\Omega \chi'_\Omega - d_\Gamma^{-1} c_\Omega \chi'_\Omega + c_\Omega \chi''_\Omega)|\nabla d_\Gamma|^2. \tag{3.2}$$

But $|\nabla d_\Gamma| = 1$ by the discussion at the beginning of Section 2 and $(\nabla^2 d_\Gamma^2)/2 \geq d - d_H$ by Corollary 2.2. Then we use the bounds on χ' and χ'' together with the lower bound $c'(s) \geq (\delta \wedge \delta')s^{-1}c(s)$ to estimate the four terms on the right-hand side of (3.2). The first two terms give positive contributions but the other terms are negative. One finds that

$$\begin{aligned}H\chi_\Omega &\geq ((d - d_H) + (\delta \wedge \delta'))(\alpha \wedge \alpha')(d_\Gamma^{-2} c_\Omega \chi_\Omega)\\ &\qquad - ((\alpha \vee \alpha') + (\alpha \vee \alpha')(\alpha \vee \alpha' + 1))(d_\Gamma^{-2} c_\Omega \chi_\Omega)\\ &= b_\alpha d_\Gamma^{-2} c_\Omega \chi_\Omega.\end{aligned}$$

Clearly, $H\chi_\Omega > 0$ if the δ, α etc are such that $b_\alpha > 0$. □

Now, assuming that α and α' are chosen to ensure that $b_\alpha > 0$, one can bound the product $\chi_\Omega^p |H\chi_\Omega|^{-p+1}$ occurring on the right-hand side of (3.1). Explicitly, one obtains

$$\chi_\Omega^p |H\chi_\Omega|^{-p+1} \leq b_\alpha^{-p+1} d_\Gamma^{-\sigma}(1 + d_\Gamma)^{-\tau}$$

with $\sigma = \alpha - (2-\delta)(p-1)$ and $\tau = (\alpha' - \alpha) + (\delta' - \delta)(p-1)$. Hence, if one chooses $\alpha = \alpha_p = (2-\delta)(p-1)$ and $\alpha' = \alpha'_p = (2-\delta')(p-1)$, one obtains the uniform bound

$$\chi_\Omega^p |H\chi_\Omega|^{-p+1} \leq b_{\alpha_p}^{-p+1}$$

as long as

$$b_{\alpha_p} = (d - d_H + (\delta \wedge \delta'))(\alpha_p \wedge \alpha'_p) - (\alpha_p \vee \alpha'_p)(\alpha_p \vee \alpha'_p + 2) > 0.$$

But this is a condition on p, δ and δ'.

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LEMMA 3.3. *If* $(d - d_H + p(\delta \wedge \delta') - 2p) \geq 2p|\delta - \delta'|(2 - \delta \vee \delta')^{-1}$, *then* $b_{\alpha_p} > 0$.

PROOF. Substituting the values of α_p and α'_p in the definition of b_α,

$$\begin{aligned} b_{\alpha_p} &= (d - d_H + (\delta \wedge \delta') - (\alpha_p \vee \alpha'_p + 2))(\alpha_p \wedge \alpha'_p) \\ &\qquad - |\alpha_p - \alpha'_p|((\alpha_p \vee \alpha'_p) + 2) \\ &\geq (p-1)((d - d_H + p(\delta \wedge \delta') - 2p)(2 - \delta \vee \delta') - 2p|\delta - \delta'|). \end{aligned}$$

Since $p > 1$, the statement of the lemma follows immediately. □

Note that the condition of the lemma is the condition posited in Theorem 1.2 for validity of the Rellich inequality.

The next lemma provides the last estimates necessary for the application of Proposition 3.1 to derive the Rellich inequality.

LEMMA 3.4. *Let* $\tilde{\chi}_\Omega = d_\Gamma^{-\alpha_p}(1 + d_\Gamma)^{-\alpha'_p + \alpha_p}$. *Assume that* $b_{\alpha_p} > 0$. *Then*

$$\tilde{\chi}_\Omega^p |H\tilde{\chi}_\Omega|^{-p+1} \leq b_{\alpha_p}^{-p+1} \quad \text{and} \quad H\tilde{\chi}_\Omega \geq b_{\alpha_p}(c_\Omega d_\Gamma^{-1})^p.$$

Moreover, $H\tilde{\chi}_\Omega^{1+\gamma} \geq 0$ *for all* $\gamma \in [0, \gamma_p]$, *where* $\gamma_p = b_{\alpha_p}/(\alpha_p \vee \alpha'_p)^2$.

PROOF. The first estimate follows from Lemma 3.2 and the choices of α_p and α'_p as discussed above. The second estimate follows from another application of Lemma 3.2 by noting that

$$\begin{aligned} H\tilde{\chi}_\Omega \geq b_{\alpha_p} d_\Gamma^{-2} c_\Omega \tilde{\chi}_\Omega &= b_{\alpha_p} d_\Gamma^{-2} d_\Gamma^{\delta} (1 + d_\Gamma)^{\delta' - \delta} d_\Gamma^{-\alpha_p} (1 + d_\Gamma)^{-(\alpha'_p - \alpha_p)} \\ &= b_{\alpha_p} d_\Gamma^{-2p} d_\Gamma^{\delta p} (1 + d_\Gamma)^{(\delta' - \delta)p} = b_{\alpha_p} (c_\Omega d_\Gamma^{-1})^p, \end{aligned}$$

where the second equality results from substituting the specific values of α_p and α'_p.

The last statement of the lemma follows by first noting that

$$\tilde{\chi}_\Omega^{1+\gamma} = d_\Gamma^{(1+\gamma)\alpha_p} (1 + d_\Gamma)^{(1+\gamma)(-\alpha'_p + \alpha_p)}.$$

Therefore, $H\tilde{\chi}_\Omega^{1+\gamma} \geq 0$ if $b_{(1+\gamma)\alpha_p} \geq 0$ by a third application of Lemma 3.2. But

$$b_{(1+\gamma)\alpha_p} = (1 + \gamma)(b_{\alpha_p} - \gamma(\alpha_p \vee \alpha'_p)^2)$$

by the definition of b_α. Therefore, $b_{(1+\gamma)\alpha_p} \geq 0$ whenever $0 \leq \gamma \leq \gamma_p$. □

At this point we have verified the conditions necessary for the application of Proposition 3.1 to H and $\tilde{\chi}_\Omega$ to obtain the Rellich inequalities of Theorem 1.2. We now evaluate (3.1) with the foregoing estimates. First, we observe that $b_{\alpha_p} > 0$ by Lemma 3.3 and the assumption of the theorem. Then it follows from the estimates of Lemma 3.4 that

$$\begin{aligned} b_{\alpha_p} \int_\Omega |c_\Omega\, d_\Gamma^{-2} \varphi|^p &\leq \int_\Omega |H\chi_\Omega||\varphi|^p \\ &\leq p^{2p}(p + \gamma_p(p-1))^{-p} \int_\Omega \chi_\Omega^p |H\chi|^{-p+1} |H\varphi|^p \\ &\leq p^{2p}(p + \gamma_p(p-1))^{-p} b_{\alpha_p}^{-p+1} \int_\Omega |H\varphi|^p. \end{aligned}$$

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Thus, by rearrangement one obtains the Rellich inequality (1.2) with

$$c_p = (p + \gamma_p(p-1))b_{\alpha_p}p^{-2}.$$

It follows from $b_{\alpha_p}, \gamma_p > 0$ that $c_p > 0$. We next argue that $c_p \le C_p$.

First,

$$b_\alpha = (d - d_H + (\delta \wedge \delta') - (\alpha \vee \alpha' + 2))(\alpha \wedge \alpha') - a_\alpha$$

with

$$\begin{aligned} a_\alpha &= (\alpha \vee \alpha')(\alpha \vee \alpha' + 2) - (\alpha \wedge \alpha')(\alpha \vee \alpha' + 2) \\ &= |\alpha - \alpha'|((\alpha \vee \alpha') + 2) \ge 0. \end{aligned}$$

Now set

$$\tilde{b}_\alpha = (d - d_H + (\delta \wedge \delta') - (\alpha \vee \alpha' + 2)).$$

Then

$$b_\alpha = (\alpha \wedge \alpha')\tilde{b}_\alpha - a_\alpha \le (\alpha \wedge \alpha')\tilde{b}_\alpha.$$

Hence, $b_{\alpha_p} \le (\alpha_p \wedge \alpha'_p)\tilde{b}_{\alpha_p}$ with equality if and only if $\alpha_p = \alpha'_p$ or, equivalently, $\delta = \delta'$. Moreover, $\gamma_p = b_{\alpha_p}/(\alpha_p \vee \alpha'_p)^2 \le \tilde{\gamma}_p$, where $\tilde{\gamma}_p = \tilde{b}_{\alpha_p}/(\alpha_p \vee \alpha'_p)$, with equality if and only if $\delta = \delta'$. Now

$$\begin{aligned} c_p &\le (p + \tilde{\gamma}_p(p-1))(\alpha_p \wedge \alpha'_p)\tilde{b}_{\alpha_p}p^{-2} \\ &\le ((\alpha_p \vee \alpha'_p)p + \tilde{b}_{\alpha_p}(p-1))\tilde{b}_{\alpha_p}p^{-2}. \end{aligned}$$

But

$$\begin{aligned} \tilde{b}_{\alpha_p} &= (d - d_H + (\delta \wedge \delta') - ((2 - \delta \wedge \delta')(p-1) + 2) \\ &= (d - d_H + p(\delta \wedge \delta') - 2p) \end{aligned}$$

and

$$\begin{aligned} (\alpha_p \vee \alpha'_p)p + \tilde{b}_{\alpha_p}(p-1) &= (2 - (\delta \wedge \delta'))p(p-1) + \tilde{b}_{\alpha_p}(p-1) \\ &= (p-1)(d - d_H). \end{aligned}$$

Combining these estimates, one has $c_p \le C_p$, where C_p is defined in Theorem 1.2.

We have avoided calculating c_p explicitly since the resulting expression is complicated and is not necessarily optimal. It is, however, straightforward to identify it from the value of b_{α_p} given prior to Lemma 3.4 and the definition of γ_p. Nevertheless, c_p does have some simple properties as a function of the degeneracy parameters δ and δ'.

Set $c_p = c_p(\delta, \delta')$ to denote the dependence on δ and δ'. Then c_p is a positive symmetric function and $\delta \in [0, 2\rangle \mapsto c_p(\delta, \delta)$ is strictly increasing. Moreover, if $c_p(\delta_0, 0) \ge 0$, then $\delta \in [0, \delta_0] \mapsto c_p(\delta, 0)$ is strictly decreasing. In particular,

$$c_p(0,0) \ge c_p(\delta, 0) \ge c_p(\delta_0, 0)$$

for all $\delta \in [0, \delta_0]$.

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These inequalities follow because

$$c_p(\delta,\delta) = (p-1)(d-d_H)(d-d_H+p\delta-2p)/p^2$$

and

$$c_p(\delta,0) = c_p(0,\delta) = (p-1)(d-d_H)((d-d_H)(1-\delta/2)-2p)(1-\delta/2)/p^2,$$

which are special cases of the general formula for c_p.

4. Optimal constants

In this section we consider the problem of deriving optimal constants in the Hardy and Rellich inequalities of Theorems 1.1 and 1.2. First, we discuss whether the constant a_p^p in Theorem 1.1 is the largest possible for the Hardy inequality. The maximal constant $\mu_p(\Omega)$ for which (1.1) is valid is given by $\mu_p(\Omega) = a_p(\Omega)^p$, where

$$a_p(\Omega) = \inf\{\|c_\Omega^{1/p}(\nabla\varphi)\|_p/\|c_\Omega^{1/p}d_\Gamma^{-1}\varphi\|_p : \varphi \in C_c^1(\Omega)\}. \tag{4.1}$$

Clearly, $a_p(\Omega) \geq a_p$ by Theorem 1.1. Therefore, optimality follows if $a_p(\Omega) \leq a_p$. Since c_Ω has a different asymptotic behaviour at the boundary Γ to that at infinity, this variational problem has two distinct elements, a local and a global. In the classical case $K = \{0\}$ a local estimate of the infimum in (4.1) can be made with a sequence of functions $\varphi_\alpha(x) = |x|^{-\alpha}\xi(|x|)$, where ξ has support in a small neighbourhood of the origin. The leading term gives a bound proportional to α^p if $\alpha < (d+\delta-p)/p$ (see, for example, [BEL15, Ch. I]). Then, by a suitable choice of localization functions ξ and a limiting argument, one concludes that $a_p(\Omega) \leq (d+\delta-p)/p$. The estimate at infinity is similar. One now chooses ξ with support in the complement of a large ball centred at the origin and by another approximation and limiting argument one obtains the upper bound $a_p(\Omega) \leq (d+\delta'-p)/p$. Then $a_p(\Omega) \leq a_p$ by taking the minimum of these bounds.

We begin the general discussion by considering comparable local estimates by the methods of Barbatis *et al.* [BFT04] as developed in Section 5 of Ward's thesis [War14]. The following theorem covers the cases with $\dim(K) < d$.

Theorem 4.1. *Adopt the assumptions of Theorem 1.1. Further, assume that* $\dim(K) < d$. *Then the optimal constant* $\mu_p(\Omega)$ *in* (1.1) *satisfies*

$$\mu_p(\Omega) \leq ((d-d_H+\delta-p)/p)^p.$$

Moreover, if $\delta \leq \delta'$, *then* $\mu_p(\Omega) = ((d-d_H+\delta-p)/p)^p$.

Proof. The theorem follows by the proof of [War14, Theorem 5.2.1] but with some modification to take into account the weighting factor c_Ω. We outline a variation of Ward's argument which is subsequently extended to give a local bound on the optimal constant in the Rellich inequality (1.2). First, we give the proof for the special case $\delta = \delta'$ or, equivalently, $c(s) = s^\delta$. Since the argument only involves functions with support in an arbitrarily small ball centred at a point of the boundary, the result can then be extended to the general weighting factor c_Ω.

The starting point of the proof is a modification of Ward's Lemma 5.1.1.

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Lemma 4.2. *Assume that* $c(s) = s^\delta$ *with* $\delta \geq 0$. *Then*

$$a_p(\Omega) \leq |(\beta + \delta - p)/p| + \|d_\Gamma^{1-\beta/p}(\nabla\varphi)\|_p / \|d_\Gamma^{-\beta/p}\varphi\|_p \tag{4.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, $\beta \geq 0$ *and* $p > 1$.

Proof. The proof follows that of Ward with φ in (4.1) replaced by $\psi\varphi$, where $\psi = d_\Gamma^{-(\beta+\delta-p)/p}$. Then one uses the Leibniz rule and the triangle inequality to deduce that

$$\begin{aligned}\|d_\Gamma^{\delta/p}(\nabla(\psi\varphi))\|_p &\leq \|d_\Gamma^{\delta/p}(\nabla\psi)\varphi\|_p + \|d_\Gamma^{\delta/p}\psi(\nabla\varphi)\|_p \\ &= |(\beta+\delta-p)/p|\,\|d_\Gamma^{-\beta/p}\varphi\|_p + \|d_\Gamma^{1-\beta/p}(\nabla\varphi)\|_p,\end{aligned}$$

where we have used the explicit form of ψ. Similarly, $\|d_\Gamma^{-1+\delta/p}\psi\varphi\|_p = \|d_\Gamma^{-\beta/p}\varphi\|_p$. The statement of the lemma follows immediately. □

The estimate for $\mu_p(\Omega)$ given in Theorem 4.1 now follows by Ward's reasoning in the proof of his Theorem 5.2.1. The idea is to construct a sequence of φ_n such that the numerator in the last term in (4.2) is bounded uniformly in n if $\beta = d - k$, with $k = \dim(\partial K) = \dim(\Gamma) = d_H$, but the denominator diverges as $n \to \infty$. This is particularly easy in the current context since we are assuming that $k = \dim(K) \leq d - 1$.

First, let $\mathbf{R}^d = \mathbf{R}^k \times \mathbf{R}^{d-k}$, where $\mathbf{R}^k$ is identified with the affine hull of K. Therefore, if one sets $x = (y, z) \in \Omega$ with $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{d-k}$, then $d_\Omega(x) = (d_K(y)^2 + |z|^2)^{1/2}$, where $d_K(y) = \inf_{y' \in K} |y - y'|$. Since $d_K(y) = 0$ if $y \in K$, it follows that $d_\Gamma(y, z) = |z|$ if $y \in K$. Secondly, define $\varphi \in C_c^\infty(\Omega)$ by setting $\varphi(y, z) = \eta(y)\chi(z)$, where $\eta \in C_c^\infty(K)$ and $\chi \in C_c^\infty(\mathbf{R}^{d-k}\backslash\{0\})$. Further, assume that χ is a radial function. Then, with $\beta = d - k$,

$$\begin{aligned}\int_\Omega d_\Gamma^{-\beta}|\varphi|^p &= \int_K dy|\eta(y)|^p \int_{\mathbf{R}^{d-k}} dz|z|^{-(d-k)}|\chi(z)|^p \\ &= a_1 \int_0^\infty dr\, r^{-1}|\chi(r)|^p.\end{aligned} \tag{4.3}$$

But

$$\begin{aligned}\int_\Omega d_\Gamma^{-\beta+p}|\nabla\varphi|^p &= \int_K dy \int_{\mathbf{R}^{d-k}} dz|z|^{-(d-k-p)}(|(\nabla\eta)(y)||\chi(z)| + |\eta(y)||(\nabla\chi)(z)|)^p \\ &\leq a_2 \int_0^\infty dr\, r^{p-1}|\chi(r)|^p + a_3 \int_0^\infty dr\, r^{p-1}|\chi'(r)|^p\end{aligned} \tag{4.4}$$

with $a_1, a_2, a_3 > 0$.

Next, consider the sequence of functions ξ_n defined on $\langle 0, \infty\rangle$ by $\xi_n(r) = 0$ if $r \leq n^{-1}$, $\xi_n(r) = \log rn/\log n$ if $n^{-1} \leq r \leq 1$ and $\xi_n = 1$ if $r \geq 1$. Then $0 \leq \xi_n \leq 1$ and the ξ_n converge monotonically upward to the identity function. Further, let ζ be a C^∞-function with $\zeta(r) = 1$ if $r \leq 1$, $\zeta(r) = 0$ if $r \geq 2$ and $0 \leq \zeta \leq 1$. Then set $\chi_n = \xi_n\zeta$.

It follows immediately that

$$\lim_{n\to\infty} \int_0^\infty dr\, r^{-1}|\chi_n(r)|^p = \infty.$$

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Moreover,

$$\int_0^\infty dr\, r^{p-1}|\chi_n(r)|^p \le \int_0^2 dr\, r^{p-1}|\xi_n(r)|^p \le \int_0^2 dr\, r^{p-1} \le 2^p p^{-1}$$

for all $n > 1$. But $\text{supp}\chi_n \subseteq [0,2]$, $\chi_n' = \xi_n'$ on $\langle 0,1]$ and $\chi_n' = \zeta'$ on $[1,2]$. Therefore,

$$\begin{aligned}\int_0^\infty dr\, r^{p-1}|\chi_n'(r)|^p &= \int_0^1 dr\, r^{p-1}|\xi_n'(r)|^p + \int_1^2 dr\, r^{p-1}|\zeta'(r)|^p \\ &= (\log n)^{-(p-1)} + a,\end{aligned}$$

where $a > 0$ is the contribution of the second integral. Since $p > 1$, the bound is uniform for all $n > 1$. Hence, if one sets $\varphi_n = \eta\chi_n$, one deduces from (4.3) and (4.4), with χ replaced by χ_n, that

$$\limsup_{n\to\infty} \int_\Omega d_\Gamma^{-\beta+p}|\nabla\varphi_n|^p \Big/ \int_\Omega d_\Gamma^{-\beta}|\varphi_n|^p = 0.$$

Therefore, replacing φ with φ_n in (4.2) and setting $\beta = d - k$,

$$a_p(\Omega) \le (d - k + \delta - p)/p = a_p.$$

This completes the proof of the upper bound for $c(s) = s^\delta$, that is, for $\delta = \delta'$.

Next, it follows by construction that

$$\text{supp}\varphi_n \subseteq \{(y,z) : y \in \text{supp}\,\eta, |z| \le 2\}.$$

The choice of the value 2 is, however, arbitrary and by rescaling the ξ_n it can be replaced by any $r > 0$ without materially affecting the argument. Then, since $|z|^\delta(1+r)^{-|\delta-\delta'|} \le c(z) \le |z|^\delta(1+r)^{|\delta-\delta'|}$ for $|z| < r$, the case of general c_Ω is reduced to the special case $\delta = \delta'$.

Finally, if $\delta \le \delta'$, it follows from Theorem 1.1 that $\mu_p(\Omega) \ge a_p^p$. Consequently, one must have equality. □

Remark 4.3. The local estimates of Theorem 4.1 remain valid in the case that $\dim(K) = d$ but $K \ne \mathbf{R}^d$. The proof is, however, rather different. It depends on the Ahlfors regularity of the boundary Γ and specifically on the estimates established in [LR16, Section 2]. The same argument also applies if Ω is convex and $\delta \ge |\delta'|$ (see Proposition 2.5).

Next, we investigate the derivation of the bounds $\mu_p(\Omega) \le ((d - d_H + \delta' - p)/p)^p$ in the setting of Theorem 4.1. These bounds require information on the global properties of K.

The dimension k of the convex set K, that is, the dimension of the affine hull A_K of K, is essentially a local concept. It carries little information about the global character of the set. For example, in two dimensions K could be a disc, an infinitely extended strip or a quadrant. But, viewed from afar, these sets would appear to have dimensions zero, one and two, respectively. This aspect of the sets is captured by the 'dimension at infinity' k_∞, which is defined by

$$k_\infty = \liminf_{r\to\infty}(\log|K\cap B_r|/\log r),$$

where $B_r = \{y \in \mathbf{R}^k : |y| < r\}$ and $|S|$ indicates the k-dimensional Lebesgue measure of the set S. The parameter k_∞ of the convex set is integer valued with $0 \leq k_\infty \leq k$. In the two-dimensional examples just cited it takes the values 0, 1 and 2, as expected. The equality $k_\infty = k$ of the global and local dimensions will be the key property in deriving the upper bounds on $\mu_p(\Omega)$. It is clearly valid if K is affine but this is not necessary.

Lemma 4.4. *Assume that $k_\infty = k$. Then*

$$\inf_{\eta\in C_c^\infty(K)} \int_K |\nabla\eta|^p \Big/ \int_K |\eta|^p = 0.$$

Proof. First, let ξ be a C^∞-function with $0 \leq \xi \leq 1$ such that $\operatorname{supp}\xi \subseteq K$ and $\xi(y) = 1$ if $d_K(y) \geq 1$, with d_K the Euclidean distance to the boundary ∂K. Secondly, let ζ_n be a sequence of C^∞-functions with $0 \leq \zeta_n \leq 1$, $\zeta_n(y)$ if $y \in B_r$ and $\zeta_n = 0$ if $y \in B^c_{r+1}$. We may assume that $\sup_n |\nabla\zeta_n| < \infty$. Now set $\eta_n = \zeta_n\xi$. Then $\eta_n \in C_c^\infty(K)$ and $\operatorname{supp}|\nabla\eta_n|$ has measure at most br^{k-1} for all $r \geq 1$ with $b > 0$ independent of r. But $\eta_n = 1$ on a set of measure cr^k with $c > 0$. Therefore,

$$\int_K |\nabla\eta_n|^p \Big/ \int_K |\eta_n|^p < ar^{-1}$$

with $a > 0$ independent of r. The lemma follows immediately. □

The following theorem establishes that $k_\infty = k$ is a sufficient condition for the expected global bounds.

Theorem 4.5. *Let K be a closed convex subset of $\mathbf{R}^d$ with $k = \dim(K) \in \{1, \ldots, d-1\}$ and with $k_\infty = k$. Then the optimal constant $\mu_p(\Omega)$ in the Hardy inequality (1.1) on $\Omega = \mathbf{R}^d\backslash K$ is given by $\mu_p(\Omega) = ((d-k+\delta\wedge\delta'-p)/p)^p$.*

Proof. First, $\mu_p(\Omega) \geq a_p^p$ with $a_p = (d-k+\delta\wedge\delta'-p)/p$ by Theorem 1.1. Therefore, it suffices to establish a matching upper bound. But the local estimates of Theorem 4.1 give the bound $\mu_p(\Omega) \leq ((d-k+\delta-p)/p)^p$. Thus, it remains to prove that $\mu_p(\Omega) \leq ((d-k+\delta'-p)/p)^p$, that is, to prove that $a_p(\Omega) \leq (d-k+\delta'-p)/p$.

Secondly, we again consider the decomposition $\mathbf{R}^d = \mathbf{R}^k \times \mathbf{R}^{d-k}$ with $K \subseteq \mathbf{R}^k$ and $\mathbf{R}^k = A_K$. Then, since $d_\Gamma(y,z) = |z|$ if $y \in K$, the weighted Hardy inequality (1.1) implies that

$$a_p(\Omega) \leq \frac{(\int_{\mathbf{R}^k} dy \int_{\mathbf{R}^{d-k}} dz\, c(|z|)|(\nabla\varphi)(y,z)|^p)^{1/p}}{(\int_{\mathbf{R}^k} dy \int_{\mathbf{R}^{d-k}} dz\, c(|z|)|z|^{-p}|\varphi(y,z)|^p)^{1/p}}$$

for all $\varphi \in C_c^1(\Omega)$ with $\operatorname{supp}\varphi \subseteq K \times \mathbf{R}^{d-k}$. Again let φ be a product $\varphi(y,z) = \eta(y)\chi(z)$ with $\eta \in C_c^\infty(K)$ but $\chi \in C_c^\infty(O_R)$, where $O_R = \{z \in \mathbf{R}^{d-k} : |z| > R\}$. Then, by the Leibniz

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rule and the triangle inequality,

$$a_p(\Omega) \le \frac{\big(\int_{O_R} dz\, c(|z|)|(\nabla\chi)(z)|^p \int_K dy|\eta(y)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)|z|^{-p}|\chi(z)|^p \int_K dy|\eta(y)|^p\big)^{1/p}} + \frac{\big(\int_{O_R} dz\, c(|z|)|\chi(z)|^p \int_K dy|(\nabla\eta)(y)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)|z|^{-p}|\chi(z)|^p \int_K dy|\eta(y)|^p\big)^{1/p}}.$$

Therefore,

$$a_p(\Omega) \le \frac{\big(\int_{O_R} dz\, c(|z|)|(\nabla\chi)(z)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)|z|^{-p}|\chi(z)|^p\big)^{1/p}} + \frac{\big(\int_{O_R} dz\, c(|z|)|\chi(z)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)|z|^{-p}|\chi(z)|^p\big)^{1/p}} \frac{\big(\int_K dy|(\nabla\eta)(y)|^p\big)^{1/p}}{\big(\int_K dy|\eta(y)|^p\big)^{1/p}}. \tag{4.5}$$

Then, taking the infimum over $\eta \in C_c^\infty(K)$, one deduces from Lemma 4.4 that

$$a_p(\Omega) \le \frac{\big(\int_{O_R} dz\, c(|z|)|(\nabla\chi)(z)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)|z|^{-p}|\chi(z)|^p\big)^{1/p}} \tag{4.6}$$

for all $\chi \in C_c^\infty(O_R)$ and all large R.

Finally, the infimum of the right-hand side of (4.6) over χ followed by the limit $R \to \infty$ gives $a_p(\Omega) \le (d - k + \delta' - p)/p$ by the global estimates for the Hardy inequality on $\mathbf{R}^{d-k}\backslash\{0\}$. The proof of the theorem now follows from this estimate combined with the observations in the first paragraph of the proof. □

Theorem 4.5 applies to the special case that K is an affine set, since the assumption $k_\infty = k$ is automatically fulfilled. The corresponding statement is an extension of a result of [SSW03]. Moreover, if K is a general closed convex set and A_K its affine hull, then the theorem identifies the constant a_p^p of Theorem 1.1 as the optimal constant $\mu_p(\mathbf{R}^d\backslash A_K)$ of the Hardy inequality (1.1) on $L_p(\mathbf{R}^d\backslash A_K)$. Therefore, one has the general conclusion that $\mu_p(\mathbf{R}^d\backslash A_K) \le \mu_p(\mathbf{R}^d\backslash K)$ for convex sets with $\dim(K) = k \in \{1, \ldots, d-1\}$. Moreover, $\mu_p(\mathbf{R}^d\backslash A_K) = \mu_p(\mathbf{R}^d\backslash K)$ if $\delta \le \delta'$ because the proof only requires a local estimate.

Next, we address the question of calculating the optimal constant in the Rellich inequality (1.2), that is, the value of $\nu_p(\Omega) = b_p(\Omega)^p$, where

$$b_p(\Omega) = \inf\{\|H\varphi\|_p/\|c_\Omega d_\Gamma^{-2}\varphi\|_p : \varphi \in C_c^2(\Omega)\}. \tag{4.7}$$

Theorem 1.2 gives the lower bound $b_p(\Omega) \ge c_p$ but this is rather complicated and not likely to be an efficient bound in general. Therefore, we consider the special case $\delta = \delta'$ with weighting factor d_Γ^δ. Then Theorem 1.2 gives the simpler bound $b_p(\Omega) \ge C_p$ with $C_p = (p-1)(d-d_H)(d-d_H+p\delta-2p)p^{-2}$. Now we establish that C_p is the optimal constant for $\delta = \delta'$ and $\dim K < d$. First, we consider the degenerate case $K = \{0\}$.

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Proposition 4.6. *If $K = \{0\}$ and $\delta = \delta' \in [0, 2\rangle$, then the optimal constant in the Rellich inequality (1.2) is given by*

$$\nu_p(\Omega) = C_p^p = ((p-1)\, d(d + p\delta - 2p)p^{-2})^p$$

for all $p > 1$ for which $d + p\delta - 2p > 0$.

Proof. It follows from Theorem 1.2, with $\delta = \delta'$, that the lower bound $\nu_p(\Omega) \geq C_p^p$ is valid. Therefore, it suffices to establish a matching upper bound. This is well known if $\delta = 0$ but the proof is almost identical for $\delta \neq 0$. First, since $K = \{0\}$, one has $d_\Gamma(x) = |x|$. Then, as $\delta = \delta'$, one can deduce an upper bound from (4.7) by a local estimate (see, for example, [BEL15, Corollary 6.3.5] for the case of the Laplacian). This is achieved by the elementary procedure used to estimate the upper bound on the Rellich constant in the one-dimensional case. One estimates with radial functions $\varphi(x) = |x|^{-\alpha}\chi(|x|)$, where $\alpha > 0$ and χ is a C^2-function with compact support near the origin. The integrability of $|H\varphi|^p$ at the origin imposes the restriction $d + p\delta - 2p > 0$. Therefore, one chooses $\alpha = (d + p\delta - 2p + \varepsilon)/p$, with $\varepsilon > 0$, and estimates as in the one-dimensional case (see [BEL15, Section 1.2]). This leads to the upper bound $\nu_p(\Omega) \leq C_p^p$. We omit the details. □

Remark 4.7. If $K = \{0\}$ and $\delta \neq \delta'$, then one can establish the upper bound $\nu_p(\Omega) \leq ((p-1)\, d(d + p(\delta \wedge \delta') - 2p)p^{-2})^p$. This follows by a local estimate, which gives the bound $((p-1)\, d(d + p\delta - 2p)p^{-2})^p$, followed by a similar estimate at 'infinity' giving the bound $((p-1)\, d(d + p\delta' - 2p)p^{-2})^p$. Then one takes the minimum of the two bounds. Unfortunately Theorem 1.2 only gives a matching lower bound if $\delta = \delta'$. If, for example, $\delta' = 0$, then the upper bound is equal to $c_p(0, 0)^p = ((p-1)\, d(d - 2p)p^{-2})^p$, where we have used the notation introduced at the end of Section 3. But Theorem 1.2 gives the lower bound $c_p(\delta, 0)^p$ under the assumption that $c_p(\delta, 0) > 0$. It follows, however, that $c_p(\delta, 0) < c_p(0, 0)$ if $\delta > 0$ by the discussion in Section 3.

Now we establish a similar conclusion for $\dim(K) \in \{1, \ldots, d-1\}$. The following result corresponds to the Rellich analogue of Theorems 4.1 and 4.5.

Theorem 4.8. *Let K be a closed convex subset of $\mathbf{R}^d$ with $k = \dim(K) \in \{1, \ldots, d-1\}$. Then the optimal constant in the Rellich inequality (1.2) satisfies the upper bound*

$$\nu_p(\Omega) \leq ((p-1)(d-k)(d-k+p\delta-2p)p^{-2})^p.$$

If, in addition, $k_\infty = k$, then

$$\nu_p(\Omega) \leq ((p-1)(d-k)(d-k+p(\delta \wedge \delta')-2p)p^{-2})^p$$

and, for $\delta = \delta'$, one has equality.

Proof. The proof follows the earlier two-step process of obtaining a local bound, dependent on δ, followed by a global bound, dependent on δ'. The local bound is independent of the assumption $k_\infty = k$.

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Step 1. The first statement of the theorem is established by a generalization of the local estimates used to prove Theorem 4.1. Since all the estimates in this first step are local, we again assume initially that $c(s) = s^\delta$.

Following the earlier proof, we choose coordinates $x = (y, z) \in \Omega$ with $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{d-k}$, where $\mathbf{R}^k$ is identified with the affine hull of K. Then $d_\Gamma(y, z) = |z|$ if $y \in K$. Again we define $\varphi \in C_c^\infty(\Omega)$ by setting $\varphi(y, z) = \eta(y)\chi(z)$, where $\eta \in C_c^\infty(K)$ and $\chi \in C_c^\infty(\mathbf{R}^{d-k}\backslash\{0\})$ is a radial function. Next, for $\alpha \geq 0$, we set $\varphi_\alpha = d_\Gamma^{-\alpha}\varphi = \eta\chi_\alpha$, where $\chi_\alpha(z) = |z|^{-\alpha}\chi(z)$. Thus, $\varphi_\alpha = d_0^{-\alpha}\varphi$, where d_0 is the operator of multiplication by $|z|$. Then

$$H\varphi_\alpha = (Hd_0^{-\alpha})\varphi + d_0^{-\alpha}(H\varphi) + 2d_0^\delta(\nabla d_0^{-\alpha}) \cdot (\nabla\varphi).$$

Therefore, one calculates that

$$|H\varphi_\alpha| \leq \alpha(d - k + \delta - \alpha - 2)\, d_0^{-\alpha-2+\delta}|\varphi| + R_\alpha$$

if $d - k + \delta - 2 > \alpha$, where

$$R_\alpha = d_0^{-\alpha}|H\varphi| + 2\alpha d_0^{-\alpha-1+\delta}|\nabla\varphi|.$$

Hence, it follows as in the proof of Lemma 4.2 that $b_p(\Omega)$, defined by (4.7), satisfies

$$b_p(\Omega) \leq \alpha(d - k + \delta - \alpha - 2) + \|R_\alpha\|_p / \|d_0^{(\delta-2)}\varphi_\alpha\|_p. \tag{4.8}$$

Now we choose $\alpha = (d - k + p\delta - 2p)/p$ and assume that $\alpha > 0$. Then the constant in the first term on the right is $((p - 1)(d - k)(d - k + p\delta - 2p)p^{-2})^p$. So, it remains to prove that the second term, with the specific choice of α, can be made insignificant by a suitable choice of a sequence of χ. First,

$$\begin{aligned} \int_\Omega d_0^{p(\delta-2)}|\varphi_\alpha|^p &= \int_K dy|\eta(y)|^p \int_{\mathbf{R}^{d-k}} dz|z|^{-p(\alpha-\delta+2)}|\chi(z)|^p \\ &= a_1 \int_0^\infty dr\, r^{-1}|\chi(r)|^p \end{aligned} \tag{4.9}$$

with $a_1 > 0$. Secondly,

$$\begin{aligned} |R_\alpha|^p &\leq a(d_0^{-p\alpha}|H\varphi|^p + d_0^{-p(\alpha-\delta+1)}|\nabla\varphi|^p) \\ &\leq a'(d_0^{-p(\alpha-\delta)}|\Delta\chi|^p|\eta|^p + d_0^{-p(\alpha-\delta+1)}(|\nabla\chi|^p|\eta|^p + |\chi|^p|\nabla\eta|^p)) \end{aligned}$$

with $a, a' > 0$. Therefore, one obtains a bound

$$\begin{aligned} \int_\Omega |R_\alpha|^p \leq a_2 \int_0^\infty dr\, r^{p-1}|\chi(r)|^p + a_3 \int_0^\infty dr\, r^{p-1}|\chi'(r)|^p \\ + a_4 \int_0^\infty dr\, r^{2p-1}|\chi''(r)|^p \end{aligned} \tag{4.10}$$

with $a_2, a_3, a_4 > 0$. This is very similar to the bounds occurring in the proof of Theorem 4.1 with the exception of the last term, which depends on χ''. If this term were absent, one could then replace χ by the sequence of functions χ_n used in the proof

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of the earlier proposition to complete the argument that $b_p(\Omega) \leq (p-1)(d-k)(d-k+p\delta-2p)p^{-2}$. But the extra term complicates things. In fact, the χ_n used earlier are not even twice differentiable. Therefore, it is necessary to make a more sophisticated choice. We now use an argument given in [RS10, Section 4].

Let χ_n be the sequence of functions on $\langle 0,\infty\rangle$ used in the proof of Theorem 4.1. The derivatives χ_n' are discontinuous at n^{-1} and at 1. The functions $\xi_n = \chi_n^2$ have similar characteristics to the χ_n except that their derivatives ξ_n' are only discontinuous at 1. Therefore, we now consider the ξ_n and modify the derivative ξ_n' by the addition of a linear function to remove the discontinuity. The modifications η_n of the derivatives are defined by $\eta_n(s) = 0$ if $s \leq n^{-1}$ or $s \geq 1$ and

$$\eta_n(s) = \xi_n'(s) - \xi_n'(1)(s-n^{-1})/(1-n^{-1})$$

if $s \in [n^{-1}, 1]$. Now η_n is continuous and we set $\zeta_n(s) = \int_0^s \eta_n$ for $s \leq 1$ and $\zeta_n(s) = \zeta_n(1)$ if $s \geq 1$. The resulting function ζ_n is twice differentiable. Finally, setting $\rho_n = \zeta_n/\zeta_n(1)$, one verifies that $0 \leq \rho_n \leq 1$, $\rho_n(s) = 0$ if $s \leq n^{-1}$ and $\rho_n(s) = 1$ if $s \geq 1$. Moreover, $\lim_{n\to\infty} \rho_n(s) = 1$ for all $s > 0$. Finally, set $\sigma_n = \rho_n \zeta$, where ζ is the cutoff function used in the proof of Theorem 4.1. Now we consider the estimates (4.9) and (4.10) with χ replaced by the sequence σ_n.

First, since $\sigma_n \to 1$ on $\langle 0, 1]$ as $n \to \infty$, it follows that $\int_0^\infty dr\, r^{-1}|\sigma_n(r)|^p \to \infty$ as $n \to \infty$ but $\int_0^\infty dr\, r^{p-1}|\sigma_n(r)|^p$ is uniformly bounded in n. Moreover, $\sigma_n' = \zeta_n(1)^{-1}\eta_n\zeta + \zeta'$ and it follows by the earlier calculation that $\int_0^\infty dr\, r^{p-1}|\sigma_n'(r)|^p$ is also uniformly bounded in n. Therefore, it remains to consider the term in (4.10) dependent on σ_n''. But $\sigma_n'' = \zeta_n(1)^{-1}(\eta_n'\zeta + \zeta') + \zeta''$. Therefore, it follows from the definition of η_n and the cutoff ζ that

$$\int_0^\infty dr\, r^{2p-1}|\sigma_n''|^p \leq a + b\int_{n^{-1}}^1 dr\, r^{2p-1}|\xi_n''(r)|^p$$

with $a, b > 0$ independent of n. Now, on $[n^{-1}, 1]$,

$$\xi_n''(r) = 2(\chi_n'(r))^2 + 2\chi_n(r)\chi_n''(r) = 2r^{-2}(1-\log rn)/(\log n)^2.$$

Therefore,

$$\int_{n^{-1}}^1 dr\, r^{2p-1}|\xi_n''(r)|^p = 2^p(\log n)^{-2p}\int_{n^{-1}}^1 dr\, r^{-1}|1-\log rn|^p \leq 2^{p-1}(\log n)^{-(p-1)}$$

and this gives a bound uniform for $n > 1$.

One now deduces that if φ_α in the bound (4.8) is replaced by $\varphi_{\alpha,n} = d_0^{-\alpha}\eta\sigma_n$, then in the limit $n \to \infty$ the second term tends to zero since the numerator is bounded uniformly for $n > 1$ and the denominator converges to infinity. Therefore, one concludes that

$$b_p(\Omega) \leq (p-1)(d-k)(d-k+p\delta-2p)p^{-2},$$

that is, one obtains the first bound of the theorem. This was, however, obtained with the assumption $c(s) = s^\delta$. But again by rescaling one can arrange that the σ_n are supported in a small interval $[0, r]$ and this allows one to reduce the general case to the special case. There is one extra small complication which did not occur in the Hardy case and that arises since the weighting factor c_Ω is positioned centrally in the operator H and is not a direct weighting of the measure. But this causes no difficulty. For example, if φ has support within distance r of the boundary, then

$$\begin{aligned}|(Hd_0^{-\alpha})\varphi| &\leq c_\Omega|\Delta d_0^{-\alpha}||\varphi| + |c'_\Omega||(\nabla d_0)\cdot(\nabla d_0^{-\alpha})||\varphi| \\ &\leq (d_0^{\delta}|\Delta d_0^{-\alpha}| + d_0^{\delta-1}|\nabla d_0^{-\alpha}|)|\varphi|(1 + r^{|\delta-\delta'|}).\end{aligned}$$

Making these modifications, one obtains the first bound of the theorem modulo an additional factor $(1 + r^{|\delta-\delta'|})$ but, since this is valid uniformly for all small $r > 0$, one can then take the limit $r \to 0$.

Step 2. Next, we assume that $k_\infty = k$ and establish the second bound in Theorem 4.8. The proof is similar to that of Theorem 4.5.

We continue to use the factorization $\mathbf{R}^d = \mathbf{R}^k \times \mathbf{R}^{d-k}$ and to set $x = (y, z) \in \Omega$ with $y \in \mathbf{R}^k$ and $z \in \mathbf{R}^{d-k}$. Then $d_\Gamma(y, z) = |z|$ if $y \in K$ and the Rellich inequality (1.2) on $L_p(\Omega)$ takes the form

$$\int_K dy \int_{\mathbf{R}^{d-k}} dz |(H\varphi)(y, z)|^p \geq c_p^p \int_K dy \int_{\mathbf{R}^{d-k}} dz\, c(|z|)^p |z|^{-2p} |\varphi(y, z)|^p$$

for all $\varphi \in C_c^2(K \times \mathbf{R}^{d-k})$. Therefore,

$$b_p(\Omega) \leq \frac{\big(\int_K dy \int_{\mathbf{R}^{d-k}} dz |(H\varphi)(y, z)|^p\big)^{1/p}}{\big(\int_K dy \int_{\mathbf{R}^{d-k}} dz\, c(|z|)^p |z|^{-2p} |\varphi(y, z)|^p\big)^{1/p}}$$

for all $\varphi \in C_c^2(K \times \mathbf{R}^{d-k})$.

Again we set $\varphi = \eta\chi$ with $\chi \in C_c^\infty(O_R)$, where $O_R = \{z \in \mathbf{R}^{d-k} : |z| > R\}$, and $\eta \in C_c^\infty(K)$. But the action of H on the product $\chi\eta$ takes the Grushin form

$$\begin{aligned}(H\varphi)(y, z) &= -\sum_{j=1}^{k} c(|z|)\chi(z)(\partial_j^2\eta)(y) - \sum_{j=k+1}^{d} (\partial_j c(|z|)\partial_j\chi)(z)\eta(y) \\ &= c(|z|)\chi(z)(\Delta\eta)(y) + (H\chi)(z)\eta(y),\end{aligned}$$

where the second line is a slight abuse of notation. This identity replaces the Leibniz rule used in the proof of Theorem 4.5.

Then, arguing as in the former proof, one obtains the estimates

$$\begin{aligned}b_p(\Omega) &\leq \frac{\big(\int_{O_R} dz |(H\chi)(z)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)^p |z|^{-2p} |\chi(z)|^p\big)^{1/p}} \\ &\quad + \frac{\big(\int_{O_R} dz\, c(|z|)^p |\chi(z)|^p\big)^{1/p}}{\big(\int_{O_R} dz\, c(|z|)^p |z|^{-2p} |\chi(z)|^p\big)^{1/p}} \frac{\big(\int_K dy |(\Delta\eta)(y)|^p\big)^{1/p}}{\big(\int_K dy |\eta(y)|^p\big)^{1/p}}\end{aligned}$$

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as a replacement for (4.5). But, since $k_\infty = k$, the infimum over η of the second term on the right-hand side is zero. This is no longer a consequence of Lemma 4.4 but it follows by identical reasoning. Hence,

$$b_p(\Omega) \leq \|H\chi\|_p / \|cd^{-2}\chi\|_p$$

for all $\chi \in C_c^\infty(O_R)$. Thus, the problem of estimating $\nu_p(\Omega)$ is reduced to a 'large-distance' estimate on the Rellich constant $\nu_p(\mathbf{R}^{d-k}\backslash\{0\})$. This follows from the standard argument sketched in the proof of Proposition 4.6. One obtains the bound

$$\nu_p(\Omega) \leq ((p-1)(d-k)(d-k+p\delta'-2p)p^{-2})^p.$$

The second statement of the theorem then follows by minimizing this bound and the local bound obtained in Step 1 of the proof.

The proof of Theorem 4.8 is completed by noting that if $\delta = \delta'$, the upper bound on $\nu_p(\Omega)$ coincides with the lower bound given by Theorem 1.2. Therefore, one has equality between $\nu_p(\Omega)$ and the bound. □

5. Conclusion

We conclude with three loosely related remarks on the weighted Rellich inequalities.

First, the inequality (1.2) for the 'weighted operator' $H = -\mathrm{div}(c_\Omega \nabla)$ was established by an extension of a result of Davies–Hinz [DH98, Theorem 4] from the Laplacian to the operator H. One can, however, also derive a 'weighted measure' inequality similar to (1.2) by applying the Davies–Hinz theorem to the Laplacian but with the Lebesgue measure weighted by c_Ω^p. Specifically, one calculates as in Section 3 setting $H = \Delta$ in (3.1) but choosing $\alpha_p = (2-\delta)(p-1) - \delta$ and $\alpha'_p = (2-\delta')(p-1) - \delta$. This gives a weighted measure Rellich inequality for a slightly different range of p. One deduces, for example, that if $\delta = \delta' \in [0, 2\rangle$, then, for $p \in \langle 2(2-\delta)^{-1}, (d-d_H)(2-\delta)^{-1}\rangle$, there is a $B_p > 0$ such that

$$\int_\Omega |d_\Gamma^\delta \Delta\varphi|^p \geq B_p \int_\Omega |d_\Gamma^{-2+\delta}\varphi|^p \tag{5.1}$$

for all $\varphi \in C_c^2(\Omega)$ (see [DH98, Section 4]). Note that the weighted operator inequality (1.2), with $\delta = \delta' \in [0, 2\rangle$, is valid for all $p \in \langle 1, (d-d_H)(2-\delta)^{-1}\rangle$. It is not clear whether (5.1) extends to all p close to one except in the case $K = \{0\}$. The latter case is thoroughly understood [MSS15].

Secondly, Theorem 1.2 establishes that the Rellich inequality is valid with a constant $c_p \leq C_p$ with equality if $\delta = \delta' \in [0, 2\rangle$. But the arguments of [Rob18] establish a more general result on $L_2(\Omega)$.

Proposition 5.1. *Adopt the assumptions of Theorem 1.2 but with $p = 2$ and $\delta, \delta' \geq 0$ satisfying $\delta + \delta' < 4$. It follows that the Rellich inequality (1.2) is valid with a constant $c_2 = C_2 = (d-d_H)(d-d_H+2(\delta \wedge \delta')-4)/4$ whenever $C_2 > 0$.*

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PROOF. The proposition is essentially a corollary of [Rob18, Theorem 1.2].

First, the Hardy inequality (1.1) of Theorem 1.1 on $L_2(\Omega)$ can be expressed as

$$\int_\Omega c_\Omega|\nabla\varphi|^2 \ge \int_\Omega |\eta\varphi|^2$$

with $\eta = a_2 c_\Omega^{1/2} d_\Gamma^{-1}$, where $a_2 = (d - d_H + (\delta \wedge \delta') - 2)/2$ and $\delta, \delta' \ge 0$. Secondly,

$$c_\Omega|\nabla\eta|^2 = a_2^2 c_\Omega^2 d_\Gamma^{-4}|1 - c'_\Omega d_\Gamma/2c_\Omega| \le (\nu/a_2^2)\eta^4,$$

where $\nu = \sup\{|1 - t/2|^2 : \delta \wedge \delta' \le t \le \delta \vee \delta'\}$. In particular, $\nu = (1 - (\delta \wedge \delta')/2)^2$ if $\delta + \delta' < 4$. Robinson [Rob18, Theorem 1.2] asserts, however, that if $\nu/a_2^2 < 1$, then the Rellich inequality (1.2) is satisfied with constant $\nu_2(\Omega) = a_2^4(1 - \nu/a_2^2)^2 = (a_2^2 - \nu)^2$. But the condition $\nu < a_2^2$ is equivalent to $d - d_H + 2(\delta \wedge \delta') - 4 > 0$ or to $C_2 > 0$. Then one calculates straightforwardly that $\nu_2(\Omega) = C_2^2$. □

In this proof the convexity of K is necessary for the existence of the Hardy inequality (1.1) but the remaining arguments are independent of this assumption. Moreover, the Rellich inequality (1.2) extends to a much larger class $\varphi \in D(\overline{H})$ by our final remark.

The L_2-Rellich inequalities established in [Rob18] are much stronger than the corresponding L_2-statement of Theorem 1.2. If $p = 2$, the Hardy inequality (1.2) gives a lower bound on the quadratic form $h(\varphi) = \int_\Omega c_\Omega|\nabla\varphi|^2$ for all $\varphi \in C_c^1(\Omega)$ and this bound extends to the closure $\overline{h}$. The latter is, however, a local Dirichlet form and it determines in a canonical manner a submarkovian operator H_F, the Friedrichs' extension of $H = -\mathrm{div}(c_\Omega\nabla)$ defined on $C_c^2(\Omega)$. Therefore, the L_2-Hardy inequality of Theorem 1.1 can be rephrased as

$$H_F \ge a_2^2 c_\Omega d_\Omega^{-2},$$

where the inequality is in the sense of ordering of positive self-adjoint operators. Theorem 1.2 of [Rob18] then establishes the Rellich inequality in the operator form

$$H_F^2 \ge c_2^2 c_\Omega^2 d_\Omega^{-4}.$$

In comparison, the Rellich inequality (1.2), after closure, gives the operator statement

$$H^*\overline{H} \ge c_2^2 c_\Omega^2 d_\Omega^{-4}.$$

Since $H_F \supseteq \overline{H}$, it follows that $(H_F)^2 \le H^*\overline{H}$ with equality if and only if H is essentially self-adjoint, that is, if and only if $H^* = \overline{H} = H_F$. It would be interesting to have a better understanding of the relationships between self-adjointness and validity of the Rellich inequality.

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